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Abstract. In this paper, we use the finite element approximation for the solution of a multidimensional nonlinear system describing the evolution of materials which can pass through several phase transitions with dissipation. The finite element error estimates are obtained.

Key Words. Phase transitions, initial-boundary value problem, finite element method, error estimates.

Mathematics Subject Classification(1991): 65M60, 65M15.

1. Introduction

Based on a special dependence of the free energy on the temperature and phase variables, taking dissipation into account, Colli and Hoffmann [4] proposed a highly nonlinear system which describes the evolution process of materials which can pass through several phase transitions. Such problems may arise from the irreversible phase change processes [1,4], supercooling and superheating effects [15], and the evolution of shape memory alloys [3,5,7,13]. A precise formulation of the related initial boundary value problem was given there in a quite general framework. In this paper, we propose a fully discrete scheme for the numerical solution of the highly nonlinear evolution system. Our main work is to derive the error estimates for the finite element approximation. We introduce the Green's operator and its finite element approximation to overcome the lower smoothness of the unknown variables of the system.

Now we formulate the nonlinear evolution model developed in [4]. Let the considered material occupy a bounded domain $\Omega \subset R^d (d \leq 3)$. The variables

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representing the state of the system are the absolute temperature θ and the internal variables $\chi_1, \chi_2, \cdots, \chi_m (m \geq 1)$ associated with the phase transitions. For instance, $\chi_1, \chi_2, \cdots, \chi_m$ may stand for the local volumetric proportions of the m different phases shown by the material, as in the Frémond model for shape memory alloys, see [3,5,7,13]. Set $\chi = (\chi_1, \chi_2, \cdots, \chi_m)$. Let $V = H_0^1(\Omega)$ related to the variable θ, suppose Dirichlet boundary condition is assigned here for θ and $X = \left(L^2(\Omega)\right)^m$ related to the internal variable χ. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in X, and by $(\cdot, \cdot)$ either the duality pairing between $H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$ and $H_0^1(\Omega)$ or the scalar product in $L^2(\Omega)$. We use $\|\cdot\|_m$ and $|\cdot|_m$, $m = 0, 1, 2$ to denote the norms and semi-norms of the usual Sobolev spaces $H^m(\Omega) = W^{m,2}(\Omega), m = 0, 1, 2$, and $\|\cdot\|_{-1} = \|\cdot\|_{H^{-1}(\Omega)}$ and $\|u\|_X = \langle u, u \rangle^{1/2}$, for $u \in X$.

Next we define some opertaors and introduce some notations used for the formulation of the problem. $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a linear, continuous and symmetric operator satisfying the coerciveness condition

$$(Lv,\, v) \geq C \, \|v\|_1^2, \ \forall \, v \in H_0^1(\Omega).$$

By $a(\cdot,\ \cdot)$ we denote the bilinear form $a(u,\ v) = (Lu,\ v),\ \forall\, u, v \in H_0^1(\Omega)$, and define the norm $\|u\|_a = \sqrt{a(u,\ u)}$, for $u \in H_0^1(\Omega)$. Then we see that there are two positive constants c_a and C_a such that

$$c_a \, \|u\|_a^2 \leq \|u\|_1^2 \leq C_a \, \|u\|_a^2. \tag{1.1}$$

By I_K we mean the indicator function of a nonempty closed convex subset $K \subset X$, i.e. $I_K(\chi) = 0$, if $\chi \in K$, $I_K(\chi) = +\infty$, if $\chi \notin K$. It is known that I_K is a proper, convex and lower semicontinuous function and its subdifferential ∂I_K is a maximum monotone operator in $X = \left(L^2(\Omega)\right)^m$. Let f be a given heat source distribution function satisfying

$$f \in W^{1,1}(0, T; H^{-1}(\Omega)). \tag{1.2}$$

Further, we introduce three nonlinear operators $A(t, \theta, \chi)$, $D(t, \theta, \chi)$ and $B(t, \theta, \chi)$ mapping $[0, T] \times H_0^1(\Omega) \times X$ into $H^{-1}(\Omega)$, $L^2(\Omega)$ and X, respectively. As in [4], we assume that

$$A \text{ is continuous and } \ D(\cdot,\, 0,\, 0) \in L^2(Q_T),\ B(\cdot,\, 0,\, 0) \in L^2(0, T; X) \tag{1.3}$$

with $Q_t = \Omega \times (0, t)$, $0 \leq t \leq T$ and that there exist two positive constants c_A, C_A, two nonnegative time functions

$$M_D,\ M_B \in L^2(0, T) \tag{1.4}$$

and three others

$$F_A \in H^1(0, T), \quad F_D \text{ and } F_B \in L^1(0, T) \tag{1.5}$$

such that for any $v, w \in H_0^1(\Omega), \lambda, \mu \in X$ and for a.e. $t, s \in (0,T)$ one has

