UCLA

COMPUTATIONAL AND APPLIED MATHEMATICS

Far Field Behavior of Slightly Compressible Flows

Lisette G. De Pillis

June 1993

CAM Report 93-15

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

FAR FIELD BEHAVIOR OF SLIGHTLY COMPRESSIBLE FLOWS

LISETTE G. DE PILLIS *

Abstract. We study the two dimensional slightly compressible Navier-Stokes equations, a standard mathematical model of low Mach number fluid flow. We analyze the far-field behavior of the velocity and pressure solutions of the equations.

The mathematical analysis is simplified by making use of a technique which allows us to expand asymptotically the slightly compressible solutions into a series of solutions of equations that model incompressible flow. This technique is valid because of the two time scales involved in the slightly compressible solutions, and because of the applicability of the bounded derivative principle.

The analysis of the behavior of the solutions in the far-field reveals that at a large enough distance from the source of the flow, the nature of the slightly compressible flow changes to that of flow which can be modeled by the wave equation. This leads to improved applications in computation.

Keywords: Navier-Stokes Equation, Compressible Flow, Incompressible Flow.

1. Introduction. In this work, we study the two dimensional slightly compressible Navier-Stokes equations, which are a standard mathematical model of low Mach number fluid flow. The Navier-Stokes equations are central to the modeling of ocean currents, weather patterns, jet engine noise, waves generated by a body moving under water, and many other physical phenomena.

Of interest to us in these phenomena are the different time scales which are involved. In nature, for example, when a weather system passes through an area, there are three time scales present. In a storm system, the actual movement of the entire system over a region of the earth can be measured in days (a slow time scale), whereas the gravity and sound waves generated by the storm must be measured in minutes and seconds, respectively (fast time scales).

A similar kind of problem involving different time scales is present in the slightly compressible Navier-Stokes equations. These equations are important in the modeling of sound propagation. In this case, there are two time scales involved. The convection scale and sound speed are on the slow and fast time scales, respectively. Whether the fast time scale waves are excited depends on how the problem is initialized. This will be discussed later in this paper.

Now consider the Cauchy problem, periodic in space, for the simplified, slightly compressible Navier-Stokes equations,

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u} + F \tag{1.1}$$

$$\epsilon^2\{p_t + (\mathbf{u}\cdot\nabla)p\} + \nabla\cdot\mathbf{u} = g, \tag{1.2}$$

with initial data

$$\mathbf{u}(x,y;0) = \mathbf{u}_0(x,y); \quad p(x,y;0) = p_0(x,y). \tag{1.3}$$

* Dept. of Mathematics, UCLA, Los Angeles, CA 90024, USA. This work was supported in part by Navy grant ONR-N00014-90-J-1382 and by NSF grant NSF-DMS90-61311.

Here, $\nu > 0$, $\epsilon > 0$, $\mathbf{u}(x,y;t) = (u(x,y;t), v(x,y;t))$ is the velocity and $p(x,y;t)$ is the pressure. Then $\nabla \cdot \mathbf{u} = u_x + v_y$ and $\nabla \times \mathbf{u} = v_x - u_y$ represent the dilatation and the vorticity, respectively. The inhomogeneous terms $F(x,y;t)$ and $g(x,y;t)$ are assumed to be C^∞-smooth. In addition, we require that the forcing function $F(x,y;t)$ decay exponentially fast in space, to ensure that the presence of forcing have no impact on the far-field decay behavior of the velocity or the pressure.

The solution $(\mathbf{u}, p)$ can be separated into a slow part and a fast part [11]. The slow part can be expanded in an asymptotic series so that the first term in the expansion is the solution of the incompressible Navier-Stokes equations. The second term in the expansion can be considered a linearized incompressible correction. This expansion can continue indefinitely. The remainder term of the asymptotic expansion will have essentially no vorticity, and will contain the fast part of the solution [11]. We note that with proper initialization, the remainder term will behave in the same way as the first terms in the expansion of the slow part. This is due in great part to the applicability of the bounded derivative principle [8, 9], which can be implemented to ensure that, within a finite time interval, the waves on the fast time scale are not excited. Details will be discussed.

One particular application of the slightly compressible Navier-Stokes equations is to model the pressure and velocity fields generated by the movement of a body submerged in a fluid (as in the case of a submarine in the ocean). Here, there will be at least two time scales involved. The movement of the fluid around the body will generate both convection and acoustic waves on the convection scale (slow part), and fast time scale waves on the sound speed scale. If the problem is not properly initialized, the fast time scale waves will be excited, but will quickly leave the domain of observation. With proper initialization, (*i.e.*, when one ensures that at least two time derivatives of the initial values are bounded), the fast time scale waves will not be excited. (There is also the possibility of a third time scale being introduced through the vibration of the body. The frequency of this time scale depends on the properties of the body.)

There are two possible approaches to solving the slightly compressible Navier-Stokes equations numerically. Assuming proper initialization, the first approach is to calculate the solution of the incompressible Navier-Stokes equations, as well as the solution of the linearized incompressible correction equation (*i.e.*, calculate the first two terms in the asymptotic expansion of the slow part of the solution). The sum of these two solutions give a good approximation to the solution of the slightly compressible Navier-Stokes equations. The second approach, which is the one we take when carrying out our numerical experiments, is simply to do a direct calculation of the slightly compressible Navier-Stokes equations. (It still is important in this case to ensure that the first couple of time derivatives of the initial data are bounded, so as not to excite the fast time scale waves.) It is reasonable to carry out a straightforward calculation of the equations, as long as the difference in magnitude of the time scales is not too large: in our case, the time scales differ only by a factor of about 10.

An issue that we consider in this work is the fact that the velocity field decays in space much more rapidly than does the pressure field. This means that at a certain distance from the body (or patch of vorticity), the velocity field will no longer be significant in magnitude, whereas the pressure field still will be relatively large. At that distance, the solution of the pressure equation actually satisfies the wave equation. This will be shown.

Our goal in this work is to discover how quickly the velocity field and pressure field modeled by the slightly compressible Navier-Stokes equations decay in space. In our analysis of decay rates, we take advantage of the fact that we can asymptotically expand out the slow part of the solution. The decay rate analysis is carried out on the first terms of the expansion of the slow part. (Again, assuming proper initialization, the fast part of the solution will not be excited within a finite time interval, so it is sufficient to focus our analysis on the slow part of the solution.) Once we know the decay rates in space of the solution components, it is straightforward to determine how far from the body we must be before we can replace the Navier-Stokes equations with the wave equation for the pressure.

Our theory and calculations confirm that the velocity field decays rapidly to zero at a distance from the body. In future works, we plan to carry out computationally the actual coupling of the wave equation with the slightly compressible Navier-Stokes equations.

The outline of the paper is as follows: In section 2, we study the decay behavior, in space, of the solutions of incompressible Euler flow, induced by a patch of vorticity. Because of the absence of diffusion in the Euler equations, the support of the vorticity remains compact in finite time.

In section 3, we introduce a rigid body into the flow. The decay rates are affected somewhat by the presence of the boundary of the body in the flow.

In section 4, diffusion and forcing are added in to the Euler equations to give us the incompressible Navier-Stokes equations. The presence of diffusion implies that the support of the vorticity will no longer remain compact in finite time. Although we no longer have compact support for the vorticity, we show in this section that vorticity does decay exponentially fast in space.

In section 5, we analyze the decay rates for the velocity and pressure fields modeled by the incompressible Navier-Stokes equations. We find that the decay behavior of the solutions of the incompressible Navier-Stokes equations is identical to that of the solutions of the incompressible Euler equations.

In section 6, we calculate the decay behavior of the space derivatives of the vorticity in the incompressible Navier-Stokes equations. We are able to determine that the vorticity *derivatives* also decay exponentially fast in space.

In section 7, we move to analyzing the decay behavior of the solutions of the slightly compressible Navier-Stokes equations. As mentioned earlier, we take advantage of the fact that we can separate the slightly compressible solution into a fast part and a slow part. Since the fast part can be suppressed by supplying appropriate initial data, we carry out our analysis on the first two terms of the expansion of the slow part of the solution. Upon determining the decay rates for the slightly compressible solution, we make use of this knowledge to show that at a certain distance from the body, the slightly compressible Navier-Stokes equations can be replaced by the wave equation for the pressure.

In section 8, we carry out numerical experiments, which confirm our analysis of the rapid decay in space of the velocity fields. Also made evident through the experiments is the fact that the pressure field does not decay nearly as rapidly as the velocity field.

In section 9, we give a brief summary of the work carried out in this paper.

2. Incompressible Euler Flow: Induced by Vortex Patch. To understand the asymptotic behavior of the pressure p and velocity $\mathbf{u} = (u, v)$ outside a patch of vorticity ω with compact support, we start by studying the incompressible Euler equations. We assume that the region of support of ω stays bounded for some finite time, $0 \leq t \leq T$. Later, we will actually calculate T in terms of distance from the origin. We also assume that at $t = 0$, $\mathbf{u}(x, y; t = 0) = \mathbf{u}(x, y)_0$ has compact support. We are interested only in velocity $\mathbf{u}(x, y; t)$ which stays bounded on all of $\Re^2$. Also, we will require that

$$\lim_{r \to \infty} \mathbf{u} = 0, \tag{2.1}$$

where $r = \sqrt{x^2 + y^2}$. The first step in our analysis will be carried out on the Euler equations in the following form:

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= 0 \\ v_t + uv_x + vv_y + p_y &= 0 \end{aligned} \tag{2.2}$$

with

$$div(\mathbf{u}) = u_x + v_y = 0. \tag{2.3}$$

2.1. Velocity Behavior: All Time. We present a theorem which will be of use to us in the discussion to follow. A proof can be found in [6, p.151]. We say that $u \in C^m$ if u has continuous derivatives up to order m.

THEOREM 1 *Let $\tilde{r} = ((x - \xi)^2 + (y - \eta)^2)^{1/2}$. Let D be a region in $\Re^2$ with boundary $\partial D = B$. Suppose we are given a function $\phi(x, y)$ which belongs to C^0 in $D \cup B$, and to C^1 in D. Then the function $u(x, y)$ defined by*

$$u(x, y) = \frac{1}{2\pi} \iint_D \phi(\xi, \eta) \log(\tilde{r}) d\xi d\eta$$

belongs to C^1 in $D \cup B$, and C^2 in D. Also, it satisfies

$$\Delta u = \phi(x, y)$$

in D. □

Now the vorticity $\omega(x, y; t)$ is defined by the relation

$$\omega = v_x - u_y. \tag{2.4}$$

This implies

$$\omega_x = v_{xx} - u_{xy} \qquad (2.5)$$
$$\omega_y = v_{xy} - u_{yy}.$$

By equation (2.3), we have

$$u_{xy} + v_{yy} = 0 \qquad (2.6)$$
$$u_{xx} + v_{xy} = 0.$$

Adding equations (2.6) to equations (2.5) yields

$$\omega_x = \Delta v \qquad (2.7)$$
$$-\omega_y = \Delta u.$$

By THEOREM 1, the solutions to these Poisson equations are given by

$$u = \frac{-1}{2\pi} \iint_D \omega_\eta \log(\tilde{r}) d\xi d\eta \qquad (2.8)$$
$$v = \frac{1}{2\pi} \iint_D \omega_\xi \log(\tilde{r}) d\xi d\eta.$$

Integration by parts yields

$$u(x,y;t) = \frac{-1}{2\pi}\left(\int_B \omega(\xi,\eta;t)\log(\tilde{r}) + \iint_D \omega(\xi,\eta;t)\frac{(y-\eta)}{\tilde{r}^2} d\xi d\eta\right) \qquad (2.9)$$
$$v(x,y;t) = \frac{1}{2\pi}\left(\int_B \omega(\xi,\eta;t)\log(\tilde{r}) + \iint_D \omega(\xi,\eta;t)\frac{(x-\xi)}{\tilde{r}^2} d\xi d\eta\right).$$

Since ω has compact support, we may choose the region D so large that $\omega \equiv 0$ on B. Thus we have

$$u(x,y;t) = \frac{-1}{2\pi} \iint_D \omega(\xi,\eta;t)\frac{(y-\eta)}{\tilde{r}^2} d\xi d\eta \qquad (2.10)$$
$$v(x,y;t) = \frac{1}{2\pi} \iint_D \omega(\xi,\eta;t)\frac{(x-\xi)}{\tilde{r}^2} d\xi d\eta.$$

Since $\omega \equiv 0$ outside region D, we have

$$\frac{-1}{2\pi} \iint_{\Re^2 - D} \omega(\xi,\eta;t)\frac{(y-\eta)}{\tilde{r}^2} d\xi d\eta = 0 \qquad (2.11)$$
$$\frac{1}{2\pi} \iint_{\Re^2 - D} \omega(\xi,\eta;t)\frac{(x-\xi)}{\tilde{r}^2} d\xi d\eta = 0.$$

Adding equations (2.11) to equation (2.10) gives us an expression for $\mathbf{u} = (u,v)$ over the entire $\Re^2$ plane:

$$u(x,y;t) = \frac{-1}{2\pi} \iint_{\Re^2} \omega(\xi,\eta;t)\frac{(y-\eta)}{\tilde{r}^2} d\xi d\eta \qquad (2.12)$$
$$v(x,y;t) = \frac{1}{2\pi} \iint_{\Re^2} \omega(\xi,\eta;t)\frac{(x-\xi)}{\tilde{r}^2} d\xi d\eta.$$

We show the uniqueness of this solution as follows: We are given (u, v) over all of $\Re^2$. Now, suppose there were a function $g = (g_1, g_2)$, such that $\Delta g = 0$ on $\Re^2$. Then it is true that

$$u(x, y; t) = \frac{-1}{2\pi} \iint_{\Re^2} \omega(\xi, \eta; t) \frac{(y-\eta)}{\tilde{r}^2} d\xi d\eta + g_1 \tag{2.13}$$
$$v(x, y; t) = \frac{1}{2\pi} \iint_{\Re^2} \omega(\xi, \eta; t) \frac{(x-\xi)}{\tilde{r}^2} d\xi d\eta + g_2,$$

would also satisfy equation (2.7). But we also require that the boundary conditions be satisfied. Since $\mathbf{u} \to 0$ as $r \to \infty$, by equation (2.1), we must also have $g \to 0$ as $r \to \infty$. By the maximum principle (*c.f.*, [6]), since g is harmonic and is equal to 0 on its boundaries, we have $g \equiv 0$ over all of $\Re^2$. Thus, the solutions (2.12) are the unique solutions satisfying equations (2.7), with zero boundary conditions.

Recall that $r = \sqrt{x^2 + y^2}$. We see from (2.12) that for large r,

$$\mathbf{u} \leq c_1/r.$$

This is consistent with results from potential theory, given a point source [7]. Intuitively it should follow that $\mathbf{u}_x \leq c_2/r^2$ and $\mathbf{u}_y \leq c_2/r^2$. We confirm this by first noting that we are interested in evaluating $\mathbf{u}_x$ and $\mathbf{u}_y$ far from the origin. Since $\tilde{r} \neq 0$ in the far-field, the expression $(y - \eta)/\tilde{r}^2$ is continuously differentiable in that region. Therefore, we can differentiate directly under the integral sign, and obtain for large r,

$$\begin{aligned} u_x &= \frac{-1}{2\pi} \iint_{\Re^2} \frac{\partial}{\partial x} \frac{(y-\eta)}{\tilde{r}^2} \omega(\xi, \eta; t) d\xi d\eta \\ &= \frac{-1}{2\pi} \iint_{\Re^2} (-2) \frac{(y-\eta)(x-\xi)}{\tilde{r}^4} \omega(\xi, \eta; t) d\xi d\eta. \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} u_y &= \frac{-1}{2\pi} \iint_{\Re^2} \frac{\partial}{\partial y} \frac{(y-\eta)}{\tilde{r}^2} \omega(\xi, \eta; t) d\xi d\eta \\ &= \frac{-1}{2\pi} \iint_{\Re^2} \left((-2) \frac{(y-\eta)^2}{\tilde{r}^4} + \frac{1}{\tilde{r}^2} \right) \omega(\xi, \eta; t) d\xi d\eta. \end{aligned} \tag{2.15}$$

A similar argument can be given for v. Thus our intuition is confirmed, and we see that

$$\mathbf{u}_x \leq c_2/r^2$$

and

$$\mathbf{u}_y \leq c_2/r^2$$

for r large.

We go one step further, and claim that since $\mathbf{u} = c_1/r$ for large r, it follows that $\mathbf{u} \leq c_3/(r+1)$ over all of $\Re^2$. This is true, because we know that there is some $M > 1$ such that for $r \geq M$, $\mathbf{u} \leq c_1/r$. We also know that $\mathbf{u}$ is bounded over all of $\Re^2$. So, for $r < M$, $\mathbf{u} \leq c_1$. Therefore, if we let $c_3 = (M+1)c_1$, we can say that everywhere on the plane,

$$\mathbf{u} \leq c_3/(r+1). \tag{2.16}$$

2.2. Pressure Behavior. In this section, we analyze the far-field behavior of the pressure p. Taking the divergence of equations (2.2), we have

$$u_{xt} + v_{yt} + u_x^2 + v_y^2 + uu_{xx} + vv_{yy} + 2u_y v_x + vu_{xy} + uv_{xy} + \Delta p = 0. \tag{2.17}$$

Recall the incompressibility condition,

$$div(\mathbf{u}) = u_x + v_y = 0. \tag{2.18}$$

Several terms drop out, and equation (2.17) becomes

$$u_x^2 + v_y^2 + 2u_y v_x + \Delta p = 0. \tag{2.19}$$

For the sake of notation, we define the following symbols:

$$D_1 = \frac{\partial}{\partial x}, \qquad D_2 = \frac{\partial}{\partial y};$$
$$u_1 = u, \qquad u_2 = v.$$

We re-write equation (2.19) as

$$\Delta p = c_1 \sum_{i,j=1}^{2} D_i D_j (u_i u_j). \tag{2.20}$$

Note that with p, as opposed to with $\mathbf{u}$, the right hand side of the Poisson equation does not have compact support. This means that we will have to examine the solution to (2.20) over all of $\Re^2$, and will now need to deal specifically with the presence of a singularity. (When dealing with $\mathbf{u}$, the singularity always lay outside the support of ω. Since our integral calculations were limited to the region of vortex support, they were unaffected by the presence of the singularity.)

Let D be a disk with boundary B and radius R centered at the origin. By Theorem 1 we have

$$\begin{aligned} p &= \lim_{R\to\infty} \frac{c_1}{2\pi}\left(\iint_D \sum_{i,j=1}^{2} D_i D_j (u_i u_j) \log(\tilde{r}) d\xi d\eta\right) \\ &= \frac{c_1}{2\pi}\left(\iint_{\Re^2} \sum_{i,j=1}^{2} D_i D_j (u_i u_j) \log(\tilde{r}) d\xi d\eta\right). \end{aligned} \tag{2.21}$$

We will handle the singularity separately. Let G be a disk of radius R_G, centered at the singularity (x, y). We denote the distance from the origin to the singularity by $\tilde{R}$. As before, the distance from the singularity to any other point in $\Re^2$ is denoted by $\tilde{r}$, and the distance from the origin to another point in $\Re^2$ is denoted by r. Then

$$\begin{aligned} p &= \frac{c_1}{2\pi}\left(\iint_{\Re^2 - G} \sum_{i,j=1}^{2} D_i D_j (u_i u_j) \log(\tilde{r}) d\xi d\eta \right. \\ &\qquad \left. + \iint_G \sum_{i,j=1}^{2} D_i D_j (u_i u_j) \log(\tilde{r}) d\xi d\eta\right) \\ &= \frac{c_1}{2\pi}(I_1 + I_2). \end{aligned} \tag{2.22}$$

Let us examine the region G about the singularity first. We integrate by parts once, then convert to polar coordinates to determine the bounding behavior of I_2.

$$
\begin{aligned}
(2.23) \qquad I_2 &= \iint_G \sum_{i,j=1}^{2} D_i D_j(u_i u_j)\log(\tilde{r})d\xi d\eta \\
&\leq \int_{\partial G} \sum_{i,j=1}^{2} D_i(u_i u_j)\log(\tilde{r}) + \iint_G \sum_{i,j=1}^{2} |D_i(u_i u_j)|\frac{1}{\tilde{r}} d\xi d\eta \\
&\leq \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \iint_G \sum_{i,j=1}^{2} |D_i(u_i u_j)|\frac{1}{\tilde{r}} d\xi d\eta \\
&\leq \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \frac{c}{(\tilde{R}-R_G)^3}\iint_G \frac{1}{\tilde{r}} d\xi d\eta \\
&= \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \frac{c}{(\tilde{R}-R_G)^3}\int_0^{2\pi}\int_0^{R_G} 1 d\tilde{r}d\theta \\
&\leq \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \frac{2\pi c R_G}{(\tilde{R}-R_G)^3} \\
&= \frac{2\pi R_G}{(\tilde{R}-R_G)^3}(\log(R_G)+c).
\end{aligned}
$$

Next we deal with I_1. Since (ξ,η) will range over $\Re^2 - G$, we have $\tilde{r} \neq 0$ throughout the domain of integration. At this point, we introduce another disk in the plane, D, of radius R_D. Let D also be centered at the singularity, but let $R_D \geq 2\tilde{R}$, so that disk G and the origin are encompassed by D. Then

$$
\begin{aligned}
(2.24) \qquad I_1 &= \iint_{\Re^2-G} \sum_{i,j=1}^{2} D_i D_j(u_i u_j)\log(\tilde{r})d\xi d\eta \\
&\leq \int_{\partial G} \sum_{i,j=1}^{2} D_i(u_i u_j)\log(\tilde{r}) + \iint_{\Re^2-G} \sum_{i,j=1}^{2} |D_i(u_i u_j)|\frac{1}{\tilde{r}} d\xi d\eta \\
&\leq \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \iint_{\Re^2-G} \sum_{i,j=1}^{2} |D_i(u_i u_j)|\frac{1}{\tilde{r}} d\xi d\eta \\
&\leq \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \int_{\partial G} \sum_{i,j=1}^{2} |u_i u_j|\frac{1}{\tilde{r}} + \iint_{\Re^2-G} \sum_{i,j=1}^{2} |(u_i u_j)|\frac{1}{\tilde{r}^2} d\xi d\eta \\
&\leq \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \frac{c}{(\tilde{R}-R_G)^2} + \iint_{\Re^2-G} \sum_{i,j=1}^{2} |(u_i u_j)|\frac{1}{\tilde{r}^2} d\xi d\eta \\
&= \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \frac{c}{(\tilde{R}-R_G)^2} + \iint_{\Re^2-D} \sum_{i,j=1}^{2} |(u_i u_j)|\frac{1}{\tilde{r}^2} d\xi d\eta \\
&\quad + \iint_{D-G} \sum_{i,j=1}^{2} |(u_i u_j)|\frac{1}{\tilde{r}^2} d\xi d\eta \\
&= \frac{2\pi R_G \log(R_G)}{(\tilde{R}-R_G)^3} + \frac{c}{(\tilde{R}-R_G)^2} + J_1 + J_2,
\end{aligned}
$$

where J_1 is the integral taken over $\Re^2 - D$ and J_2 is the integral taken over the annulus $D - G$. Let us first examine J_1, the region outside of the disk D. Since the distances $\tilde{R}$, $\tilde{r}$, and r represent the positive lengths of the three sides of the triangle connecting the origin, the singularity, and some point outside the disk D, it is clear that the following inequalities hold:

$$\begin{aligned} r + \tilde{R} &> \tilde{r} \\ \Rightarrow r &> \tilde{r} - \tilde{R} \\ \Rightarrow \frac{1}{r} &< \frac{1}{\tilde{r} - \tilde{R}}. \end{aligned} \tag{2.25}$$

We can then use $1/(\tilde{r} - \tilde{R})$ to dominate the behavior of $\mathbf{u} \leq c/r$ in J_1. Also, since we are only interested in the region lying outside D, we see that

$$\begin{aligned} \tilde{r} &> \tilde{R} \\ \Rightarrow \tilde{r} &> \tilde{r} - \tilde{R} > 0 \\ \Rightarrow \frac{1}{\tilde{r}} &< \frac{1}{\tilde{r} - \tilde{R}}. \end{aligned} \tag{2.26}$$

This means we can also use $1/(\tilde{r} - \tilde{R})$ to dominate the behavior of $1/\tilde{r}$ in J_1. We can bound J_1 as follows:

$$\begin{aligned} J_1 &= \iint_{\Re^2 - D} \sum_{i,j=1}^{2} |(u_i u_j)| \frac{1}{\tilde{r}^2} d\xi d\eta \\ &= \int_0^{2\pi} \int_{R_D}^{\infty} \sum_{i,j=1}^{2} |(u_i u_j)| \frac{1}{\tilde{r}} d\tilde{r} d\theta \\ &\leq 2\pi c_1 \int_{R_D}^{\infty} \frac{1}{r^2} \frac{1}{\tilde{r}} d\tilde{r} \\ &\leq 2\pi c_1 \int_{R_D}^{\infty} \frac{1}{(\tilde{r} - \tilde{R})^3} d\tilde{r} \\ &\leq \frac{c_2}{(R_D - \tilde{R})^2}. \end{aligned} \tag{2.27}$$

Next, when dealing with J_2 over the disk D, we introduce a larger disk D' centered at the origin, with radius $R_{D'}$ large enough to encompass all of disk D. Recall that $\mathbf{u} \leq c_3/(1 + r)$ over the entire $\Re^2$ plane, even over the support of the vorticity. We use this fact when examining the bounding behavior of J_2:

$$\begin{aligned} J_2 &= \iint_{D-G} \sum_{i,j=1}^{2} |(u_i u_j)| \frac{1}{\tilde{r}^2} d\xi d\eta \\ &\leq \frac{1}{R_G^2} \iint_{D-G} \sum_{i,j=1}^{2} |(u_i u_j)| d\xi d\eta \\ &\leq \frac{1}{R_G^2} \iint_{D'} \sum_{i,j=1}^{2} |(u_i u_j)| d\xi d\eta \end{aligned} \tag{2.28}$$

$$
\begin{aligned}
&= \frac{1}{R_G^2}\int_0^{2\pi}\int_0^{R_{D'}}\sum_{i,j=1}^{2}|(u_iu_j)|\tilde{r}d\tilde{r}d\theta \\
&\leq \frac{2\pi c_3}{R_G^2}\int_0^{R_{D'}}\frac{\tilde{r}}{(1+\tilde{r})^2}d\tilde{r} \\
&\leq \frac{2\pi c_3}{R_G^2}\int_0^{R_{D'}}\frac{(1+\tilde{r})}{(1+\tilde{r})^2}d\tilde{r} \\
&= \frac{2\pi c_3}{R_G^2}\int_0^{R_{D'}}\frac{1}{(1+\tilde{r})}d\tilde{r} \\
&\leq \frac{2\pi c}{R_G^2}\log(R_{D'}).
\end{aligned}
$$

Next, we set $\tilde{R} = r$, and choose R_G, R_D, and $R_{D'}$ to be of the same order as r. For example, we can choose $R_G = r/2$, $R_D = 2r$ and $R_{D'} = 3r$. Then, for some constant c large enough, we find

(2.29)
$$
\begin{aligned}
J_1 &\leq \frac{c}{r^2} \\
J_2 &\leq \frac{c\log(r)}{r^2}
\end{aligned}
$$

which gives

$$I_1 \leq \frac{c\log(r)}{r^2}.$$

Similarly, we find that

$$I_2 \leq \frac{c\log(r)}{r^2}.$$

Putting all the information together gives

(2.30)
$$
\begin{aligned}
p &= \frac{c_1}{2\pi}(I_1 + I_2) \\
&\leq \frac{c\log(r)}{r^2}.
\end{aligned}
$$

Since the right hand side of (2.30) goes to zero as $r \to \infty$, we may require that

$$\lim_{r\to\infty} p = 0.$$

Thus, by the same argument given for the velocity $\mathbf{u}$, we know that the solution given in equation (2.21) is the only solution for p satisfying the homogeneous boundary conditions at infinity.

