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FINITE VOLUME METHODS WITH LOCAL REFINEMENT FOR CONVECTION–DIFFUSION PROBLEMS

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ABSTRACT. Based on approximation of the balance relation for convection-diffusion problems, finite difference schemes on rectangular locally refined grids are derived and studied. A priori estimates and error bounds in discrete H^1-norm are provided. Numerical examples demonstrating the accuracy of the schemes for a variety of model convection-diffusion problems are presented and discussed.

1. INTRODUCTION

The finite volume method (also called control volume or balance method) has been used in many applications as a systematic approach for effective discretization of conservation law equations (cf., e.g. Patankar and Spalding [13] for fluid flows). Pioneering work in this area for one-dimensional elliptic and parabolic equations with piece-wise smooth coefficients has been done by Samarskii in the early 60-ies (for comprehensive presentation see e.g. Samarskii [14]). Among the characterization, Samarskii proved that the conservation property is a necessary condition for the convergence of finite difference solutions for problems with discontinuous coefficients. Recently this approach has been augmented by new techniques and results by Manteuffel and White [10], Weiser and Wheeler [18], Bank and Rose [3], Hackbusch [6], Cai, Mandel and McCormick [4], McCormick [11].

An important feature of finite volume approach is that the approximations satisfy exactly cell-by-cell conservation law (of mass, heat, momentum, etc.). A key aspect here, is the choice of finite volumes (control volumes). In this context we distinguish two principal ways of choosing these volumes: by cell-centered or vertex-centered grids.

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Vertex-centered grid approximations are studied by Hackbusch [6], Bank and Rose [3], Heinrich [7], Cai, Mandel and McCormick [4], McCormick [11], Samarskii, Lazarov and Makarov [15], where the basic theory of the stability and the convergence analysis is developed.

Convergence and superconvergence analysis of cell-centered approximations for elliptic equations on rectangles has been presented by Weiser and Wheeler in [18], where the relations of the constructed schemes with the mixed finite element discretizations is used. This method has been used by Pedrosa [12] for efficient computation of fluid flows in porous media, and by Ewing, Lazarov and Vassilevski [5], and Vassilevski, Petrova and Lazarov [17] for elliptic equations for rectangular and triangular cell-centered grids with local refinement.

While solving two and three–dimensional problems a substantial reduction in computer resources can be achieved in exploiting the local properties of the solution if the discretization method takes advantage of these local properties. Grid refinement procedures that consist of underlying coarse grid and patches of locally refined grids (possibly in more than one level), have been used and discussed by many authors. This approach has been also widely used in reservoir modeling (see, e.g., Pedrosa [12] and references there).

Two important issues in this refinement approach have to be addressed: accurate treatment of the interface between the coarse and fine regions, and efficient solution methods for the resulting composite grid algebraic problem. In this paper we address the first issue - construction of conservative cell-centered approximations on locally refined grids for convection-diffusion second order elliptic equations that have optimal order of convergence and satisfy the discrete maximum principle.

In order to produce monotone scheme for convection-diffusion problems a variety of upwinding strategies have been used for a long time. However due to their first order of accuracy there have been several attempts to modify them in order to improve the accuracy, cf. e.g. Samarskii [14], Spalding [16], Il'in [8], Axelsson and Gustafson [2]. These approaches intend to construct second order accurate schemes while retaining unconditionally stable in maximum norm. Many of these investigations are done under rather demanding assumptions for the solution (to have four continuous derivatives). In this paper, while keeping required regularity of the solution to H^s for $3/2 \leq s \leq 3$, we derive monotone approximations on grids with local patch refinement. We estimate the error in the natural discrete H^1-norm and prove convergence rates of $O(h^{1/2})$, or $O(h^{3/2})$ depending upon the interpolation (piece-wise constant or linear) used and also depending if the simple or modified upwinding strategies are used. These result can be considered as an extension of a previous paper by Lazarov, Mishev and Vassilevski [9] and of the paper by Ewing, Lazarov and Vassilevski [5] for the symmetric case.

The remainder of the paper is organized as follows. In §1.1 the problem is formulated and in §1.2 the basic notations are introduced. §2 contains the derivations of the schemes. §3 deals with the main properties of the discrete problems. The error analysis is presented in §4. At the end in §5 we provide extensive computer experiments for a variety of convection- diffusion problems, including convection dominated ones, in support of our theoretical results and to assess the applicability of the derived schemes and error bounds. Some technical details are given in Appendices A and B.

1.1. Boundary value problem. We use the standard notation for Sobolev spaces [1]:

$$W_p^m(\Omega) = \{u \in L^p(\Omega) \, : \, D^\alpha u \in L^p(\Omega) \, , \, |\alpha| \leq m\} \, , \quad m \geq 0 \, , \, 1 \leq p \leq \infty$$

and $W_2^m(\Omega) = H^m(\Omega)$. The norm in $H^m(\Omega)$ is denoted $\|.\|_{m,\Omega}$ and defined by

$$\|u\|_{m,\Omega} \equiv \left(\sum_{i=0}^{m} |u|_{i,\Omega}^2 \right)^{1/2}, \qquad |u|_{i,\Omega} \equiv \left(\sum_{|\alpha|=i} \|D^\alpha u\|_{0,\Omega}^2 \right)^{1/2},$$

$$\|u\|_{m,\infty,\Omega} \equiv \max_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha u| \, ,$$

where $\|.\|_{0,\Omega}$ is the standard L^2-norm in Ω. We also use Sobolev spaces with real index $m > 0$ [1].

We consider the following convection-diffusion boundary value problem: find a function $u(x)$ which satisfies the following differential equation and boundary condition:

$$\left\{ \begin{array}{rcll} \mathbf{div}(-a(x)\nabla u(x) + \underline{b}(x)u(x)) & = & f(x) & \text{in } \Omega \\ u(x) & = & 0 & \text{on } \Gamma \end{array} \right. \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain and $\Gamma = \partial\Omega$. The coefficients $a(x)$ and $\underline{b}(x) = (b_1(x), b_2(x))$ are supposed to fulfill for some constants a_0 and β_0, β_1 the conditions

(i) $a(x) \geq a_0 > 0$, $a(x) \in W_\infty^1(\Omega)$,

(ii) $| b_i(x) | \leq \beta_1$, $b_i \in W_\infty^1(\Omega)$,

and in order to obtain coercivity is sufficient to hold

(iii) $(\nabla.\underline{b}(x)) \geq \beta_0 > 0$.

The function $f(x)$ is given in Ω and $f(x) \in L^2(\Omega)$.

From (iii) we obtain that the bilinear form arising from convection-diffusion equation (1) is a $H_0^1(\Omega)$–elliptic. Hence the continuous problem has a unique solution.

1.2. Notations. We suppose that Ω is a rectangle with sides parallel to the axes x_1 and x_2. Extensions to the case of more general domains can be accomplished using the technique described in Samarskii, Lazarov and Makarov [15, Chapter III, p. 123] or by using triangular cells, cf. Vassilevski, Petrova and Lazarov [17] for the selfadjoint case.

We consider the case of cell-centered grids, which owing to their good conservation properties, are very popular in reservoir simulation, weather prediction, heat transfer etc. We cover the plane R^2 by square cells with sides of length h. The grid points are the centers of the cells (see, Fig. 2). We suppose that the Dirichlet boundary Γ passes through the grid points, as shown in Fig. 2.

The grid points are denoted by $x = (x_1, x_2) = (x_{1,i}, x_{2,j}) = (ih, jh)$, where $i, j = 0, 1, 2, ..., N$ are integer indices. We introduce the following notations for various grids in $\overline{\Omega}$:

$$\overline{\omega} = \left\{(x_{1,i}, x_{2,j}) \in \overline{\Omega} : i, j = 0, 1, 2, ..., N\right\};$$

$$\omega = \overline{\omega} \cap \Omega, \quad \gamma = \overline{\omega} \backslash \omega;$$

$$\omega_i^{\pm} = \omega \cup \gamma_i^{\pm}\ , \text{ where } \ \gamma_i^{\pm} = \{x \in \gamma : cos(x_i, \underline{n}) = \pm 1\}\ ,\ i = 1,\ 2,$$

here $\underline{n}$ is the unit outer vector normal to the boundary Γ.

Functions defined for $x \in \omega$ are called grid functions. We consistently use the dual notation for the value of the function y at the grid point $x = (x_{1,i}, x_{2,j})$; $y(x) = y(x_{1,i}, x_{2,j}) = y_{i,j}$ and in the points $(x_{1,i}, x_{2,j} \pm h/2) = (x_{1,i}, x_{2,j\pm 1/2})$ and $(x_{1,i} \pm h/2, x_{2,j}) = (x_{1,i\pm 1/2}, x_{2,j})$, $y_{i,j\pm 1/2} = y(x_{1,i}, x_{2,j\pm 1/2})$, $y_{i\pm 1/2,j} = y(x_{1,i\pm 1/2}, x_{2,j})$.

