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with Highly Oscillatory Coefficients**

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# MULTIGRID METHODS FOR DIFFERENTIAL EQUATIONS WITH HIGHLY OSCILLATORY COEFFICIENTS\*

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## SUMMARY

New coarse grid multigrid operators for problems with highly oscillatory coefficients are developed. These types of operators are necessary when the characters of the differential equations on coarser grids or longer wavelengths are different from that on the fine grid. Elliptic problems for composite materials and different classes of hyperbolic problems are practical examples.

The new coarse grid operators can be constructed directly based on the homogenized differential operators or hierarchally computed from the finest grid. Convergence analysis based on the homogenization theory is given for elliptic problems with periodic coefficients and some hyperbolic problems. These are classes of equations for which there exists a fairly complete theory for the interaction between shorter and longer wavelengths in the problems. Numerical examples are presented.

## INTRODUCTION

Multigrid methods are usually not so effective when applied to problems for which the standard coarse grid operators have significantly different properties from those of the fine grid operators [1,3,7-9,11-12]. In some of these problems the coarse grid operators should be constructed based on other principles than just simple restriction from the finest grid. Elliptic and parabolic equations with strongly variable coefficients and some hyperbolic equations are such problems. One feature in these problems is that the smallest eigenvalues

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do not correspond to very smooth eigenfunctions. It is thus not easy to represent these eigenfunctions of the coarser grids.

We shall investigate elliptic equations with highly oscillatory coefficients,

$$\sum_j \frac{\partial}{\partial x_j} a_j^\epsilon(x) \frac{\partial}{\partial x_j} u_\epsilon(x) = f(x), \quad a_j^\epsilon(x) = a_j(x, \frac{x}{\epsilon}) \quad (1)$$

with  $a_j(x, y)$  strictly positive, continuous and 1-periodic in  $y$ . This is one class of the problems discussed above for which there exists a fairly complete analytic theory such that a rigorous treatment is possible. This homogenization theory describes the dependence of the large scale features in the solutions on the smaller scales in the coefficients [2,11]. We shall consider model problems but there are also important practical applications of these equations in the study of elasticity and heat conduction for composite materials.

In this paper we analyse the convergence of multigrid methods for equation (1) by introducing new coarse grid operators, based on local or global homogenized form of the equation. We consider only two level multigrid methods. For full multigrid or with more general coefficients the homogenized operator can be numerically calculated from the finer grids based on local solution of the so called cell problem [2].

In a number of numerical tests we compare the convergence rate for different choices of parameter and coarse grid operators applied to a two dimensional elliptic model problem.

The convergence rate is also analyzed theoretically for a one dimensional problem. If, for example, the oscillatory coefficient is replaced by its average, the direct estimate for multigrid convergence rate is not asymptotically better than just using the damped Jacobi smoothing operator. The homogenized coefficient reduces the number of smoothing operation from  $O(h^{-2})$  to  $O(h^{-10/7} \log h)$ . When  $h/\epsilon$  belongs to the set of *Diophantine numbers* [4], ergodic mixing improves the estimate to  $O(h^{-6/5} \log h)$ . The step size is  $h$  and the wave length in the oscillating coefficient  $\epsilon$ .

These results carry over to some but not all hyperbolic problems. A numerical study of using hyperbolic time stepping with multigrid in order to compute a steady state give similar results to the elliptic case.

## TWO DIMENSIONAL ELLIPTIC PROBLEMS

Elliptic problems on the form (1) will be considered,

$$-\nabla \cdot a^\epsilon(x, y) \nabla u_\epsilon = f(x, y), \quad (x, y) \in \Omega = [0, 1] \times [0, 1], \quad (2)$$

subject to Dirichlet boundary condition  $u_\epsilon|_{\partial\Omega} = 0$ . The function  $a^\epsilon(x, y) = a(x/\epsilon, y/\epsilon)$  is

strictly positive and 1-periodic in  $x$  and  $y$ . From homogenization theory [2] follows,

$$\max_{(x,y) \in \Omega} |u_\epsilon - u| \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

where  $u$  satisfies the following effective equation,

$$-A_{11} \frac{\partial^2 u}{\partial x^2} - (A_{12} + A_{21}) \frac{\partial^2 u}{\partial y \partial x} - A_{22} \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega, \quad (3)$$

subject to the same boundary condition. Here,

$$A_{ij} = \int a(s_1, s_2) \left( \delta_{ij} - \frac{\partial \kappa_j}{\partial s_i} \right) ds_1 ds_2, \quad i, j = 1, 2,$$

and the periodic functions  $\kappa_j$  are given by,

$$-\nabla_s \cdot a(s_1, s_2) \nabla_s \kappa_j = \frac{\partial a(s_1, s_2)}{\partial s_j}, \quad j = 1, 2.$$