If we wish to find a bound for the decay behavior of the gradient of the pressure, we replace equation (2.20) with the following equations:

(2.31)
$$\Delta p_x = c_1 \sum_{i,j=1}^{2} D_iD_jD_1(u_iu_j)$$

(2.32)
$$\Delta p_y = c_1 \sum_{i,j=1}^{2} D_iD_jD_2(u_iu_j).$$

It is then straightforward to find estimates on p_x and p_y in the same way we found estimates on p itself. Working through the estimates, one finds

$$\nabla p \leq \frac{c\log(r)}{r^3}. \tag{2.33}$$

2.3. Velocity Behavior: Finite Time. In this section, we find the far-field behavior of $\mathbf{u} = (u, v)$ along its characteristic lines. We follow a particle along a characteristic from a point at distance r_0 from the origin (at time $t = 0$), to a point no closer than $r_0/2$ from the origin. We find T in terms of r_0 such that for $0 \leq t \leq T$ we know the particle is no closer than $r_0/2$ to $(0,0)$. To do this, we first examine the total derivative of $\mathbf{u}$, assuming $(x, y; t)$ space, so that x and y are t-dependent:

$$\begin{aligned} \frac{du}{dt} &= u_t + (\frac{dx}{dt})u_x + (\frac{dy}{dt})u_y \\ \frac{dv}{dt} &= v_t + (\frac{dx}{dt})v_x + (\frac{dy}{dt})v_y \end{aligned} \tag{2.34}$$

Now assume that p is a given function representing pressure. We know by equation (2.30) that the behavior of p is dominated by $(\frac{c\log(r)}{r^2})$. In addition, from equation (2.33) it follows that at a point which is large distance r_0 away from the origin, and for some appropriate constant c, the behavior of p_x and p_y is given asymptotically by $(\frac{c\log(r_0)}{r_0^3})$.

Substituting the asymptotic behavior of the gradient of the pressure in the far-field into equation (2.2) gives:

$$\begin{aligned} u_t + uu_x + vu_y &= \frac{c\log(r_0)}{r_0^3} \\ v_t + uv_x + vv_y &= \frac{c\log(r_0)}{r_0^3}. \end{aligned} \tag{2.35}$$

From equations (2.35) and (2.34) we find that

$$\begin{aligned} \frac{du}{dt} &= \frac{c\log(r_0)}{r_0^3} \\ \frac{dv}{dt} &= \frac{c\log(r_0)}{r_0^3} \end{aligned} \tag{2.36}$$

on the characteristic line which satisfies

$$\begin{aligned} \frac{dx}{dt} &= u \\ \frac{dy}{dt} &= v. \end{aligned} \tag{2.37}$$

Let us impose some initial conditions, namely,

$$\begin{aligned} u(x, y; t = 0) &= u_0(x, y), \\ v(x, y; t = 0) &= v_0(x, y). \end{aligned} \tag{2.38}$$

These are equivalent to

$$\mathbf{u}(x, y; t = 0) = \mathbf{u}_0(x, y). \tag{2.39}$$

We also have

$$\begin{aligned} x(t = 0) &= x_0, \\ y(t = 0) &= y_0. \end{aligned} \tag{2.40}$$

Then integrating equations (2.36) and (2.37) with respect to time yields

$$\begin{aligned} \mathbf{u} &= \int \frac{d\mathbf{u}}{dt} dt \\ &= \int \frac{c \log(r_0)}{r_0^3} dt \\ &= \frac{c \log(r_0)}{r_0^3} t + \mathbf{u}_0 \end{aligned}$$

on the line parameterized by

$$\begin{aligned} x(t) &= \frac{c \log(r_0)}{r_0^3} t^2 + u_0 t + x_0 \\ y(t) &= \frac{c \log(r_0)}{r_0^3} t^2 + v_0 t + y_0. \end{aligned} \tag{2.41}$$

In the far-field, then, we have the behavior of $\mathbf{u}$ dominated by

$$\mathbf{u} \leq \frac{c \log(r_0)}{r_0^3} t + \mathbf{u}_0,$$

for an appropriate constant c.

Now we want to find a time T in terms of r_0 such that for $0 \leq t \leq T$, the particle is no closer than $r_0/2$ from the origin. If we set $x = r_0/2$ and $y = r_0/2$ in equation (2.41), and solve for t, the requirement will be satisfied. First, we can see that for large t, the behavior of x and y in equation (2.41) is dominated by the quadratic term. (In addition, recall that we made the assumption that $\mathbf{u}_0$ has compact support, so for large r_0, the $\mathbf{u}_0 t$ term in (2.41) vanishes.) Thus, we assume a large t, and let

$$x(t) = y(t) = \frac{c \log(r_0)}{r_0^3} t^2. \tag{2.42}$$

Setting $x = y = r_0/2$, and solving for t, gives

$$\begin{aligned} t^2 &= \frac{r_0^4}{2c \log(r_0)} \\ \Rightarrow \quad t &\leq \frac{c r_0^2}{(\log(r_0))^{1/2}} = T. \end{aligned}$$

Next, we use the same method of analysis for the vorticity ω in order to see that the support of ω stays bounded for $0 \le t \le T = \frac{cr_0^2}{(\log(r_0))^{1/2}}$. Taking the curl of the incompressible Euler equations (2.2), gives an equation for the vorticity:

$$\omega_t + u\omega_x + v\omega_y = 0. \tag{2.43}$$

Taking the total derivative of ω in (x, y, z) space gives

$$\frac{d\omega}{dt} = \omega_t + \frac{dx}{dt}\omega_x + \frac{dy}{dt}\omega_y. \tag{2.44}$$

From equations (2.43) and (2.44) we see that

$$\frac{d\omega}{dt} = 0$$

along

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v. \tag{2.45}$$

Since the vorticity follows the same characteristic lines as the velocity, we then know that for some starting point r_0 from the origin, at which vorticity $\omega = 0$, the vorticity will continue to be zero valued, at least for $0 \le t \le T = \frac{cr_0^2}{(\log(r_0))^{1/2}}$.

Since this analysis is valid for any starting point r_0 far away from the origin, it is also valid for any r far from the origin. Thus, we have shown that for finite time t, $0 \le t \le T = \frac{cr^2}{(\log(r))^{1/2}}$, the support of ω stays bounded, and the behavior of velocity $\mathbf{u}$ in the far-field is dominated by

$$\mathbf{u}(x, y; t) \le c\log(r)/r^3,$$

for an appropriate constant c. We note that it can also be shown in the far-field that,

$$\mathbf{u}_x,\ \mathbf{u}_y \le c\log(r)/r^4, \quad \text{and} \quad \mathbf{u}_{xx},\ \mathbf{u}_{xy}, \ldots \le c\log(r)/r^5, \tag{2.46}$$

and so forth, by carrying out the same analysis on the respective derivatives of our original equations.

3. Incompressible Euler Flow: Induced by a Rigid Body. Up to this point, we have been looking at a flow field containing a simple vorticity patch with compact support. In this section we wish to examine the behavior of the flow field induced by a body submerged in a non-viscous fluid. (We are still working in 2 dimensions.) We assume a body which can be conformally mapped to a disk, and thus carry out our analysis on a circular body.

Our analysis will follow along lines of reasoning similar to those in the previous section, except that we now must take into account the values of pressure and velocity on the boundary ∂B of the body B. The analysis we carry out will be valid whether or not the surface of the body is vibrating. If it is not vibrating, then we consider velocity to be zero on the boundary (or surface) of the body. If it is vibrating, then we simply consider the velocity to be a function of position and time on the boundary of the body.

Inverse distance decay behavior for the velocity components in all time is still valid, so we move directly to the analysis of the behavior of the pressure field.

3.1. Pressure Behavior. We assume that the function representing velocity is analytic, and therefore can be expanded in a Taylor series. Assume the body B is centered at the origin, and is of radius a. We carry out our analysis in polar coordinates (r, θ). Over $\Re^2$ containing B, we want to solve:

$$\begin{aligned} \Delta p = & \; c_1 \textstyle\sum_{i,j=1}^2 D_i D_j (u_i u_j) \quad \text{on} \;\; \Re^2 - B. \\ p(r,\theta) = & \; f(\theta) \quad \text{on} \;\; \partial B. \end{aligned} \tag{3.1}$$

The first thing we do is extend the function $\mathbf{u}$ into the region of the body. Since we assume $\mathbf{u}$ is reasonably smooth, (that is, for some large m, $\mathbf{u} \in C^m$) this extension can be carried out by means of a simple $m-$term Taylor series expansion. Let $\mathbf{u}_E$ be the extension of $\mathbf{u}$. We need not concern ourselves with the behavior of $\mathbf{u}_E$ as it nears the origin. This is because we can define a cutoff function $\chi(r)$ which drops rapidly to zero as it approaches the origin, and multiply it by $\mathbf{u}_E$ to create a new function $\mathbf{u}^\star$, which we can work with in place of $\mathbf{u}_E$. We define χ as follows:

$$\chi(r) = \begin{cases} e^{1-(a/r)} & \text{for } r \le a \\ 1 & \text{for } r > a \end{cases} \tag{3.2}$$

Thus, $\mathbf{u}^\star$ will become

$$\mathbf{u}^\star = \begin{cases} \chi(r)\mathbf{u}_E(r,\theta) & \text{for } r < a \\ \mathbf{u}(r,\theta) & \text{for } r \ge a \end{cases} \tag{3.3}$$

In order to solve equation (3.1) with boundary conditions, we first solve the following equation, without boundary conditions imposed:

$$\Delta p = c_1 \sum_{i,j=1}^{2} D_i D_j (u^\star{}_i u^\star{}_j) \quad \text{on} \;\; \Re^2. \tag{3.4}$$

Let $p_c(r,\theta)$ be the solution of equation (3.4). Potential theory [7] tells us that $\mathbf{u}^\star$ is bounded by c/r, for some constant c, as $r \to \infty$. It is also clear that for $r < a$, we can bound $\mathbf{u}^\star$ by $c/(1+r)$, by the definition of $\mathbf{u}^\star$. Since $\mathbf{u}^\star$ is dominated by the same behavior that dominates $\mathbf{u}$ in equation (2.16), over the whole plane, we know that the bound we found for p in equation (2.30) will also be valid for the solution p_c of equation (3.4). Thus we have

$$p_c(r,\theta) \le \frac{c \log(r)}{r^2}. \tag{3.5}$$

It is clear that p_c satisfies the differential equation in (3.1), but that the boundary conditions are not yet satisfied.

We next concern ourselves with finding a harmonic solution p_h of the Laplace equation satisfying special boundary conditions. Solve

$$\begin{aligned} \Delta p = & \; 0 \quad \text{on} \;\; \Re^2, \\ p(r,\theta) = & \; f(\theta) - p_c(a,\theta) \quad \text{on} \;\; \partial B. \end{aligned} \tag{3.6}$$

In polar coordinates, the differential equation (3.6) reads,

$$p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} = 0. \tag{3.7}$$

Clearly, the solution $p_h(r,\theta)$ of equation (3.7) must be 2π-periodic in θ. We also require that p_h stay finite for all $r > a$.

Assume that

$$p(r,\theta) = R(r)\Theta(\theta), \tag{3.8}$$

and use the technique of separation of variables to solve (3.7). For completeness, we carry out the calculations here. Differentiating p in equation (3.8), and substituting the results back into equation (3.7) gives rise to two ordinary differential equations:

$$r^2R'' + rR' - cR = 0 \tag{3.9}$$

$$\Theta'' + c\Theta = 0 \tag{3.10}$$

for some constant c. It can be shown that c must be real. See, for example, [2, p.507]. We therefore need only consider three separate cases: $c < 0$, $c = 0$, and $c > 0$.

Let us first look at the case where $c < 0$. For some $\lambda > 0$, we let $c = -\lambda^2$. Then equation (3.10) becomes

$$\Theta'' - \lambda^2\Theta = 0, \tag{3.11}$$

which has the general solution

$$\Theta = c_1e^{\lambda\theta} + c_2e^{-\lambda\theta}. \tag{3.12}$$

Recall, however, that we require that $p_h(r,\theta)$ be 2π-periodic in θ. Thus, Θ must be 2π-periodic. This will only happen if $c_1 = c_2 = 0$. So the case $c < 0$ only gives the trivial solution

$$p(r,\theta) = 0.$$

Next we examine the case $c = 0$. Equation (3.10) becomes

$$\Theta'' = 0,$$

which has the general solution

$$\Theta = c_1 + c_2\theta.$$

The requirement that Θ be 2π-periodic forces $c_2 = 0$. The solution becomes

$$\Theta(\theta) = c_1. \tag{3.13}$$

When $c = 0$, equation (3.9) becomes

$$r^2R'' + rR' = 0. \tag{3.14}$$

This is an Euler type equation, and has the general solution

$$R(r) = k_1 + k_2 \log(r).$$

Recall that we require $p = R\Theta$ to be finite for $r \geq a$, which means we require $R(r) < \infty$ for $r \geq a$. Therefore, we must discard the $\log(r)$ term, and force $k_2 = 0$. This gives

$$R(r) = k_1.$$

We now can say that for $c = 0$,

$$p(r, \theta) = 1. \tag{3.15}$$

In the case where $c > 0$, we let $c = \lambda^2$. Then equation (3.9) becomes

$$r^2 R'' + rR' - \lambda^2 R = 0.$$

This is again an Euler type equation, and has a general solution of the form

$$R(r) = k_1 r^{\lambda} + k_2 r^{-\lambda}$$

The condition that $R(r)$ be finite for all $r \geq a$ forces $k_1 = 0$, since $\lambda > 0$. Thus,

$$R(r) = k_2 r^{-\lambda}.$$

Now for $c = \lambda^2 > 0$, equation (3.10) becomes

$$\Theta'' + \lambda^2 \Theta = 0.$$

The general solution is of the form

$$\Theta(\theta) = c_1 \sin(\lambda\theta) + c_2 \cos(\lambda\theta).$$

Since Θ must be 2π-periodic, λ must be a positive integer n. We then have fundamental solutions in the case $c = \lambda^2 > 0$,

$$\begin{aligned} p_n &= r^{-n} \sin(n\theta) \\ q_n &= r^{-n} \cos(n\theta). \end{aligned}$$

Taking a linear combination of all fundamental solutions gives

$$p_h(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^{-n}(c_n \cos(n\theta) + k_n \sin(n\theta)). \tag{3.16}$$

On the boundary of B, we see from equations (3.6) and (3.16) that

$$\begin{aligned} p_h(a, \theta) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} a^{-n}(c_n \cos(n\theta) + k_n \sin(n\theta)) \\ &= f(\theta) - p_c(a, \theta). \end{aligned}$$

We assume that both $f(\theta)$ and $p_c(r,\theta)$ are expressible in terms of a Fourier series, so the term $f(\theta) - p_c(a,\theta)$ is also expressible in terms of a Fourier series. Our solution p_h already takes the form of a Fourier series, so we simply choose the coefficients of the series to make the boundary condition consistent. The coefficients c_n and k_n become

$$\begin{aligned} c_n &= \frac{a^n}{\pi}\int_0^{2\pi} (f(\theta) - p_c(a,\theta))\cos(n\theta)d\theta \\ k_n &= \frac{a^n}{\pi}\int_0^{2\pi} (f(\theta) - p_c(a,\theta))\sin(n\theta)d\theta. \end{aligned}$$

Letting

$$p(r,\theta) = p_c(r,\theta) + p_h(r,\theta) \tag{3.17}$$

gives the complete solution $p(r,\theta)$ of the boundary value problem (3.1). Looking at the bounding behavior of $p(r,\theta)$, we see from equations (3.5), (3.16), and (3.17), that

$$p(r,\theta) \le c_1\left(1 + \frac{1}{r} + \frac{\log(r)}{r^2} + \frac{1}{r^2} + \cdots\right). \tag{3.18}$$

Therefore, the gradient of p is bounded by

$$\nabla p(r,\theta) \le c_2\left(\frac{1}{r^2} + \frac{\log(r)}{r^3} + \frac{1}{r^3} + \cdots\right), \tag{3.19}$$

for some constant c_2 independent of r. This bound for the gradient of p can be found formally by replacing the Laplace equation in (3.1) with the following two equations:

$$\Delta p_x = c_1 \sum_{i,j=1}^{2} D_i D_j D_1(u_i u_j) \tag{3.20}$$

$$\Delta p_y = c_1 \sum_{i,j=1}^{2} D_i D_j D_2(u_i u_j). \tag{3.21}$$

Carrying out arguments almost identical to those for estimating the decay behavior of p, we find estimates for the decay behavior of p_x and p_y. These then lead to the final bound for ∇p in equation (3.19).

For large r, the behavior of $\nabla p(r,\theta)$ is dominated by the largest term in its dominating series. So for large r we have

$$\nabla p(r,\theta) \le \frac{c}{r^2}. \tag{3.22}$$

We will use this dominating behavior in the next section, where we will analyze the far-field behavior of $\mathbf{u}$, assuming the pressure p is given.

3.2. Velocity Behavior: Finite Time. As in section 2.3, we use characteristics to determine the behavior of **u** within a finite time period. We follow precisely those arguments outlined in section 2.3, substituting only the new behavior of the pressure p in the presence of a body submerged in the fluid.

Equations (2.2) and (3.22) tell us that in the far-field,

$$\begin{aligned} u_t + uu_x + vu_y &= \frac{c}{r^2} \\ v_t + uv_x + vv_y &= \frac{c}{r^2}. \end{aligned} \tag{3.23}$$

The analysis of characteristics shows that

$$\begin{aligned} \frac{du}{dt} = \frac{c}{r^2} \\ \frac{dv}{dt} = \frac{c}{r^2} \end{aligned} \tag{3.24}$$

on

$$\begin{aligned} \frac{dx}{dt} = u \\ \frac{dy}{dt} = v. \end{aligned} \tag{3.25}$$

Integrating equations (3.24) and (3.25) with respect to time, and imposing the same initial conditions as in equations (2.39) and (2.40), gives

$$\begin{aligned} u = c\frac{t}{r^2} + u_0 \\ v = c\frac{t}{r^2} + v_0 \end{aligned} \tag{3.26}$$

on

$$\begin{aligned} x(t) = c\frac{t^2}{r^2} + u_0 t + x_0 \\ y(t) = c\frac{t^2}{r^2} + v_0 t + y_0. \end{aligned} \tag{3.27}$$

To determine the the time interval in which the behavior of **u** remains unchanged in nature, we restrict the particle to moving no closer than distance $r/2$ from the origin. Once again, we assume the behaviors of $x(t)$ and $y(t)$ are dominated by their quadratic terms. Setting

$$x = y = \frac{r}{2}$$

and solving for t shows us that for

$$0 \le t \le cr^{3/2} = T,$$

the behavior of $\mathbf{u}$ in the far-field can be described by

$$\mathbf{u} \leq c/r^2,$$

for some independent constant c. In addition, it can be shown (by taking the appropriate derivatives of equation (3.23) and following the same line of arguments) that in the far-field, the behavior of the space derivatives of the velocity can be given by

$$\mathbf{u}_x,\ \mathbf{u}_y \leq c/r^3 \tag{3.28}$$

$$\mathbf{u}_{xx},\ \mathbf{u}_{xy}, \ldots \leq c/r^4 \tag{3.29}$$

and so forth.

4. Incompressible Navier-Stokes Flow: Vorticity Behavior. Up to this point we have considered the incompressible Euler equations (2.2). We now add a diffusion term and a forcing term back into the equations, to obtain the incompressible Navier-Stokes equations,

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u} + \mathbf{F}, \quad \nu > 0 \tag{4.1}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{4.2}$$

Here, $\mathbf{F}(x,y;t) = (f_1(x,y;t), f_2(x,y;t))$. We require that the components f_1 and f_2 of $\mathbf{F}$, as well as all derivatives of the components, decay rapidly enough in space so as not to interfere with the decay behavior of the velocity, pressure, or vorticity. We therefore require that $\mathbf{F}$ decay exponentially fast in space. The exact decay rate bound will be determined according to need.

With diffusion present, we can no longer assume that the vorticity $\omega(x,y;t)$ has compact support in finite time. We intend to show, however, that the vorticity will decay, as a function of distance from the origin, at a rapid enough rate so as to contribute only negligibly to our bounds on the far-field behavior of velocity and pressure. Once this is shown, the same types of bounding estimates that were carried out on the incompressible Euler equations will also be valid for the incompressible Navier-Stokes equations.

Taking the curl of equation (4.1) gives us the following equation for the vorticity,

$$\omega_t + u\omega_x + v\omega_y = \nu\Delta\omega + f, \tag{4.3}$$

where we let $f(x,y;t) = (f_2)_x - (f_1)_y$. Thus, f itself decays exponentially in space.