$$\big(A(t,v,\lambda) - A(t,w,\lambda),\ v-w\big) \geq c_A\|v-w\|_0^2 \tag{1.6a}$$
$$\|A(t,v,\lambda) - A(t,v,\mu)\|_0^2 \leq C_A\|\lambda-\mu\|_X^2 \tag{1.6b}$$
$$\|D(t,v,\lambda) - D(t,w,\mu)\|_0 \leq M_D(t)\big(\|v-w\|_0 + \|\lambda-\mu\|_X\big) \tag{1.6c}$$
$$\|B(t,v,\lambda) - B(t,w,\mu)\|_X \leq M_B(t)\big(\|v-w\|_0 + \|\lambda-\mu\|_X\big) \tag{1.6d}$$
$$\|A(t,v,\lambda) - A(s,v,\lambda)\|_0 \leq |F_A(t) - F_A(s)|\big(\|v\|_1 + \|\lambda\|_X + 1\big) \tag{1.6e}$$
$$\|D(t,v,\lambda) - D(s,v,\lambda)\|_0 \leq |F_D(t) - F_D(s)|\big(\|v\|_1 + \|\lambda\|_X + 1\big) \tag{1.6f}$$
$$\|B(t,v,\lambda) - B(s,v,\lambda)\|_X \leq |F_B(t) - F_B(s)|\big(\|v\|_1 + \|\lambda\|_X + 1\big) \tag{1.6g}$$

Finally we assume

$$A(0,\,\cdot,\,0) : H^1(\Omega) \to L^2(\Omega) \text{ is bounded.} \tag{1.7}$$

and

$$\theta^0 \in H_0^1(\Omega),\ \chi^0 = (\chi_1^0, \chi_2^0, \cdots, \chi_m^0) \in X. \tag{1.8}$$

We are now in a position to state our problem

Problem (P): Find $\theta \in H^1\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;H_0^1(\Omega)\big), \chi \in H^1\big(0,T;X\big), \eta \in L^2\big(0,T;X\big)$ such that $\theta(0) = \theta^0, \chi(0) = \chi^0$ and for a.e. $t \in (0,T)$

$$\frac{d}{dt}A(t,\theta(t),\chi(t)) + L\,\theta(t) = D(t,,\theta(t),\chi(t)) + f(t) \text{ in } H^{-1}(\Omega) \tag{1.9a}$$
$$\frac{d}{dt}\chi(t) + \eta(t) = B(t,\theta(t),\chi(t)) \quad \text{in } X \tag{1.9b}$$
$$\eta(t) \in \partial I_K(\chi(t)). \tag{1.9c}$$

About the solutions of Problem (P) we have

Lemma 1.1[4]. *Under the assumptions (1.1)–(1.8) Problem (P) has a unique solution* (θ, χ, η).

Remark 1.1: Our results of the paper are easily to generalize to the case of I_K replaced by some other more general proper, convex and lower semicontinuous function Φ as in [4]. But for important practical problems, we often have $\Phi = I_K$, e.g., all three problems described in [4, Section 2].

Remark 1.2: If the absolute temerature θ is assumed with the third kind of boundary conditions, the above space $V = H_0^1(\Omega)$ is then replaced by $H^1(\Omega)$, it does not affect our results.

2. Finite element method

In this section we construct a fully discrete approximation of Problem (P). We shall use backward difference scheme to discretize the problem in time. Let M be a positive integer and $\tau = T/M$ be the time step size. For any $n = 1, 2, \cdots, M$, we denote $t^n = n\tau$ and $I^n = (t^{n-1}, t^n]$. For a given sequence $\{u^n\}_{n=0}^M \subset L^2(\Omega)$, we define

$$\partial_\tau u^n = \frac{u^n - u^{n-1}}{\tau}, \quad \tilde{u}^n = \frac{1}{\tau} \int_{I^n} u(t)\, dt, \; n = 1, 2, \cdots, M.$$

For a continuous mapping $u : [0, T] \to L^2(\Omega)$, we define $u^n = u(\cdot, n\tau)$, $0 \le n \le M$.