For the numerical examples we shall choose a special case with diagonal oscillatory coefficient,

$$a^\epsilon(x, y) = a\left(\frac{x-y}{\epsilon}\right). \quad (4)$$

From (3), we know that the corresponding homogenized equation is,

$$-\frac{(\mu + \bar{a})}{2} \frac{\partial^2 u}{\partial x^2} + (\mu - \bar{a}) \frac{\partial^2 u}{\partial x \partial y} - \frac{(\mu + \bar{a})}{2} \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (5)$$

where  $\mu = m(1/a^\epsilon)^{-1}$  and  $\bar{a} = m(a^\epsilon)$ . Here, the mean value  $m(f)$  of a  $\epsilon$ -periodic function is defined as,

$$m(f) = \frac{1}{\epsilon} \int_0^\epsilon f(x) dx.$$

For convenience, we introduce a brief notation of a  $N \times N$  block tridiagonal matrix  $T$ ,

$$T = \text{Tridiag}[T_{i-1}, T_i, T_{i+1}]_{(N \times N)} = \begin{bmatrix} T_{11} & T_{12} & & & \\ T_{21} & T_{22} & T_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & & T_{NN-1} & T_{NN} \end{bmatrix}.$$

## Numerical Algorithm

The discretization of (2) combined with (4) is

$$-D_+^x a_{ij}^h D_-^x u_{ij}^h - D_+^y b_{ij}^h D_-^y u_{ij}^h = f_{ij}^h. \quad (6)$$

where  $a_{ij}^h = a^\epsilon(x_i - \frac{h}{2} - y_j)$ ,  $b_{ij}^h = a^\epsilon(x_i - y_j + \frac{h}{2})$ ,  $i, j = 0, \dots, N$ .  $D_+$  and  $D_-$  are forward and backward divided differences, respectively;  $h = \frac{1}{N}$  denotes the grid size. Using vector notation, we can rewrite (6) as

$$L_{\epsilon, h} U_{\epsilon, h} = F_{\epsilon, h}$$

where

$$L_{\epsilon, h} = \frac{1}{h^2} \text{Tridiag}[B_{j-1}^h, A_j^h, B_j^h]_{(N-1) \times (N-1)} \quad (7)$$

$$A_j^h = \text{Tridiag}[-a_{i-1, j}^h, a_{i-1, j}^h + a_{ij}^h + b_{ij}^h + b_{ij-1}^h, -a_{ij}^h]_{(N-1) \times (N-1)}$$

$B_j^h$  is a diagonal matrix, denoted by  $B_j^h = \text{Diag}[-b_{i, j}^h]_{(N-1) \times (N-1)}$  and

$$U_{\epsilon, h} = (u_{11}^h, u_{21}^h, \dots, u_{N-1, 1}^h, \dots, u_{1, N-1}^h, u_{2, N-1}^h, \dots, u_{N-1, N-1}^h)^T$$

$$F_{\epsilon, h} = (f_{11}^h, f_{21}^h, \dots, f_{N-1, 1}^h, \dots, f_{1, N-1}^h, f_{2, N-1}^h, \dots, f_{N-1, N-1}^h)^T$$

For simplicity, we only consider the two-grid method. Denote the full iteration operator of this method by  $M$ . It is defined by,

$$M = S^\gamma (I - I_H^h L_H^{-1} I_h^H L_{\epsilon, h}) S^\gamma, \quad (8)$$

where the restriction and interpolation operators are given, as denoted below, by the weighting restriction and bilinear interpolation operators, respectively,

$$I_h^H = \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} H \\ \\ h \end{bmatrix}, \quad I_H^h = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} h \\ \\ H \end{bmatrix}.$$

The smoothing iteration operator  $S$  is based on the damped Jacobi iteration,

$$S = I - \omega h^2 L_{\epsilon, h}. \quad (9)$$

*The coarse grid operators  $L_H$  is one of the following operators:*

Global Homogenized operator: which is the discretized form of (5)

$$-0.5(\mu + \bar{a})D_+^x D_-^x u_{ij} + (\mu - \bar{a})D_0^x D_0^y u_{ij} - 0.5(\mu + \bar{a})D_+^y D_-^y u_{ij} = f_{ij}.$$