Recall our basic assumptions:

- The velocity $\mathbf{u} = (u,v)$ is bounded, *i.e.*, for some finite constants $\mathbf{u}_0$ and $\mathbf{u}_1$, $\mathbf{u}_0 < \mathbf{u} < \mathbf{u}_1$.
- The vorticity $\omega(x,y;t)$ has compact support and is bounded at time $t = 0$.
- The vorticity $\omega(x,y;t)$ is bounded in terms of its initial data $\omega(x,y;0)$. The same holds for all space derivatives of $\omega(x,y;t)$;

The assumptions on the vorticity clearly imply that

$$\lim_{(x,y)\to\infty} \omega(x,y;t) = 0.$$

We next determine how quickly ω decays to zero at infinity. There are four separate cases to examine:

1. For some δ such that $\delta > 0$, $u < -\delta$. This means on the (x, y) plane, the flow is moving to the left.
2. For some δ such that $u_1 > \delta > 0$, $u > \delta$. This means on the (x, y) plane, the flow is moving to the right.
3. For some γ such that $\gamma > 0$, $v < -\gamma$. This means on the (x, y) plane, the flow is moving downwards.
4. For some γ such that $v_1 > \gamma > 0$, $v > \gamma$. This means on the (x, y) plane, the flow is moving upwards.

Case 1: Suppose $u < -\delta$ $(\delta > 0)$.

Claim 1: We claim that $\omega(x, y; t)$ decays exponentially in x as $x \to \infty$.

Proof of Claim 1: Assume that (u, v) is given. Let

$$\omega(x,y;t) = e^{-\tilde{\delta}x}\tilde{\omega}(x,y;t), \tag{4.4}$$

for some $\tilde{\delta} > 0$ to be determined. We note that since $\omega(x, y; 0)$ is bounded and has compact support, then $\tilde{\omega}(x, y; 0)$ must also be bounded and have compact support. Plug equation (4.4) back into the vorticity equation (4.3), and we have

$$\tilde{\omega}_t - (\tilde{\delta}u + \nu\tilde{\delta}^2)\tilde{\omega} + (u + 2\tilde{\delta}\nu)\tilde{\omega}_x + v\tilde{\omega}_y - \nu\Delta\tilde{\omega} - \tilde{f} = 0. \tag{4.5}$$

Here, $\tilde{f} = e^{\tilde{\delta}x}f$. So, we now stipulate the following:

$$|f(x,y;t)| < e^{-\alpha x}, \quad \forall x > 0,$$

for some α such that $\alpha > \tilde{\delta}$. This ensures that $\tilde{f}$ decays exponentially in space as well.

If $\tilde{\omega}$ attains a maximum, the maximum must be attained at time $t = 0$. This can be shown as follows:

We know that there exists some r_0 such that for all $r > r_0$, $|\tilde{f}| < |(\tilde{\delta}u + \nu\tilde{\delta}^2)\tilde{\omega}|$. (This is because we require $\tilde{f}$ to decay exponentially fast to zero in space, whereas $\nu\tilde{\delta}^2$ is a constant.) Therefore, we examine all points in space further away from the origin than r_0. Suppose the maximum in this region is attained at some point $(x_0, y_0; t_0)$. Without loss of generality, we may assume that $\tilde{\omega}(x_0, y_0; t_0) > 0$. Then at this maximum point, we have

$$\tilde{\omega}_x = \tilde{\omega}_y = 0,$$
$$\Delta\tilde{\omega} < 0.$$

We thus have

$$\tilde{\omega}_t = \underbrace{(\tilde{\delta}u + \nu\tilde{\delta}^2)\tilde{\omega}}_{\leq 0 \text{ if } \tilde{\delta}=\delta/\nu} + \underbrace{(\nu)\Delta\tilde{\omega}}_{<0} + \underbrace{\tilde{f}}_{|\tilde{f}|<|\tilde{\delta}u+\nu\tilde{\delta}^2|}. \tag{4.6}$$

(Note that because of the magnitude restrictions we put on $\tilde{f}$, the sign of $\tilde{f}$ has no impact on the sign of $\tilde{\omega}_t$.) Therefore, given the fact that $u < -\delta$, if we choose

$$\tilde{\delta} = \frac{\delta}{\nu},$$

we know that

$$\tilde{\omega}_t(x, y; t) < 0.$$

This means that the maximum must occur at $t = 0$, which gives

$$|\tilde{\omega}(x, y; t)|_\infty \leq |\tilde{\omega}(x, y; 0)|_\infty < \infty. \tag{4.7}$$

Since $\tilde{\omega}(x, y; t)$ is bounded, it is then clear that because we have

$$\omega = e^{-\tilde{\delta}x}\tilde{\omega}, \tag{4.8}$$

$\omega(x, y; t)$ decays exponentially as $x \to \infty$ for $u < -\delta$.

Case 2: Suppose for $u_1 > \delta > 0$, $u > \delta$.

Claim 2: We claim that, with the introduction of a moving coordinate system, $\omega(x, y; t)$ decays exponentially in x as $x \to \infty$. (That is, for any fixed time t, we can show exponential decay in space).

Proof of Claim 2: For our coordinate system to move at constant speed with respect to time, we have for some x' and t', and some constant speed c,

$$\begin{aligned} x &= x' + ct \quad \Rightarrow \quad x' = x - ct, \\ t &= t'. \end{aligned}$$

We make a change of variables in equation (4.3): let $\omega = \omega(x', y; t)$. Then

$$\frac{\partial t'}{\partial t} = 1; \quad \frac{\partial x'}{\partial t} = -c,$$

and

$$\frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \omega}{\partial x'}\frac{\partial x'}{\partial t}.$$

Then we have

$$\begin{aligned} \omega_t &= \omega_{t'} - c\omega_{x'}, \\ \omega_x &= \omega_{x'}, \\ \omega_{xx} &= \omega_{x'x'}. \end{aligned} \tag{4.9}$$

Substituting the values from equations (4.9) into equation (4.3) gives

$$\omega_{t'} + (u - c)\omega_{x'} + v\omega_y = \nu(\omega_{x'x'} + \omega_{yy}) + f = \nu\nabla\omega + f. \tag{4.10}$$

If we now choose

$$c > u + \delta,$$

we have

$$(u - c) < -\delta$$

We have now reduced Case 2 to Case 1, in which we know the solution of equation (4.3) decays exponentially as $x \to \infty$.

Note, also, that this last argument actually can be applied to all u for $u_0 < u < u_1$. As a matter of fact, we are free to allow δ to be as large as we like. This means we can always have $u < \delta$, and the arguments will still hold. Therefore, it is possible to choose one constant speed at which to move our coordinate system in time, so that at any fixed time t, and for any bounded wave speed u, the vorticity decays exponentially as $x \to \infty$.

Now, with regard to Case 3 and Case 4, the same arguments as given in Cases 1 and 2, respectively, can be applied to show that for a fixed time t, the vorticity decays exponentially as $y \to \infty$.

We have now shown, given the assumptions outlined at the beginning of this section, that the vorticity $\omega(x, y; t)$, with diffusion present, decays exponentially in space as we move away from the origin. Therefore, there are positive constants c and k, such that

$$|\omega(x, y; t)|_\infty \leq ce^{-kr} \tag{4.11}$$

for $r = (x^2 + y^2)^{1/2}$, c determined by the constant $|\omega(x, y; 0)|_\infty$, and $k > 0$.

5. Incompressible Navier-Stokes Flow: Pressure and Velocity . In this section, we follow the same line of reasoning that is used in sections 2.2 and 2.3, as well as in section 3, to determine the far-field behavior of the solutions of the incompressible Navier-Stokes equations (4.1). We show that the presence of viscosity and exponentially decaying forcing in these equations does not affect the far-field behavior of the pressure and velocity. That is, the far-field behavior of the incompressible Navier-Stokes solutions is the same as that for the Euler Equations solutions. This is because we were able to show, in section 4, that although the vorticity $\omega(x, y; t)$ no longer has compact support, its far-field behavior is dominated by exponential decay.

5.1. Velocity Behavior: All Time. In this section we will show that even with viscosity and forcing present, the inverse-distance decay behavior of the velocity is still valid. In fact, the presence of exponentially decaying forcing is not even relevant in the arguments to follow. What is important, is that in the presence of diffusion, the vorticity patch no longer has compact support for $t > 0$.

As was done in previous sections, once we demonstrate $1/r$ decay behavior for $\mathbf{u}$ in all time, we determine the far-field behavior of the pressure p, and we then go back and find the bound for the far-field behavior of $\mathbf{u}$ in finite time.

As done in section 2 (see equation (2.7), which is still valid in the presence of diffusion and forcing), we start by solving the Poisson equations

$$
\begin{aligned}
-\omega_y &= \Delta u \\
\omega_x &= \Delta v.
\end{aligned}
\tag{5.1}
$$

We want to solve these over all of $\Re^2$, but as before, we first solve over the finite disk D with boundary $B = \partial D$, and then allow the radius of D, r_D, to go to ∞.

We will focus on solving equation (5.1) for the first component u in velocity vector $\mathbf{u} = (u, v)$, since the procedure for solving for v is analogous. Solving with Theorem 1, and integrating by parts leads to the expression

$$
u(x,y;t) = \frac{-1}{2\pi}\left(\int_B \omega(\xi,\eta;t)\log(\tilde{r}) + \iint_D \omega(\xi,\eta;t)\frac{y-\eta}{\tilde{r}^2}d\xi d\eta\right), \tag{5.2}
$$

with $\tilde{r} = ((x-\xi)^2 + (y-\eta)^2)^{1/2}$. As previously stated, since we actually want to solve the problem over all of $\Re^2$, we let the radius of disk D, r_D, go to ∞.

In section 2, when we allowed ω to have compact support, we knew that since $\omega \equiv 0$ outside a large region, we could set

$$
\lim_{r_D\to\infty}\int_B \omega\log(\tilde{r}) = 0. \tag{5.3}
$$

That gave

$$
u(x,y;t) = \frac{-1}{2\pi}\iint_{\Re^2}\omega(\xi,\eta;t)\frac{y-\eta}{\tilde{r}^2}d\xi d\eta. \tag{5.4}
$$

In addition, because of the compact support of $\omega(\xi,\eta;t)$, we had no problem with encountering the singularity $(\xi,\eta) = (x,y)$, as long as we assumed (x,y) outside the support of ω. At this point we were able to see clearly that $u(x,y;t) \sim c/r$ for large $r = (x^2+y^2)^{1/2}$.

Now, however, with viscosity present, we no longer have the assumption of compact support for vorticity ω. What we do have is the exponential decay of ω for large r. We must now take into account the presence of the singularity at $(\xi,\eta) = (x,y)$, which might possibly fall on the boundary B or inside the disk D.

In the case where the singularity potentially falls on the boundary B, we know that since the radius of D is being extended to ∞, we can just continue to extend the radius of D so that the singularity is always contained in D, and bounded away from the boundary B.

We now introduce

Lemma 1 *Let $r = (x^2+y^2)^{1/2}$. Let disk D have radius r_D and boundary ∂D. Given function $f(x,y)$ such that*

$$
|f(x,y)| \leq r^{-\alpha},
$$

for large r, then for all real α, such that $\alpha > 1$, we have

$$
\lim_{r_D\to\infty}\int_{\partial D} f(x,y) = 0.
$$

Proof Integrating $f(x,y)$ about the boundary of disk D gives

$$\begin{aligned}\int_{\partial D} f(x,y) &= \int_{-r_D}^{+r_D} f(x,(r_D^2-x^2)^{1/2}) - f(x,-(r_D^2-x^2)^{1/2})dx\\ &\leq \int_{-r_D}^{+r_D} 2|(x^2+(r_D^2-x^2))^{-\alpha/2}|dx\\ &= 2\int_{-r_D}^{+r_D} r_D^{-\alpha}dx\\ &= 4r_D^{1-\alpha}\end{aligned}$$

Clearly,

$$\lim_{r_D\to\infty} 4r_D^{1-\alpha}=0,$$

since $\alpha > 1$. This implies

$$\lim_{r_D\to\infty}\int_{\partial D} f(x,y)=0.$$

QED □

Because of the exponential decay in space of vorticity ω, we have at least

$$\omega(\xi,\eta;t)\log(\tilde{r}) \leq 1/\tilde{r}$$

for large $\tilde{r}$. Therefore, if we allow disk D to be centered at singularity (x,y), Lemma 1 applies, and we have

$$\lim_{r_D\to\infty}\int_B \omega\log(\tilde{r})=0. \tag{5.5}$$

Now we have come to the point in our case where we can say

$$u(x,y;t)=\frac{-1}{2\pi}\iint_{\Re^2}\omega(\xi,\eta;t)\frac{y-\eta}{\tilde{r}^2}d\xi d\eta. \tag{5.6}$$

We note that since $(y-\eta)\leq((y-\eta)^2+(x-\xi)^2)^{1/2}=\tilde{r}$,

$$u(x,y;t)\leq c\iint_{\Re^2}\omega(\xi,\eta;t)\frac{1}{\tilde{r}}d\xi d\eta. \tag{5.7}$$

We still have the potential problem of encountering the singularity within our region of integration.

To deal with the presence of the singularity, we treat it in much the same way we treated the presence of the singularity when estimating the pressure p in section 2. Split $\Re^2$ into three regions: a disk D_1 centered at the origin, with large enough radius r_{D_1} so that outside D_1, ω can be approximated by exponential decay; a disk D_2, centered at the singularity, with radius r_{D_2}; and the remainder, a region $D_0=\Re^2-D1-D2$. We will choose the radii of D_1 and D_2 so that $r_{D_1} + r_{D_2} < r$, where $r = (x^2+y^2)^{1/2}$ is the distance to the singularity. Then we have

$$u(x,y;t) \leq c\left(\underbrace{\iint_{D_0}\omega(\xi,\eta;t)\frac{1}{\tilde{r}}d\xi d\eta}_{I_0} + \underbrace{\iint_{D_1}\omega(\xi,\eta;t)\frac{1}{\tilde{r}}d\xi d\eta}_{I_1} + \underbrace{\iint_{D_2}\omega(\xi,\eta;t)\frac{1}{\tilde{r}}d\xi d\eta}_{I_2}\right).$$

By equation (4.11), we know that there are some constants c and k such that

$$|\omega|_\infty \leq ce^{-kr}, \tag{5.8}$$

with $r = (x^2+y^2)^{1/2}$ the distance from the origin.

We first examine integral I_2. Within disk D_2 we can bound the maximum norm of ω by $ce^{-k(r-r_{D_2})}$. Substituting in this bound for ω, and converting to polar coordinates gives

$$\begin{aligned} I_2 &\leq c_1 e^{-k(r-r_{D_2})}\iint_{D_2}\frac{1}{\tilde{r}}d\xi d\eta \\ &= c_1 e^{-k(r-r_{D_2})}\int_0^{2\pi}\int_0^{r_{D_2}} 1 d\tilde{r}d\theta \\ &= c_2 e^{-k(r-r_{D_2})}\,\tilde{r}\Big]_0^{r_{D_2}} \\ &= C r_{D_2} e^{-k(r-r_{D_2})} \end{aligned} \tag{5.9}$$

for some constant C.

Next, for I_0, we again make use of the facts that we can bound ω by exponential decay, and that $\tilde{r}$ never gets smaller than r_{D_2}. Then we have

$$\begin{aligned} I_0 &= \iint_{\Re^2-D_1-D_2}\frac{\omega}{\tilde{r}}d\xi d\eta \\ &\leq \frac{1}{r_{D_2}}\iint_{\Re^2-D_1-D_2}\omega d\xi d\eta \\ &\leq \frac{1}{r_{D_2}}\iint_{\Re^2-D_1}\omega d\xi d\eta \\ &\leq \frac{c_1}{r_{D_2}}\int_0^{2\pi}\int_{r_{D_1}}^{\infty} e^{-kr} r dr d\theta \\ &= \frac{c_2}{r_{D_2}}\int_{r_{D_1}}^{\infty} e^{-kr} r dr \\ &= \frac{c_2}{r_{D_2}}\left(\frac{-1}{k}re^{-kr}\Big]_{r_{D_1}}^{\infty} - \int_{r_{D_1}}^{\infty}\frac{-1}{k}e^{-kr}dr\right) \\ &= \frac{c_2}{r_{D_2}}\left(\frac{1}{k}r_{D_1}e^{-kr_{D_1}} - \frac{1}{k^2}e^{-kr}\Big]_{r_{D_1}}^{\infty}\right) \\ &= \frac{c_2}{r_{D_2}}\left(\frac{1}{k}r_{D_1}e^{-kr_{D_1}} - \frac{1}{k^2}e^{-kr_{D_1}}\right) \\ &\leq C\frac{r_{D_1}}{r_{D_2}}e^{-kr_{D_1}} \end{aligned}$$

for some constant C.

Finally, for integral I_1, since we never range outside the disk D_1, we can make $(x,y) >> (\xi,\eta)$. Then it is clear that

$$I_1 \le C/r, \tag{5.10}$$

for some constant C.

Now we choose $r_{D_1} = r/3$ and $r_{D_2} = r/3$, for example. (All that is required is that the radii of the disks are proportional to r, and that their sum is less than r.) Then for some appropriate constants c_1, c and k, we have

$$\begin{aligned} u(x,y;t) &\le I_0 + I_1 + I_2 \\ &\le c_1\left(e^{-kr} + \frac{1}{r} + re^{-kr}\right) \\ &\le \frac{c}{r}. \end{aligned} \tag{5.11}$$

Thus we have shown c/r decay behavior for u with viscosity. It follows that the far-field behavior of $\mathbf{u}$ is given by

$$\mathbf{u} \le c/r.$$

In section 2 we discussed the fact that since $\mathbf{u}$ had c/r asymptotic behavior, $\mathbf{u}_x$ and $\mathbf{u}_y$ had c/r^2 asymptotic behavior. In that case, we could reasonably claim that the integrand of equation (2.12) never became singular, since the vorticity had compact support (and thus the variables of integration (ξ,η) could always be bounded away from the singularity at (x,y)). Thus, (2.12) was continuously differentiable, so we could differentiate under the integral sign, and easily show $1/r^2$ behavior for $\mathbf{u}_x$, $\mathbf{u}_y$.

With viscosity now present, and the support of vorticity no longer compact, we show explicitly that differentiation under the integral sign is still allowable.

We prove the following theorem (a closely related theorem and proof can be found in [6, pp.151-156]):

THEOREM 2 *Define the function $u_i(x,y)$ to be*

$$u_i(x,y) = \frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)K_i(\xi,\eta;x,y)\,d\xi d\eta, \qquad i = 1,2. \tag{5.12}$$

Let

$$\begin{aligned} K_1(\xi,\eta;x,y) &= \log\tilde{r}, \\ K_2(\xi,\eta;x,y) &= \frac{\partial}{\partial\xi}\log\tilde{r} = \frac{(y-\eta)}{\tilde{r}^2}, \end{aligned}$$

where

$$\tilde{r} = \sqrt{(x-\xi)^2 + (y-\eta)^2}. \tag{5.13}$$

Then:

(1) *For $g(x,y)$ continuous in $\Re^2$, the function $u(x,y)$ satisfies*

$$(u_i(x,y))_x = -\frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{\partial}{\partial\xi}K_i(\xi,\eta;x,y)d\xi d\eta. \tag{5.14}$$

(2) *Suppose $|g(\xi,\eta)||K_i(\xi,\eta;x,y)| < 1/\tilde{r}$ for large $\tilde{r}$, i.e., for large $\tilde{r}$ we have*

$$\begin{aligned} |g(\xi,\eta)| &< 1/(\tilde{r}\log\tilde{r}), \quad i=1,\\ |g(\xi,\eta)| &< \tilde{r}/(y-\eta), \quad i=2. \end{aligned}$$

Provided that $g(x,y)$ belongs to C^1 (i.e., $g(x,y)$ has a continuous derivative) in $\Re^2$, the function $u(x,y)$ satisfies

$$(u_i(x,y))_{xx} = \frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{\partial^2}{\partial^2\xi}K_i(\xi,\eta;x,y)d\xi d\eta, \tag{5.15}$$

Proof We prove part (1) of the theorem by first verifying that the first order derivatives of u can be obtained by differentiating equation (5.12) under the integral sign. We do this indirectly. For this situation, it is only necessary to require that the function $g(x,y)$ be continuous (in C^0) in $\Re^2$. Define the functions

$$u_i^\epsilon = \frac{1}{2\pi}\iint_{\Re^2} K_i^\epsilon(\tilde{r})g(\xi,\eta)d\xi d\eta \tag{5.16}$$

where

$$K_1^\epsilon(\tilde{r}) = \begin{cases} \frac{1}{2}\left(\frac{\tilde{r}^2}{\epsilon^2}-1\right)+\log\epsilon & \text{for } \tilde{r}\le\epsilon\\ \log\tilde{r} & \text{for } \tilde{r}>\epsilon, \end{cases} \tag{5.17}$$

and

$$K_2^\epsilon(\tilde{r}) = \begin{cases} \frac{1}{2}\left(\frac{\tilde{r}^2}{\epsilon^2}-1\right)+\frac{y-\eta}{\epsilon^2} & \text{for } \tilde{r}\le\epsilon\\ (y-\eta)/\tilde{r}^2 & \text{for } \tilde{r}>\epsilon. \end{cases} \tag{5.18}$$

The auxiliary functions K_i^ϵ differ from the fundamental solutions, K_i, only in a circular neighborhood $\mathcal{N}_\epsilon$ about (x,y) where in contrast to K_i they remain bounded. (It is easily verified that K_i^ϵ belongs to C^1.)

We then have the following estimates:

$$\begin{aligned} |u_1 - u_1^\epsilon| &\le \frac{1}{2\pi}\iint_{\Re^2} |(\log\tilde{r} - K_1^\epsilon)g(\xi,\eta)|d\xi d\eta\\ &\le \frac{1}{2\pi}\iint_{\mathcal{N}_\epsilon} |g|\left(|\log\tilde{r}| + \frac{1}{2}\frac{\tilde{r}^2}{\epsilon^2} + \frac{1}{2} + |\log\epsilon|\right)d\xi d\eta\\ &\le \frac{M}{2\pi}\int_0^{2\pi}\int_0^{\epsilon}\left(|\tilde{r}\log\tilde{r}| + \frac{1}{2}\frac{\tilde{r}^3}{\epsilon^2} + \frac{1}{2}\tilde{r} + |\tilde{r}\log\epsilon|\right)d\tilde{r}d\theta\\ &\le 2M(\epsilon^2\log\epsilon + \epsilon^2) \end{aligned}$$

where M is a bound for g. It follows that for $\epsilon\to 0$, the u_1^ϵ converge uniformly to u_1 in $\Re^2$.

Also,

$$\begin{aligned}
|u_2 - u_2^\epsilon| &\leq \frac{1}{2\pi}\iint_{\Re^2} |(\frac{y-\eta}{\tilde{r}^2} - K_2^\epsilon) g(\xi,\eta)| d\xi d\eta \\
&\leq \frac{1}{2\pi}\iint_{\mathcal{N}_\epsilon} |g| \left(|\frac{y-\eta}{\tilde{r}^2}| + \frac{1}{2}\frac{\tilde{r}^2}{\epsilon^2} + \frac{1}{2} + |\frac{y-\eta}{\epsilon^2}| \right) d\xi d\eta \\
&\leq \frac{M}{2\pi}\int_0^{2\pi}\int_0^\epsilon \left(|\frac{y-\eta}{\tilde{r}}| + \frac{1}{2}\frac{\tilde{r}^3}{\epsilon^2} + \frac{1}{2}\tilde{r} + |\tilde{r}\frac{y-\eta}{\epsilon^2}| \right) d\tilde{r} d\theta \\
&\leq M\int_0^\epsilon \left(1 + \frac{\tilde{r}^3}{2\epsilon^2} + \frac{\tilde{r}}{2} + \frac{\tilde{r}^2}{\epsilon^2} \right) dr \\
&\leq M(\epsilon + \epsilon^2).
\end{aligned}$$

It follows, once again, that for $\epsilon \to 0$, the u_2^ϵ converge uniformly to u_2 in $\Re^2$.