For a given function $y(x)$, $x \in \overline{\omega}$ we use the following discrete inner products and norms:

$$(y, v) = \sum_{x\in\omega} h^2 y(x) v(x),\ \|y\|_{0,\omega} = (y, y)^{\frac{1}{2}};$$

$$(y, v]_s = \sum_{x\in\omega_s^+} h^2 y(x) v(x)\,,\ \|y]|_s = (y, y]_s^{\frac{1}{2}}\,, \quad s = 1,\ 2\,.$$

We introduce the following finite differences for a grid function $y(x)$:

(*i*) forward difference $\Delta_1 y_{i,j} = y_{i+1,j} - y_{i,j}$ and divided forward difference $y_{x_1} = \Delta_1 y/h$;

(*ii*) backward difference $\overline{\Delta}_1 y_{i,j} = y_{i,j} - y_{i-1,j}$ and divided backward difference $y_{\overline{x}_1} = \overline{\Delta}_1 y/h$;

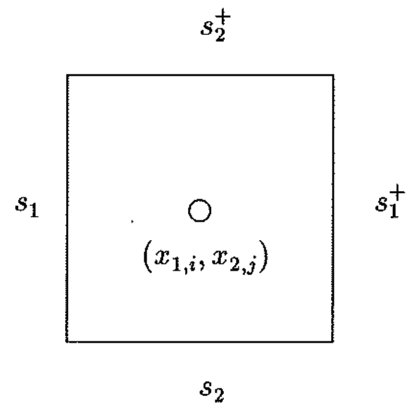
We also introduce the discrete analogue of H^1 -norm:

$$|y|^2_{1,\omega} = \|y_{\overline{x}_1}]|_1^2 + \|y_{\overline{x}_2}]|_2^2\,,$$

$$\|y\|^2_{1,\omega} = |y|^2_{1,\omega} + \|y\|^2_{0,\omega}\,.$$

Any grid function $y(x)$ can be considered as an element of a vector space of dimension equal to n, the number of the grid points in ω. In this case, we denote $y(x)$ by $\mathbf{y} \in R^n$ and consider it as an n-dimensional column vector. Then $\mathbf{y}^T$ will be the row vector transpose of $\mathbf{y}$.

FIGURE 1. Cell $e(x)$



2. Approximation of the differential equation.

First we consider a uniform mesh without local refinement. Let the domain Ω be covered by a set of rectangular cells e. The finite difference approximation is derived from the balance equation. We integrate (1) over each cell e

$$\int_e div[(-a(x)\nabla u(x) + \underline{b}(x)u(x)]\, dx = \int_e f(x)\, dx$$

and then using the Green's formula we get

$$\int_{\partial e} [-a\nabla u.\underline{n} + u\underline{b}.\underline{n}]\, d\gamma = \int_e f(x)\, dx \tag{2}$$

where $\underline{n}$ is the unit outward vector normal to the boundary of e. Splitting $\partial e = s_1^+ \cup s_2^+ \cup s_1 \cup s_2$ (see Fig. 1) this identity can be written in the form:

$$\begin{aligned} \int_{\partial e} W\, d\gamma + \int_{\partial e} V\, d\gamma \;=\; & \int_{s_1^+} W\, d\gamma + \int_{s_1^+} V\, d\gamma - \int_{s_1} W\, d\gamma - \int_{s_1} V\, d\gamma \\ & + \int_{s_2^+} W\, d\gamma + \int_{s_2^+} V\, d\gamma - \int_{s_2} W\, d\gamma - \int_{s_2} V\, d\gamma \end{aligned} \tag{3}$$

where we have denoted by

$$W = -a(\gamma)\nabla u(\gamma).\underline{n} \quad \text{and} \quad V = \underline{b}(\gamma).\underline{n}u(\gamma)\,, \quad \text{for } \gamma \in s_1^+,\, s_2^+, s_1,\, s_2.$$

The approximations that we exploit for $\int_{s_l^+} V_l^+\, d\gamma$ and $\int_{s_l} V_l\, d\gamma$ lead to upwind difference schemes. For the integrals $\int_{s_l^+} W_l^+\, d\gamma$ and $\int_{s_l} W_l\, d\gamma$ we use two different approaches. The first one as used in [14] and [5] reads

$$(4)\qquad \begin{aligned} w_l^+(x) &\equiv w_{l,i,j}^+ = -k_{l,i,j}^+ \triangle_l y_{i,j}\,,\ l=1\,,\,2\,,\\ w_l(x) &\equiv w_{l,i,j} = -k_{l,i,j}\overline{\triangle}_l y_{i,j}\,,\ l=1\,,\,2\,,\end{aligned}$$

where

$$k_{1,i,j} = \left(\frac{1}{h}\int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s,x_{2,j})}\right)^{-1}, \ k_{1,i,j}^+ = k_{1,i+1,j}$$

$$k_{2,i,j} = \left(\frac{1}{h}\int_{x_{2,j-1}}^{x_{2,j}} \frac{ds}{a(x_{1,i},s)}\right)^{-1}, \ k_{2,i,j}^+ = k_{2,i,j+1}$$

and

$$\triangle_1 y_{i,j} = y_{i+1,j} - y_{i,j}\,,\ \overline{\triangle}_1 y_{i,j} = y_{i,j} - y_{i-1,j}\,.$$
$$\triangle_2 y_{i,j} = y_{i,j+1} - y_{i,j}\,,\ \overline{\triangle}_2 y_{i,j} = y_{i,j} - y_{i,j-1}\,.$$

The second one gives a modified upwind difference scheme **MUDS** [2] , [9]

$$(5)\qquad \begin{aligned} w_{l,i,j}^+ &= -\frac{k_{l,i,j}^+}{1+|B_{l,i,j}^+|/k_{l,i,j}^+}\triangle_l y_{i,j}\,,\ l=1,\,2\,,\\ w_{l,i,j} &= -\frac{k_{l,i,j}}{1+|B_{l,i,j}|/k_{l,i,j}}\overline{\triangle}_l y_{i,j}\,,\ l=1,\,2\,,\end{aligned}$$

where

$$B_{1,i,j}^+ = \frac{b_{1,i+1/2,j}h}{2}\,, \qquad B_{1,i,j} = \frac{b_{1,i-1/2,j}h}{2}\,,$$
$$B_{2,i,j}^+ = \frac{b_{2,i,j+1/2}h}{2}\,, \qquad B_{2,i,j} = \frac{b_{2,i,j-1/2}h}{2}\,.$$

The integrals $\int_{s_l^+} V_l^+\,d\gamma$ and $\int_{s_l} V_l\,d\gamma$ are approximated as follows

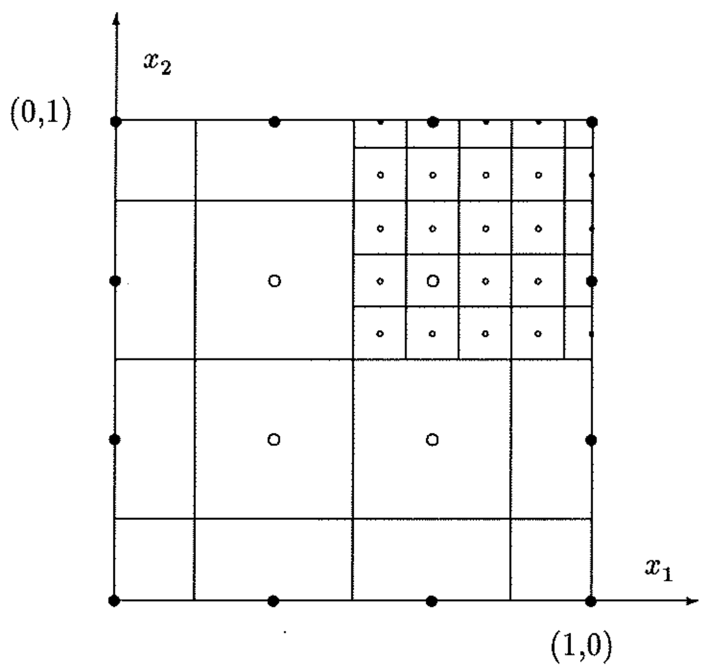
$$(6)\qquad \begin{aligned} v_l^+(x) &\equiv v_{l,i,j} = (B_{l,i,j}^+ - |B_{l,i,j}^+|)y_{i+1,j} + (B_{l,i,j}^+ + |B_{l,i,j}^+|)y_{i,j}\,,\\ v_l(x) &\equiv v_{l,i,j} = (B_{l,i,j} - |B_{l,i,j}|)y_{i,j} + (B_{l,i,j} + |B_{l,i,j}|)y_{i-1,j}\,.\end{aligned}$$

In this way we get two different schemes—upwind difference scheme (**UDS**) based on formulas (4), (6) and modified upwind difference scheme (**MUDS**) based on formulas (5), (6).

Now we consider the case with local refinement, where some of the cells are refined into a number of fine grid cells and introduced as a grid points the centers of the new finer cells (see Fig. 2). The subregion covered by the refined grid is denoted by Ω_2 and the remaining part of Ω is denoted by Ω_1, i.e., $\bar{\Omega} = \bar{\Omega}_1 \bigcup \bar{\Omega}_2$. We assume that the cells are squares. There are cells of two different sizes: coarse grid cells of size h_c and fine grid cells of size $h_f = \frac{1}{m}h_c$, where m is a given positive integer.

The centers of the coarse grid cells contained in Ω define the coarse grid, which is denoted by $\tilde{\omega}$. The set of coarse grid points in Ω_2 is designated by $\tilde{\omega}_2$, i.e., $\tilde{\omega}_2 = \tilde{\omega} \bigcap \Omega_2$.

FIGURE 2. Composite cell-centered mesh



The coarse grid points in Ω_1 and the fine grid points in Ω_2 define the composite grid denoted by ω. The grid points of the composite grid next to the boundary between Ω_1 and Ω_2 we will call irregular. All remaining grid points will be called regular.

From now on we will consider only the terms of the difference schemes in the x_1–direction. In the other direction the corresponding expressions are derived similarly.