In space we use the piecewise linear finite element approximation to Problem (P). For convenience we assume the domain Ω is polygonal ($n \le 2$) or polyhedral ($n = 3$). Suppose now for each parameter $h > 0$ we are given a family of quasi-uniform triangulation $\mathcal{T}_h = \{\Delta_i\}$ of Ω, see [2]. We define the finite element space

$$V_h = \{v \in C(\overline{\Omega}); v\,|_{\Delta_i} \text{ is linear, for all } \Delta_i \in \mathcal{T}_h\}$$

and

$$V_h^0 = V_h \cap H_0^1(\Omega), \quad K_h = (V_h)^m \cap K.$$

Before giving our discrete problem, we first define three discrete projection operators. P_h is a projection from the space $X = \left(L^2(\Omega)\right)^m$ into the finite dimensional closed convex subset K_h defined by

$$\big\langle P_h\psi, v_h - P_h\psi \big\rangle \ge \big\langle \psi, v_h - P_h\psi \big\rangle, \; \forall\, v_h \in K_h, \; \psi \in X \tag{2.1}$$

and Q_h, T_h are L^2-projections from $L^2(\Omega)$ into V_h^0 and V_h, respectively:

$$(Q_h\, \psi,\, v_h) = (\psi,\, v_h), \; \forall\, v_h \in V_h^0, \; \psi \in L^2(\Omega), \tag{2.2a}$$

$$(T_h\, \phi,\, v_h) = (\phi,\, v_h), \; \forall\, v_h \in V_h, \; \phi \in L^2(\Omega). \tag{2.2b}$$

The projection operators Q_h, P_h and T_h have the properties

Lemma 2.1. *For any $v \in H_0^1(\Omega)$, $\psi \in X$, and $\phi \in H^1(\Omega)$ we have*

$$\|v - Q_h\, v\|_s \le C\, h^{1-s} |v|_1, \; s = -1, 0, 1, \tag{2.3a}$$

$$\|\phi - T_h\, \phi\|_0 \le C\, h^{1-s} \|\phi\|_{1-s}, \; s = 0, 1, \tag{2.3b}$$

$$\|\phi - T_h\, \phi\|_{(H^1(\Omega))'} \le C\, h^{2-s} \|\phi\|_{1-s}, \; s = 0, 1, \tag{2.3c}$$

$$\|\psi - P_h\, \psi\|_X = \inf_{\psi_h \in K_h} \|\psi - \psi_h\|_X. \tag{2.3d}$$

Proof. (2.3d) follows immediately from the definition of P_h, while (2.3a) with $s = 0, 1$ can be found in [14, 16], and the case $s = -1$ can be proved directly from the dual argument and the result (2.3a) with $s = 0$. Analogously, we can show (2.3b, c).

Now we can state our discrete approximation to Problem (P) as follows

Finite element problem (FEP): For $n = 1, 2, \cdots, M$, find $\theta_h^n \in V_h^0$, $\chi_h^n \in (V_h)^m$ such that

$$\theta_h^0 = Q_h\theta^0, \quad \chi_h^0 = (T_h\chi_1^0, \cdots, T_h\chi_m^0), \tag{2.4a}$$

$$(\partial_\tau A_h^n,\, v_h) + a(\theta_h^n,\, v_h) = \frac{1}{\tau}\int_{I^n} \big(D(t, \theta_h^{n-1}, \chi_h^{n-1}),\, v_h\big)dt$$
$$+ \frac{1}{\tau}\int_{I^n} \big(f(t),\, v_h\big)dt, \ \forall v_h \in V_h^0, \tag{2.4b}$$

$$\partial_\tau \chi_h^n + \eta_h^n = \frac{1}{\tau}\int_{I^n} B(t, \theta_h^{n-1}, \chi_h^{n-1})dt, \ \text{ in } (V_h)^m, \tag{2.4c}$$

$$\eta_h^n \in \partial I_{K_h}(\chi_h^n) \tag{2.4d}$$

where $A_h^n = A(t^n, \theta_h^n, \chi_h^n)$, and I_{K_h} is the indicator function of the closed convex subset K_h.

Due to the lower smoothness of the solution (θ, χ) of Problem (P), see the definition of the problem, we are not able to expect the H^1-norm error estimates, e.g., as in [8,9]. In this paper, we shall give the L^2-norm error estimates for the absolute temperature θ and the internal variables $\chi_i, i = 1, 2, \cdots, m$. In order to reach the purpose, we introduce here Green operator $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ defined by

$$a(G\,\psi,\, v) = (\psi,\, v), \ \forall \psi \in H^{-1}(\Omega), \ v \in H_0^1(\Omega) \tag{2.5}$$

and its finite element approximation $G_h : H^{-1}(\Omega) \to V_h^0$ defined by

$$a(G_h\,\psi,\, v_h) = a(G\,\psi,\, v_h), \ \forall \psi \in H^{-1}(\Omega), \ v_h \in V_h^0. \tag{2.6}$$

We assume in the paper that Green operator G is regular, referring to [2], that is, for any $\psi \in L^2(\Omega)$ we have $G\psi \in H^2(\Omega)$ and $\|G\psi\|_2 \le C\,\|\psi\|_0$. Under this assumption we have [2, 12]

$$\|G\psi - G_h\psi\|_s \le C\,h^{2-(r+s)}\|\psi\|_{-r}, \ 0 \le r, s \le 1 \tag{2.7}$$

where $\|\cdot\|_s$ and $\|\cdot\|_{-r}$ are the norms of fractional Sobolev spaces $H^s(\Omega) \equiv [H^1(\Omega),\, L^2(\Omega)]_{1-s}$ and $H^{-r}(\Omega) \equiv [L^2(\Omega),\, H^{-1}(\Omega)]_r$, respectively, see [11].