Now, since the K_i^ϵ are continuously differentiable, we may differentiate equation (5.16) under the integral sign to obtain

$$\frac{\partial u_i^\epsilon}{\partial x} = \frac{1}{2\pi}\iint_{\Re^2} \frac{\partial K_i^\epsilon}{\partial x} g(\xi,\eta) d\xi d\eta. \tag{5.19}$$

Notice that

$$\frac{\partial}{\partial x} K_1^\epsilon(\xi,\eta;x,y) = \frac{\partial}{\partial x} K_2^\epsilon(\xi,\eta;x,y).$$

Let us now consider the candidate Φ for the derivative of u_i $(i = 1,2)$ with respect to x, obtained by differentiating (5.12) under the integral sign; that is, consider the convergent integral

$$\Phi(x,y) \equiv \frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{(\xi - x)}{\tilde{r}^2} d\xi d\eta. \tag{5.20}$$

Then we get the following estimate (for $i = 1,2$):

$$\begin{aligned}
|\frac{\partial u_i^\epsilon}{\partial x} - \Phi| &\leq \frac{1}{2\pi}\iint_{\Re^2} \left| \left(\frac{\partial K_i^\epsilon}{\partial x} - \frac{(\xi - x)}{\tilde{r}^2} \right) g(\xi,\eta) \right| d\xi d\eta \\
&\leq \frac{1}{2\pi}\iint_{\mathcal{N}_\epsilon} \left| \frac{\partial K_i^\epsilon}{\partial x} - \frac{(\xi - x)}{\tilde{r}^2} \right| |g| d\xi d\eta \\
&\leq \frac{M}{2\pi}\int_0^{2\pi}\int_0^\epsilon |\cos\theta| \left(\frac{\tilde{r}^2}{\epsilon^2} + 1 \right) d\tilde{r} d\theta \\
&\leq M\int_0^\epsilon \left(\frac{\tilde{r}^2}{\epsilon^2} + 1 \right) d\tilde{r} \\
&\leq \frac{4}{3} M\epsilon.
\end{aligned}$$

That is, for $\epsilon \to 0$, the $\frac{\partial u_{i\epsilon}}{\partial x}$ converge uniformly to Φ in $\Re^2$. Then it follows, since $u_{i\epsilon} \to u_i$, that $\frac{\partial u_i}{\partial x}$ exists, is continuous in $\Re^2$, and is given by Φ.

Thus, it is established that

$$(u_i)_x = \frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{\partial}{\partial x}K_i(\xi,\eta;x,y)d\xi d\eta \tag{5.21}$$

or

$$(u_i)_x = -\frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{\partial}{\partial \xi}K_i(\xi,\eta;x,y)d\xi d\eta, \tag{5.22}$$

where we make use of the fact that $\frac{\partial K_i(\xi,\eta;x,y)}{\partial x} = -\frac{\partial K_i(\xi,\eta;x,y)}{\partial \xi}$. Part (1) of Theorem 2 has been proved.

We now prove part (2) of the theorem: to compute the second order derivative of u_i from equation (5.22), all we require is that we have $g \in C^1$. We also require that, for $i = 1$, g decay at least as rapidly as $1/(\tilde{r}\log\tilde{r})$ as $\tilde{r} \to \infty$, and that, for $i = 2$, g be bounded by $\tilde{r}/(y-\eta)$ (which is > 1) as $\tilde{r} \to \infty$. We can apply Lemma 1 to discard boundary terms, and integrating by parts gives:

$$(u_i)_x = \frac{1}{2\pi}\iint_{\Re^2} K_i(\xi,\eta;x,y)\frac{\partial g(\xi,\eta)}{\partial \xi}d\xi d\eta. \tag{5.23}$$

Since $g \in C^1$, we have that at least $\frac{\partial g(\xi,\eta)}{\partial \xi} \in C^0$. Equation (5.23) is now in the same form as equation (5.12). Thus, by the same arguments, we know that we can differentiate under the integral sign to obtain

$$\begin{aligned}
(u_i)_{xx} &= \frac{\partial}{\partial x}\frac{1}{2\pi}\iint_{\Re^2} K_i(\xi,\eta;x,y)\frac{\partial g(\xi,\eta)}{\partial \xi}d\xi d\eta \\
&= -\frac{\partial}{\partial x}\frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{\partial K_i(\xi,\eta;x,y)}{\partial \xi}d\xi d\eta \\
&= -\frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{\partial}{\partial x}\frac{\partial K_i(\xi,\eta;x,y)}{\partial \xi}d\xi d\eta \\
&= \frac{1}{2\pi}\iint_{\Re^2} g(\xi,\eta)\frac{\partial^2 K_i(\xi,\eta;x,y)}{\partial \xi^2}d\xi d\eta,
\end{aligned}$$

and the theorem is proved. QED □

Note that the same proof techniques can be used to show that analogous results hold for first and second derivatives in *both* x and y.

We now apply Theorem 2 to the velocity $u(x,y;t)$, where the vorticity $\omega(x,y;t)$ replaces $g(x,y)$ in the theorem, and the kernel of the integral is K_2 in the theorem. All the requirements of the hypotheses are met, so we *can* differentiate equation (5.6) under the integral sign to obtain the following expressions for the derivatives of the velocity (see also equation (2.14)):

$$\begin{aligned}
u_x &= \frac{1}{2\pi}\iint_{\Re^2} \omega(\xi,\eta;t)\frac{\partial}{\partial \xi}\frac{(y-\eta)}{\tilde{r}^2}d\xi d\eta \\
&= \frac{1}{\pi}\iint_{\Re^2} \frac{(y-\eta)(x-\xi)}{\tilde{r}^4}\omega(\xi,\eta;t)d\xi d\eta
\end{aligned} \tag{5.24}$$

and

$$(5.25)\qquad u_{xx} = \frac{-1}{2\pi}\iint_{\Re^2}\omega(\xi,\eta;t)\frac{\partial^2}{\partial\xi^2}\frac{(y-\eta)}{\tilde{r}^2}d\xi d\eta$$

$$(5.26)\qquad = \frac{-1}{\pi}\iint_{\Re^2}\omega(\xi,\eta;t)\frac{\partial}{\partial\xi}\frac{(y-\eta)(x-\xi)}{\tilde{r}^4}d\xi d\eta$$

$$(5.27)\qquad = \frac{-1}{\pi}\iint_{\Re^2}\omega(\xi,\eta;t)\left(\frac{4(y-\eta)(x-\xi)^2}{\tilde{r}^6}-\frac{(y-\eta)}{\tilde{r}^4}\right)d\xi d\eta.$$

We return now to the determination of the decay behavior of the derivatives of the velocity component $u(x,y;t)$. When determining the asymptotic behavior of the velocity u itself, we saw from equation (5.6) that equation (5.7) followed, *i.e.*, we knew

$$(5.28)\qquad u(x,y;t)\leq c\iint_{\Re^2}\frac{1}{\tilde{r}}\omega(\xi,\eta;t)d\xi d\eta.$$

From here, it was argued from equation (5.7) through equation (5.11) that as $r\to\infty$, we have

$$(5.29)\qquad u(x,y;t)\leq\frac{c}{r}.$$

Similarly, from equations (5.24) and (5.25), we have the bounds

$$u_x\leq c\iint_{\Re^2}\frac{1}{\tilde{r}^2}\omega(\xi,\eta;t)d\xi d\eta$$

and

$$u_{xx}\leq c\iint_{\Re^2}\frac{1}{\tilde{r}^3}\omega(\xi,\eta;t)d\xi d\eta.$$

From this point we argue in the same way we argued for the velocity u itself that

$$(5.30)\qquad u_x(x,y;t)\leq\frac{c}{r^2}$$

and

$$(5.31)\qquad u_{xx}(x,y;t)\leq\frac{c}{r^3}$$

as $r\to\infty$.

Nearly identical arguments, and the same decay rates, hold for the derivatives of u in y, as well as for the x and y derivatives of the second velocity component v.

5.2. Pressure Behavior. The analogous equation to (2.2), and the scalar equivalent to (4.1) is

$$(5.32)\qquad \begin{aligned} u_t+uu_x+vu_y+p_x &= \nu\Delta u+f_1\\ v_t+uv_x+vv_y+p_y &= \nu\Delta v+f_2.\end{aligned}$$

We take the divergence of these equations, and recall the incompressibility condition,

$$div(\mathbf{u}) = div(u, v) = u_x + v_y = 0. \tag{5.33}$$

Because the velocity has zero divergence, terms on the left hand side of the equation will cancel (as in section 2), and we get

$$u_x^2 + v_y^2 + 2u_x v_y + \Delta p = \nu \left((\Delta u)_x + (\Delta v)_y \right) + \nabla \cdot \mathbf{F}. \tag{5.34}$$

We note that equation (2.7) is unaffected by the presence of viscosity and forcing, since it comes about by the zero divergence of the velocity. Equation (2.7) applied to the right hand side yields

$$u_x^2 + v_y^2 + 2u_x v_y + \Delta p = \nu \left((-\omega_y)_x + (\omega_x)_y \right) + \nabla \cdot \mathbf{F}, \tag{5.35}$$

and therefore

$$u_x^2 + v_y^2 + 2u_x v_y + \Delta p = \nabla \cdot \mathbf{F}. \tag{5.36}$$

Notice that equation (5.36) is of the same form as equation (2.19), except for the presence of the exponentially decaying right hand side. The presence of $\nabla \cdot \mathbf{F}$ does not contribute to the far-field behavior of the pressure p, since we have required $\nabla \cdot \mathbf{F}$ to decay so rapidly in space. Thus, the estimates on the decay behavior of the pressure and the pressure gradient induced by a vorticity patch, given in (2.30) and (2.33), are still valid. In addition, the estimates on the decay behavior of the pressure and the pressure gradient induced by a rigid body, given in (3.18) and (3.19), are also valid in the presence of diffusion and exponentially decaying forcing.

To summarize, equation (4.1) yields the following far-field decay behavior estimates for the pressure gradient (for some constant c, and for $r = (x^2 + y^2)^{1/2}$):

Pressure induced by a vorticity patch:

$$\nabla p \leq \frac{c \log r}{r^3}. \tag{5.37}$$

Pressure induced by a rigid body:

$$\nabla p \leq \frac{c}{r^2}. \tag{5.38}$$

5.3. Velocity Behavior: Finite Time. Turning now to the determination of the far-field behavior of the velocity along its characteristic lines, we refer back to the arguments given in section 2.3. The line of reasoning with viscosity present is almost identical to that when it is absent. There is only one minor point to observe. Instead of using equation (2.2) as our starting point, we use equation (5.32),

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= \nu \Delta u + f_1 \\ v_t + uv_x + vv_y + p_y &= \nu \Delta v + f_2. \end{aligned}$$

Decay behavior for the pressure gradient, in the absence or presence of a rigid body, respectively, was given in equations (5.37) and (5.38).

The behavior of the pressure gradient in either case dominates the c/r^3 behavior of the Laplacian and the exponential decay behavior of the forcing on the right hand side. Thus, in the absence of a rigid body, equations (2.35) are still valid with viscosity and forcing present.

Also, for flow induced by a rigid body, equation (3.23) still holds in the presence of viscosity and forcing. The arguments put forth in section 3 (determining the behavior of pressure and velocity induced by a rigid body) are also applicable with viscosity and forcing present, again because the exponential decay behavior of the vorticity and the forcing does not contribute to the overall decay behavior of the pressure and the velocity.

Therefore, the arguments that follow after equations (2.35) and (3.23), to determine the decay behavior of velocity in finite time, will be identical.

To summarize, we have shown that in finite time, and with viscosity and exponentially decaying forcing present, the velocity $\mathbf{u}(x, y; t)$ induced by a vorticity patch can be bounded by decay behavior

$$\mathbf{u}(x, y; t) \leq c \log(r)/r^3, \tag{5.39}$$

and velocity $\mathbf{u}(x, y; t)$ induced by a rigid body can be bounded by decay behavior

$$\mathbf{u}(x, y; t) \leq c/r^2. \tag{5.40}$$

This is the same decay behavior we found for the Euler equations.

6. Incompressible Navier-Stokes Flow: Vorticity Derivatives. Having shown exponential decay for the vorticity ω of incompressible Navier-Stokes flow, we wish to show that both $D\omega$ and $D^2\omega$ decay exponentially in space, as well. Here we define

$$D\omega = \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \mathbf{z}, \tag{6.1}$$

and

$$D^2\omega = \begin{pmatrix} \omega_{xx} \\ \omega_{yx} \\ \omega_{xy} \\ \omega_{yy} \end{pmatrix} = \begin{pmatrix} z^1_x \\ z^2_x \\ z^1_y \\ z^2_y \end{pmatrix} = \begin{pmatrix} w^1 \\ w^2 \\ w^3 \\ w^4 \end{pmatrix} = \mathbf{w}. \tag{6.2}$$

Recall that in section 5, we were able to use the exponential decay behavior of the vorticity $\omega(x, y; t)$ to show algebraic decay behavior for $\mathbf{u}$ and its first and second derivatives. We now make use of the algebraic decay behavior of the derivatives of $\mathbf{u}$ to show exponential decay behavior for the *derivatives* of $\omega(x, y; t)$.

We first examine $D\omega$. Taking the x and y derivatives of equation (4.3), we find

$$\omega_{xt} + u\omega_{xx} + v\omega_{xy} = \nu\Delta\omega_x + f_x - (u_x\omega_x + v_x\omega_y), \tag{6.3}$$

$$\omega_{yt} + u\omega_{xy} + v\omega_{yy} = \nu\Delta\omega_y + f_y - (u_y\omega_x + v_y\omega_y). \tag{6.4}$$

(Recall the stipulation that f and its derivatives be chosen to decay exponentially fast in space.) Making the substitutions given by the definition of $D\omega$ above, these two equations are the same as

$$z_t^1 + uz_x^1 + vz_y^1 = \nu\Delta z^1 + f_x - (u_x z^1 + v_x z^2), \tag{6.5}$$

$$z_t^2 + uz_x^2 + vz_y^2 = \nu\Delta z^2 + f_y - (u_y z^1 + v_y z^2). \tag{6.6}$$

Define the vector function $\Phi(x, y)$ to be

$$\Phi(x,y) = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}. \tag{6.7}$$

Equations (6.5) and (6.6) can now be re-written in vector form, which gives

$$\mathbf{z}_t + u\mathbf{z}_x + v\mathbf{z}_y = \nu\Delta\mathbf{z} - \Phi\cdot\mathbf{z} + \nabla f. \tag{6.8}$$

We are now in a position to carry out the same steps we carried out in section 4 for showing the exponential decay behavior of $\mathbf{z}$. We start by examining

Case 1: Suppose $u < \delta$ (for some $\delta > 0$). Let

$$\mathbf{z}(x,y;t) = e^{-\tilde{\delta}x}\tilde{\mathbf{z}}(x,y;t), \tag{6.9}$$

for some $\tilde{\delta} > 0$ to be determined. Recall that $\tilde{\mathbf{z}}(x, y; t = 0)$ is bounded with compact support. Substituting this back into equation (6.8) yields

$$\tilde{\mathbf{z}}_t - (\tilde{\delta}u + \nu\tilde{\delta}^2)\tilde{\mathbf{z}} + \Phi\cdot\tilde{\mathbf{z}} + (u + 2\tilde{\delta}\nu)\tilde{\mathbf{z}}_x + v\tilde{\mathbf{z}}_y - \nu\Delta\tilde{\mathbf{z}} - \tilde{f} = 0. \tag{6.10}$$

Here, $\tilde{f} = e^{\tilde{\delta}x}\nabla f$. We stipulate the following:

$$|\nabla f| < e^{-\alpha x}, \quad \forall x > 0, \tag{6.11}$$

for some α such that $\alpha > \tilde{\delta}$. This ensures the exponential decay in space of $\tilde{f}$. As pointed out in section 4, there exists an r_1 such that for all points at least distance r from the origin, where $r > r_1$, we know

$$|\tilde{f}| < |(\tilde{\delta}u + \nu\tilde{\delta}^2)\tilde{\mathbf{z}}|.$$

In addition, because the elements of Φ are made up of the first derivatives of velocity $\mathbf{u}$, we know that Φ decays like c/r^2 in space. Thus, there exists an r_2 such that for all $r > r_2$, each element of Φ is smaller in magnitude than $|\tilde{\delta}u + \nu\tilde{\delta}^2|$. Therefore, we let $r_0 = \max(r_1, r_2)$, and examine all points further from the origin than r_0. We suppose the maximum in this region is attained at some point $(x_0, y_0; t_0)$. Without loss of generality, we assume that $\tilde{\mathbf{z}}(x_0, y_0; t_0) > 0$. Then, at this maximum point,

$$\tilde{\mathbf{z}}_x = \tilde{\mathbf{z}}_y = 0, \tag{6.12}$$

$$\Delta\tilde{\mathbf{z}} < 0. \tag{6.13}$$

This yields

$$\tilde{\mathbf{z}}_t = \underbrace{(\tilde{\delta}u + \nu\tilde{\delta}^2)\tilde{\mathbf{z}}}_{\leq 0 \text{ if } \tilde{\delta}=\delta/\nu} + \underbrace{(\nu)\nabla\tilde{\mathbf{z}}}_{<0} + \underbrace{\Phi \cdot \mathbf{z}}_{|\Phi|_\infty < |\tilde{\delta}u+\nu\tilde{\delta}^2|} + \underbrace{\tilde{f}}_{|\tilde{f}|<|(\tilde{\delta}u+\nu\tilde{\delta}^2)\tilde{\mathbf{z}}|} . \tag{6.14}$$

(Note that because of the magnitude restrictions we put on $\tilde{f}$, the sign of $\tilde{f}$ has no impact on the sign of $\tilde{\mathbf{z}}_t$. In addition, because Φ decays to zero in space, the sign of the Φ term also has no impact on the sign of $\tilde{\mathbf{z}}_t$.) Therefore, given the fact that $u < -\delta$, if we choose

$$\tilde{\delta} = \frac{\delta}{\nu},$$

we know that

$$\tilde{\mathbf{z}}_t(x, y; t) < 0.$$

This means that the maximum must occur at $t = 0$, which gives

$$|\tilde{\mathbf{z}}(x, y; t)|_\infty \leq |\tilde{\mathbf{z}}(x, y; 0)|_\infty < \infty. \tag{6.15}$$

Since $\tilde{\mathbf{z}}(x, y; t)$ is bounded, it is then clear that because

$$\mathbf{z} = e^{-\tilde{\delta}x}\tilde{\mathbf{z}}, \tag{6.16}$$

$\mathbf{z}(x, y; t)$ decays exponentially as $x \to \infty$ for $u < -\delta$.

All the arguments given in section 4 for the velocity (u, v) taken at varying speeds can now be applied directly here. Therefore, there exist positive constants c and k such that

$$|D\omega|_\infty = |\mathbf{z}(x, y; t)|_\infty \leq ce^{-kr} \tag{6.17}$$

for $r = (x^2 + y^2)^{1/2}$, c determined by the constant $|\mathbf{z}(x, y; 0)|_\infty$, and $k > 0$.

We can similarly show exponential decay behavior for $D^2\omega$. Taking all the second derivatives of equation (4.3) yields

$$\begin{aligned}
z^1_{xt} + uz^1_{xx} + vz^1_{xy} &= \nu\Delta z^1_x + f_{xx} - (u_{xx}z^1 + u_x z^1_x + v_{xx}z^2 + v_x z^2_x + u_x z^1_x + v_x z^1_y) \\
z^2_{xt} + uz^2_{xx} + vz^2_{xy} &= \nu\Delta z^2_x + f_{xy} - (u_{yx}z^1 + u_y z^1_x + v_{yx}z^2 + v_y z^2_x + u_x z^2_x + v_x z^2_y) \\
z^1_{yt} + uz^1_{xy} + vz^1_{yy} &= \nu\Delta z^1_y + f_{xy} - (u_{xy}z^1 + u_x z^1_y + v_{xy}z^2 + v_x z^2_y + u_y z^1_x + v_y z^1_y) \\
z^2_{yt} + uz^2_{xy} + vz^2_{yy} &= \nu\Delta z^2_y + f_{yy} - (u_{yy}z^1 + u_y z^1_y + v_{yy}z^2 + v_y z^2_y + u_y z^2_x + v_y z^2_y),
\end{aligned}$$

which leads to

$$\begin{aligned}
w^1_t + uw^1_x + vw^1_y &= \nu\Delta w^1 + f_{xx} - \mathbf{u}_{xx} \cdot \mathbf{z} - (2u_x z^1_x + v_x z^2_x + v_x z^1_y) \\
w^2_t + uw^2_x + vw^2_y &= \nu\Delta w^2 + f_{xy} - \mathbf{u}_{xy} \cdot \mathbf{z} - (u_y z^1_x + v_x z^2_y) \\
w^3_t + uw^3_x + vw^3_y &= \nu\Delta w^3 + f_{xy} - \mathbf{u}_{xy} \cdot \mathbf{z} - (u_y z^1_x + v_x z^2_y) \\
w^4_t + uw^4_x + vw^4_y &= \nu\Delta w^4 + f_{yy} - \mathbf{u}_{yy} \cdot \mathbf{z} - (u_y z^2_x + u_y z^1_y + 2v_y z^2_y).
\end{aligned}$$

Define the four-element vectors ψ_i to be

$$\begin{aligned}\psi_1 &= (2u_x, u_y, u_y, 0)\\ \psi_2 &= (v_x, 0, 0, u_y)\\ \psi_3 &= (v_x, 0, 0, u_y)\\ \psi_4 &= (0, v_x, v_x, 2v_y).\end{aligned}$$

Define the vector function $\Psi(x,y)$ to be

$$\Psi(x,y) = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}, \tag{6.18}$$

and the vector function $\mathbf{Uz}(x,y)$ to be

$$\mathbf{Uz}(x,y) = \begin{pmatrix}\mathbf{u}_{xx}\cdot\mathbf{z}\\ \mathbf{u}_{xy}\cdot\mathbf{z}\\ \mathbf{u}_{xy}\cdot\mathbf{z}\\ \mathbf{u}_{yy}\cdot\mathbf{z}\end{pmatrix}. \tag{6.19}$$

Then the vector form for the equations for $D^2\omega$ becomes

$$\mathbf{w}_t + u\mathbf{w}_x + v\mathbf{w}_y = \nu\Delta\mathbf{w} + \nabla\nabla f - \mathbf{Uz} - \Psi\cdot\mathbf{w}. \tag{6.20}$$

We choose f to decay at a rapid enough exponential rate so that its second derivatives (and thus $\nabla\nabla f$) decay at a rapid enough exponential rate. Also, we have already shown that $\mathbf{z}$ decays exponentially fast in space, and that the second derivatives of $\mathbf{u}$ decay at most like c/r^2. Let

$$g = \nabla\nabla f - \mathbf{Uz}.$$

Now g is itself an exponentially decaying function. Therefore, analyzing (6.20) is equivalent to analyzing the behavior of the following equation:

$$\mathbf{w}_t + u\mathbf{w}_x + v\mathbf{w}_y = \nu\Delta\mathbf{w} + g - \Psi\cdot\mathbf{w}. \tag{6.21}$$

Note that each element of $\Psi(x,y)$ decays at least as quickly as c/r^2, so $\Psi(x,y)\to 0$ as $r\to\infty$. It is now clear that equation (6.21) is of the same form as equation (6.8). Therefore, all the arguments applied to equation (6.8) can also be applied to equation (6.21), and consequently to equation (6.20). It follows that there exist positive constants c and k such that for large enough r,

$$|D^2\omega|_\infty = |\mathbf{w}(x,y;t)|_\infty \le ce^{-kr}. \tag{6.22}$$

Thus, the exponential decay behavior of ω, as well as that of its first and second space derivatives, has been shown.

7. Slightly Compressible Navier-Stokes Flow. In this section we make use of our knowledge of the far-field decay behavior of the solutions of the incompressible Navier-Stokes equations, to determine the far-field decay behavior of the simplified, slightly compressible Navier-Stokes equations given by

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u} + \mathbf{F} \tag{7.1}$$

$$\epsilon^2\{p_t + (\mathbf{u}\cdot\nabla)p\} + \nabla\cdot\mathbf{u} = 0. \tag{7.2}$$

Here, $\nu > 0$, $\epsilon > 0$. The forcing function $\mathbf{F}(x,y;t)$, as well as all its derivatives, are assumed to decay exponentially fast in space and to be C^∞-smooth.

The discussion that follows of the decomposition of the slightly compressible solution, can also be found in [11]. Formally allowing $\epsilon \to 0$, the limiting equations become

$$\mathbf{U}_t + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla P = \nu\Delta\mathbf{U} + \mathbf{F}, \tag{7.3}$$

$$\nabla\cdot\mathbf{U} = 0. \tag{7.4}$$

These limiting equations now describe incompressible flow. For the limiting equations to be valid, however, proper initial data must be specified for consistency. Conditions for the initial data will be presented below.

We will discuss the Cauchy problem in $\Re^2$. For equations (7.1) and (7.2), we give the initial conditions

$$\mathbf{u}(x,y;0) = \mathbf{u}_0(x,y); \quad p(x,y;0) = p_0(x,y). \tag{7.5}$$

For (7.3) and (7.4), we can only prescribe initial velocity

$$\mathbf{U}(x,y;0) = \mathbf{U}_0(x,y), \tag{7.6}$$

where

$$\nabla\cdot\mathbf{U}_0 = 0 \tag{7.7}$$

is required for consistency.