Written in matrix form,

$$L_H = \frac{1}{H^2} \text{Tridiag}[B_{j-1}^H, A_j^H, B_j^H]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)}, \quad (10)$$

where

$$A_j^H = \text{Tridiag}\left[-\frac{\mu + \bar{a}}{2}, 2(\mu + \bar{a}), -\frac{\mu + \bar{a}}{2}\right]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)},$$

$$B_j^H = \text{Tridiag}\left[-\frac{\bar{a} - \mu}{4}, -\frac{\mu + \bar{a}}{2}, \frac{\bar{a} - \mu}{4}\right]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)}.$$

Local Homogenized operator:  $L_H$  has the same form as (10), except the entries for  $A_j^H, B_j^H$  coming from the local discretized values of  $a_\epsilon(x - y)$ ,

$$L_H = \frac{1}{H^2} \text{Tridiag}[B_{j-1}^H, A_j^H, B_j^H]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)}, \quad (11)$$

where

$$A_j^H = \text{Tridiag}[-a_{i-1,j}^H, a_{i-1,j}^H + a_{ij}^H + b_{ij-1}^H + b_{ij}^H, -a_{ij}^H]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)},$$

$$B_j^H = \text{Tridiag}[-c_{i-1}^H, -b_{ij}^H, c_i^H]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)},$$

with

$$b_{ij}^H = \frac{b_{ij}^h + b_{ij-1}^h + 2\delta(b_{ij}^h, b_{ij-1}^h)}{4}, \quad a_{ij}^H = \frac{a_{ij}^h + a_{ij-1}^h + 2\delta(a_{ij}^h, a_{i-1j}^h)}{4}, \quad c_i^H = \frac{\mu - \bar{a}}{2}.$$

$\delta(c_1, c_2)$  is defined to be  $\frac{2c_1 c_2}{c_1 + c_2}$ .

Reduced Local Homogenized operator:  $L_H$  has the same form as in (11), except here we ignore the cross term  $D_0^x D_0^y$ . That means  $B_j^H$  is a diagonal matrix,  $B_j^H = \text{Diag}[-b_{ij}^H]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)}$ ,

$$L_H = \frac{1}{H^2} \text{Tridiag}[B_{j-1}^H, A_j^H, B_j^H]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)} \quad (12)$$

Sampling operator:  $L_{\epsilon,H}$  has the exact form as  $L_{\epsilon,h}$ , but values  $a_{ij}, b_{ij}$  are defined on the coarse grids,

$$L_H = L_{\epsilon,H} \quad (13)$$

Variational operator:

$$L_H = I_h^H L_{\epsilon,h} I_H^h \quad (14)$$

## Numerical Results

In practice, it is not always easy to calculate the spectral radius  $\rho(M)$ . Therefore, we study the mean rate [14] of convergence under different coarse grid operators  $L_H$ . The mean rate of convergence is defined by

$$\rho = \left( \frac{\|L_{\epsilon,h} u^i - f_h\|_h}{\|L_{\epsilon,h} u^1 - f_h\|_h} \right)^{\frac{1}{i-1}} \quad (15)$$

where  $i$  is the smallest integer satisfying  $\|L_{\epsilon,h} u^i - f_h\|_h \leq 1 \times 10^{-6}$ .

In Figure 1,  $a^\epsilon(x-y) = 2.1 + 2 \sin(2\pi(x-y)/\epsilon)$ . We plot  $\rho$  defined by (15) as a function of  $\gamma$  by taking  $\epsilon = \sqrt{2}h$ , and  $\omega$  in (9) is 0.095.