We require that the finite difference schemes fulfill a conservation law. Hence from

$$\begin{aligned}\int_{s^+_{1,i-1,j+1}} \left(W_1^+ + V_1^+\right) d\gamma &= \int_{s_{1,i,j}} (W_1 + V_1)\, d\gamma + \int_{s_{1,i,j+1}} (W_1 + V_1)\, d\gamma \\ &\quad + \int_{s_{1,i,j+2}} (W_1 + V_1)\, d\gamma\end{aligned}$$

we have

$$w^+_{1,i-1,j+1} + v^+_{1,i-1,j+1} = w_{1,i,j} + v_{1,i,j} + w_{1,i,j+1} + v_{1,i,j+1} + w_{1,i,j+2} + v_{1,i,j+2}\,.$$

There exist various ways to approximate the fluxes $w_{1,i,j+l}$ and $v_{1,i,j+l}$, $l = 0, 1, 2$. Next we consider two simple ways based on constant and linear interpolation.

2.1. Constant Approximation. We suppose that the grid function $y(x)$, $x \in \omega$ is extended in Ω as a constant over each cell $e(x)$, $x \in \omega$. We have to consider the modification of our finite volume schemes that have to be made along the interface between Ω_1 and Ω_2, i.e., at the irregular points. We use formula for non uniform mesh (see Fig. 3) where for definiteness we assume $h_c = 3h_f$

$$w_{1,i,j+l} = -\frac{2h_f}{h_c + h_f} k_{1,i,j+l} \overline{\Delta}_l y_{i,j+l}\,,\ l = 0, 1, 2,$$

here

$$k_{1,i,j+l} = \left(\frac{2}{h_c + h_f} \int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s, x_{2,j})} \right)^{-1},\ k^+_{1,i,j} = k_{1,i+1,j}$$

and

$$\overline{\Delta}_1 y_{i,j+l} = y_{i,j+l} - y_{i-1,j+l} = y_{i,j+l} - y_{i-1,j+1}\,.$$

Note that (x_{i-1}, y_{j+l}), $l = 0, 2$ are "slave" nodes and $y_{i-1,j+1} = y_{i-1,j+l}$, $l = 0, 2$ because we use constant interpolation. Since $h_c = 3h_f$ we get

$$w_{1,i,j+l} = -\frac{1}{2} k_{1,i,j+l} \overline{\Delta}_1 y_{i,j+l}\,. \tag{7}$$

and

$$v_{1,i,j+l} = (B_{1,i,j+l} - |B_{1,i,j}|) y_{i,j+l} + (B_{1,i,j+l} + |B_{1,i,j}|) y_{i-1,j+1}\,. \tag{8}$$

Because of the poor approximation properties we do not consider constant approximation for **MUDS**.

2.2. Linear Approximation. We use this approximations for **MUDS** in irregular points. In this case is supposed that $y(x)$, $x \in \omega$ is interpolated linearly between any two neighboring coarse grid nodes. For simplicity of presentation we confine again only with the case $h_c = 3h_f$. We will need values of y at the points $(x_{1,i-1}, x_{2,j})$ and $(x_{1,i-1}, x_{2,j+2})$ which are not grid ones (see Fig. 3). To get them we use the following linear interpolation

$$
\begin{aligned}
y_{i-1,j} &= \frac{2}{3}y_{i-1,j+1} + \frac{1}{3}y_{i-1,j-1}\,, \\
y_{i-1,j+2} &= \frac{2}{3}y_{i-1,j+1} + \frac{1}{3}y_{i-1,j+4}\,.
\end{aligned} \tag{9}
$$

We sketch the derivation of **MUDS** at the irregular points (see [9] for a detailed derivation of **MUDS** at the regular points). First we write the standard central finite difference scheme, i.e., $\int_s V$ is approximated by the analog of central differences

$$
\begin{aligned}
\int_{s_{1,i,j}} (W+V)\,ds &= -\frac{2h_f}{h_c+h_f}k_{1,i,j}\left[y_{i,j} - y_{i-1,j}\right] \\
&\quad + b_{1,i-1/2,j}h_f\left[\frac{h_f y_{i-1,j} + h_c y_{i,j}}{h_f+h_c}\right] + O(h^2) \\
&= -\frac{1}{2}k_{1,i,j}\left[y_{i,j} - y_{i-1,j}\right] + \frac{B_{1,i,j}}{2}\left[y_{i-1,j} + 3y_{i,j}\right] + O(h^2)\,.
\end{aligned}
$$

Next we substitute $y_{i-1,j}$ from (9) and represent the terms approximating $\int_s V$ in an upwind manner

$$
\begin{aligned}
\int_{s_{1,i,j}} (W+V)\,ds &= -\frac{1}{2}k_{1,i,j}\left(y_{i,j} - \frac{2}{3}y_{i-1,j+1} - \frac{1}{3}y_{i-1,j-1}\right) \\
&\quad + \frac{B_{1,i,j}}{2}\left(\frac{2}{3}y_{i-1,j+1} + \frac{1}{3}y_{i-1,j-1} + 3y_{i,j}\right) + O(h^2) \\
&= -\frac{1}{2}k_{1,i,j}\left[\left(y_{i,j} - y_{i-1,j+1}\right) + \frac{1}{3}\left(y_{i-1,j+1} - y_{i-1,j-1}\right)\right] \\
&\quad + \left(B_{1,i,j} - |B_{1,i,j}|\right)y_{i,j} \\
&\quad + \left(B_{1,i,j} + |B_{1,i,j}|\right)\left(\frac{2}{3}y_{i-1,j+1} + \frac{1}{3}y_{i-1,j-1}\right) \\
&\quad + \left(\frac{B_{1,i,j}}{2} + |B_{1,i,j}|\right)\left(y_{i,j} - y_{i-1,j+1}\right) \\
&\quad + \frac{1}{3}\left(\frac{B_{1,i,j}}{2} + |B_{1,i,j}|\right)\left(y_{i-1,j+1} - y_{i-1,j-1}\right) + O(h^2)\,.
\end{aligned}
$$

Finally, we get

$$\begin{aligned}\int_{s_{1,i,j}} (W+V)\,ds &= -\frac{1}{2}\left[k_{1,i,j} - (2|B_{1,i,j}| + B_{1,i,j}\right] \overline{\triangle}_1 y_{i,j} \\ &\quad -\frac{1}{6}\left[k_{1,i,j} - (2|B_{1,i,j}| + B_{1,i,j}\right] \overline{\triangle}_2 y_{i-1,j+1} \\ &\quad + \left(B_{1,i,j} - |B_{1,i,j}|\right) y_{i,j} \\ &\quad + \left(B_{1,i,j} + |B_{1,i,j}|\right)\left(\frac{2}{3} y_{i-1,j+1} + \frac{1}{3} y_{i-1,j-1}\right) + O(h^2).\end{aligned}$$

In order to obtain upwind scheme we approximate the first term in the above formula

$$\begin{aligned}k_{1,i,j} - \left(2|B_{1,i,j}| + B_{1,i,j}\right) &= \frac{k_{1,i,j}}{1 + \left(2|B_{1,i,j}| + B_{1,i,j}\right)/k_{1,i,j}} \\ &\quad - \frac{\left(2|B_{1,i,j}| + B_{1,i,j}\right)^2}{k_{1,i,j} + 2|B_{1,i,j}| + B_{1,i,j}} \\ &= \frac{k_{1,i,j}}{1 + \left(2|B_{1,i,j}| + B_{1,i,j}\right)/k_{1,i,j}} + O(h^2).\end{aligned}$$

In the last step we have taken into account that $B_1 = O(h)$. In this way we define the approximate fluxes w and v as follows:

$$\begin{aligned}w_{1,i,j} &= -\frac{1}{2}\frac{k_{1,i,j}}{1 + \left(2|B_{1,i,j}| + B_{1,i,j}\right)/k_{1,i,j}} \overline{\triangle}_1 y_{i,j} \\ &\quad -\frac{1}{6}\left(\frac{k_{1,i,j}}{1 + \left(2|B_{1,i,j}| + B_{1,i,j}\right)/k_{1,i,j}}\right) \overline{\triangle}_2 y_{i-1,j+1} \\ w_{1,i,j+1} &= -\frac{1}{2}\cdot\frac{k_{1,i,j+1}}{1 + \left(2|B_{1,i,j+1}| + B_{1,i,j+1}\right)/k_{1,i,j+1}} \overline{\triangle}_1 y_{i,j+1} \\ w_{1,i,j+2} &= -\frac{1}{2}\cdot\frac{k_{1,i,j+2}}{1 + \left(2|B_{1,i,j+2}| + B_{1,i,j+2}\right)/k_{1,i,j+2}} \overline{\triangle}_1 y_{i,j+2} \\ &\quad -\frac{1}{6}\cdot\frac{k_{1,i,j+2}}{1 + \left(2|B_{1,i,j+2}| + B_{1,i,j+2}\right)/k_{1,i,j+2}} \triangle_2 y_{i-1,j+1}\end{aligned} \tag{10}$$

and

$$(11)\quad \begin{aligned} v_{1,i,j} &= \left(B_{1,i,j} - |B_{1,i,j}|\right) y_{i,j} + \left(B_{1,i,j} + |B_{1,i,j}|\right) y_{i-1,j+1} \\ &\quad -\frac{1}{3}\left(B_{1,i,j} + |B_{1,i,j}|\right) \overline{\triangle}_2 y_{i-1,j+1}\,, \\ v_{1,i,j+1} &= \left(B_{1,i,j+1} - |B_{1,i,j+1}|\right) y_{i,j+1} + \left(B_{1,i,j+1} + |B_{1,i,j+1}|\right) y_{i-1,j+1} \\ v_{1,i,j+2} &= \left(B_{1,i,j+2} - |B_{1,i,j+2}|\right) y_{i,j+2} + \left(B_{1,i,j+2} + |B_{1,i,j+2}|\right) y_{i-1,j+1} \\ &\quad +\frac{1}{3}\left(B_{1,i,j+2} + |B_{1,i,j+2}|\right) \triangle_2 y_{i-1,j+1}\,. \end{aligned}$$