It is easy to check that there exist two constants C_0 and C_1 such that

$$a(G_h\psi,\, G_h\psi) \le C_1\|\psi\|_{-1}^2, \ C_0\|\psi\|_{-1}^2 \le a(G\psi,\, G\psi) \le C_1\|\psi\|_{-1}^2. \tag{2.8}$$

In the following sections, the letter C is always used to denote a positive constant, which may possibly be different at each occurrence, but which depends only on the given data and is independent of the time step size τ and the mesh step size h.

3. Existence, uniqueness of the discrete solutions and their a priori estimates

Our purpose of this section is to demonstrate

Theorem 3.1. *Under the assumptions (1.1)–(1.8) there exists one and only one solution* $(\theta_h^n,\, \chi_h^n)$ *of Problem (FEP), for each* $1 \le n \le M$. *Moreover, we have the a priori estimates*

$$\max_{1\le n\le M} \|\theta_h^n\|_a^2 + \max_{1\le n\le M} \|\chi_h^n\|_X^2 + \tau \sum_{n=1}^{M} \|\partial_\tau \chi_h^n\|_X^2 + \tau \sum_{n=1}^{M} \|\partial_\tau \theta_h^n\|_0^2 \le C, \quad (3.1)$$

$$\max_{1\le n\le M} \|A_h^n\|_0^2 \le C. \quad (3.2)$$

Proof. By the definition of subdifferential we know $(2.4c, d)$ is equivalent to the variational inequality: Find $\chi_h^n \in (V_h)^m$ satisfying

$$\langle \partial_\tau \chi_h^n,\, \lambda_h - \chi_h^n \rangle + I_{K_h}(\lambda_h) - I_{K_h}(\chi_h^n) \ge \frac{1}{\tau} \int_{I^n} \langle B(t, \theta_h^{n-1}, \chi_h^{n-1}), \lambda_h - \chi_h^n \rangle \, dt,$$
$$\forall\ \lambda_h \in (V_h)^m. \quad (3.3)$$

It is easily seen that (3.3) has a unique solution $\chi_h^n \in K_h$, by the standard variational inequality theory, see Glowinski [6]. Thus η_h^n is also uniquely determined by $(2.4c)$. The same argument as in [4, P.286] gives the existence of a unique solution θ_h^n in $(2.4b)$.

Now we prove (3.1) and (3.2). Letting $v_h = \theta_h^n - \theta_h^{n-1}$ in $(2.4b)$, summing up for $n = 1, 2, \cdots, k \le M$ and using $(1.6a)$, one obtains that

$$\begin{aligned}
&\tau\, c_A \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0^2 + \frac{1}{2}\|\theta_h^k\|_a^2 - \frac{1}{2}\|\theta_h^0\|_a^2 \\
&\le \sum_{n=1}^{k} \big(A(t^{n-1}, \theta_h^{n-1}, \chi_h^{n-1}) - A(t^n, \theta_h^{n-1}, \chi_h^{n-1}),\, \partial_\tau \theta_h^n\big) \\
&+ \sum_{n=1}^{k} \big(A(t^n, \theta_h^{n-1}, \chi_h^{n-1}) - A(t^n, \theta_h^{n-1}, \chi_h^n),\, \partial_\tau \theta_h^n\big) \\
&+ \sum_{n=1}^{k} \frac{1}{\tau} \int_{I^n} \big(f(t),\, \theta_h^n - \theta_h^{n-1}\big)\, dt + \sum_{n=1}^{k} \int_{I^n} \big(D(t, \theta_h^{n-1}, \chi_h^{n-1}),\, \partial_\tau \theta_h^n\big)\, dt \\
&=: (\mathrm{I})_1 + (\mathrm{I})_2 + (\mathrm{I})_3 + (\mathrm{I})_4. \qquad (3.4)
\end{aligned}$$