If an initial velocity $\mathbf{u}_0$ for (7.1), (7.2) is given, we construct $\mathbf{U}_0$ so that

$$\begin{aligned} \nabla\cdot\mathbf{U}_0 &= 0, & \nabla\times\mathbf{U}_0 &= \nabla\times\mathbf{u}_0 \\ (1,U_0)_{L_2} &= (1,u_0)_{L_2}, & (1,V_0)_{L_2} &= (1,v_0)_{L_2}. \end{aligned} \tag{7.8}$$

Here, we use the notation

$$(f,g)_{L_2} = \iint_{\Re^2} f(x,y)g(x,y)\,dx\,dy$$

to denote the L_2-scalar product. According to [11], the incompressible problem (7.3), (7.4) and (7.6) has a solution $(\mathbf{U},P)$ in $0 \le t < \infty$ which is unique up to a time dependent function $\bar{P}(t)$, which can be added to P. We fix $\bar{P}(t)$, a constant in space, so that

$$(7.9) \qquad \begin{aligned} (1, p_0 - P(\cdot,0) - \bar{P}(0))_{L_2} &= 0, \\ (1, P_t + \bar{P}_t + (\mathbf{U}\cdot\nabla)P)_{L_2} &= 0, \quad t \geq 0. \end{aligned}$$

A theorem in [11] states the following:

Theorem 3 *Assume the initial data satisfy*

$$\nabla \cdot \mathbf{u}_0 = \mathcal{O}(\epsilon), \quad p_0 = P(\cdot,0) + \bar{\mathrm{P}}(0) + \mathcal{O}(1).$$

For any $T > 0$ and $0 < \epsilon \leq \epsilon_0(T)$, the compressible problem (7.1), (7.2), (7.5) has a unique solution in $0 \leq t \leq T$. It can be written in the form

$$(7.10) \qquad \begin{aligned} \mathbf{u} &= \mathbf{U} + \mathbf{u}_1 + \mathcal{O}(\epsilon^2), \\ p &= P + \bar{\mathrm{P}}(t) + p_1 + \mathcal{O}(\epsilon), \end{aligned}$$

where $(\mathbf{u}_1, p_1)$ are highly oscillatory in time. The functions $(\mathbf{u}_1, \epsilon p_1)$ and their space derivatives can be estimated by the initial data

$$\mathbf{u}_0 - \mathbf{U}_0, \quad \epsilon(p_0 - P(\cdot,0) - \bar{\mathrm{P}}(0))$$

and their space derivatives. □

According to [11], $\mathbf{u}_1$ and p_1 represent sound waves which oscillate on the fast time scale t/ϵ. (To first order, they do not create vorticity.) The $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon)$ terms in (7.10) contain the result of the interaction between the fast and slow time scales. We may continue to expand out the slow part of the solution of the slightly compressible Navier-Stokes equations from these terms. $\mathbf{U}$ and P, which are the solutions of (7.3), (7.4), are the first terms in the asymptotic expansion of the solution of the slow part. We have already analyzed the decay behavior of these first terms in the expansion.

To understand the decay behavior of the complete solution of the slightly compressible Navier-Stokes equations, it remains only to carry out the decay behavior analysis on the second term in the asymptotic expansion of the slow part. The reason for this will be made clear presently.

Our next step, then, is to find the second term in the asymptotic expansion. We do this as follows:

Define variables $\mathbf{u}', p'$ by

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad p = P + \bar{P}(t) + p'.$$

Plugging back into equations (7.1) and (7.2) we obtain

$$(7.11) \qquad \mathbf{u}'_t + (\mathbf{U}\cdot\nabla)\mathbf{u}' + (\mathbf{u}'\cdot\nabla)\mathbf{U} + (\mathbf{u}'\cdot\nabla)\mathbf{u}' + \nabla p' = \nu\Delta\mathbf{u}'$$

$$(7.12) \qquad \epsilon^2\{p'_t + (\mathbf{U}\cdot\nabla)p' + (\mathbf{u}'\cdot\nabla)P + (\mathbf{u}'\cdot\nabla)p'\} + \nabla\cdot\mathbf{u}' = \epsilon^2 g_1$$

where

$$(7.13) \qquad g_1 = -\{P_t + \bar{P}_t + (\mathbf{U}\cdot\nabla)P\}.$$

By (7.9), $(1, g_1)_{L_2} = 0$. The initial conditions for $\mathbf{u}'$ and p' are

$$\mathbf{u}' = \mathbf{u}_0 - \mathbf{U}_0, \qquad p' = p_0 - P(\cdot, 0) - \bar{P}(0) \quad \text{at} \quad t = 0.$$

We now determine the first term in the slow part of $\mathbf{u}'$ and p'. We write:

$$\mathbf{u}' = \epsilon^2 \mathbf{U}_1 + \mathbf{u}'', \qquad p' = \epsilon^2 \left(P_1 + \bar{P}_1(t) \right) + p'',$$

where we define $\mathbf{U}_1$, P_1 as the solution of the linearized incompressible problem

$$\mathbf{U}_{1t} + (\mathbf{U} \cdot \nabla)\mathbf{U}_1 + (\mathbf{U}_1 \cdot \nabla)\mathbf{U} + \nabla P_1 = \nu \Delta \mathbf{U}_1 \tag{7.14}$$

$$\nabla \cdot \mathbf{U}_1 = \epsilon^2 g_1. \tag{7.15}$$

with

$$(1, P_1(\cdot, t))_{L_2} = 0$$

and

$$\mathbf{U}_1 = \mathbf{U}_{1,0} \quad \text{at} \quad t = 0.$$

Here, the initial data are defined as the solution of

$$\nabla \cdot \mathbf{U}_{1,0} = g_1(\cdot, 0), \qquad \nabla \times \mathbf{U}_{1,0} = 0, \qquad (1, U_{1,0})_{L_2} = (1, V_{1,0})_{L_2} = 0.$$

Then the "error" or "remainder" terms satisfy the compressible correction equations

$$\begin{aligned}
\mathbf{u}''_t + (\mathbf{U}^{(1)} \cdot \nabla)\mathbf{u}'' + (\mathbf{u}'' \cdot \nabla)\mathbf{U}^{(1)} + (\mathbf{u}'' \cdot \nabla)\mathbf{u}'' + \nabla p'' = \nu \Delta \mathbf{u}'' + \epsilon^4 \mathbf{F}_2, \\
\epsilon^2 \left\{ p''_t + (\mathbf{U}^{(1)} \cdot \nabla) p'' + (\mathbf{u}'' \cdot \nabla) P^{(1)} + (\mathbf{u}'' \cdot \nabla) p'' \right\} + \nabla \cdot \mathbf{u}'' = \epsilon^4 g_2,
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{U}^{(1)} &= \mathbf{U} + \epsilon^2 \mathbf{U}_1, \\
P^{(1)} &= P + \epsilon^2 P_1, \\
\mathbf{F}_2 &= -(\mathbf{U}_1 \cdot \nabla)\mathbf{U}_1, \\
g_2 &= -\left\{ P_{1t} + \bar{P}_{1t} + (\mathbf{U} \cdot \nabla)P_1 + (\mathbf{U}_1 \cdot \nabla)P \right\} - \epsilon^2 (\mathbf{U}_1 \cdot \nabla)P_1.
\end{aligned}$$

$\bar{P}_1(t)$ is chosen so that $(1, g_2(\cdot, t))_{L_2} = 0$. Note that the equations for $\mathbf{u}''$ and p'' have the same structure as the equations for $\mathbf{u}'$ and p', except that the inhomogeneous terms have been reduced to $\mathcal{O}(\epsilon^4)$. It is shown in [11] that this expansion process can continue indefinitely. Each new term in the expansion will satisfy a linearized incompressible differential equation which is of the same form as that which the previous term satisfies (this is true from the second term in the expansion on). The inhomogeneous terms in the successive differential equations for the error (or compressible correction) can be reduced to arbitrarily high order in ϵ. Note, however, that the solution of the error equations will be small only if the initial data are properly specified.

The following theorem from [11] describes the form of the solutions of the slightly compressible Navier-Stokes equations with two terms of the slow part expanded out, and with proper initialization:

THEOREM 4 *If the initial data for (7.1), (7.2) are chosen such that two time derivatives of the solution are bounded independently of* ϵ *at time* $t = 0$, *then*

$$\begin{aligned} \mathbf{u} &= \mathbf{U} + \epsilon^2 \mathbf{U}_1 + \mathbf{u}_1 + \mathcal{O}(\epsilon^4), \\ p &= P + \bar{\mathrm{P}}(t) + \epsilon^2 \left(P_1 + \bar{\mathrm{P}}_1(t)\right) + p_1 + \mathcal{O}(\epsilon^3). \end{aligned}$$

Here, $\mathbf{U}_1$, P_1 *are solutions of the linearized incompressible equations, and* $\mathbf{u}_1 = \mathcal{O}(\epsilon^2)$, $p_1 = \mathcal{O}(\epsilon)$ *are highly oscillatory in time. The highly oscillatory part is suppressed further if more than two time derivatives stay bounded at* $t = 0$. □

In other words, the first two terms in the asymptotic expansion of the solution of the slightly compressible Navier-Stokes equations give the solution of the slow part, up to $\mathcal{O}(\epsilon^4)$ (or $\mathcal{O}(\epsilon^3)$ for pressure). The fast part can be suppressed to arbitrarily high order of ϵ with proper initialization.

Now, because each subsequent term in the expansion of the slow part will have the same form as its previous term (from the second term on), we see that it suffices to understand the decay behavior of the first and second terms of the asymptotic expansion of the slow part of the slightly compressible solution. This is why, *with proper initialization to suppress the fast part*, it is valid to claim that the solutions of the slightly compressible Navier-Stokes equations will exhibit the same asymptotic decay behavior as the sum of the first two terms in the asymptotic expansion of the slow part of the total solution.

At this point, we carry out the analysis of the decay behavior of the second term in the expansion of the slow part.

The first step is to note that we can directly infer the decay rate behavior of the function g_1 in (7.13), and its derivatives, from the already known behavior of the incompressible velocity and pressure solutions. We do this now. From equation (2.20) we get an equation for the time derivative of incompressible pressure,

$$\Delta P_t = c_1 \sum_{i,j=1}^{2} D_i D_j \frac{\partial}{\partial t}(U_i U_j). \tag{7.16}$$

From (7.3) and (7.4) it is clear that $\mathbf{U}_t$ decays at most like ∇P, which in turn decays at most like c/r^2 (see equation (5.38)). Using this decay behavior for $\mathbf{U}_t$ in equation (7.16), and carrying out the same analysis as is done in sections 2 and 3, reveals that P_t also decays like c/r^2. It follows that g_1 decays like $\bar{P}_t + c/r^2$ (where $\bar{P}$ is a constant in space). By similar arguments, we can find decay rates on ∇P_t and $\nabla(\mathbf{U} \cdot \nabla)P$, which show that for large r,

$$\nabla g_1 \leq c/r^3. \tag{7.17}$$

In addition, following a line of reasoning analogous to that above, one can determine the decay behavior for the derivative of g_1, and show that for large r,

$$(g_1)_t \leq c/r^4. \tag{7.18}$$

7.1. Vorticity Behavior: Linearized Incompressible Part. Our next step is to show that the vorticity for the linearized incompressible term (or second term in the expansion of the slow part) of the slightly compressible flow decays exponentially fast in space, as it did for the non-linear incompressible flow. If we define the vorticity W_1 to be the curl of the velocity (in two dimensions), then

$$W_1(x,y;t) = V_{1x} - U_{1y} \tag{7.19}$$

together with equations (7.14), (7.15) and (7.13) give the vorticity equation for the linearized incompressible term in the solution expansion:

$$W_{1t} + UW_{1x} + VW_{1y} + W(\mathbf{U}_1 \cdot \nabla + \nabla \cdot \mathbf{U}_1) = \nu \Delta W_1, \tag{7.20}$$

where $\mathbf{U} = (U,V)$ and $W = V_x - U_y$ are the velocity and vorticity of the incompressible flow equations (7.3)-(7.4), respectively.

Carrying out an analysis almost identical to that of sections 4 and 6, we first assume that for some

$$\delta > 0,$$

we have

$$U < -\delta.$$

For some $\tilde{\delta} > 0$, we let

$$W_1 = e^{-\tilde{\delta}x}\tilde{\omega}$$

and substitute this into our equation for vorticity. This leads to the following equation for $\tilde{\omega}$:

$$\begin{aligned} \tilde{\omega}_t &= (\tilde{\delta}U + \nu\tilde{\delta}^2)\tilde{\omega} \\ &\quad -(U + 2\nu\tilde{\delta})\tilde{\omega}_x - (V)\tilde{\omega}_y \\ &\quad +\nu\Delta\tilde{\omega} - \tilde{f}, \end{aligned} \tag{7.21}$$

where

$$\tilde{f} = (\mathbf{U}_1 \cdot \nabla + \nabla \cdot \mathbf{U}_1)e^{\tilde{\delta}x}W. \tag{7.22}$$

Since W and its derivatives decay exponentially fast in space (see section 6), and since $\mathbf{U}_1$ and $\nabla \cdot \mathbf{U}_1$ are known to be bounded in space, we can assume that $\tilde{f}$ itself decays exponentially fast in space. This can be guaranteed as long as $\tilde{\delta}$ is chosen so that W and ∇W decay faster than $e^{-\tilde{\delta}x}$.

At this point, the equation for $\tilde{\omega}$ has the same form as equation (4.5) in section 4. Once we go far enough away from the origin so that the behavior of $\tilde{f}$ is dominated by that of the other terms, the arguments given in section 4 for the boundedness of $\tilde{\omega}$ will apply here as well. Choosing

$$\tilde{\delta} = \delta/\nu,$$

we can show that $\tilde{\omega}(x, y; t)$ is bounded in the maximum norm. Consequently, W_1 decays exponentially in the x-direction in space in the case where $U < -\delta$. As done in section 4, the introduction of a moving coordinate system allows us to show that for U, V at any speed, there exist constants c and k such that

$$|W_1|_\infty \leq ce^{-kr} \tag{7.23}$$

for $r = (x^2 + y^2)^{1/2}$, with c determined by the constant $|W_1(x, y; 0)|_\infty$ and $k > 0$.

7.2. Velocity Behavior: Linearized Incompressible Part – All Time. Now that we have shown exponential decay behavior for the vorticity W_1 of the second term in the expansion of the slightly compressible flow solution, we wish to analyze the decay rate for the linearized incompressible velocity term, $\mathbf{U}_1 = (U_1, V_1)$.

We start by finding the Poisson equation to describe $\mathbf{U}_1$. From (7.19) we have

$$\begin{aligned} W_{1x} &= V_{1xx} - U_{1xy} \\ W_{1y} &= V_{1xy} - U_{1yy}. \end{aligned}$$

From (7.15) we get

$$\begin{aligned} U_{1xy} + V_{1yy} &= \epsilon^2 (g_1)_y \\ U_{1xx} + V_{1xy} &= \epsilon^2 (g_1)_x. \end{aligned}$$

From these equations we find our Poisson equations

$$\Delta U_1 = W_{1y} + \epsilon^2 (g_1)_x \tag{7.24}$$

$$\Delta V_1 = W_{1x} + \epsilon^2 (g_1)_y. \tag{7.25}$$

We solve only for U_1, since the procedure for finding V_1 is analogous. Theorem 1 tells us that the solution U_1, over some region $D \in \Re^2$, can be written

$$U_1(x, y; t) = -\frac{1}{2\pi} \iint_D W_{1\eta} \log(\tilde{r}) d\xi d\eta + \frac{1}{2\pi} \iint_D \epsilon^2 (g_1)_\xi \log(\tilde{r}) d\xi d\eta \tag{7.26}$$

$$= I_1 + I_2 \tag{7.27}$$

for

$$\tilde{r} = ((x - \xi)^2 + (y - \eta)^2)^{1/2}.$$

Integrating I_1 by parts gives

$$I_1 = -\frac{1}{2\pi} \left(\int_{\partial D} W_1(\xi, \eta; t) \log(\tilde{r}) + \iint_D W_1(\xi, \eta; t) \frac{(y - \eta)}{\tilde{r}^2} d\xi d\eta \right) \tag{7.28}$$

In the limit as the radius of D goes to ∞, the boundary term goes to zero. Therefore, the solution of the Poisson equation for U_1 over all of $\Re^2$ can be written

$$(7.29)\qquad \begin{aligned} U_1(x,y;t) &= -\frac{1}{2\pi}\iint_{\Re^2} W_1\frac{(y-\eta)}{\tilde{r}^2}d\xi d\eta + \frac{1}{2\pi}\iint_{\Re^2}\epsilon^2(g_1)_\xi\log(\tilde{r})d\xi d\eta \\ &= I_1^\infty + I_2^\infty. \end{aligned}$$

We know that W_1 is at least bounded in space (equation (7.23)), and that ∇g_1 decays like c/r^3 (equation (7.17)). By carrying out estimate calculations in the same way as was done in earlier sections (see, for example, section 5), we can show that, at worst,

$$\begin{aligned} I_1^\infty &\leq c/r \\ I_2^\infty &\leq c/r^2. \end{aligned}$$

Thus, for large r, we have shown that

$$(7.30)\qquad \mathbf{U}_1(x,y;t) \leq c/r.$$

This bound can, in fact, be made sharper, if we take into account the known decay behavior of the vorticity W_1 from (7.23). A sharper bound for this term will not necessarily be of use to us, however, in our overall estimates.

Note that the elements of the expression for U_1 (7.29) meet the hypotheses of THEOREM 2, so it can also be readily shown that

$$(7.31)\qquad \mathbf{U}_{1x}(x,y;t) \leq c/r^2,$$
$$(7.32)\qquad \mathbf{U}_{1xx}(x,y;t) \leq c/r^3.$$

It is clear that analogous results will hold for first and second order derivatives in both x and y.

7.3. Pressure Behavior: Linearized Incompressible Part. We find an expression for the pressure P_1 of the linearized incompressible term by taking the divergence of (7.14), and making use of (7.15):

$$(7.33)\qquad \begin{aligned} \Delta P_1 = &-\Big(\epsilon^2(g_1)_t + (U+U_1)\epsilon^2(g_1)_x + (V+V_1)\epsilon^2(g_1)_y \\ &+2U_xU_{1x} + 2V_yV_{1y} + 2U_yV_{1x} + 2V_xU_{1y}\Big) \\ &+\nu\Delta\epsilon^2 g_1. \end{aligned}$$

Note that each term in the expression for ΔP_1 is bounded by at least c/r^4 for large r. (See, for example, equations (7.17), (7.18).) Thus, one can make use of the arguments similar to those of sections 2 and 3 to show that for large r, the gradient of the pressure of the fast part is given by at most

$$(7.34)\qquad \nabla P_1 \leq c/r^2.$$

7.4. Velocity Behavior: Linearized Incompressible Part – Finite Time. In this section we make use of the method of characteristics to determine an improved decay rate for the linearized incompressible velocity $\mathbf{U}_1 = (U_1, V_1)$ in finite time. From equation (7.14) we know that

$$U_{1t} + UU_{1x} + VU_{1y} = G^1(x, y; t) \tag{7.35}$$
$$V_{1t} + UV_{1x} + VV_{1y} = G^2(x, y; t) \tag{7.36}$$

where

$$G^1(x, y; t) = -U_1 U_x - V_1 U_y + \nu \Delta U_1 - P_{1x} \tag{7.37}$$
$$G^2(x, y; t) = -U_1 V_x - V_1 V_y + \nu \Delta V_1 - P_{1y}. \tag{7.38}$$

Notice that each element of the functions G^1 and G^2 decays at most like c/r^2.

Now, the total derivative of $\mathbf{U}_1$ can be written as

$$\begin{aligned} \frac{dU_1}{dt} &= U_{1t} + (\frac{dx}{dt})U_{1x} + (\frac{dy}{dt})U_{1y} \\ \frac{dV_1}{dt} &= V_{1t} + (\frac{dx}{dt})V_{1x} + (\frac{dy}{dt})V_{1y}. \end{aligned} \tag{7.39}$$

If we have velocity written as

$$\frac{dx}{dt} = U, \quad \frac{dy}{dt} = V, \tag{7.40}$$

then standard characteristics arguments tell us that

$$\frac{dU_1}{dt} = c/r^2, \quad \frac{dV_1}{dt} = c/r^2. \tag{7.41}$$

Equations (7.40) and (7.41) are now of the same form as equations (3.24) and (3.25). Therefore, the arguments that follow (in section 3.2) are identical to those we would use here. In the presence of a rigid body, then, the fast part of the velocity is bounded by

$$\mathbf{U}_1 \leq c/r^2 \tag{7.42}$$

for large r and for some finite time t $(0 \leq t \leq T < \infty)$.

7.5. Slightly Compressible Flow: Total Decay Behavior . In the previous sections, we learned that the second term in the asymptotic expansion of the solution of the slightly compressible equations exhibits the same decay rates (in pressure and velocity) as the solutions of the incompressible equations. As implied by THEOREM 4, assuming proper initialization, the sum of first two terms in the asymptotic expansion will give us the solution of the slightly compressible flow, up to $\mathcal{O}(\epsilon^4)$. The first term in the expansion is the solution of the simplified incompressible Navier-Stokes equations

(7.3), (7.4), and the second term in the expansion is the solution of the linearized incompressible Navier-Stokes equations (7.14), (7.15).

Therefore, for large enough $r = (x^2 + y^2)^{1/2}$, and for an appropriate constant c, the decay behavior of the compressible velocity in all time can be given by

$$\mathbf{u} = \mathbf{U} + \epsilon^2 \mathbf{U}_1 + \mathbf{u}_1 + \mathcal{O}(\epsilon^4) \leq c/r, \tag{7.43}$$

(see equations (5.11) and (7.30)). Recall that because we can expand out terms from the slow part of the solution indefinitely, and since each term in the expansion is the solution of linearized incompressible flow equations, it can be shown that the compressible correction will also decay in the same way as the first two terms in the expansion.

The decay behavior of the gradient of the pressure is given by

$$\nabla p = \nabla(P + \epsilon^2 P_1 + p_1 + \mathcal{O}(\epsilon^3)) \leq c/r^2, \tag{7.44}$$

(see equations (5.38) and (7.34)). The decay behavior of the compressible velocity in finite time can be given by

$$\mathbf{u} = \mathbf{U} + \epsilon^2 \mathbf{U}_1 + \mathbf{u}_1 + \mathcal{O}(\epsilon^4) \leq c/r^2, \tag{7.45}$$

(see equations (5.40) and (7.42)). Similarly, it also follows that

$$D\mathbf{u} \leq c/r^3 \tag{7.46}$$

$$D^2\mathbf{u} \leq c/r^4. \tag{7.47}$$

Recall that the highly oscillatory parts of the slightly compressible solution can be reduced to an arbitrarily high order of ϵ with the proper initialization. Bounding the first two time derivatives, for example, gives us

$$\mathbf{u}_1 = \mathcal{O}(\epsilon^2) \quad \text{and} \quad p_1 = \mathcal{O}(\epsilon).$$

Knowing this decay behavior reveals to us that at some point in space, the nature of the slightly compressible Navier-Stokes flow changes to that of flow that can be modeled by the wave equation.

To see this, we again examine the relative sizes of each term in the Navier-Stokes equations at a large distance from the source of the flow (assumed to be centered at the origin). For some appropriate constant c, we have

$$\begin{aligned} 0 = u_t + uu_x + vu_y + p_x - \nu\Delta u &\leq u_t + c/r^5 + c/r^5 + p_x - c/r^4, \\ 0 = v_t + uv_x + vv_y + p_y - \nu\Delta y &\leq v_t + c/r^5 + c/r^5 + p_y - c/r^4, \\ 0 = p_t + up_x + vp_y + (u_x + v_y)/\epsilon^2 &\leq p_t + c/r^4 + c/r^4 + (u_x + v_y)/\epsilon^2. \end{aligned}$$

We see from the above that we may discard the convection terms at large distances, with an error of at most $\mathcal{O}(1/r^4)$. The equations that now model the approximate flow behavior at large distances are taken to be

$$\begin{aligned} u_t + p_x &= 0, \\ v_t + p_y &= 0, \\ p_t + (u_x + v_y)/\epsilon^2 &= 0. \end{aligned}$$

Solving these equations is equivalent to solving the wave equation for the pressure p. (We may also view it as solving the wave equation for the divergence of the velocity $\mathbf{u}$.) To see this more clearly, we take the space derivatives of the velocity equations, and the time derivative of the pressure equation, which yields:

$$\begin{aligned} u_{xt} &= -p_{xx}, \\ v_{yt} &= -p_{yy}, \\ p_{tt} &= -(u_{xt} + v_{yt})/\epsilon^2. \end{aligned}$$

Substituting the first two equations into the third gives

$$p_{tt} = \frac{1}{\epsilon^2}\Delta p. \tag{7.48}$$

In the far-field, once the velocity components have essentially decayed to zero, the pressure component is still relatively large. Therefore, it is possible to model the flow in the far-field simply by solving the wave equation for the pressure p. Inflow boundary conditions for the pressure can be taken to be those numerical values calculated for p in the near-field.