3. Formulation of the Discrete Problems.

Two difference schemes derived in Section 2 can be written in the general form

$$\begin{cases} \sum_{x\in\omega}\sum_{l=1}^{2}\left(w_l^+(x) - w_l(x)\right) + \left(u_l^+(x) - u_l(x)\right) = \int_e f(x)\,ds & \text{in } \Omega \\ y(x) = g(x) & \text{on } \Gamma \end{cases}$$

For **MUDS** the approximate fluxes $w_l^+(x)$, $w_l(x)$, $u_l^+(x)$ and $u_l(x)$ are defined by (5), (6) at the regular and by (10) and (11) at the irregular points. In matrix terms we write

$$(12)\qquad A\mathbf{y} = \mathbf{f}$$

where in the right-hand side f we have taken into account the boundary conditions. We will denote A_0 the matrix of the constant approximation (7), (8) in irregular points and (4), (6) in regular points; for this scheme we denote

$$(13)\qquad A_0\mathbf{y} = \mathbf{f}$$

Consider $x = (x_{1,i-1}, x_{2,j+1})$ (see Fig. 3). We let

$$B^+_{1,i-1,j+1} = B_{1,i,j} + B_{1,i,j+1} + B_{1,i,j+2}\,.$$

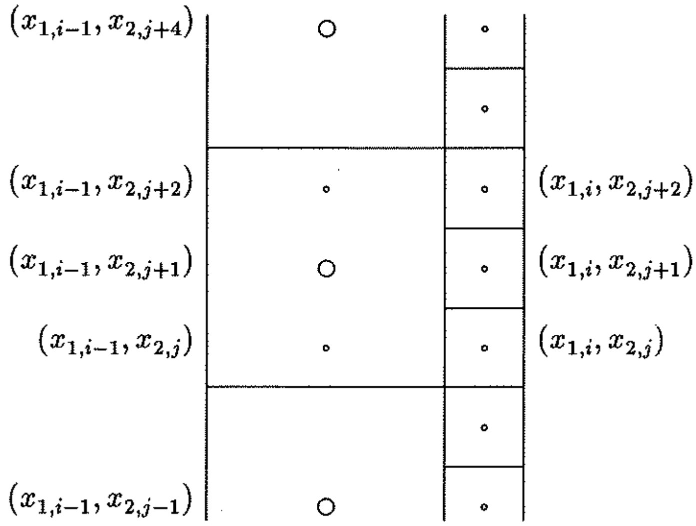
We will use the following auxiliary result.

Lemma 3.1. *Let $\underline{b}(x) \in (W^{1+\alpha}_\infty(\Omega))^2$, $\alpha > 0$ and $(\nabla.\underline{b}(x)) \geq \beta_0 = Const > 0$. Then there exists a h_0 such that for $h < h_0$ it holds*

$$\left[(B^+_{1,i,j} - B_{1,i,j}) + (B^+_{2,i,j} - B_{2,i,j})\right] \geq c_0 h^2\,,$$

where $c_0 = \beta_0 - O(h^\alpha)$, $0 < \alpha \leq 1$.

FIGURE 3. Irregular cell $e(x_{1,i-1}, x_{2,j+1})$



Proof. For the regular points see Lazarov, Mishev and Vassilevski [9]. For the irregular point $x = (x_{1,i-1}, x_{2,j+1})$ and the adjacent points $(x_{1,i},\ x_{2,j})$, $(x_{1,i}, x_{2,j+1})$ and $(x_{1,i}, x_{2,j+2})$ in the refined region we consider the linear functional l

$$\begin{aligned} l(b_1) \;=\; & \frac{b_{1,i-1/2,j} + b_{1,i-1/2,j+1} + b_{1,i-1/2,j+2} - 3b_{1,i-3/2,j+1}}{3h_c} \\ & - \frac{\partial b_1(x_{1,i-1}, x_{2,j+1})}{\partial x_1} . \end{aligned}$$

The functional l is bounded for $b_1 \in W^1_\infty(e(x_{1,i}, x_{2,j}))$ and vanishing for all polynomials of first degree. Hence

$$|l(b_1)| \le C h^\alpha |b_1|_{1+\alpha,\infty,e}\,,\ 0 < \alpha \le 1\,.$$

A similar inequality holds for b_2. Using the triangle inequality and the assumption $\nabla.\underline{b} \ge \beta_0$ the desired inequality is obtained. □

Our goal now is to show that both schemes have unique solution. First we investigate some properties of **UDS** in the following lemma.

Lemma 3.2. *Let $z(x)$ and $y(x)$ be grid functions satisfying $y(x)$, $z(x) = 0$ on Γ. If*

A_0 is the matrix defined by (4), (6), (7) and (8) then the following formula holds

$$
\begin{aligned}
(14)\qquad z^T A_0 y \;=\; & -\sum_{x\in\omega}\sum_{l=1}^{2} w_l(x)\overline{\triangle}_l z(x) \\
& +\sum_{x\in\omega}\sum_{l=1}^{2} |B_l(x)|\overline{\triangle}_l y(x)\overline{\triangle}_l z(x) \\
& +\sum_{x\in\omega}\sum_{l=1}^{2} \left(B_l^+(x)-B_l(x)\right) y(x)z(x) \\
& +\sum_{x\in\omega}\sum_{l=1}^{2} B_l(x)\left(z(x)\overline{\triangle}_l y(x) - y(x)\overline{\triangle}_l z(x)\right) .
\end{aligned}
$$

(The proof is provided in Appendix A.)

If we set $\mathbf{z} = \mathbf{y}$ in (14) and use Lemma 3.1 we immediately obtain the following result.

Corollary 3.1. *If $\underline{b} \in \left(W_\infty^{1+\alpha}(\Omega)\right)^2$, $\alpha > 0$ then the matrix A_0 is positive real, i.e. its symmetric part is a positive definite matrix, and hence the* ***UDS*** *defined by (4), (6), (7) and (8) has unique solution. Moreover we have the discrete H_0^1-coercivity estimate*

$$\boldsymbol{y}^T A_0 \boldsymbol{y} \geq C\|y\|_{1,\omega}^2 .$$

We can write matrix A_0 in the following form

$$A_0 = A_0^{(1)} + A_0^{(2)}$$

where $A_0^{(1)}$ corresponds to the diffusion part and $A_0^{(2)}$ corresponds to the remaining convection part. For $A_0^{(1)}$ we have

$$
\begin{aligned}
(15)\qquad \mathbf{z}^T A_0^{(1)}\mathbf{y} \;&=\; -\sum_{x\in\omega} w_1(x)\overline{\triangle}_1 z(x) + w_2(x)\overline{\triangle}_2 z(x) \\
&=\; \sum_{x\in\omega}\left(\alpha_1\overline{\triangle}_l y\overline{\triangle}_l z + \alpha_2\overline{\triangle}_2 y\overline{\triangle}_2 z\right) ,
\end{aligned}
$$

where

$$\alpha_1 = \alpha_{1,i,j} = \begin{cases} k_{1,i,j}/2 & \text{for } j \geq 0,\ i = 0, \\ k_{1,i,j} & \text{for the remaining indices} \end{cases}$$

$$\alpha_2 = \alpha_{2,i,j} = \begin{cases} k_{2,i,j}/2 & \text{for } i \geq 0,\ j = 0, \\ k_{2,i,j} & \text{for the remaining indices} \end{cases}$$

(See Fig. 4.) In the same way we split the matrix A arising from linear approximation into two parts

$$A = A^{(1)} + A^{(2)}.$$

For $A^{(1)}$ we get

$$\begin{aligned}\mathbf{z}^T A^{(1)}\mathbf{y} &= \sum_{x\in\omega}\left(\beta_1\overline{\Delta}_l y\overline{\Delta}_l z + \beta_2\overline{\Delta}_2 y\overline{\Delta}_2 z\right)\\ &+\frac{1}{6}\sum_{j=1,4,7,\dots}\left[\beta_{1,0,j-1}\overline{\Delta}_2 y_{-1,j}\overline{\Delta}_1 z_{0,j-1} - \beta_{1,0,j+1}\Delta_2 y_{-1,j}\overline{\Delta}_1 z_{0,j+1}\right]\\ &+\frac{1}{6}\sum_{i=1,4,7,\dots}\left[\beta_{2,i-1,0}\overline{\Delta}_1 y_{i,-1}\overline{\Delta}_2 z_{i-1,0} - \beta_{2,i+1,0}\Delta_1 y_{i,-1}\overline{\Delta}_2 z_{i+1,0}\right]\end{aligned}$$

where

$$\beta_1 = \beta_{1,i,j} = \begin{cases}\tilde{k}_{1,i,j}/2 & \text{for } j\ge 0,\ i=0,\\ \hat{k}_{1,i,j} & \text{for the remaining indices}\end{cases}$$

$$\beta_2 = \beta_{2,i,j} = \begin{cases}\tilde{k}_{2,i,j}/2 & \text{for } i\ge 0,\ j=0,\\ \hat{k}_{2,i,j} & \text{for the remaining indices}\end{cases}$$

and

$$\tilde{k}_{l,i,j} = \frac{k_{l,i,j}}{1+\left(2|B_l|+B_l\right)/k_{l,i,j}}\,,\qquad \hat{k}_{l,i,j} = \frac{k_{l,i,j}}{1+|B_l|/k_{l,i,j}}$$