By applying (1.5), $(1.6e)$ and Young's inequality, the term $(\mathrm{I})_1$ can be estimated as follows

$$|(\mathrm{I})_1| \le \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0 \big(\|\theta_h^{n-1}\|_1 + \|\chi_h^{n-1}\|_X + 1\big) |F_A(t^n) - F_A(t^{n-1})|$$

$$\leq \frac{1}{6} c_A \tau \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0^2 + C \sum_{n=1}^{k} (\|\theta_h^{n-1}\|_1^2 + \|\chi_h^{n-1}\|_X^2) \|\frac{dF_A}{dt}\|^2_{L^2(I^n)}, \quad (3.5)$$

but from (1.6b) and Young's inequality, we get the estimation of $(\mathrm{I})_2$

$$\left|(\mathrm{I})_2\right| \leq \frac{1}{6} c_A \tau \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0^2 + \frac{2C_A}{c_A} \tau \sum_{n=1}^{k} \|\partial_\tau \chi_h^n\|_X^2. \quad (3.6)$$

By using (1.1), (1.2) and Lemma 2.1, we obtain for the term $(\mathrm{I})_3$ that

$$\begin{aligned}
\left|(\mathrm{I})_3\right| &\leq \Big| \sum_{n=1}^{k} \frac{1}{\tau} \Big(\int_{I^n} (f(t),\, \theta_h^n) dt - \int_{I^n} (f(t),\, \theta_h^{n-1}) dt \Big) \Big| \\
&\leq \frac{1}{\tau} \Big| \int_{I^k} (f(t),\, \theta_h^k) dt \Big| + \frac{1}{\tau} \Big| \int_{I^1} (f(t),\, \theta_h^0) dt \Big| \\
&\quad + \Big| \frac{1}{\tau} \sum_{n=1}^{k-1} \int_{I^n} (f(t+\tau) - f(t),\, \theta_h^n)\, dt \Big| \\
&\leq C(\|\theta^0\|_1 + \|\theta_h^k\|_1) \|f\|_{L^\infty(0,T;H^{-1}(\Omega))} + \sum_{n=1}^{k-1} \|\theta_h^n\|_1 \int_{t^{n-1}}^{t^{n+1}} \|f_t\|_{-1}\, dt \\
&\leq C + \frac{1}{4} \|\theta_h^k\|_a^2 + \frac{1}{8} \max_{1 \leq n \leq k-1} \|\theta_h^n\|_a^2. \quad (3.7)
\end{aligned}$$

The term $(\mathrm{I})_4$ can be treated by using (1.3), (1.4) and (1.6c),

$$\begin{aligned}
\left|(\mathrm{I})_4\right| &\leq \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0 \int_{I^n} \Big[\|D(t,0,\,0)\|_0 + M_D(t) (\|\theta_h^{n-1}\|_0 + \|\chi_h^{n-1}\|_X) \Big]\, dt \\
&\leq C + \frac{1}{6} c_A \tau \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0^2 + C \sum_{n=1}^{k} \|M_D\|^2_{L^2(I^n)} (\|\theta_h^{n-1}\|_0^2 + \|\chi_h^{n-1}\|_X^2). \quad (3.8)
\end{aligned}$$

Thus it follows from (1.1), (3.4)–(3.8) and Lemma 2.1 that

$$\begin{aligned}
\frac{1}{2} c_A \tau \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0^2 + \frac{1}{4} \|\theta_h^k\|_a^2 &\leq C + \frac{1}{8} \max_{1 \leq n \leq k-1} \|\theta_h^n\|_a^2 + \frac{2C_A}{c_A} \tau \sum_{n=1}^{k} \|\partial_\tau \chi_h^n\|_X^2 \\
&\quad + C \sum_{n=1}^{k} (\|\theta_h^{n-1}\|_1^2 + \|\chi_h^{n-1}\|_X^2)(\|M_D\|^2_{L^2(I^n)} + \|dF_A/dt\|^2_{L^2(I^n)}). \quad (3.9)
\end{aligned}$$

Next we turn to the estimation of χ_h^n. In view of the definition of I_{K_h} it is immediate to deduce from (3.3) that the solution χ_h^n of (2.4c) solves the variational inequality: $\chi_h^n \in K_h$ satisfying

$$\langle \partial_\tau \chi_h^n,\, \chi_h^n - \lambda_h^n \rangle \leq \frac{1}{\tau} \int_{I^n} \langle B(t, \theta_h^{n-1}, \chi_h^{n-1}),\, \chi_h^n - \lambda_h^n \rangle\, dt, \ \forall\, \lambda_h^n \in K_h. \quad (3.10)$$