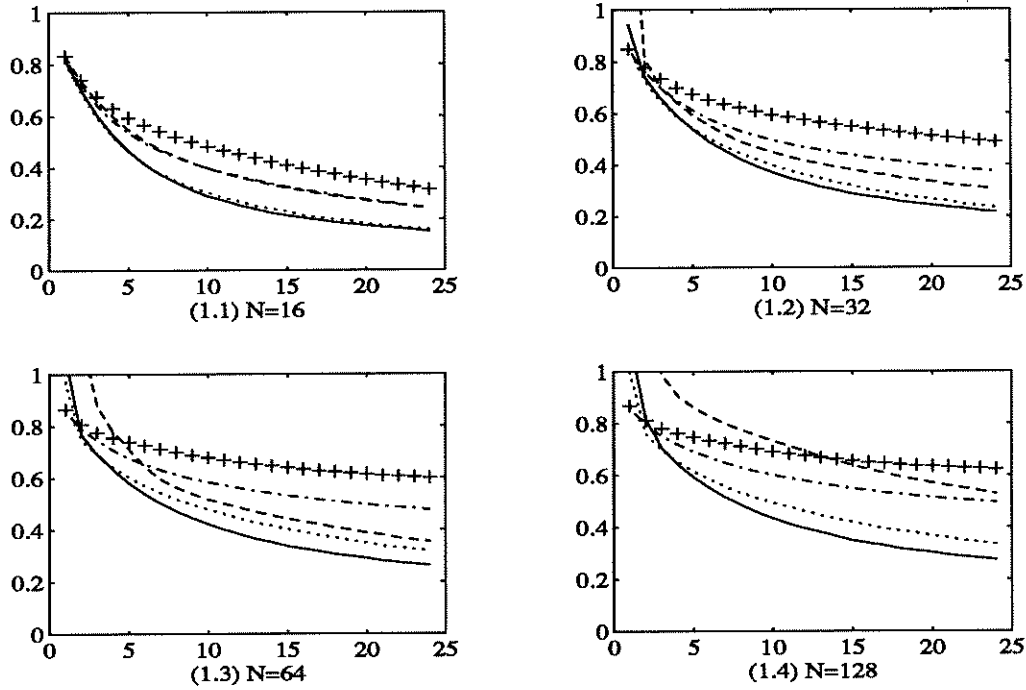


Figure 1:  $\rho$  as a function of  $\gamma$ . Dotted line is for (10), solid line for (11), dashed line for (13), and dashdot for (12), + for (14). (1.1)-(1.4) are for different number of grid points  $N$ .

It is clear that the coarse grid operators derived from the homogenized forms (10) and (11) are superior. The effect is more pronounced for large  $\gamma$  when the eigenspace corresponding to large eigenvalues of  $L_{\epsilon,h}$  is essentially eliminated. For the practical low  $\gamma$  case, a study of the impact of the choices of  $I_H^h$  and  $I_h^H$  is needed. In this paper we are concentrating on the asymptotic behavior (large  $\gamma$ ). Different  $I_H^h$  and  $I_h^H$  operators are briefly discussed for the one-dimensional problem.

In Figure 2, we plot  $\rho$  as a function of the variable  $\alpha$ , where  $L_H(\alpha)$  comes from the discretized operator  $-a_{ij}^H D_+^x D_-^x + \alpha(\mu - \bar{a}) D_0^x D_0^y - b_{ij}^H D_+^y D_-^y$ ,  $\omega = 0.095$  and  $\epsilon = \sqrt{2}h$ .

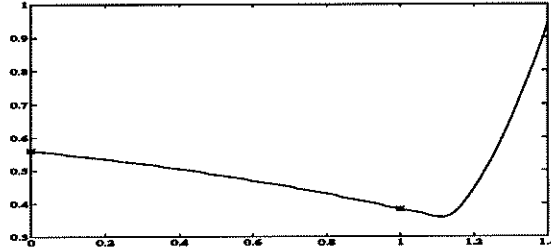


Figure 2:  $\rho$  as a function of  $\alpha$ . Here  $N = 64$  and  $\gamma = 12$ . "—" denote  $\rho$  under the different choice of Normal and Local Homogenized coarse grid operator, respectively.

From Figure 2, we get further evidence of the importance of using the correct homogenized operator. Techniques based on one-dimensional analysis does not contain the mixed derivative term [1].

In order to isolate the influence of the coarse grid approximation we have kept the smoothing operator fixed. It obviously also affect the performance. If we use Gauss Seidel iteration method in (9), the convergence rate can be improved. In Table 1, we test the same coefficient  $a^\epsilon(x-y) = 2.1 + 2 \sin(2\pi(x-y)/\epsilon)$ . Taking  $N = 128$ ,  $\epsilon = \sqrt{2}h$ , we compare the convergence rate by choosing damped Jacobi iteration and Gauss Seidel iteration.

$\gamma$	5	6	7	8	9	10	11	12
Jacobi	.5929	.5519	.5173	.4863	.4579	.4349	.4140	.3950
G-S	.4545	.4221	.3922	.3703	.3491	.3304	.3158	.3008

Table 1: Spectral radius, two dimensional case

## ONE DIMENSIONAL PROBLEMS



The one dimensional equation is useful as a model for which a more complete mathematical analysis is possible.