Applying the Cauchy inequality to the $\mathbf{z}^T A^{(1)}\mathbf{y}$ and taking into account that $\tilde{k}_{l,i,j}$ and $\hat{k}_{l,i,j}$ are less than $k_{l,i,j}$ we get

$$|\mathbf{z}^T A^{(1)}\mathbf{y}| \le \left(\frac{7}{6}+C_2 h\right)\left(\mathbf{z}^T A_0^{(1)}\mathbf{z}\right)^{1/2}\left(\mathbf{y}^T A_0^{(1)}\mathbf{y}\right)^{1/2},$$

where the constant C_2 depends on the values of the coefficient $a(x)$ only locally, i.e., cell by cell. To derive a lower bound we need the inequality

$$\frac{Pk_{l,i,j}-k_{l,i,j}-|B_{l,i,j}|}{k_{l,i,j}+|B_{l,i,j}|} > 0\,,\quad P := 1+\sup_{\substack{x\in\omega\\ l=1,2}}\frac{|b_l(x)|h_c}{2k_l(x)}\,.$$

Consider auxiliary matrix $A_*^{(1)}$ obtained by replacing in (15) the coefficients $\alpha_1\,,\alpha_2$ with β_1, β_2. For z = y combining

$$\left(\frac{5}{6}-C_1 h\right)\mathbf{y}^T A_*^{(1)}\mathbf{y} \le \mathrm{v}^T A^{(1)}\mathrm{v} \quad\text{and}\quad P^{-1}\mathbf{y}^T A_0^{(1)}\mathbf{y} \le \mathbf{y}^T A_*^{(1)}\mathbf{y}$$

we get

$$P^{-1}\left(\frac{5}{6}-C_1 h\right)\mathbf{y}^T A_0^{(1)}\mathbf{y} \le \mathbf{y}^T A^{(1)}\mathbf{y} \le \left(\frac{7}{6}+C_2 h\right)\mathbf{y}^T A_0^{(1)}\mathbf{y}\,. \tag{16}$$

For the constant C_1 is also valid the remark above. The derivation of Lemma 3.2, (14) and (11) gives us

$$\begin{aligned}\mathbf{z}^T A^{(2)}\mathbf{y} \;=\; & \mathbf{z}^T A_0^{(2)}\mathbf{y} + \frac{1}{3}\sum_{j=1,4,7,\dots} \Big[\big(B_{1,0,j} + |B_{1,0,j}|\big)\,\overline{\triangle}_2 y_{-1,j}\overline{\triangle}_1 z_{0,j} \\ & - \big(B_{1,0,j+2} + |B_{1,0,j+2}|\big)\,\triangle_2 y_{-1,j}\overline{\triangle}_1 z_{0,j+2}\Big] \\ & + \frac{1}{3}\sum_{i=1,4,7,\dots} \Big[\big(B_{2,i,0} + |B_{2,i,0}|\big)\,\overline{\triangle}_1 y_{i,-1}\overline{\triangle}_2 z_{i,0} \\ & - \big(B_{2,i+2,0} + |B_{2,i+2,0}|\big)\,\triangle_1 y_{i,-1}\overline{\triangle}_2 z_{i+2,0}\Big]\,.\end{aligned}$$

Similarly as (16) was derived we find

$$\left(\frac{1}{3} - C_3 h\right)\mathbf{y}^T A_0^{(2)}\mathbf{y} \le \mathbf{y}^T A^{(2)}\mathbf{y} \le \left(\frac{5}{3} + C_4 h\right)\mathbf{y}^T A_0^{(2)}\mathbf{y}$$

The above inequalities show the following theorem.

Theorem 3.1. *If $\underline{b}(x) \in \left(W_\infty^{1+\alpha}(\Omega)\right)^2$, $\alpha > 0$ then **MUDS** defined by (5), (6), (10) and (11) has a unique solution. Moreover the following inequalities hold*

$$\gamma_1 \boldsymbol{y}^T A_0 \boldsymbol{y} \le \boldsymbol{y}^T A \boldsymbol{y} \le \gamma_2 \boldsymbol{y}^T A_0 \boldsymbol{y},$$

$$|\boldsymbol{v}^T A \boldsymbol{y}| \le \gamma_2 (\boldsymbol{v}^T A_0 \boldsymbol{v})^{1/2} (\boldsymbol{y}^T A_0 \boldsymbol{y})^{1/2},$$

where A_0 is the matrix of constant approximation, and A is the matrix of linear approximation.

Remark 3.1. *P is in fact local Peclet number plus 1 and γ_2 depends on P. This shows that condition number of the matrix $A_0^{-1}A$ can become very large when P is a large number.*

Corollary 3.2. *If $\underline{b}(x) \in \left(W_\infty^{1+\alpha}(\Omega)\right)^2$, $\alpha > 0$ then the matrix A is positive real. Moreover the discrete H_0^1-coercivity holds*

$$\boldsymbol{y}^T A \boldsymbol{y} \ge C\|y\|_{1,\omega}^2\,.$$

4. Error estimates

The error analysis presented here is done in the general framework of the methods developed in [15] and [5]. We consider only the case when $a(x) \equiv 1$. Let

$$z(x) = y(x) - u(x)\,,\; x \in \omega$$

be the error of the finite difference method. Substituting $y = z + u$ in (12) (13) we obtain

$$Az = \mathrm{f} - Au \equiv \psi. \tag{17}$$

Then using (3)–(17) we transform ψ in the following form

$$\sum_{l=1}^{2}\left\{\left[\int_{s_l^+} -\frac{\partial u}{\partial x_l}\,d\gamma - w_l^+\right] - \left[\int_{s_l} -\frac{\partial u}{\partial x_l}\,d\gamma - w_l\right]\right\}$$
$$+\sum_{l=1}^{2}\left\{\left[\int_{s_l^+} b_l u\,d\gamma - u_l^+\right] - \left[\int_{s_l} b_l u\,d\gamma - u_l\right]\right\} \equiv \psi_1 + \psi_2 = \psi$$

where the local truncation error ψ has been split up into two terms

$$\psi_1 \equiv \sum_{l=1}^{2}\left(\eta_l^+(x) - \eta_l(x)\right)\,,\; \psi_2 \equiv \sum_{l=1}^{2}\left(\mu_l^+(x) - \mu_l(x)\right)\,,$$

$$\eta_l = \int_{s_l} -\frac{\partial u}{\partial x_l}\,d\gamma - w_l\,,\; \mu_l = \int_{s_l} b_l u\,d\gamma - u_l \tag{18}$$

Here ψ_1 is the error of approximation of first derivatives, and ψ_2 is the error of approximation of the second derivatives.

Note that the components of the local truncation error η_l and μ_l are defined on the shifted grids ω_l^+, $l = 1, 2$. Using summation by parts and the Schwarz inequality, we get

$$\begin{aligned}
(\psi_2, z) &= \sum_{l=1}^{2}\sum_{x\in\omega}\left[\eta_l^+(x) - \eta_l(x)\right] z(x) \\
&= -\sum_{l=1}^{2}\sum_{x\in\omega_l^+}\eta_l(x)\overline{\Delta}_l z(x) \\
&\le \left(\sum_{l=1}^{2}\sum_{x\in\omega_l^+}\eta_l^2(x)\right)^{1/2}\left(\sum_{l=1}^{2}\sum_{x\in\omega_l^+}\overline{\Delta}_l^2 z(x)\right)^{1/2} \\
&\le \left(\|\eta_1]|_1 + \|\eta_2]|_2\right)\|z\|_{1,\omega}\,.
\end{aligned}$$

Likewise

$$(\psi_2, z) \le \left(\|\mu_1]|_1 + \|\mu_2]|_2\right)\|z\|_{1,\omega}$$

Summarizing these results and using the corollaries 1 and 2 we obtain the following main result.

Lemma 4.1. *The error* $z(x) = y(x) - u(x)$, $x \in \omega$ *of all considered finite difference schemes satisfies the a priori estimate*

$$\|z\|_{1,\omega} \leq C \sum_{l=1}^{2} (\|\eta_l]|_l + \|\mu_l]|_l) \tag{19}$$

where the components η_l, μ_l, $l = 1, 2$ *of the local truncation error are defined by* (18) *with approximate fluxes* w_l^+, w_l, v_l^+, v_l, $l = 1, 2$ *determined by* (4), (6), (7) *and* (8) *for* ***UDS*** *and* (5), (6), (10) *and* (11) *for* ***MUDS***. *(The constant* C *does not depend on* h *or* z.*)*

In order to use the estimate (19) of Lemma 3.2 we have to bound the norms of η_l, μ_l, $l = 1, 2$ defined by (18). We state the estimates for the local truncation error components in regular points, proved in Ewing, Lazarov and Vassilevski [5]

$$|\eta_l(x)| \leq C h^{m-1} |u|_{m,\bar{e}}, \ \frac{3}{2} < m \leq 3, \tag{20}$$

and in Lazarov, Mishev and Vassilevski [9]

$$|\mu_l(x)| \leq \begin{cases} C h^m \|b_l\|_{1,\infty,\Omega} |u|_{m,\bar{e}} & \text{for } \mathbf{MUDS}, \\ C\left[h|b_l|_{0,\infty,\Omega}|u|_{1,\bar{e}} + h^m \|b_l\|_{1,\infty,\Omega}|u|_{m,\bar{e}}\right] & \text{for } \mathbf{UDS}, \end{cases} \tag{21}$$

where $1 < m \leq 2$; $\bar{e} = e_{i-1,j} \bigcup e_{i,j}$ for $l = 1$ and $\bar{e} = e_{i,j-1} \bigcup e_{i,j}$ for $l = 2$.