8. Numerical Calculations. In this section, we calculate numerically the solution (u, v, p) of the slightly compressible Navier-Stokes equations (7.1), (7.2). To this end, we make use of a grid generation package, CMPGRD [3], and a generalized differential equations solver CGCNS [5] (which is still under development). The solver is written specifically to solve problems on computational grids generated by CMPGRD. The class of problems that are meant to be solved by CGCNS are systems of parabolic and hyperbolic equations which can be written in the form

$$\frac{\partial \mathbf{u}}{\partial t} = f(\mathbf{u}, \mathbf{u}_{x_i}, \mathbf{u}_{x_i x_j}, \cdots).$$

When discretized in space, the PDE system becomes a system of ODE's of the form

$$\frac{d\mathbf{U}}{dt} = \mathcal{F}(\mathbf{U}),$$

where $\mathbf{U}$ is a vector of all solution values at all grid points. The main functions of the code can be broken down into the following four subsections:

1. Initialization and Setup
2. Time Stepping
3. Computation of $\mathcal{F}(\mathbf{U})$
4. Saving Results

Item (3) is application dependent, and all sections of code pertaining to item (3) were modified to model our specific equations. Item (2) is mostly application independent, except for the actual determination of the size of the time step to be taken, Δt. The determination of the time step depends on the set of equations being solved. We describe in section 8.1 how Δt is calculated.

The actual time-stepping routine implemented in our experiments is the second-order accurate midpoint rule. For the equation

$$\frac{d\mathbf{U}}{dt} = \mathcal{F}(\mathbf{U}),$$

the midpoint rule is defined to be

$$\begin{aligned} \mathbf{U}^* &= \mathbf{U}(t) + \frac{\Delta t}{2}\mathcal{F}(\mathbf{U}(t)) \\ \mathbf{U}(t+\Delta t) &= \mathbf{U}(t) + \Delta t \mathcal{F}(\mathbf{U}^*). \end{aligned}$$

8.1. Determining the Time Step. In this section, we determine analytically the maximum size of the time step allowed when numerically calculating the solution (u, v, p) of the slightly compressible Navier-Stokes equations

$$\begin{aligned} u_t &= -uu_x - vu_y - p_x + \nu\Delta u \\ v_t &= -uv_x - vv_y - p_y + \nu\Delta v \\ p_t &= -up_x - vp_y - u_x/\epsilon^2 - v_y/\epsilon^2. \end{aligned} \tag{8.1}$$

For the sake of analysis, we freeze the coefficients. We discard the viscosity term, since the viscosity coefficient ν is small compared with the other terms, and will not significantly impact the outcome of our analysis. We then work with

$$\begin{aligned} u_t &= -Uu_x - Vu_y - p_x \\ v_t &= -Uv_x - Vv_y - p_y \\ p_t &= -Up_x - Vp_y - u_x/\epsilon^2 - v_y/\epsilon^2 \end{aligned} \tag{8.2}$$

where U and V can be considered constants.

In order to make use of CMPGRD in conjunction with the calculation of the solution to our equations, we must start by converting our equations from Euclidean coordinates (x, y) to computational coordinates (r, s). It is assumed that there exists a smooth mapping with inverse from (x, y) space to (r, s) space. Conversion formulae are as follows:

$$\begin{aligned} \frac{\partial}{\partial x} &= r_x\frac{\partial}{\partial r} + s_x\frac{\partial}{\partial s} \\ \frac{\partial}{\partial y} &= r_y\frac{\partial}{\partial r} + s_y\frac{\partial}{\partial s} \\ \frac{\partial^2}{\partial x^2} &= r_x^2\frac{\partial^2}{\partial r^2} + 2r_xs_x\frac{\partial^2}{\partial r\partial s} + s_x^2\frac{\partial^2}{\partial s^2} + r_{xx}\frac{\partial}{\partial r} + s_{xx}\frac{\partial}{\partial s} \\ \frac{\partial^2}{\partial y^2} &= r_y^2\frac{\partial^2}{\partial r^2} + 2r_ys_y\frac{\partial^2}{\partial r\partial s} + s_y^2\frac{\partial^2}{\partial s^2} + r_{yy}\frac{\partial}{\partial r} + s_{yy}\frac{\partial}{\partial s} \end{aligned}$$

where

$$\begin{aligned} r_{xx} &= r_x\frac{\partial r_x}{\partial r} + s_x\frac{\partial r_x}{\partial r} \\ &\approx r_x\left(\frac{r_x(i+1,j) - r_x(i-1,j)}{2\Delta r}\right) + s_x\left(\frac{r_x(i,j+1) - r_x(i,j-1)}{2\Delta s}\right) \end{aligned}$$

The frozen coefficient Navier-Stokes equations transformed to computational space then become

$$
\begin{aligned}
u_t &= -(Ur_x + Vr_y)u_r - (r_x)p_r - (Us_x + Vs_y)u_s - (s_x)p_s \\
v_t &= -(Ur_x + Vr_y)v_r - (r_y)p_r - (Us_x + Vs_y)v_s - (s_y)p_s \\
p_t &= -(Ur_x + Vr_y)p_r - (r_x/\epsilon^2)u_r - (r_y/\epsilon^2)v_r \\
&\quad -(Us_x + Vs_y)p_s - (s_x/\epsilon^2)u_s - (s_y/\epsilon^2)v_s
\end{aligned}
\tag{8.3}
$$

To write this system of equations in matrix form, we define W to be the vector of solution components,

$$W = (u, v, p)^T.$$

Then system (8.3) is equivalent to

$$W_t = AW_r + BW_s \equiv P\left(\frac{\partial}{\partial \mathbf{r}}\right) W \tag{8.4}$$

with smooth 2π-periodic initial data

$$W(\mathbf{r}, 0) = f(\mathbf{r}).$$

The differential operator $P(\frac{\partial}{\partial \mathbf{r}})$, with $\mathbf{r} = (r_1, r_2) = (r, s)$, is defined by equation (8.4). Matrices A and B are

$$A = \begin{bmatrix} -(Ur_x + Vr_y) & 0 & -r_x \\ 0 & -(Ur_x + Vr_y) & -r_y \\ -(r_x/\epsilon^2) & -(r_y/\epsilon^2) & -(Ur_x + Vr_y) \end{bmatrix}$$

$$B = \begin{bmatrix} -(Us_x + Vs_y) & 0 & -s_x \\ 0 & -(Us_x + Vs_y) & -s_y \\ -(s_x/\epsilon^2) & -(s_y/\epsilon^2) & -(Us_x + Vs_y) \end{bmatrix}$$

At this point, we discretize our equations in space only (the method of lines). We introduce the following notation:

Let $h = 2\pi/(N+1)$, N a natural number, denote the grid size in both the r and s directions. A two-dimensional grid is defined by the grid points

$$\mathbf{r}_j = (j_1 h, j_2 h) \qquad j_\nu = 0, \pm 1, \pm 2, \cdots$$

We define the grid functions to be

$$W_j = W(\mathbf{r}_j).$$

For the sake of the analysis, we assume the grid function to be 2π-periodic. Thus,

$$W_j = W(\mathbf{r}_j) = W(j_1h + 2\pi, j_2h + 2\pi) = W_{j+N+1}.$$

Define the linear translation operators E_r and $E_s$by

$$\begin{aligned}(E_r^pW)_{j1,j2} &= W_{j1+p,j2} \\ (E_s^pW)_{j1,j2} &= W_{j1,j2+p}\end{aligned}$$

We then define the forward, backward, and central difference operators in terms of the translation operator. The forward, backward, and central difference operators are defined respectively (in the r-direction) by

$$\begin{aligned}D_{+r} &= (E_r^1 - E_r^0)/h \\ D_{-r} &= (E_r^0 - E_r^{-1})/h \\ D_{0r} &= (E_r^1 - E_r^{-1})/2h = (D_{+r} + D_{-r})/2\end{aligned}$$

Analogous definitions hold in the s-direction.

Before applying the difference operators to our equations, we first consider their application to the restriction of $e^{i\langle\omega,\mathbf{r}\rangle}$ to the grid. (Here, $\langle\omega,\mathbf{r}\rangle = \omega_1 r + \omega_2 s$.) We consider the discretization in the r-direction only, since the discretization procedure in the s-direction is analogous. We find that

$$\begin{aligned}hD_{r+}e^{i\langle\omega,\mathbf{r}\rangle} &= (e^{i\omega_1 h} - 1)e^{i\langle\omega,\mathbf{r}\rangle} \\ hD_{r-}e^{i\langle\omega,\mathbf{r}\rangle} &= (1 - e^{-i\omega_1 h})e^{i\langle\omega,\mathbf{r}\rangle} \\ 2hD_{r0}e^{i\langle\omega,\mathbf{r}\rangle} &= 2i\sin(\omega_1 h)e^{i\langle\omega,\mathbf{r}\rangle}\end{aligned}$$

The forward and backward difference operators are first order accurate in space, and the central difference operator is second order accurate in space.

Discretizing equation (8.4) with central differencing yields

$$W_t = AD_{r0}W + BD_{s0}W, \tag{8.5}$$

with initial data

$$W^0_{j1,j2} = f_{j1,j2}.$$

We first consider initial data that consist of one wave,

$$f_{j1,j2} = \frac{1}{\sqrt{2\pi}}e^{i\langle\omega,\mathbf{r}_j\rangle}\hat{f}(\omega).$$

We assume the solution can be expressed as a single Fourier component, and make the *ansatz*

$$W_j = \frac{1}{\sqrt{2\pi}}e^{i\langle\omega,\mathbf{r}_j\rangle}\widehat{W}(\omega).$$

Substituting this back into equation (8.5) yields

$$\widehat{W}_t = (iA\sin(\omega_1 h)/h + iB\sin(\omega_2 h)/h)\widehat{W}(\omega, t).$$

We see that application of the difference operators in Fourier space is equivalent to multiplication by the *symbol* $P(i\omega)$, which is defined to be

$$P(i\omega) \equiv iA\sin(\omega_1 h)/h + iB\sin(\omega_2 h)/h.$$

We have then obtained the ODE

$$\widehat{W}_t = P(i\omega)\widehat{W}(\omega, t),$$

the solution of which is given by

$$\widehat{W}(\omega, t) = e^{P(i\omega)t}\widehat{W}_0.$$

The solution in Euclidean space is then represented by

$$W(\mathbf{r}, t) = e^{P(i\omega)t}e^{i\langle\omega,\mathbf{r}_j\rangle}\widehat{W}_0.$$

The eigenvalues of the symbol $P(i\omega)$ are given by

$$\begin{aligned}
\lambda_1 &= -i\frac{\left(\sin(\omega_1 h)Ur_x + \sin(\omega_1 h)Vr_y + \sin(\omega_2 h)Us_x + \sin(\omega_2 h)Vs_y\right)}{h} \\
\lambda_2, \lambda_3 &= \lambda_1 \pm \frac{\sqrt{(a+b+c+d+e+f)}}{(h\epsilon)}
\end{aligned}$$

where

$$\begin{aligned}
a &= r_y^2\left(\cos(\omega_1 h)^2 - 1\right) \\
b &= s_y^2\left(\cos(\omega_2 h)^2 - 1\right) \\
c &= r_x^2\left(\cos(\omega_1 h)^2 - 1\right) \\
d &= s_x^2\left(\cos(\omega_2 h)^2 - 1\right) \\
e &= -2\sin(\omega_1 h)r_y\sin(\omega_2 h)s_y \\
f &= -2\sin(\omega_1 h)r_x\sin(\omega_2 h)s_x
\end{aligned}$$

We now find an upper bound on the magnitude of the largest eigenvalue of $P(i\omega)$: Let U_{max}, V_{max} be the magnitudes of the largest values of u and v taken over all possible (x, y). Then, taking an upper bound over all possible wavelengths ω gives

$$|\lambda_1| \le \frac{U_{max}(|r_x| + |s_x|) + V_{max}(|r_y| + |s_y|)}{h} \equiv \lambda_{1max},$$

and

$$|\lambda_2|, |\lambda_3| \le \lambda_{1max} + \frac{\sqrt{r_y^2 + s_y^2 + r_x^2 + s_x^2 + 2|r_y||s_y| + 2|r_x||s_x|}}{h\epsilon} \equiv \lambda_{max}.$$

Next we must determine an appropriate time step Δt for our numerical method. If λ_{max} is the magnitude of the largest eigenvalue of the symbol $P(i\omega)$, taken over all wavelengths ω, then for some appropriate constant C_{cfl}, we choose Δt so that

$$\Delta t \lambda_{max} = C_{cfl}.$$

The constant C_{cfl} is determined by the time-stepping routine implemented. For second order Adams-Bashforth, for example, it is safe to choose $C_{cfl} = 1/4$. For the midpoint rule, we choose $C_{cfl} = 1/2$. (See, for example, [1].) In other words, we choose $\Delta t \lambda_{max}$ so that it lies within the stability region of the time-stepping method chosen.

8.2. Numerical Experiments. In this section we examine the numerical outcome of directly solving the two-dimensional slightly compressible Navier-Stokes equations (7.1), (7.2), in a finite time interval. We are interested in simulating the flow in an infinite domain induced by a vorticity patch with compact support.

In our experiments, we carry out our calculations over two-dimensional square regions of increasing size.

The first step in the computational process is to calculate appropriate initial conditions. We specify a stream function, ϕ, to have compact support. Then we write $u(x,y;t=0) = u_0(x,y)$, $v(x,y;t=0) = v_0(x,y)$, and over the entire computational domain we set

$$u_0 = -\phi_y, \qquad v_0 = \phi_x.$$

Initial velocity then has compact support. (It follows that the initial vorticity also has compact support.) In addition, the initial velocity has zero divergence ($\nabla \cdot \mathbf{u}_0 = 0$). The initial pressure, $p(x,y;t=0) = p_0(x,y)$, then satisfies the following elliptic equation:

$$\Delta p_0 = -u_{0x}^2 - 2u_{0y}v_{0x} - v_{0y}^2.$$

Prescribing the initial velocity so that it has zero divergence at initial time, also ensures that two time derivatives of the pressure component are bounded independently of ϵ at initial time. For the first time derivative of pressure this is clear directly from the pressure equation

$$\epsilon^2 \left\{ p_t + up_x + vp_y \right\} + u_x + v_y = 0.$$

At time $t = 0$, we have

$$\epsilon^2 p_t = -\epsilon^2 \left\{ up_x + vp_y \right\} = \epsilon^2 \mathcal{O}(1).$$

(Recall that all space derivatives of (u, v, p) are bounded for all times.) To see that the second time derivative of the pressure is also bounded, we take one time derivative of the pressure equation to obtain

$$\epsilon^2 p_{tt} = -\epsilon^2 \left\{ (up_x)_t + (vp_y)_t \right\} - (u_x + v_y)_t.$$

First note that at $t = 0$,

$$(u_x + v_y)_t = 0.$$

This can be seen by taking the space derivatives of the velocity equations, and making use of the elliptic equation for the pressure p. Next we examine the term

$$(up_x)_t + (vp_y)_t = u_t p_x + v_t p_y + up_{xt} + vp_{yt}.$$

Each term in

$$u_t p_x + v_t p_y$$

is bounded independently of ϵ at $t = 0$, so it only remains to consider whether the same holds for

$$up_{xt} + vp_{yt}.$$

Taking the appropriate space derivatives of the pressure equation yields

$$up_{xt} + vp_{yt} = H(u, v, p) - \frac{1}{\epsilon^2} u(u_x + v_y)_x - \frac{1}{\epsilon^2} v(u_x + v_y)_y,$$

where $H(u, v, p)$ is an expression in u, v, p and their space derivatives. At time $t = 0$, the divergence terms are zero, which shows that at $t = 0$,

$$up_{xt} + vp_{yt} = \mathcal{O}(1).$$

Thus, at $t = 0$ we have,

$$\epsilon^2 p_{tt} = \epsilon^2 \mathcal{O}(1),$$

indicating the boundedness of the second derivative of the pressure at initial time.

Once we have been able to bound the time derivatives at initial time, it follows that the time derivatives are bounded independently of ϵ for all time. Choosing initial data in this way is called *initialization by the bounded derivative principle.* For details, see [8, 9, 10].

By properly initializing our initial data the way we have, Theorem 4 would lead us to expect that a portion of the fast part of our solution not be excited. In addition, the fast part of the velocity components should be suppressed by a factor of ϵ more than the fast part of the pressure component. In fact, by Theorem 4 we know that the fast part of the velocity will be $\mathcal{O}(\epsilon^2)$, whereas the fast part of the pressure will be $\mathcal{O}(\epsilon)$.

To solve for the initial value of p_0 over the entire computational domain, we use an elliptic PDE solver CGES [4]. Initial boundary conditions are set to be $u_0 = 0$, $v_0 = 0$, $p_0 = 0$, which is consistent with the fact that each initial function has compact support over the domain.

Once we have calculated appropriate initial values u_0, v_0, and p_0, we feed them into the Navier-Stokes equations solver. We calculate results from time $t = 0$ to time

$t = 0.5$. We run the calculations specifying Mach number to be 0.1 (*i.e.*, $\epsilon^2 = .01$), and viscosity coefficient to be 0.1. The time step Δt varies with the spatial discretization.

Mathematically, it is only necessary to specify two boundary conditions: one for the first velocity component u, and one for the second velocity component v. The pressure component p, on the other hand, satisfies the pressure equation, which assumes a known velocity field. The pressure equation specifies the change in pressure of a fluid particle along its trajectory. Therefore, if the particle is leaving the computational domain (as it does in our examples), its pressure must be calculated as part of the solution, and cannot be imposed arbitrarily.

Although it is not necessary from a *mathematical* point of view to specify p on the computational boundaries, the particular solver that we use makes it necessary, from a *computational* point of view, to specify some boundary condition for p. To this end, we make use of an extra line of "fictitious" grid points just outside our true computational grid. The equation for the pressure p is applied at the computational boundary, and then the value of p at the fictitious grid points is extrapolated from the value of p computed at the line of grid points just inside the computational boundary. This process for computing p at the boundary is equivalent to specifying that the normal component of p at the boundary be zero. Thus, the boundary conditions specified for our calculations are

$$u_{bddy} = 0$$
$$v_{bddy} = 0$$
$$\frac{\partial p_{bddy}}{\partial n} = 0$$

We assert that it is reasonable to specify zero Dirichlet boundary conditions for the velocity components, since, as we have seen, the velocity decays at least like c/r^2 in space. (Note that it would not be reasonable to specify zero boundary conditions for the pressure, since the pressure induced by a vorticity patch decays only like c/r, which is not rapid enough on a relatively small computational domain.)

One thing we are looking for in our experiments is evidence of the different time scales involved in the computation of the slightly compressible Navier-Stokes equations. By THEOREM 4 above, we expect the pressure component still to contain some small element in the solution that varies on the fast time scale. We will see evidence of these different time scales in the experiments that follow. Notice especially in experiments 2 and 3 how quickly the fast part of the pressure component spreads over the domain in time, as compared with the slow spread of the velocity components, in which the fast part has been well suppressed. The fast part of the pressure is still very small in magnitude, however.

In each experiment, the parameters listed are specified as follows:

- $\epsilon^2 = (\text{Mach Number})^2 = 0.01 \;\Rightarrow\; \epsilon = 0.1$.
- ν =Viscosity Coefficient= 0.1.
- Time Interval: $t \in [0.0, 0.5]$.
- Time Step Size: $\Delta t \approx 0.0017$.

- Space Step Size: $\Delta x = \Delta y = .05$.
- Initial Stream Function:

$$\phi = ae^{br},$$

where

$$a = 1/50; \quad b = -50; \quad r = x^2 + y^2.$$

- Initial Velocity Components:

$$\begin{aligned} u_0 &= -\phi_y = -2abye^{br} \\ v_0 &= \phi_x = 2abxe^{br} \end{aligned}$$

- Initial Pressure: Solve for p given u and v where

$$\Delta p_0 = -u_{0x}^2 - 2u_{0y}v_{0x} - v_{0y}^2.$$

For all experiments, we keep the space size of the mesh constant so that the time-step remains about the same, despite the varying sizes of the physical domain over which the problem is computed.

For each surface plot presented, the maximum and minimum values of the function are labeled directly on the plot.

8.2.1. Experiment 1: Flow in a 2×2 Square. Experiment 1 is computed over the square $(x, y) \in [-1...1, -1...1]$. In this 2×2 square, the mesh is 40×40 (*i.e.*, 1600 grid points).

Initial maximum and minimum values are given by:

$$\begin{aligned} p : (min, max) &= (-0.197E+00, 0.405E-03) \\ u : (min, max) &= (-0.383E+00, 0.383E+00) \\ v : (min, max) &= (-0.383E+00, 0.383E+00) \end{aligned}$$

Final maximum and minimum values are given by:

$$\begin{aligned} p : (min, max) &= (-0.367E-01, 0.371E-02) \\ u : (min, max) &= (-0.137E+00, 0.137E+00) \\ v : (min, max) &= (-0.137E+00, 0.137E+00) \end{aligned}$$

Figure 8.1 is a surface plot of the initial velocity component u. Figure 8.2 is the surface plot of velocity component u at time $t = 0.5$. Figures 8.3 and 8.4, respectively, are the surface plots of of the initial and final velocity component v. We see that the velocity "blobs" have diffused somewhat, but that there has not been much convection over time. The zero boundary conditions for the velocity are seen to be appropriate.

Figures 8.5 through 8.10 are the surface plots of the pressure, from initial pressure at time $t = 0.0$, to final pressure at time $t = 0.5$, with increments of $t = 0.1$ in between. The initial pressure "sink" diffuses somewhat, and a small component of pressure is seen to spread rapidly over the domain through time. A clear pattern of spread is not yet discernible, so we run the experiment on a larger domain next.

8.2.2. Experiment 2: Flow in a 4×4 Square. Experiment 2 is computed over the square $(x, y) \in [-2...2, -2...2]$. In this 4×4 square, the mesh is 80×80 (*i.e.*, 6400 grid points).

Initial maximum and minimum values are given by:

$$\begin{aligned} p : (min, max) &= (-0.198E + 00, 0.390E - 03) \\ u : (min, max) &= (-0.383E + 00, 0.383E + 00) \\ v : (min, max) &= (-0.383E + 00, 0.383E + 00) \end{aligned}$$

Final maximum and minimum values are given by:

$$\begin{aligned} p : (min, max) &= (-0.274E - 01, 0.837E - 04) \\ u : (min, max) &= (-0.136E + 00, 0.136E + 00) \\ v : (min, max) &= (-0.136E + 00, 0.136E + 00) \end{aligned}$$

Figure 8.11 is a surface plot of the initial velocity component u. Figure 8.12 is the surface plot of velocity component u at time $t = 0.5$. Figures 8.13 and 8.14, respectively, are the surface plots of of the initial and final velocity component v. Once again, we see that the velocity "blobs" have diffused somewhat, but that there has not been much convection over time. The zero boundary conditions for the velocity are still appropriate.

Figures 8.15 through 8.20 are the surface plots of the pressure, from initial pressure at time $t = 0.0$, to final pressure at time $t = 0.5$, with increments of $t = 0.1$ in between. The initial pressure "sink" again diffuses somewhat, even slightly more than on the 2×2 region. This is because there is not as much reflection of pressure from the boundaries in this case. Again, a small component of the pressure is seen to spread rapidly over the domain through time, and a pattern of spread is becoming discernible. We will still run the experiment on another larger domain.

8.2.3. Experiment 3: Flow in a 8×8 Square. Experiment 3 is computed over the square $(x, y) \in [-4...4, -4...4]$. In this 8×8 square, the mesh is 160×160 (*i.e.*, 25600 grid points).