Consider the following one-dimensional elliptic boundary value problem with a periodic oscillatory coefficient,

$$-\frac{d}{dx}a^\epsilon(x)\frac{du_\epsilon}{dx} = 1, \quad 0 < x < 1, \quad u_\epsilon(0) = u_\epsilon(1) = 0, \quad (16)$$

where  $a^\epsilon(x) = a(\frac{x}{\epsilon})$  and satisfies the same assumption as above. As  $\epsilon \rightarrow 0$ ,  $u_\epsilon$  converges strongly in  $L_\infty$  to the solution  $u$  of the homogenized equation,

$$-\alpha\frac{d^2u}{dx^2} = 1, \quad 0 < x < 1, \quad \alpha = m(1/a^\epsilon)^{-1}. \quad (17)$$

Subject to the boundary conditions  $\phi(0) = \phi(1) = 0$ .

### Numerical Algorithm

Let the difference approximation of (16) be of the form:

$$-a^\epsilon(x_{j+\frac{1}{2}})(u_{j+1}^h - u_j^h) + a^\epsilon(x_{j-\frac{1}{2}})(u_j^h - u_{j-1}^h) = 1, \quad j = 1, \dots, N-1 \quad (18)$$

In matrix form, (18) can be written as

$$L_{\epsilon,h}u^h = 1, \quad u^h = (u_1^h \dots, u_{N-1}^h)^T$$

where

$$L_{\epsilon,h} = \frac{1}{h^2} \text{Tridiag}[-a_{i-1}, a_{i-1} + a_i, a_{i+1}]_{(N-1) \times (N-1)} \quad (19)$$

with  $a_j = a^\epsilon(x_j - \frac{h}{2})$ .

We first consider a two-grid method by applying standard restriction, standard interpolation operators and Jacobi smoothing iteration.

*The coarse grid operator  $L_H$  will be one of the following:*

Homogenized operator:

$$L_H = \frac{m(1/a^\epsilon)^{-1}}{H^2} \text{Tridiag}[-1, 2, -1]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)} \quad (20)$$

Averaged operator:

$$L_H = \frac{m(a^\epsilon)}{H^2} \text{Tridiag}[-1, 2, -1]_{(\frac{N}{2}-1) \times (\frac{N}{2}-1)} \quad (21)$$

Sampling operator:  $L_{\epsilon,H}$  has the exact form as  $L_{\epsilon,h}$ , but only every second  $a_j$  value is used,

$$L_H = L_{\epsilon,H} \quad (22)$$

Variational operator:

$$L_H = I_h^H L_{\epsilon,h} I_H^h. \quad (23)$$

### Convergence Theory

The theorem below on the convergence rate is too pessimistic in the number of smoothing iterations necessary. However, the analysis still gives insight into the convergence process and the role of homogenization. With  $L_H$  replaced by averaging (21) the same analysis results in  $\gamma = O(h^{-2})$  which means that multigrid does not improve the rate of convergence over just using Jacobi iterations. This follows from the effect of the oscillations on the lower eigenmodes. It should also be noticed that in the case (ii), the solution of  $L_{\epsilon,h}$  are much closer to those of  $L_H$ , see [11].

**Theorem 1** *Let  $M$  be defined as in (8), with  $L_H$  defined by (11). There exists a constant  $C$  such that,*

$$\rho(M) \leq \rho_0 < 1,$$

*when either one of the following condition is satisfied:*

(i)  $\gamma \geq Ch^{-1-3/7} \ln h$

(ii) *the ratio of  $h$  to  $\epsilon$  belongs to the set of Diaphantine number, and  $\gamma \geq Ch^{-1-1/5} \ln h$ .*

For details of the proof, see [10]. An outline is as follow. Separate the complete eigenspace of  $L_{\epsilon,h}$  into two orthogonal subspaces, the space of low eigenmodes and that of high eigenmodes. After several Jacobi smoothing iteration in the fine grid level, the high eigenmodes of the error are reduced, and only the low eigenmodes are left. Combining eigenvalue analysis with homogenization theory [11], one may realize that the low eigenmodes of the original discrete operator are close to those of the corresponding homogenized operator. We then approximate them by the corresponding homogenized eigenmodes and correct these in the coarse grid level.

### Numerical Results

In Figure 3.1 and Figure 3.2,  $a^\epsilon(x) = 2.1 + 2\sin(2\pi x/\epsilon)$ . We plot the analogous graph to Figure 1. Here  $\epsilon = \sqrt{2}h$  and  $\omega$  in (9) is 0.1829. In Figure 3.3 and Figure 3.4,  $a^\epsilon(x) = 2.1 + 2\sin(2\pi x/\epsilon + \pi/4)$ . Here  $\epsilon = 4h$  and  $\omega = 0.1585$ .