Now we consider the components of the local truncation error for the **MUDS** at the irregular points $(x_{1,i}, x_{2,j+l})$, $l = 0, 1, 2$. We remark here that we split the schemes into two parts only for convenience of the analysis. We replace (10) by

$$\begin{aligned}
w_{1,i,j} &= -\frac{1}{2}\left(\frac{1}{1+2|B_{1,i,j}|+B_{1,i,j}} + 2|B_{1,i,j}| + B_{1,i,j}\right) \\
&\quad \times \left[y_{i,j} - \frac{2}{3}y_{i-1,j+l} - \frac{1}{3}y_{i-1,j-1}\right], \\
w_{1,i,j+1} &= -\frac{1}{2}\left(\frac{1}{1+2|B_{1,i,j+1}|+B_{1,i,j+1}} + 2|B_{1,i,j+1}| + B_{1,i,j+1}\right) \\
&\quad \times \left[y_{i,j+1} - y_{i-1,j+1}\right], \\
w_{1,i,j+2} &= -\frac{1}{2}\left(\frac{1}{1+2|B_{1,i,j+2}|+B_{1,i,j+2}} + 2|B_{1,i,j+2}| + B_{1,i,j+2}\right) \\
&\quad \times \left[y_{i,j+2} - \frac{2}{3}y_{i-1,j+1} - \frac{1}{3}y_{i-1,j+4}\right],
\end{aligned}$$

and (11) by

$$\begin{aligned} v_{1,i,j} &= \frac{B_{1,i,j}}{2}\left[\frac{2}{3}y_{i-1,j+1}+\frac{1}{3}y_{i-1,j-1}+3y_{i,j}\right], \\ v_{1,i,j+1} &= \frac{B_{1,i,j+1}}{2}\left[y_{i-1,j+1}+3y_{i,j+l}\right], \\ v_{1,i,j+2} &= \frac{B_{1,i,j+2}}{2}\left[\frac{2}{3}y_{i-1,j+1}+\frac{1}{3}y_{i-1,j+4}+3y_{i,j+2}\right]. \end{aligned}$$

Note that $w_{1,i,j+l}+v_{1,i,j+l}$, $l=0,\,1,\,2$ is not changed. Consider η_1. By construction

$$\left(\frac{1}{1+2|B_{1,i,j+l}|+B_{1,i,j+l}}+2|B_{1,i,j+l}|+B_{1,i,j+l}\right)=1+C_1(x)h^2,$$

where $C_1(x)\sim b_1^2(x)$. Then in the point $(x_{1,i},x_{2,j})$ we have

$$\begin{aligned} \eta_1(x_{1,i},x_{2,j}) &= -\int_{s(i,j)}\frac{\partial u}{\partial x_1}\,d\gamma+w_1(x) \\ &= -\int_{s(i,j)}\frac{\partial u}{\partial x_1}\,d\gamma+\frac{1}{2}(1+C_1h^2)\left[u_{i,j}-\frac{2}{3}u_{i-1,j+1}-\frac{1}{3}u_{i-1,j-1}\right]. \end{aligned}$$

Taking into account

$$\left|u_{i,j}-\frac{2}{3}u_{i-1,j+1}-\frac{1}{3}u_{i-1,j-1}\right|\le C(|u|_{1,\bar e}+h^{m-1}|u|_{m,\bar e}),\ 1<m\le 2$$

and the estimate (see [5])

$$\left|\int_{s(i,j)}\frac{\partial u}{\partial x_1}\,d\gamma-\frac{1}{2}\left[u_{i,j}-\frac{2}{3}u_{i-1,j+1}-\frac{1}{3}u_{i-1,j-1}\right]\right|\le Ch^{m-1}|u|_{m,\bar e},\ \frac{3}{2}<m\le 2,$$

we get

$$|\eta_1(x_{1,i},x_{2,j})|\le Ch^{m-1}|u|_{m,\bar e},\ \frac{3}{2}<m\le 2. \tag{22}$$

With the similar argument we obtain the estimate (22) for $\eta(x_{1,i},x_{2,j+l})$, $l=1,\,2$. The inequalities (20) and (22) imply

$$\begin{aligned} \sum_{x\in\omega}\eta_1^2(x) &\le Ch^{m-1}\left(\sum_{x\in\Omega_h}|u|^2_{m,\bar e(x)}+\sum_{x\in\omega}h^{2\alpha}|u|^2_{m+\alpha,\bar e(x)}\right) \\ &\le Ch^{m-1}(|u|^2_{m,\Omega_h}+h^{2\alpha}|u|^2_{m+\alpha,\Omega}), \end{aligned}$$

here Ω_h is a strip with a width $4h$ around the interface between Ω_1 and Ω_2 (coarse and fine grid regions) and $\frac{3}{2}<m\le 2$, $0\le\alpha\le 1$.

The first term in the right is estimated by the well-known Il'in's inequality [15], [5]

$$\|u\|_{0,\Omega_\delta} \leq C\delta^\alpha \|u\|_{\alpha,\Omega}\,, \quad 0 \leq \alpha < \frac{1}{2}\,,$$

where Ω_δ is a strip in Ω with a width δ. Therefore, we have

$$\|\eta_1]|_1 = \left(\sum_{x\in\omega} \eta_1^2(x)\right)^{1/2} \leq Ch^{m-1}\|u\|_{m,\Omega}\,, \quad \frac{3}{2} < m < \frac{5}{2}\,. \tag{23}$$

In a similar way we can estimate $\eta_2(x)$.

For the component $\mu_1(x)$ we prove in the Appendix B the upper bound

$$\|\mu_1]|_1 \leq Ch^m \|b_1\|_{1,\infty,\Omega}\|u\|_{m,\Omega}\,, \quad 1 < m \leq 2. \tag{24}$$

Summarizing these results we get

Theorem 4.1. *If the solution of the problem (1) is H^m–regular, $\frac{3}{2} < m < \frac{5}{2}$ then for the **MUDS** is valid*

$$\|y-u\|_{1,\omega} \leq Ch^{m-1}\left[1 + h^\delta\left(\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega}\right)\right]\|u\|_{m,\Omega}\,.$$

Here

$$\delta = \begin{cases} 1 & \frac{3}{2} < m \leq 2, \\ 3-m & 2 \leq m < \frac{5}{2}. \end{cases}$$

With the same approach one can prove the following result for **UDS**.

Theorem 4.2. *If the solution $u(x)$ of the problem (1) is H^m–regular, $\frac{3}{2} < m \leq 3$ then for the **UDS** is valid*

$$\|y-u\|_{1,\omega} \leq Ch^{1/2}\left[1 + h^{1/2}(\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega})\right]\|u\|_{m,\Omega}$$

5. Numerical results

In this section on the basis of model examples we study the error behavior of all considered schemes. We consider three test problems. In first two examples we solved (1) with the velocity field

$$b_1 = (1 + x\cos(\alpha))\cos(\alpha)\,, \quad b_2 = (1 + y\sin(\alpha))\sin(\alpha)\,, \tag{25}$$

where the angle was $\alpha = 15^0$.

Problem 1. *Consider a smooth solution with a diffusion coefficient $a(x) = 1$*

$$u(x) = \begin{cases} 10\,\exp(-\frac{c^2}{c^2-r^2}), & r < c, \\ 0, & r \geq c, \end{cases}$$

where $c = 0.125$, $r^2 = (x-x_0)^2 + (y-y_0)^2$, $x_0 = 0.8$, $y_0 = 0.7$.

TABLE 1. Problem 1, MUDS

n_c	h_c/h_f	*error* (1)	*order*	*error* (2)	*order*
10	1	$0.284.10^{+1}$		$0.284.10^{+1}$	
	3	$0.870.10^{0}$		$0.285.10^{+1}$	
	5	$0.365.10^{0}$		$0.291.10^{+1}$	
	7	$0.250.10^{0}$		$0.296.10^{+1}$	
20	1	$0.132.10^{+1}$	1.105	$0.132.10^{+1}$	1.105
	3	$0.398.10^{0}$	1.128	$0.135.10^{+1}$	1.078
	5	$0.172.10^{0}$	1.085	$0.136.10^{+1}$	1.094
	7	$0.825.10^{-1}$	1.599	$0.138.10^{+1}$	1.100
40	1	$0.745.10^{0}$	0.825	$0.745.10^{0}$	0.825
	3	$0.119.10^{0}$	1.742	$0.681.10^{0}$	0.987
	5	$0.459.10^{-1}$	1.906	$0.694.10^{0}$	0.971
	7	$0.236.10^{-1}$	1.806	$0.700.10^{0}$	0.979
80	1	$0.232.10^{0}$	1.683	$0.232.10^{0}$	1.683
	3	$0.328.10^{-1}$	1.859	$0.243.10^{0}$	1.487
	5	$0.119.10^{-1}$	1.948	$0.249.10^{0}$	1.479
	7	$0.610.10^{-2}$	1.952	$0.252.10^{0}$	1.474
160	1	$0.691.10^{-1}$	1.747	$0.691.10^{-1}$	1.747

We choose two different domains $\Omega = \Omega_2^{(l)}$, $l = 1, 2$ for local refinement to investigate the influence of the interpolation along the boundary of Ω_2. When the support of $u(x)$ is in $\Omega_2^{(1)} = \{0.5 \le x \le 1\,,\ 0.5 \le y \le 1\}$, the error caused by the interpolation is eliminated and we get approximately second order of convergence. This shows that when $|u|_{1,\Omega_1}$ is comparatively small we can expect good results using schemes with local refinement. The worst possible case is when the solution $u(x)$ has a large gradient along the boundary of Ω_2. We tested this case for a subdomain $\Omega_2^{(2)} = \{0.7 \le x \le 1\,,\ 0.7 \le y \le 1\}$. The results show $O(h^{3/2})$ convergence rate in the discrete H^1–norm, i.e., we lose half of order of accuracy which is in agreement with Theorem 4.1.