By substituting $\lambda_h^n = \chi_h^{n-1} \in K_h$ in (3.10), adding it to the inequality

$$\frac{1}{2}\|\chi_h^n\|_X^2 - \frac{1}{2}\|\chi_h^{n-1}\|_X^2 \le \langle \chi_h^n, \chi_h^n - \chi_h^{n-1} \rangle_X$$

and then summing the resultant inequality for $n = 1, 2, \cdots, k \le M$ and using (1.3), (1.6d) one comes to

$$\begin{aligned}
&\tau \sum_{n=1}^{k} \|\partial_\tau \chi_h^n\|_X^2 + \|\chi_h^n\|_X^2 - \|\chi_h^0\|_X^2 \le 2\tau \sum_{n=1}^{k} \|\chi_h^n\|_X^2 + \sum_{n=1}^{k} 2 \int_{I^n} \|B(t, \theta_h^{n-1}, \chi_h^{n-1})\|_X^2 \, dt \\
&\le C + 2\tau \sum_{n=1}^{k} \|\chi_h^n\|_X^2 + 4 \sum_{n=1}^{k} \|M_B\|_{L^2(I^n)}^2 (\|\theta_h^{n-1}\|_0^2 + \|\chi_h^{n-1}\|_X^2). \qquad (3.11)
\end{aligned}$$

Now we multiply (3.11) by constant $\frac{3C_A}{c_A}$, sum it with (3.9) to obtain

$$\begin{aligned}
e_k :\equiv & \frac{1}{2} c_A \tau \sum_{n=1}^{k} \|\partial_\tau \theta_h^n\|_0^2 + \frac{1}{4}\|\theta_h^k\|_a^2 + \frac{C_A}{c_A} \tau \sum_{n=1}^{k} \|\partial_\tau \chi_h^n\|^2 + \frac{3C_A}{c_A} \|\chi_h^k\|_X^2 \\
&\le C + C\tau \sum_{n=1}^{k} \|\chi_h^k\|_X^2 + \frac{1}{8} \max_{1 \le n \le k} \|\theta_h^n\|_a^2 \\
&\quad + C \sum_{n=1}^{k-1} \alpha_{n+1} (\|\theta_h^n\|_1^2 + \|\chi_h^n\|_X^2) \qquad (3.12)
\end{aligned}$$

with $\alpha_n = \|M_D\|_{L^2(I^n)}^2 + \|M_B\|_{L^2(I^n)}^2 + \|\frac{dF_A}{dt}\|_{L^2(I^n)}^2$, and therefore

$$\sum_{n=1}^{M} \alpha_n = \|M_D\|_{L^2(0,T)}^2 + \|M_B\|_{L^2(0,T)}^2 + \|\frac{dF_A}{dt}\|_{L^2(0,T)}^2 \le C, \qquad (3.13)$$

that is,

$$e_k \le C + \frac{1}{2} \max_{1 \le n \le k} e_n + C\Big(\tau \sum_{n=1}^{k} e_n + \sum_{n=1}^{k-1} \alpha_{n+1} e_n\Big)$$

where we have used Lemma 2.1. By taking the maximum of e_n between $1 \le n \le k$, we see

$$e_k \le C(1 + \tau \sum_{n=1}^{k} e_n + \sum_{n=1}^{k-1} \alpha_{n+1} e_n).$$

Thus (3.1) follows from Gronwall's inequality and (3.13).

(3.2) is a direct consequence of (3.1), (1.6e), (1.7) and the following equality

$$A_h^n = [A(t^n, \theta_h^n, \chi_h^n) - A(t^n, \theta_h^n, 0)] + [A(t^n, \theta_h^n, 0) - A(0, \theta_h^n, 0)] + A(0, \theta_h^n, 0). \quad (3.14)$$

So we have completed the proof of Theorem 3.1.

4. Finite element error estimates

We devote this section to derive the finite element error estimates of the problem (FEP) defined in Section 2. Let (χ, θ) and (θ_h^n, χ_h^n) be the solutions of (1.9a, b, c) and (2.4a, b, c, d), respectively. Set $e_A^n = A^n - A_h^n = A(t^n, \theta^n, \chi^n) - A(t^n, \theta_h^n, \chi_h^n)$, $e_\theta^n = \theta^n - \theta_h^n$ and $e_\chi^n = \chi^n - \chi_h^n$.