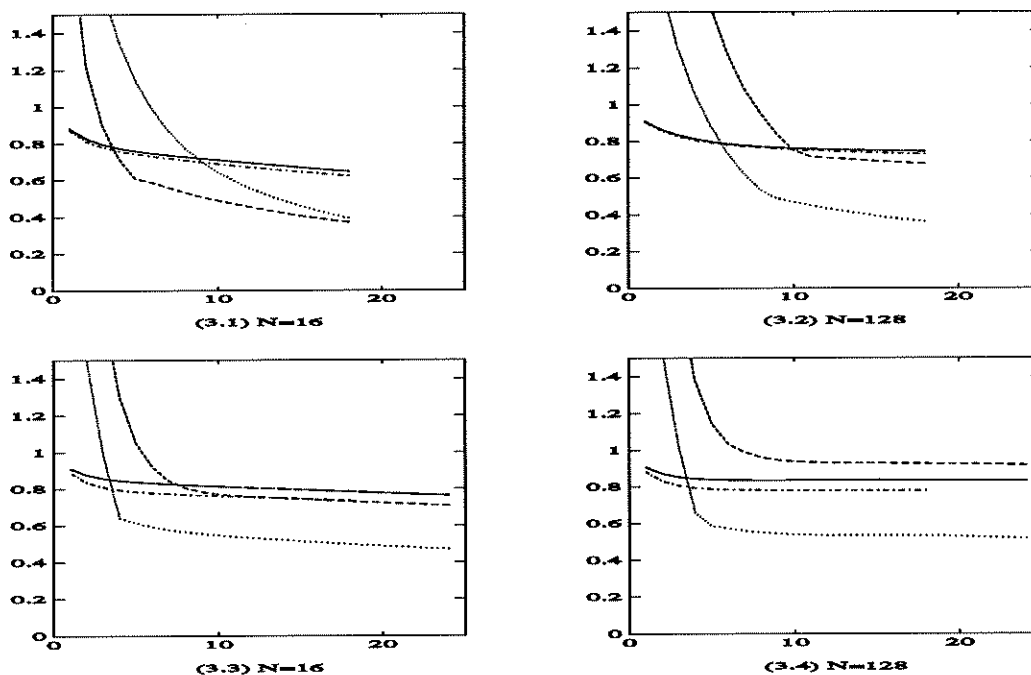


Figure 3:  $\rho$  as a function of  $\gamma$ . Dotted line is for (20), solid line for (21), dashed line for (22), and dashdot for (23). (3.1)-(3.4) are for different number grid points  $N$ .

In Figure 4, with the assumptions in Figure 3.3-3.4, we plot  $a^\epsilon(x)$  and the approximation of (18) under the choices of coefficients in Figure 3.

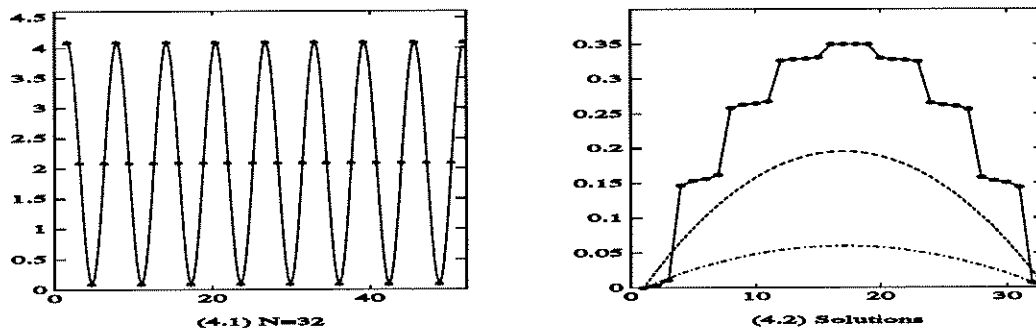


Figure 4: (4.1) and (4.2) are the graphs for  $a^\epsilon(x)$ , where  $*$  are the discretized values. (4.2) is the solution. Dashed line is for (17). Dashdot line is for  $-m(a^\epsilon)u_{xx} = 1$  and line with circles is for (18).

In Figure 5, we plot  $\rho$  as a function of the variable  $\alpha$ , where  $L_H = \alpha\Delta_H$ . In (5.1),  $a^\epsilon(x) = 2.1 + 2\sin(2\pi x/\epsilon)$ ,  $\omega = 0.1829$  and  $\epsilon = \sqrt{2}h$ ; In (5.2),  $a^\epsilon(x) = 20.1$  if  $x/\epsilon - [x/\epsilon] \in (0.7, 0.9)$ ; otherwise,  $0.1$ ,  $\omega = 0.0373$  and  $\epsilon = 4h$ .