Initial maximum and minimum values are given by:

$$\begin{aligned} p : (min, max) &= (-0.198E + 00, 0.392E - 03) \\ u : (min, max) &= (-0.383E + 00, 0.383E + 00) \\ v : (min, max) &= (-0.383E + 00, 0.383E + 00) \end{aligned}$$

Final maximum and minimum values are given by:

$$\begin{aligned} p : (min, max) &= (-0.247E - 01, 0.332E - 03) \\ u : (min, max) &= (-0.136E + 00, 0.136E + 00) \\ v : (min, max) &= (-0.136E + 00, 0.136E + 00) \end{aligned}$$

Figure 8.21 is a surface plot of the initial velocity component u. Figure 8.22 is the surface plot of velocity component u at time $t = 0.5$. Figures 8.23 and 8.24, respectively, are the surface plots of of the initial and final velocity component v. Here, too, we see that the velocity "blobs" have diffused somewhat, but otherwise there is not much spread of velocity over the domain. The zero boundary conditions for the velocity continue to be appropriate.

Figures 8.25 through 8.30 are the surface plots of the pressure, from initial pressure at time $t = 0.0$, to final pressure at time $t = 0.5$, with increments of $t = 0.1$ in between. The initial pressure "sink" again diffuses somewhat, at about the same rate as it does on the 4×4 region. A small component of the pressure is seen to spread rapidly over the domain through time. This is consistent with the assertions of THEOREM 4 of section 7, and our discussion in section 7.5, where it was predicted that the fast part of the velocity solution would be reduced to $\mathcal{O}(\epsilon^2)$, but the fast part of the pressure solution would only be reduced to $\mathcal{O}(\epsilon)$. A pattern of spread of the small fast part is now more clear. We observe a circular pressure wave (small in magnitude) spreading out from the pressure sink.