We first integrate (1.9a) over I^n and take a test function $Ge_A^n \in H_0^1(\Omega)$ to get

$$\tau(\partial_\tau A^n, Ge_A^n) + \tau a(\tilde{\theta}^n, Ge_A^n) = \tau(\tilde{f}^n, Ge_A^n) + \int_{I^n} \big(D(t, \theta(t), \chi(t)), Ge_A^n\big)\, dt. \quad (4.1)$$

Letting $v_h = \tau G_h e_A^n \in V_h^0$ in (2.4b) gives

$$\tau(\partial_\tau A_h^n, G_h e_A^n) + \tau a(\theta_h^n, G_h e_A^n) = \tau(\tilde{f}^n, G_h e_A^n) + \int_{I^n} \big(D(t, \theta_h^{n-1}, \chi_h^{n-1}), G_h e_A^n\big)\, dt. \quad (4.2)$$

Then subtracting (4.1) from (4.2) and summing the resultant equation for $n = 1, 2, \cdots, k \leq M$ implies that

$$\begin{aligned}
&(\mathrm{II})_1 + (\mathrm{II})_2 + (\mathrm{II})_3 \\
&=: \tau \sum_{n=1}^{k} (\partial_\tau e_A^n, Ge_A^n) + \tau \sum_{n=1}^{k} a\big(\tilde{\theta}^n - \theta_h^n, Ge_A^n\big) + \tau \sum_{n=1}^{k} a\big(\theta_h^n, (G - G_h)e_A^n\big) \\
&= \tau \sum_{n=1}^{k} \big(\partial_\tau A_h^n, (G_h - G)e_A^n\big) + \tau \sum_{n=1}^{k} \big(\tilde{f}^n, (G - G_h)e_A^n\big) \\
&\quad + \sum_{n=1}^{k} \int_{I^n} \Big[\big(D(t, \theta(t), \chi(t)),\, Ge_A^n\big) - \big(D(t, \theta_h^{n-1}, \chi_h^{n-1}),\, G_h e_A^n\big)\Big]\, dt \\
&=: (\mathrm{III})_1 + (\mathrm{III})_2 + (\mathrm{III})_3. \qquad (4.3)
\end{aligned}$$

Next we estimate $(\mathrm{II})_i$, $(\mathrm{III})_i$, $i = 1, 2, 3$ in (4.3), one by one.

Obviously, from the definition of G and G_h, we know $(\mathrm{II})_3 = 0$, but for $(\mathrm{II})_1$ we have from (2.5) that

$$(\mathrm{II})_1 \geq \sum_{n=1}^{k} \Big(\frac{1}{2}\|Ge_A^n\|_a^2 - \frac{1}{2}\|Ge_A^{n-1}\|_a^2\Big) = \frac{1}{2}\|Ge_A^k\|_a^2 - \frac{1}{2}\|Ge_A^0\|_a^2. \quad (4.4)$$

By using $(1.6a)$ we deduce that

$$
\begin{aligned}
(\mathrm{II})_2 &= \tau \sum_{n=1}^{k} (A^n - A_h^n, \tilde{\theta}^n - \theta_h^n) \\
&= \tau \sum_{n=1}^{k} \big(A^n - A(t^n, \theta_h^n, \chi^n),\, e_\theta^n\big) + \tau \sum_{n=1}^{k} (A^n - A_h^n, \tilde{\theta}^n - \theta^n) \\
&\quad + \tau \sum_{n=1}^{k} \big(A(t^n, \theta_h^n, \chi^n) - A(t^n, \theta_h^n, \chi_h^n),\, e_\theta^n\big) \\
&\geq \tau c_A \sum_{n=1}^{k} \|e_\theta^n\|_0^2 + \tau \sum_{n=1}^{k} (A^n - A_h^n, \tilde{\theta}^n - \theta^n) \\
&\quad + \tau \sum_{n=1}^{k} \big(A(t^n, \theta_h^n, \chi^n) - A(t^n, \theta_h^n, \chi_h^n),\, e_\theta^n\big) \\
&=: c_A \tau \sum_{n=1}^{k} \|e_\theta^n\|_0^2 + (\mathrm{II})_2^1 + (\mathrm{II})_2^2.
\end{aligned} \tag{4.5}
$$

From $(3.2), (1.6b)$ and Young's inequality we get

$$
\big|(\mathrm{II})_2^1\big| \leq \tau \sum_{n=1}^{k} \|A^n - A_h^n\|_0 \, \|\theta^n - \tilde{\theta}^n\|_0 \leq C \tau^{\frac{3}{2}} \sum_{n=1}^{k} \Big(\int_{I^n} \|\theta_t\|_0^2 \, dt \Big)^{\frac{1}{2}} \leq C\tau \tag{4.6}
$$

and

$$
\big|(\mathrm{II})_2^2\big| \leq \tau \sqrt{C_A} \sum_{n=1}^{k} \|e_\chi^n\|_X \, \|e_\theta^n\|_0 \leq \frac{1}{2} c_A \tau \sum_{n=1}^{k} \|e_\theta^n\|_0^2 + \frac{C_A}{2c_A} \tau \sum_{n=1}^{k} \|e_\chi^n\|_X^2. \tag{4.7}
$$