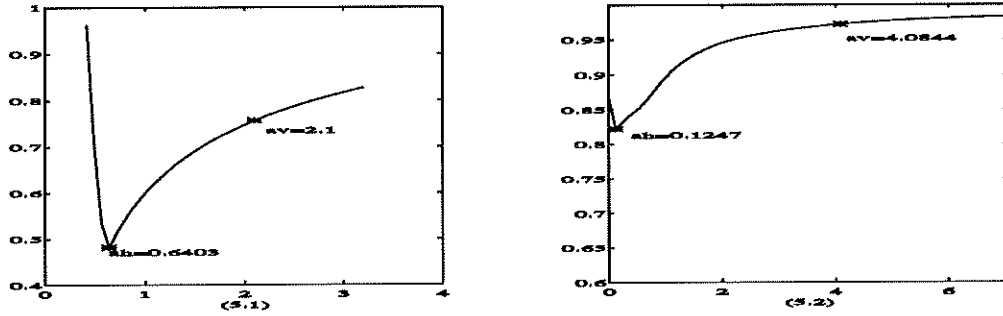


Figure 5:  $\rho$  as a function of  $\alpha$ . The homogenized value  $ah = m(1/a^\epsilon)^{-1}$  and the arithmetic value  $av = m(a^\epsilon)$  are given. Here  $\gamma = 10$  and  $N = 256$ .

In Figure 6, we present the convergence  $u_\epsilon \rightarrow u$ , as  $\epsilon \rightarrow 0$  by giving the numerical solutions of (16) and (17). Recall that our goal is to solve the oscillatory problem and to use the homogenized operator only for the coarse grids.

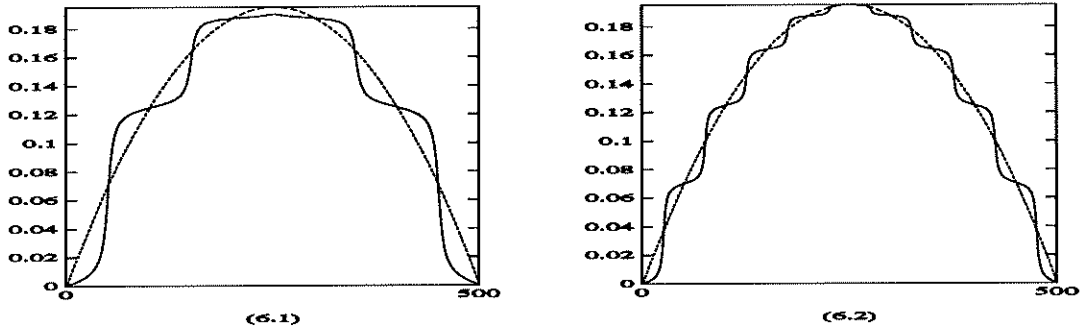


Figure 6: Solid lines are the approximations for (18), dashed lines are the solutions for (17), respectively, when  $\epsilon = 0.2$  in (6.1) and  $\epsilon = 0.1$  in (6.2). Here  $N = 500$ .

## HYPERBOLIC PROBLEMS

Time evolution of a hyperbolic differential equation can be used for steady state computations. This is common in computational fluid dynamics, [6]. In multigrid this means that hyperbolic timestepping replaces the smoothing step. There are fundamental difference with standard multigrid for elliptic problems but some of our earlier discussions carry over to the hyperbolic case. The dissipative mechanism for hyperbolic problems are mainly the boundary conditions. Consider using the model problem,

$$\frac{\partial^2 u_\epsilon}{\partial t^2} - \frac{\partial}{\partial x} a^\epsilon(x) \frac{\partial u_\epsilon}{\partial x} = f(x), \quad 0 \leq x \leq 1 \quad (24)$$

as the smoothing equation in multigrid for the numerical solution of (16), subject to the boundary conditions

$$u_\epsilon(0) = 0, \quad \frac{\partial u_\epsilon(1)}{\partial x} = 0. \quad (25)$$

The equation (24) must have boundary conditions which are dissipative but reduces to (25) at steady state, see [5],

$$u_\epsilon(0, t) = 0, \quad \frac{\partial u_\epsilon(1, t)}{\partial t} + \sqrt{a^\epsilon(x)} \frac{\partial u_\epsilon(1, t)}{\partial x} = 0. \quad (26)$$

The initial condition should support the transport of the residual to the dissipative boundary  $x = 1$ ,

$$u_\epsilon(x, \tilde{t}) = u_\epsilon^0(x) \quad \text{given,}$$

$$u_\epsilon(x, \tilde{t} + \Delta t) = u_\epsilon^0(x) - \Delta t \sqrt{a^\epsilon(x)} D_0^x u_\epsilon^0(x).$$

Note that the initial condition approximates the transport equation  $u_t + \sqrt{a} u_x = 0$ . The difference approximation of (24) needs a low level of numerical dissipation.