Problem 2. *Consider a solution* $u \in H^m(\Omega)$, $m < \frac{5}{2}$ *which support is in* $\Omega_2 = \{0.5 \le x \le 1\,,\ 0.5 \le y \le 1\}$ *and a smooth coefficient* $a(x)$,

$$a(x) = [1 + 10(x^2 + y^2)]^{-1}\ , \quad u(x) = \phi(x)\psi(y)\,,$$

$$\phi(x) = \begin{cases} \sin^2\left(\pi\frac{x-d_1}{1-d_1}\right), & x \in (d_1, 1), \\ 0, & \textit{otherwise}, \end{cases}$$

$$\psi(y) = \begin{cases} \sin^2\left(\pi\frac{y-d_2}{1-d_2}\right), & y \in (d_2, 1), \\ 0, & \textit{otherwise}, \end{cases}$$

where $d_1 = d_2 = 0.875$.

TABLE 2. Problem 2

n_c	h_c/h_f	UDS	MUDS	N
10	1	$0.128.10^{0}$	$0.101.10^{0}$	100
	3	$0.855.10^{-1}$	$0.494.10^{-1}$	331
	5	$0.606.10^{-1}$	$0.193.10^{-1}$	804
	7	$0.470.10^{-1}$	$0.120.10^{-1}$	1519
20	1	$0.138.10^{0}$	$0.174.10^{0}$	400
	3	$0.545.10^{-1}$	$0.149.10^{-1}$	1261
	5	$0.355.10^{-1}$	$0.569.10^{-2}$	3004
	7	$0.265.10^{-1}$	$0.447.10^{-2}$	5629
40	1	$0.743.10^{-1}$	$0.836.10^{-1}$	1600
	3	$0.311.10^{-1}$	$0.599.10^{-2}$	4921
	5	$0.194.10^{-1}$	$0.248.10^{-2}$	11604
	7	$0.141.10^{-1}$	$0.786.10^{-3}$	21649
80	1	$0.443.10^{-1}$	$0.466.10^{-1}$	6400
	3	$0.165.10^{-1}$	$0.197.10^{-2}$	19441
	5	$0.101.10^{-1}$	$0.850.10^{-3}$	45604
	7	$0.293.10^{-2}$	$0.207.10^{-3}$	84889

We compare the H^1 error for both schemes, **UDS** and **MUDS**. In the last column of Table 2 the number of unknowns N is shown. It is clear from the results in Table 2 that **MUDS** is superior to **UDS** and it is also seen that a prescribed accuracy can be achieved for less unknowns when local refinement is used.

Problem 3. *Consider a smooth solution u with a boundary layer along line $x = 1$,*

$$u(x) = 4xy(1-y)\left(1 - \frac{\exp(x/\varepsilon) - 1}{\exp(1/\varepsilon) - 1}\right),$$

a coefficient $a(x) = \varepsilon$ and two different velocity fields. First is (25) and second is

$$b_1 = 2y(1-x^2) + 0.1x\,, \quad b_2 = -2x(1-y^2) + 0.1y\,, \tag{26}$$

We refine in the strip along the boundary layer $\Omega_2 = \{0.7 \le x \le 1\,,\ 0 \le y \le 1\}$. The objective is to compare the behavior of the finite difference scheme (**MUDS**) with and without refinement. We report the discrete L^∞, L^2 and H^1 norm in the first, second and third row in Table 3 correspondingly. For mildly dominated convection ($\varepsilon = 10^{-2}$) the scheme with local refinement shows better accuracy for both velocity fields.

TABLE 3. Problem 3, $\underline{b}(x)$ defined by (25)

n_c	h_c/h_f	*norm*	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
40	1	L^∞	$0.214.10^{-4}$	$0.207.10^{0}$	$0.623.10^{-2}$
		L^2	$0.840.10^{-5}$	$0.241.10^{-1}$	$0.148.10^{-2}$
		H^1	$0.489.10^{-4}$	$0.842.10^{0}$	$0.106.10^{-1}$
	3	L^∞	$0.263.10^{-2}$	$0.580.10^{-1}$	$0.252.10^{-1}$
		L^2	$0.111.10^{-2}$	$0.807.10^{-2}$	$0.722.10^{-2}$
		H^1	$0.633.10^{-2}$	$0.478.10^{0}$	$0.506.10^{0}$
80	1	L^∞	$0.565.10^{-5}$	$0.215.10^{0}$	$0.624.10^{-2}$
		L^2	$0.222.10^{-5}$	$0.195.10^{-1}$	$0.941.10^{-3}$
		H^1	$0.138.10^{-4}$	$0.893.10^{0}$	$0.360.10^{-1}$

TABLE 4. Problem 3, $\underline{b}(x)$ defined by (26)

n_c	h_c/h_f	*norm*	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
40	1	L^∞	$0.244.10^{-2}$	$0.166.10^{0}$	$0.125.10^{0}$
		L^2	$0.121.10^{-2}$	$0.311.10^{-1}$	$0.211.10^{-1}$
		H^1	$0.536.10^{-2}$	$0.377.10^{0}$	$0.363.10^{0}$
	3	L^∞	$0.216.10^{-2}$	$0.417.10^{-1}$	$0.316.10^{0}$
		L^2	$0.111.10^{-2}$	$0.115.10^{-1}$	$0.281.10^{-1}$
		H^1	$0.600.10^{-2}$	$0.110.10^{0}$	$0.179.10^{+1}$
80	1	L^∞	$0.123.10^{-2}$	$0.698.10^{-1}$	$0.247.10^{-1}$
		L^2	$0.615.10^{-3}$	$0.143.10^{-1}$	$0.232.10^{-1}$
		H^1	$0.280.10^{-2}$	$0.175.10^{0}$	$0.132.10^{+1}$

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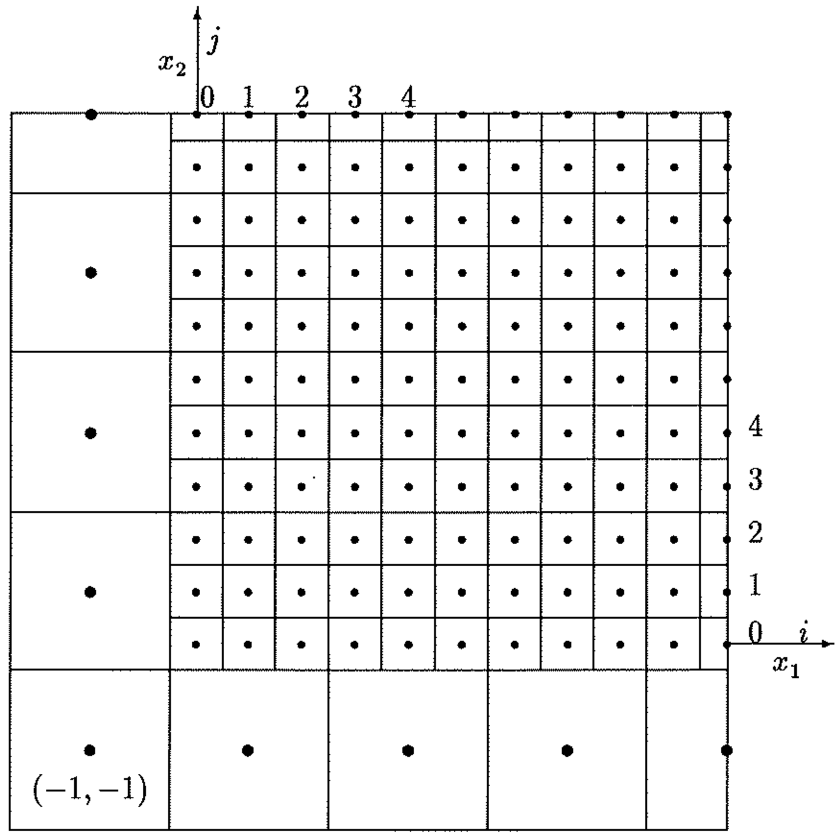
FIGURE 4. Composite cell-centered mesh



6. APPENDIX A

We prove Lemma 3.1 for the case shown on the Fig. 4.
Consider the following inner product

$$\begin{aligned} z^T A_0 y &= \sum_{x\in\omega} z(x) \sum_{l=1}^{2} \left[\left(w_l^+(x) - w_l(x) \right) + \left(v_l^+(x) - v_l(x) \right) \right] \\ &= \sum_{l=1}^{2} \sum_{x\in\omega} z(x) \left[\left(w_l^+(x) - w_l(x) \right) + \left(v_l^+(x) - v_l(x) \right) \right] \\ &= \sum_{l=1}^{2} I_l. \end{aligned}$$

We represent the term I_1 for the case of Figure 4 in the form

$$I_1 = \sum_{j<0}\sum_{\forall i} + \sum_{j>0}\sum_{i<0} + \sum_{j\geq 0}\sum_{i\geq 0}$$

or

$$I_1 = A_1 + A_2 + B_1 + B_2 + C_1 + C_2$$

$$(.)_1 = \sum_{l=1}^{2} w_l^+ - w_l\,,\ (.)_2 = \sum_{l=1}^{2} v_l^+ - v_l\,.$$

Expressions for A_1, B_1 and C_1 were derived in [5]. For A_2 we have

$$\begin{aligned} A_2 &= \sum_{j<0}\sum_{\forall i}\left[\left(B_{1,i,j}^+ - |B_{1,i,j}^+|\right) y_{i+1,j} + \left(B_{1,i,j}^+ + |B_{1,i,j}^+|\right) y_{i,j}\right] z_{i,j} \\ &\quad - \sum_{j<0}\sum_{\forall i}\left[\left(B_{1,i,j} - |B_{1,i,j}|\right) y_{i,j} + \left(B_{1,i,j} + |B_{1,i,j}|\right) y_{i-1,j}\right] z_{i,j} \end{aligned}$$

and after using partial summation we get [9]

$$\begin{aligned} A_2 &= \sum_{j<0}\sum_{\forall i} |B_1(i,j)| \overline{\triangle}_1 y_{i,j} \overline{\triangle}_1 z_{i,j} \\ &\quad + \sum_{j<0}\sum_{\forall i} B_{1,i,j}\left(z_{i,j}\overline{\triangle}_1 y_{i,j} - y_{i,j}\overline{\triangle}_1 z_{i,j}\right) + \sum_{j<0}\sum_{\forall i}\left(B_{1,i,j}^+ - B_{1,i,j}\right) y_{i,j} z_{i,j}\,. \end{aligned}$$