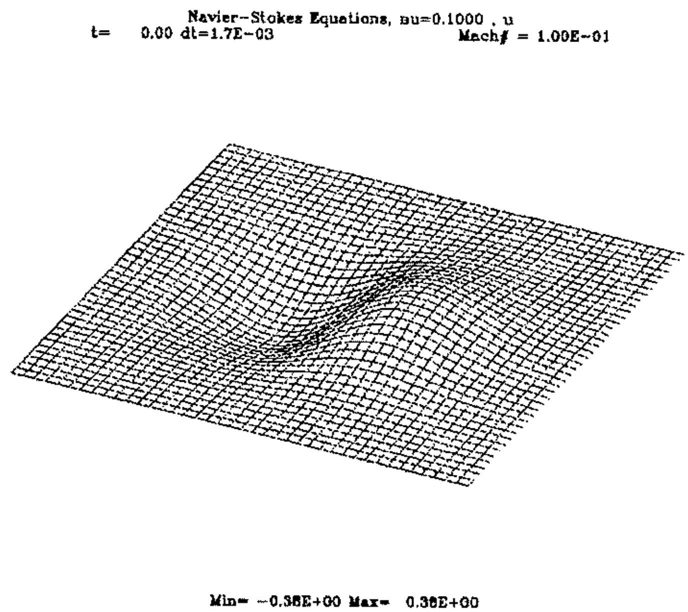

FIG. 8.1. *Region 2×2 - Initial Velocity u, 40×40 Mesh*

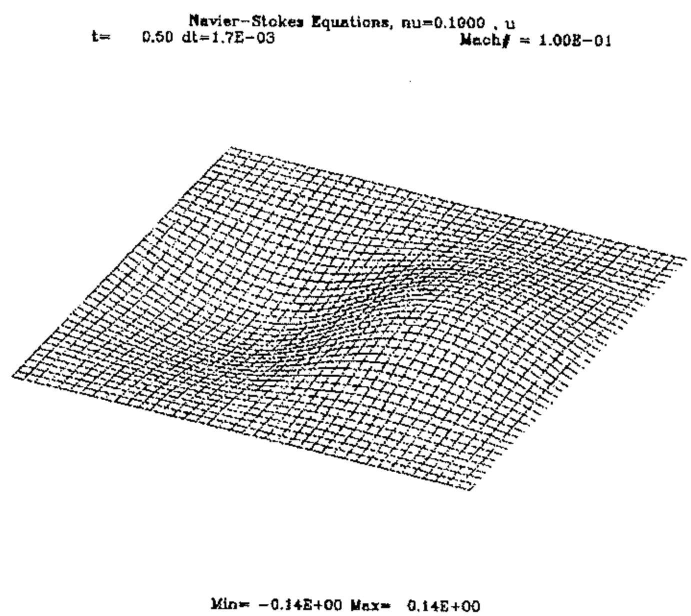

FIG. 8.2. *Region 2×2 - Final Velocity u, 40×40 Mesh*

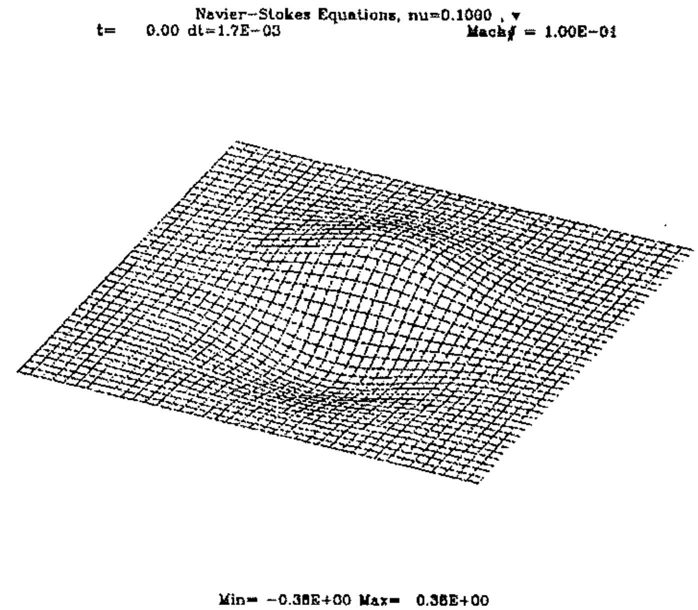

FIG. 8.3. *Region* 2×2 *- Initial Velocity* v*,* 40×40 *Mesh*

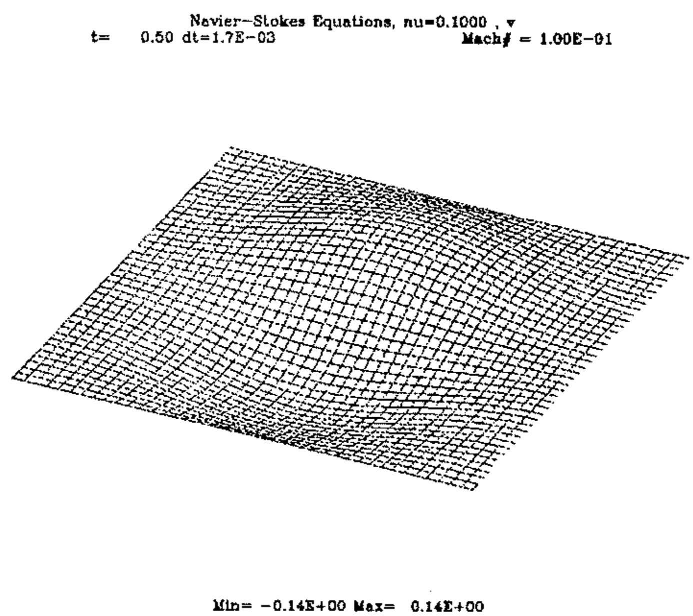

FIG. 8.4. *Region* 2×2 *- Final Velocity* v*,* 40×40 *Mesh*

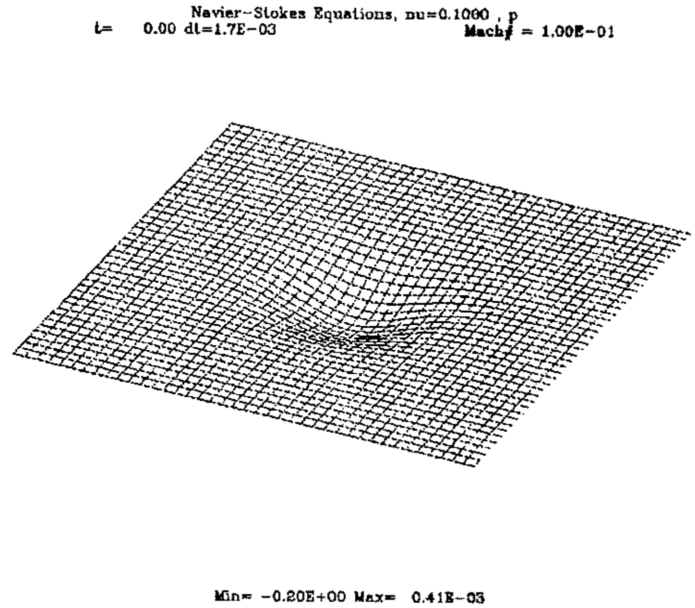

FIG. 8.5. *Region* 2×2 *- Initial Pressure p,* 40×40 *Mesh*

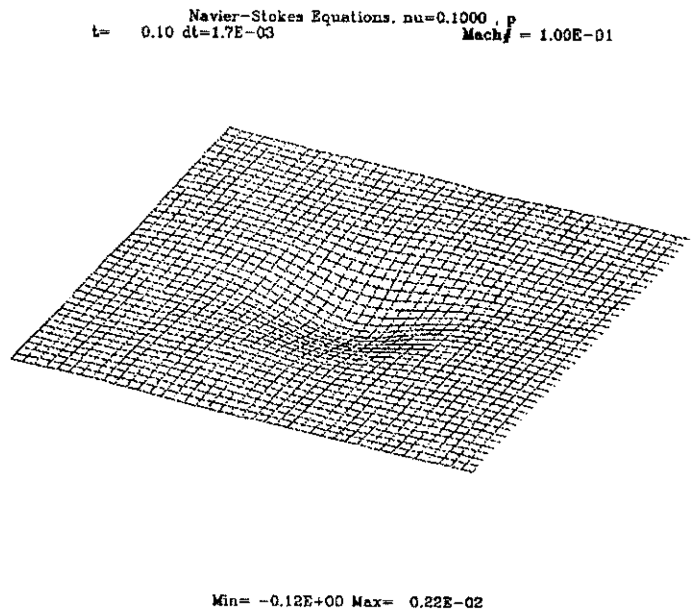

FIG. 8.6. *Region* 2×2 *- Pressure p,* $t = 0.1$, 40×40 *Mesh*

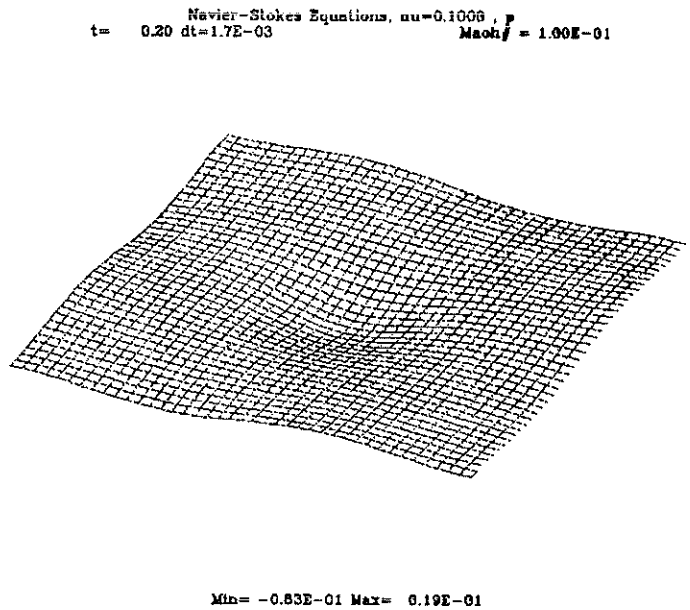

FIG. 8.7. *Region* 2×2 *– Pressure* p, $t = 0.2$, 40×40 *Mesh*

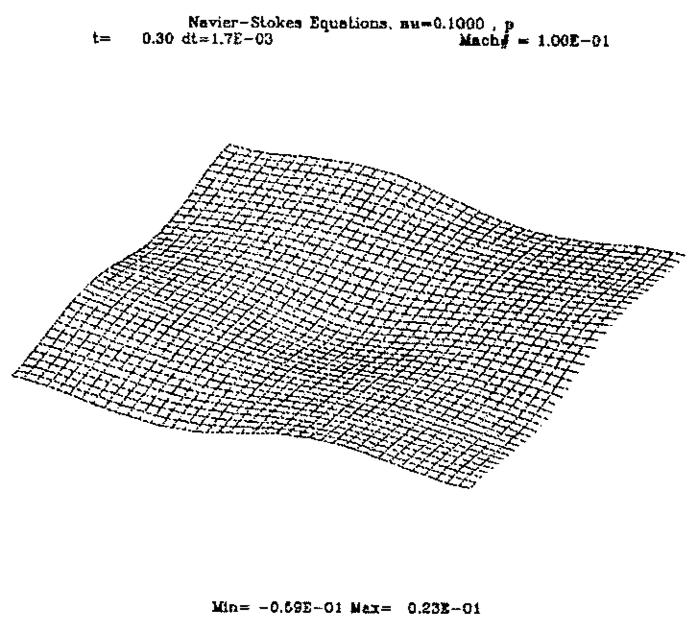

FIG. 8.8. *Region* 2×2 *– Pressure* p, $t = 0.3$, 40×40 *Mesh*

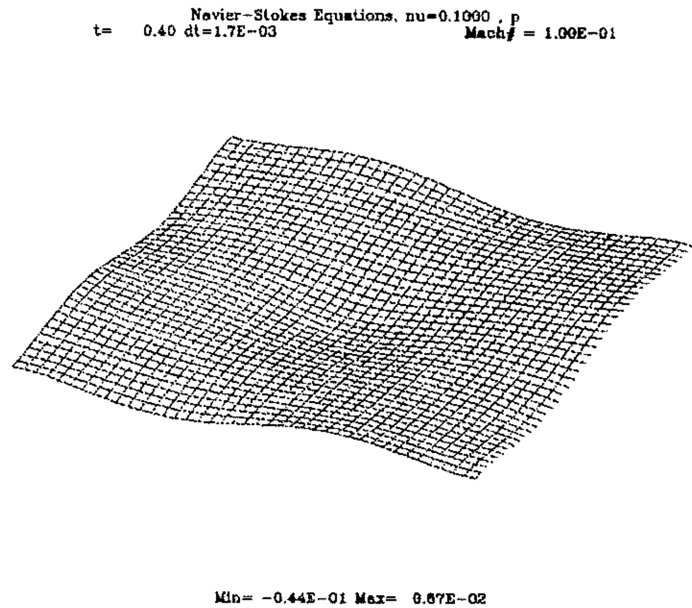

FIG. 8.9. *Region* 2×2 *- Pressure p,* $t = 0.4$, 40×40 *Mesh*

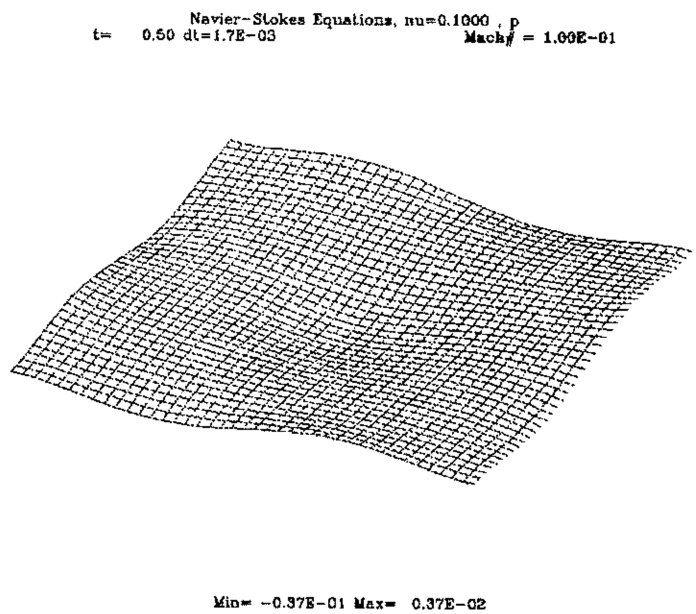

FIG. 8.10. *Region* 2×2 *- Pressure p,* $t = 0.5$, 40×40 *Mesh*

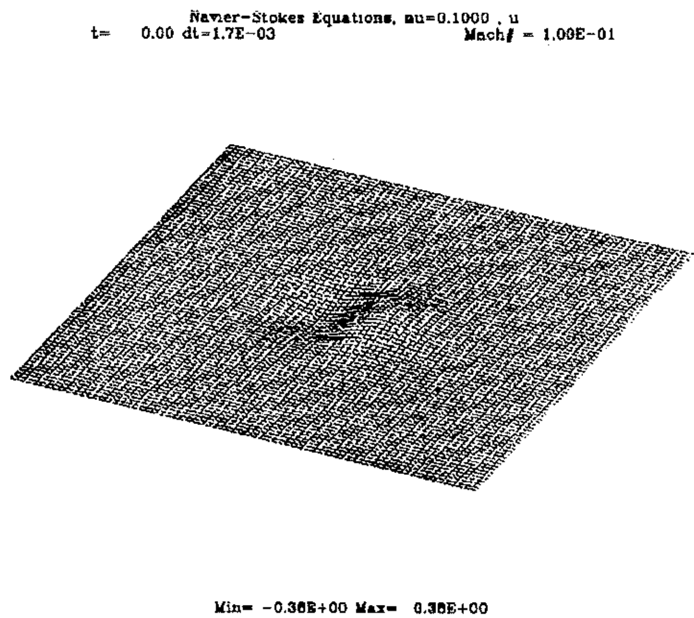

FIG. 8.11. *Region 4 × 4 – Initial Velocity u, 80 × 80 Mesh*

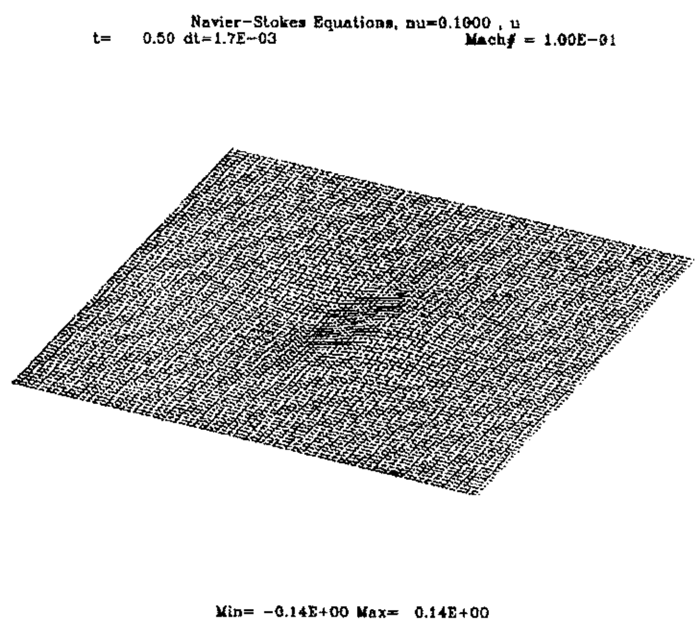

FIG. 8.12. *Region 4 × 4 – Final Velocity u, 80 × 80 Mesh*

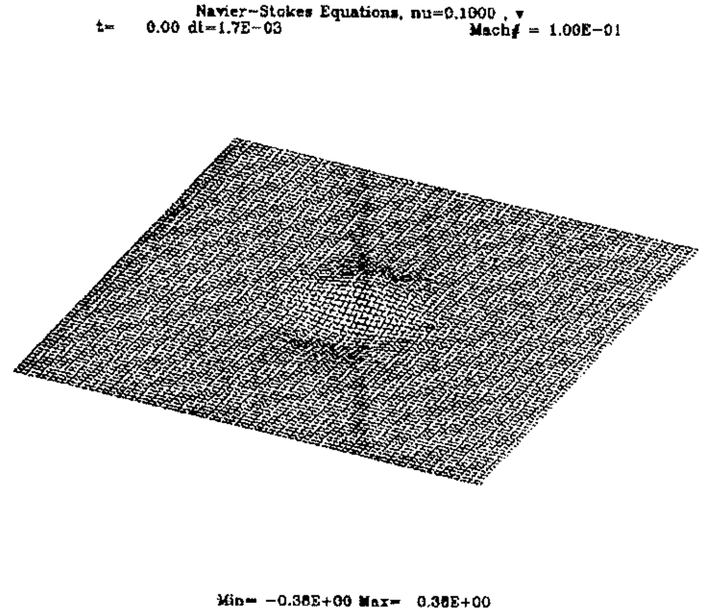

FIG. 8.13. *Region 4 × 4 – Initial Velocity v, 80 × 80 Mesh*

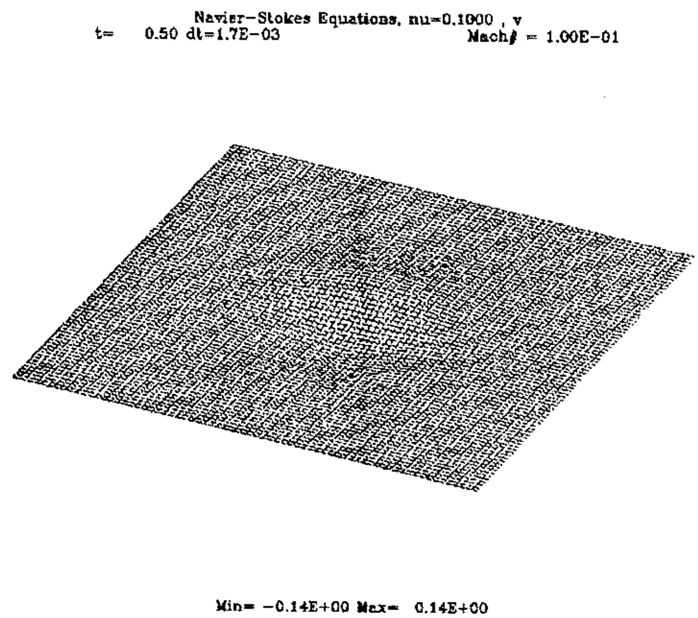

FIG. 8.14. *Region 4 × 4 – Final Velocity v, 80 × 80 Mesh*

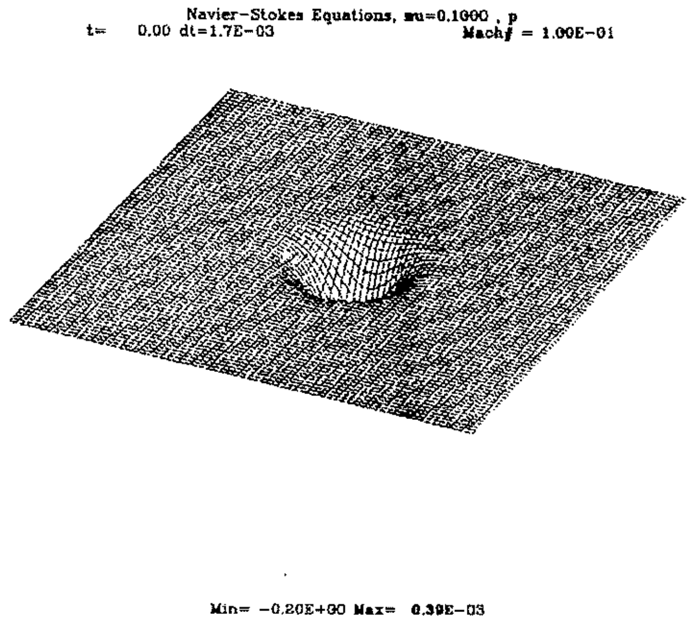

FIG. 8.15. *Region* 4×4 *- Initial Pressure p,* 80×80 *Mesh*

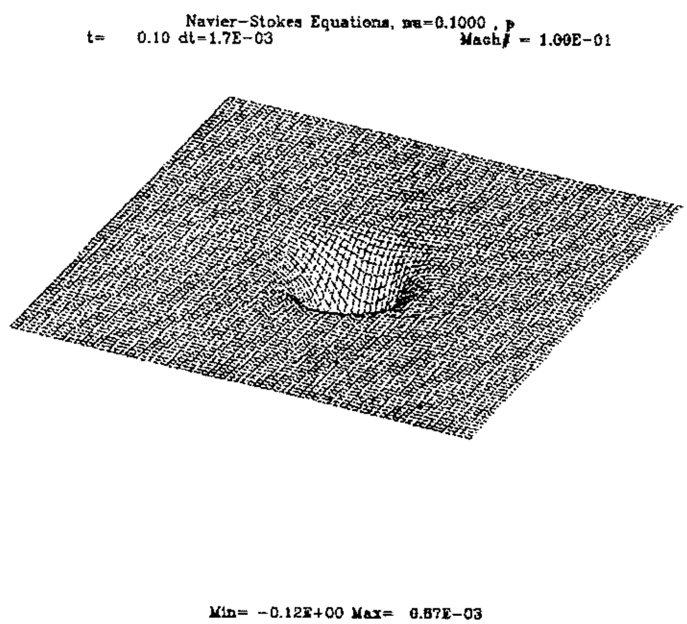

FIG. 8.16. *Region* 4×4 *- Pressure p,* $t = 0.1$, 80×80 *Mesh*

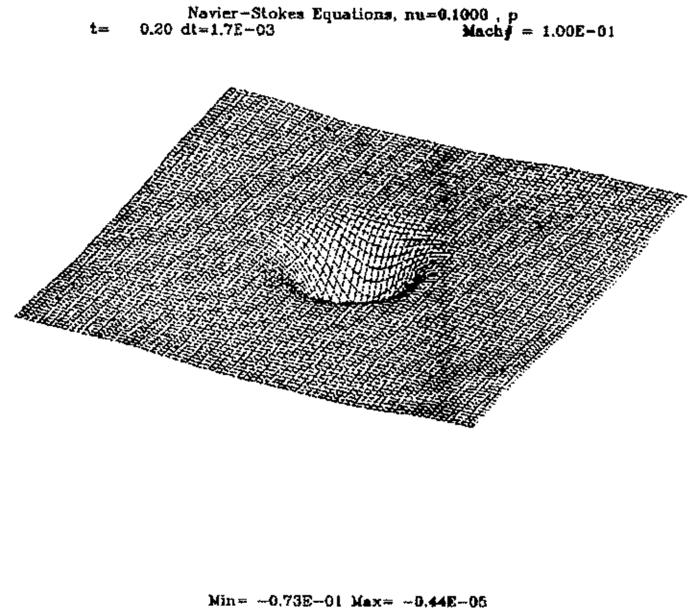

FIG. 8.17. *Region* 4×4 *– Pressure* p*,* $t = 0.2$*,* 80×80 *Mesh*

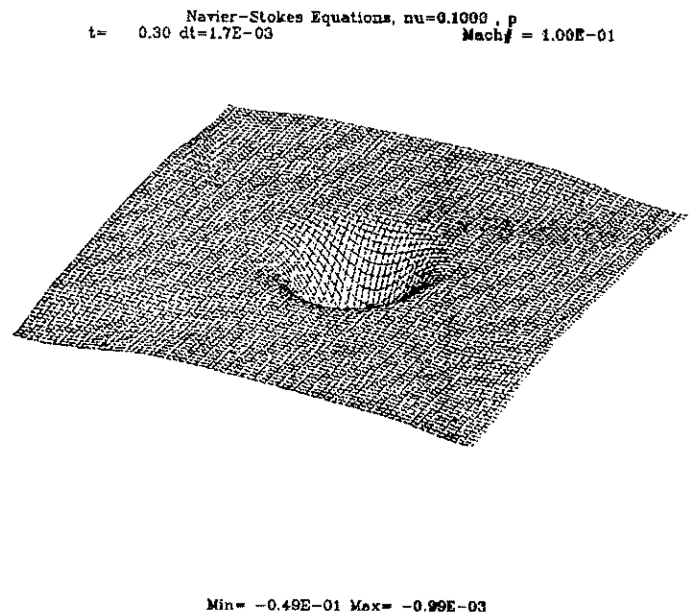

FIG. 8.18. *Region* 4×4 *– Pressure* p*,* $t = 0.3$*,* 80×80 *Mesh*

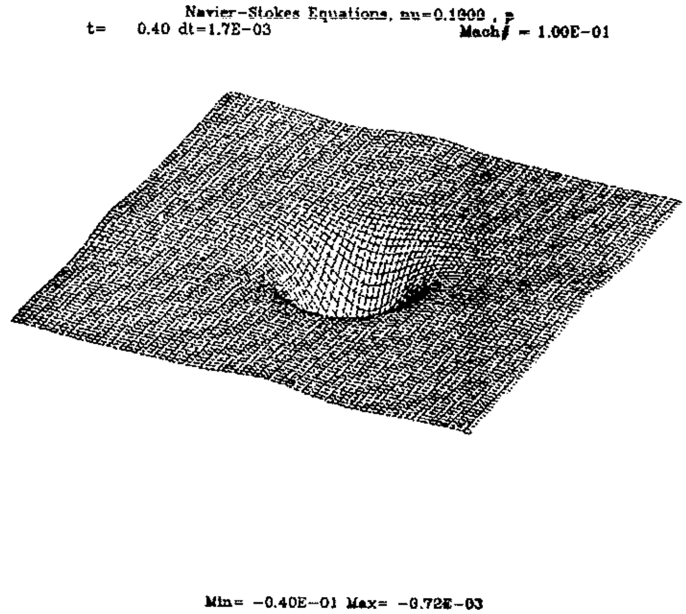

FIG. 8.19. *Region* 4×4 *- Pressure* p, $t = 0.4$, 80×80 *Mesh*

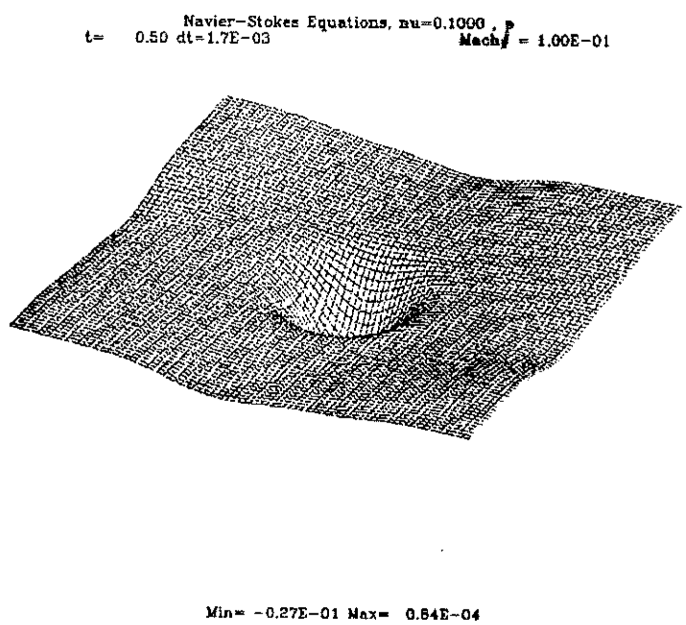

FIG. 8.20. *Region* 4×4 *- Pressure* p, $t = 0.5$, 80×80 *Mesh*

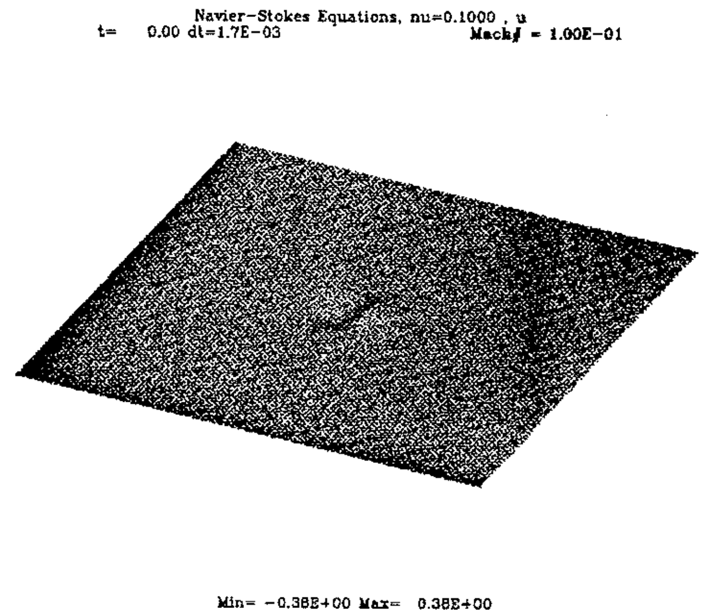

FIG. 8.21. *Region* 8×8 *– Initial Velocity* u*,* 160×160 *Mesh*

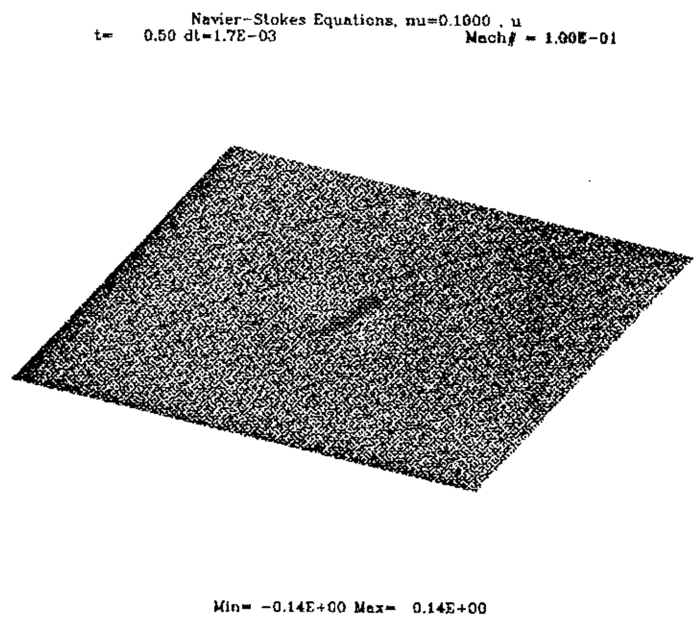

FIG. 8.22. *Region* 8×8 *– Final Velocity* u*,* 160×160 *Mesh*

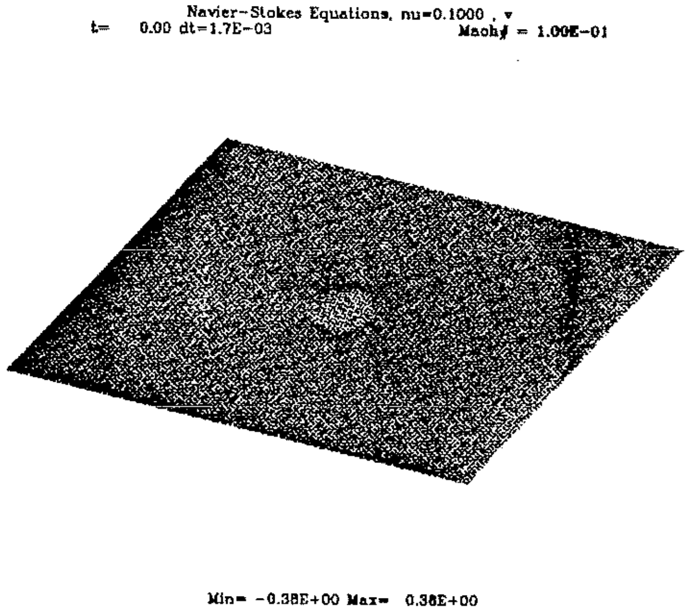

FIG. 8.23. *Region* 8×8 - *Initial Velocity v*, 160×160 *Mesh*

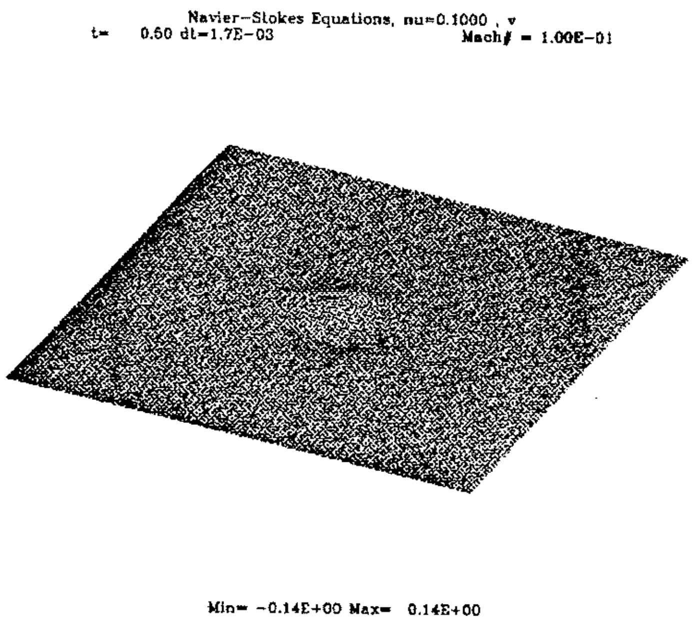

FIG. 8.24. *Region* 8×8 - *Final Velocity v*, 160×160 *Mesh*

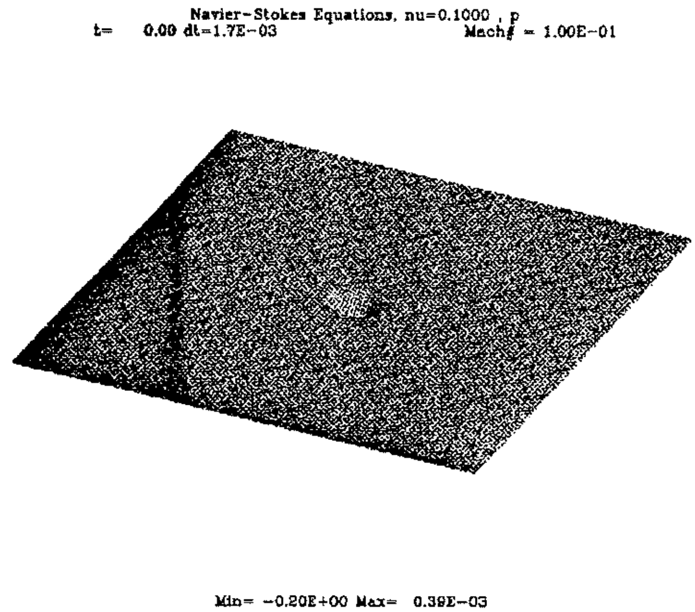

FIG. 8.25. *Region* 8×8 *- Initial Pressure p,* 160×160 *Mesh*

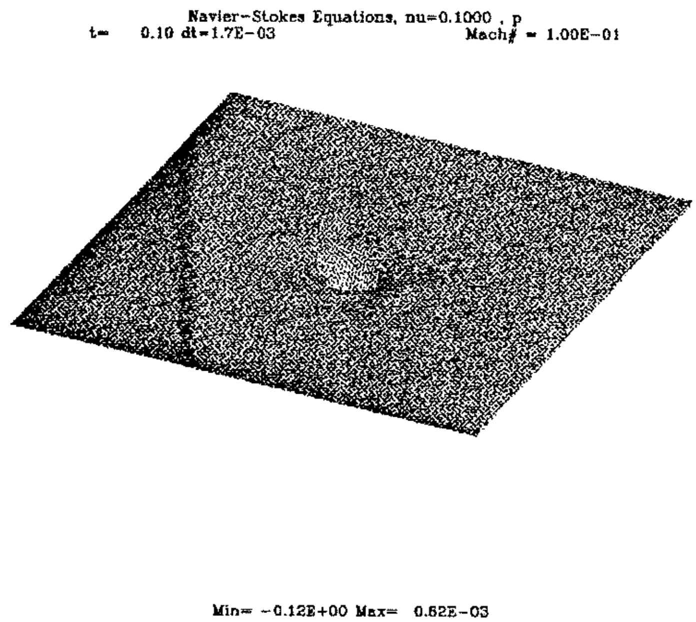

FIG. 8.26. *Region* 8×8 *- Pressure p,* $t = 0.1$, 160×160 *Mesh*

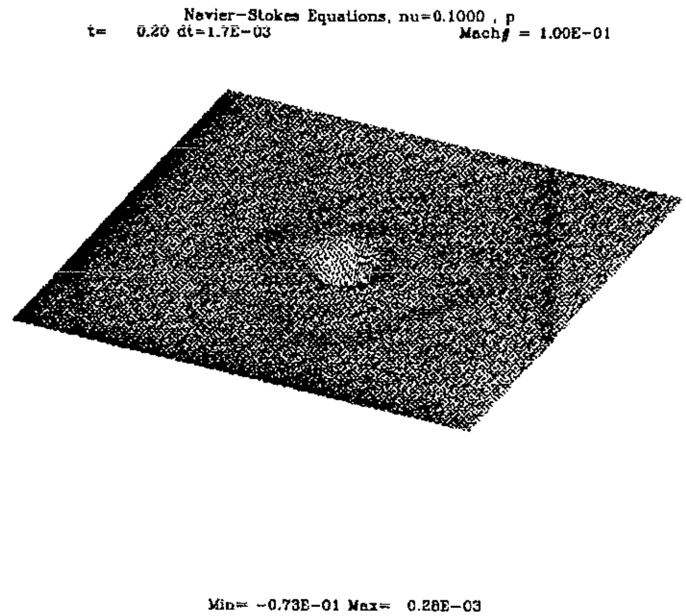

FIG. 8.27. *Region* 8×8 - *Pressure* p, $t = 0.2$, 160×160 *Mesh*

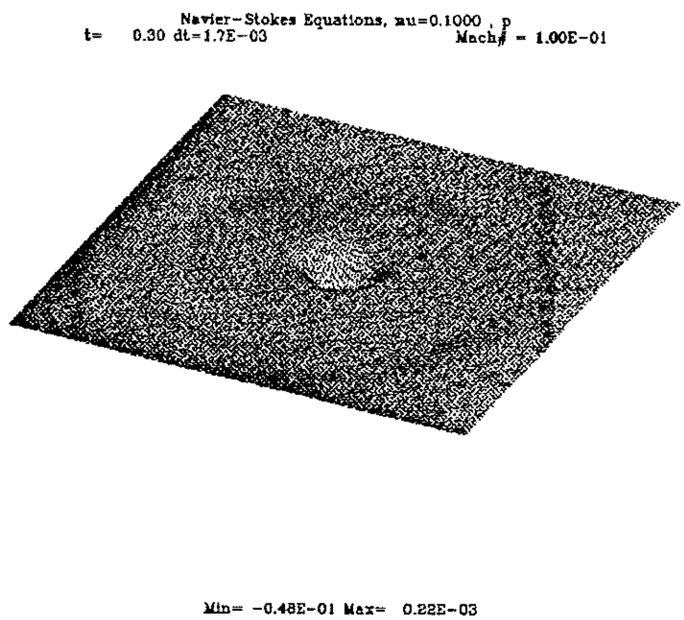

FIG. 8.28. *Region* 8×8 - *Pressure* p, $t = 0.3$, 160×160 *Mesh*

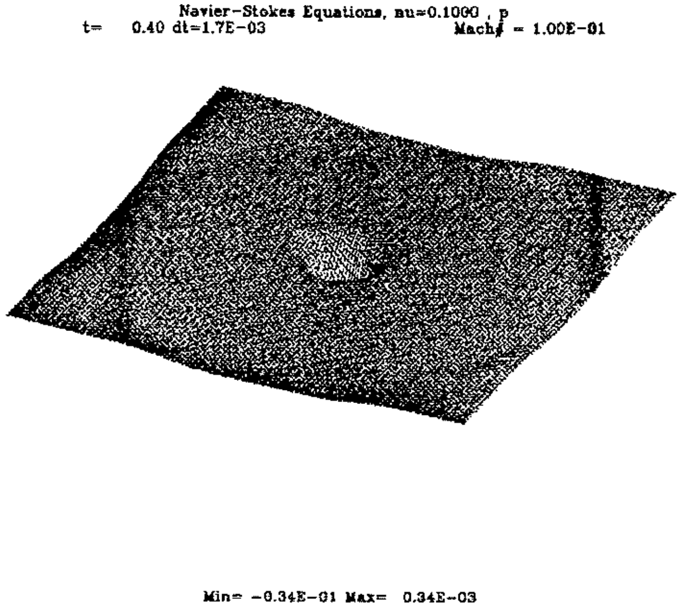

FIG. 8.29. *Region* 8×8 - *Pressure p*, $t = 0.4$, 160×160 *Mesh*

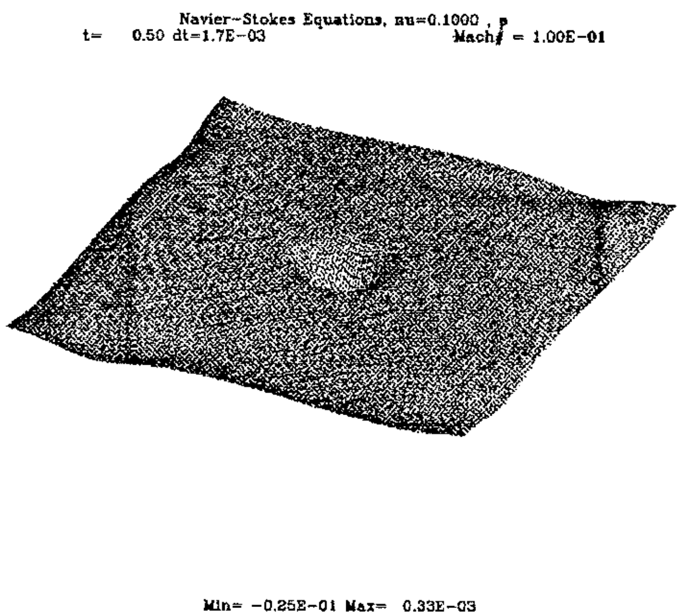

FIG. 8.30. *Region* 8×8 - *Pressure p*, $t = 0.5$, 160×160 *Mesh*

9. Summary. In this paper, we carried out a mathematical analysis of the decay rates in space of the solutions of the slightly compressible Navier-Stokes equations. This was accomplished by decomposing the slightly compressible solutions into a sum of solutions of incompressible Navier-Stokes equations, and a compressible correction term. Since the incompressible solutions vary on a slow time scale, and since, with proper initialization, we can ensure that we do not excite the fast time scale solutions, all decay rate analysis was done on the slow part of the solutions of the slightly incompressible flow equations. This technique simplifies the analysis process considerably.

We discovered that if we considered the solutions within a finite time interval, we could achieve sharper bounds on the space decay rates than otherwise could be achieved when considering the solutions throughout all time. This is useful in a computational context, in that, numerically, we always will be calculating our solutions over a finite time interval. Our theory allows us to determine how large that time interval can become before the decay rates no longer are valid.

We performed numerical experiments which bore out our assertion that the decay rates for the velocity of the slightly compressible flow are relatively rapid: rapid enough to allow us to simulate an infinite domain problem over a finite domain without specifying especially complicated outflow boundary conditions. We showed that the velocity components of slightly compressible flow decay much more rapidly in space than does the pressure component. Velocity, in fact, will decay to zero at a large enough distance, whereas the pressure component continues to be relatively large. We determined, therefore, that reasonable outflow boundary conditions could consist of specifying zero Dirichlet boundary conditions for both velocity components, and extrapolating the pressure at the boundary.

Were we to increase the size of our computational domain, so as to be able to determine the solution in the far-field, it was shown analytically that while the velocity solution would essentially be zero, the pressure solution could be well approximated by the solution of the wave equation. Inflow boundary data for the pressure in the far-field could be taken to be those numerical values calculated for the pressure on the outflow boundary of the near-field.

We plan to continue these numerical experiments in future work. The first step would involve carrying out the numerical coupling of the slightly compressible Navier-Stokes equations with the wave equation.

REFERENCES

[1] K.E. Atkinson, *An Introduction to Numerical Analysis*, John Wiley and Sons, 1978.

[2] W.E. Boyce and R.C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, Third Edition, John Wiley and Sons, 1977.

[3] D.L. Brown, G. Chesshire, and W.D. Henshaw, *Getting Started with CMPGRD, Introductory User's Guide and Reference Manual*, report LA-UR-89-1294, Los Alamos National Laboratory, 1989.

[4] W.D. Henshaw, *CGES: A Composite Grid Equation Solver*, IBM research report, IBM Research Division, Yorktown Heights, New York, 1993.

[5] W.D. Henshaw and D. W. Schwendeman, *CGCNS: A Solver for the Compressible Navier-Stokes (and Other Systems) on Overlapping Grids*, IBM research report, IBM Research Division, Yorktown Heights, New York, 1993.

[6] F. John, *Partial Differential Equations* , Springer Verlag, 1971.

[7] O.D. Kellogg, *Foundations of Potential Theory* , Dover Publications, Inc., 1954.

[8] H.-O. Kreiss, "Problems with Different Time Scales for Ordinary Differential Equations", *SIAM J. Numer. Anal.*, **16** (1979), pp.980-998.

[9] H.-O. Kreiss, "Problems with Different Time Scales for Partial Differential Equations", *Comm. Pure Appl. Math.*, **33** (1980), pp.399-439.

[10] H.-O Kreiss, *Problems with Different Time Scales*, Academic Press, Inc., 1985.

[11] H.-O. Kreiss, J. Lorenz, M.J. Naughton, "Convergence of the Solutions of the Compressible to the Incompressible Navier-Stokes Equations", *Advances in Applied Mathematics*, **12** (1991), pp.187-214.