The homogenization theory of [2] is also valid for equation of the type (24). A numerical indication is seen in Figure 7.

The positive effect of multigrid on the convergence rate does not carry over to problems for which the steady state is hyperbolic or contains hyperbolic components. If,

$$\frac{\partial u_\epsilon}{\partial t} + \alpha \frac{\partial u_\epsilon}{\partial x} + \beta \frac{\partial u_\epsilon}{\partial y} = 0$$

is used for the equation,

$$\alpha \frac{\partial u_\epsilon}{\partial x} + \beta \frac{\partial u_\epsilon}{\partial y} = 0, \quad x, y \in [0, 1],$$

$u_\epsilon$  is 1-periodic in  $y$ ,  $u_\epsilon(0, y, t) = a^\epsilon(y)$ .

The coarse grid operator must resolve all scales of  $a^\epsilon$  to required accuracy in order to produce multigrid speed up. More on this phenomena will be reported elsewhere.

### Numerical Results

In Figure 7, take 50 smoothing steps. Coefficient  $a(x/\epsilon)$  is the same as in Figure 1.

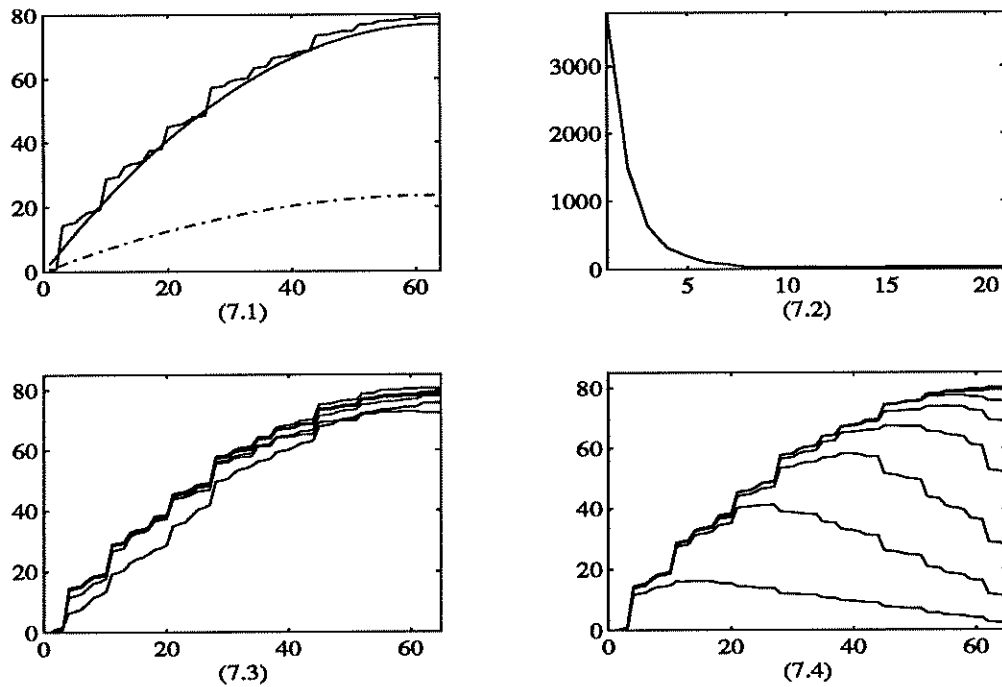


Figure 7: (7.1) Solutions: Solid line is the solution of steady state; Dashed line for homogenized solution; Dashed dot line for average solution. (7.2) Residue as function of two level multigrid cycles. (7.3) Approximate solutions after each two level cycle. (7.4) Approximate solutions for time evolution equation.

### CONCLUSION

Elliptic equations and some hyperbolic equations with highly oscillatory coefficients have been studied. We have shown that the homogenized form of the equations are very useful in the design of coarse grid operators for multigrid.

The evidence is from a sequence of numerical examples with strongly variable coefficients and to some extent from theoretical analysis. The result is clear in the asymptotic regime of many smoothing iterations.

The impact on the coarse grid operator from the numerical truncation error and the interpolation operator needs to be assessed in order to improve the performance in the regime of very few smoothing iterations per cycle.

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