In the same way

$$\begin{aligned} B_2 &= \sum_{j>0}\sum_{i<0}\left[v_{1,i,j}^+ - v_{1,i,j}\right] z_{i,j} = \sum_{j>0}\left[v_{1,-1,j}^+ - v_{1,-1,j}\right] z_{-1,j} \\ &\quad + \sum_{j>0}\sum_{i<-1} |B_{1,i,j}| \overline{\triangle}_1 y_{i,j} \overline{\triangle}_1 z_{i,j} - \sum_{j>0} |B_{1,-2,j}^+| \left(y_{-1,j} - y_{-2,j}\right) z_{-2,j} \\ &\quad + \sum_{j>0}\sum_{i<-1} B_{1,i,j}\left(z_{i,j}\overline{\triangle}_1 y_{i,j} - y_{i,j}\overline{\triangle}_1 z_{i,j}\right) + \sum_{j>0} B_{1,-2,j}^+ y_{-1,j} z_{-2,j} \\ &\quad + \sum_{j>0}\sum_{i<-1}\left(B_{1,i,j}^+ - B_{1,i,j}\right) y_{i,j} z_{i,j}\,. \end{aligned}$$

Using the fact that $B_{1,-1,j} = B_{1,-2,j}^+$ we finally get

$$\begin{aligned} B_2 &= \sum_{j>0}\sum_{i<0} |B_{1,i,j}| \overline{\triangle}_1 y_{i,j} \overline{\triangle}_1 z_{i,j} + \sum_{j>0}\sum_{i<0} B_{1,i,j}\left(z_{i,j}\overline{\triangle}_1 y_{i,j} - y_{i,j}\overline{\triangle}_1 z_{i,j}\right) \\ &\quad + \sum_{j>0}\sum_{i<-1}\left(B_{1,i,j}^+ - B_{1,i,j}\right) y_{i,j} z_{i,j} + \sum_{j>0} v_{1,-1,j}^+ z_{-1,j} - \sum_{j>0} B_{1,-1,j} y_{-1,j} z_{-1,j}\,. \end{aligned}$$

Expression for C_2 is derived similarly

$$\begin{aligned} C_2 &= \sum_{j\geq 0}\sum_{i\geq 0}|B_{1,i,j}\overline{\triangle}_1 y_{i,j}\overline{\triangle}_1 z_{i,j} + \sum_{j\geq 0}\sum_{i\geq 0} B_{1,i,j}\left(z_{i,j}\overline{\triangle}_1 y_{i,j} - y_{i,j}\overline{\triangle}_1 z_{i,j}\right) \\ &+ \sum_{j\geq 0}\sum_{i\geq 0}\left(B^+_{1,i,j} - B_{1,i,j}\right) y_{i,j} z_{i,j} + \sum_{j\geq 0}\left(B^+_{1,0,j} y_{0,j} z_{0,j} - v_{1,0,j} z_{0,j}\right) \end{aligned}$$

Summarizing these results and taking into account the equalities

$$v^+_{1,-1,j+1} = v_{1,0,j} + v_{1,0,j+1} + v_{1,0,j+2}$$

$$B^+_{1,-1,j+1} = B_{1,0,j} + B_{1,0,j+1} + B_{1,0,j+2}$$

we get the assertion of the lemma.

7. Appendix B

Here we investigate the local truncation errors μ_1, μ_2 and prove the inequality (24). For the component $\mu_1(x)$ we have

$$\begin{aligned} \mu_1(x) &= \int_{x_{2,j}+(l-0.5)h}^{x_{2,j}+(l+0.5)h} b_1(x_{1,i-1/2}, s) u(x_{1,i-1/2}, s)\, ds \\ &- \left(\frac{b_{1,i-1/2,j+l} h_f}{2} - \frac{|b_{1,i-1/2,j+l}| h_f}{2}\right) u_{i,j+l} \\ &- \left(\frac{b_{1,i-1/2,j+l} h_f}{2} + \frac{|b_{1,i-1/2,j+l}| h_f}{2}\right) u_{i-1,j+1} \end{aligned} \tag{27}$$

Using the equality

$$\begin{aligned} &\left(\frac{b_{1,i-1/2,j+l}}{2} - \frac{|b_{1,i-1/2,j+l}|}{2}\right) u_{i,j+l} + \left(\frac{b_{1,i-1/2,j+l}}{2} + \frac{|b_{1,i-1/2,j+l}|}{2}\right) u_{i-1,j+1} \\ &= b_{1,i-1/2,j+l}\left(\frac{3u_{i,j+l} + u_{i-1,j+1}}{4}\right) - \left(\frac{b_{1,i-1/2,j+l}}{4} + \frac{|b_{1,i-1/2,j+l}|}{2}\right)\overline{\triangle}_1 u_{i,j+l} \\ &= b_{1,i-1/2,j+l}\left(\frac{3u_{i,j+l} + u_{i-1,j+l}}{4}\right) + b_{1,i-1/2,j+l}\left(\frac{u_{i-1,j+1} - u_{i-1,j+l}}{4}\right) \\ &- \left(\frac{b_{1,i-1/2,j+l}}{4} + \frac{|b_{1,i-1/2,j+l}|}{2}\right)\overline{\triangle}_1 u_{i,j+l} \end{aligned}$$

we represent formula (27) in the form

$$\begin{aligned}(28)\qquad \mu_1(x) \;=\; & \Bigg[\int_{x_{2,j}+(l-0.5)h}^{x_{2,j}+(l+0.5)h} b_1(x_{1,i-1/2},s)u(x_{1,i-1/2},s)\,ds \\ & \quad - b_{1,i-1/2,j+l}h_f\left(\frac{3u_{i,j+l}+u_{i-1,j+l}}{4}\right)\Bigg] \\ & -b_{1,i-1/2,j+l}h_f\left(\frac{u_{i-1,j+1}-u_{i-1,j+l}}{4}\right) \\ & +\left(\frac{b_{1,i-1/2,j+l}h_f}{4}+\frac{|b_{1,i-1/2,j+l}|h_f}{2}\right)\overline{\triangle}_1 u_{i,j+l}\,.\end{aligned}$$

Thus yields

$$\begin{aligned}(29)\qquad |\mu_1(x)| \;\le\; & |l(b_1,u)| + \frac{3h_f}{4}|b_{1,i-1/2,j+l}||\overline{\triangle}_1 u_{i,j+l}| \\ & +\frac{h_f}{4}|b_{1,i-1/2,j+l}|\left|u_{i-1,j+1}-u_{i-1,j+l}\right|,\end{aligned}$$

where the bilinear functional $l(b_1, u)$ is defined by

$$\begin{aligned}(30)\qquad l(b_1,u) \;=\; & \int_{x_{2,j}+(l-0.5)h}^{x_{2,j}+(l+0.5)h} b_1(x_{1,i-1/2},s)u(x_{1,i-1/2},s)\,ds \\ & -b_1(x_{1,i-1/2},x_{2,j+l})h_f\left(\frac{3u_{i,j+l}+u_{i-1,j+l}}{4}\right).\end{aligned}$$

We consider $u_{i,j+l}-u_{i-1,j+1}$ as a linear functional of u for a fixed $x\in\omega^+$. This functional is bounded in $H^m(\overline{e})$, $1<m\le 3$ and vanishes for all polynomials of zero degree. Therefore, by the corollary of the Bramble-Hilbert lemma we get

$$(31)\qquad |u_{i,j+l}-u_{i-1,j+1}| \le C(|u|_{1,\overline{e}}+h^{m-1}|u|_{m,\overline{e}}),\ 1<m\le 3.$$

Hence for the second term in the inequality (29) we get

$$\frac{3h_f}{4}|b_{1,i-1/2,j+l}||\overline{\triangle}_1 u_{i,j+l}| \le Ch|b_1|_{0,\infty,\Omega}(|u|_{1,\overline{e}}+h^{m-1}|u|_{m,\overline{e}}),\ 1<m\le 3.$$

Similarly we estimate the third term in (29) by

$$\begin{aligned}&\frac{h_f}{4}|b_{1,i-1/2,j+l}|\left|u_{i-1,j+1}-u_{i-1,j+l}\right| \\ &\quad \le Ch|b_1|_{0,\infty,\Omega}(|u|_{1,\overline{e}}+h^{m-1}|u|_{m,\overline{e}}),\ 1<m\le 3.\end{aligned}$$

The functional $l(b_1,u)$ is estimated in the following lemma, proved in Lazarov, Mishev and Vassilevski [9].

Lemma 7.1. *If the solution of problem (1) is H^m-regular, $1 < m$, then for the bilinear functional $l(b_1, u)$ defined by (30) the following estimate is valid:*

$$|l(b_1, u)| \leq Ch^m \|b_1\|_{1,\infty,\Omega} \|u\|_{m,\bar{e}},\ 1 < m \leq 2.$$

Above remarks give us the upper bound for $|\mu_1(x)|$ which coincides with the estimates (21) for the regular points.

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