Hence from (4.3)–(4.7) we have derived

$$
\frac{1}{2}\|Ge_A^k\|_a^2 + \frac{1}{2} c_A \tau \sum_{n=1}^{k} \|e_\theta^n\|_0^2 \leq C\tau + \frac{1}{2}\|Ge_A^0\|_a^2 + \frac{C_A}{2c_A} \tau \sum_{n=1}^{k} \|e_\chi^n\|_X^2 + \sum_{i=1}^{3} (\mathrm{III})_i. \tag{4.8}
$$

Now we analyse $(\mathrm{III})_1, (\mathrm{III})_2$ and $(\mathrm{III})_3$. For $(\mathrm{III})_3$ we rewrite it as follows

$$
\begin{aligned}
(\mathrm{III})_3 &= \sum_{n=1}^{k} \int_{I^n} \big(D(t, \theta, \chi) - D(t, \theta_h^{n-1}, \chi_h^{n-1}),\, Ge_A^n\big) \, dt \\
&\quad + \sum_{n=1}^{k} \int_{I^n} \big(D(t, \theta_h^{n-1}, \chi_h^{n-1}), (G - G_h) e_A^n\big) \, dt \\
&=: (\mathrm{III})_3^1 + (\mathrm{III})_3^2.
\end{aligned} \tag{4.9}
$$

By applying (1.6c), (1.1), Lemma 2.1 and Young's inequality we have

$$
\begin{aligned}
\left|(\mathrm{III})_3^1\right| &\le \sum_{n=1}^{k} \int_{I^n} \|D(t,\theta,\chi) - D(t,\theta_h^{n-1},\chi_h^{n-1})\|_0 \, \|Ge_A^n\|_0 \, dt \\
&\le C \sum_{n=1}^{k} \|Ge_A^n\|_a \int_{I^n} M_D(t)\big(\|\theta - \theta^{n-1}\|_0 + \|\chi - \chi^{n-1}\|_X\big)\, dt \\
&\quad + C \sum_{n=1}^{k} \|Ge_A^n\|_a \int_{I^n} M_D(t)\big(\|e_\theta^{n-1}\|_0 + \|e_\chi^{n-1}\|_X\big)\, dt \\
&\le C\,\tau + C\tau \sum_{n=1}^{k-1} \|e_\chi^n\|_X^2 + C \sum_{n=1}^{k} \|M_D\|_{L^2(I^n)}^2 \|Ge_A^n\|_a^2 + \frac{1}{4} c_A \tau \sum_{n=1}^{k-1} \|e_\theta^n\|_0^2,
\end{aligned}
\tag{4.10}
$$

while from (1.3), (2.8), (2.7), (3.1) and (1.6c) we obtain

$$
\begin{aligned}
\left|(\mathrm{III})_3^2\right| &\le \sum_{n=1}^{k} \int_{I^n} \|D(t,\theta_h^{n-1},\chi_h^{n-1})\|_0 \, \|(G - G_h)e_A^n\|_0 \, dt \\
&\le C\,h \sum_{n=1}^{k} \|e_A^n\|_{-1} \int_{I^n} \Big[\|D(t,0,0)\|_0 + M_D(t)(\|\theta_h^{n-1}\|_0 + \|\chi_h^{n-1}\|_X)\Big]\, dt \\
&\le C\,h\sqrt{\tau} \sum_{n=1}^{k} \|Ge_A^n\|_a \big(D(t,0,0)_{L^2(I^n;L^2(\Omega))} + \|M_D\|_{L^2(I^n)}\big) \\
&\le C\,h^2 + C\,\tau \sum_{n=1}^{k} \|Ge_A^n\|_a^2.
\end{aligned}
\tag{4.11}
$$

So we have deduced from (4.9)–(4.11) that

$$
\begin{aligned}
\left|(\mathrm{III})_3\right| \le C\,(h^2 + \tau) + C \sum_{n=1}^{k} \big(\tau + \|M_D\|_{L^2(I^n)}^2\big) \|Ge_A^n\|_a^2 + C\,\tau \sum_{n=1}^{k-1} \|e_\chi^n\|_X^2 \\
+ \frac{1}{4} c_A \tau \sum_{n=1}^{k} \|e_\theta^n\|_0^2.
\end{aligned}
\tag{4.12}
$$

The term $(\mathrm{III})_1$ can be estimated from (2.5), (2.6), (2.7) and (3.2) as follows

$$
(\mathrm{III})_1 = \tau \sum_{n=1}^{k} \big(\partial_\tau A_h^n,\, (G_h - G)e_A^n\big) = \tau \sum_{n=1}^{k} a\left(G\partial_\tau A_h^n, (G_h - G)e_A^n\right)
$$

$$
\begin{aligned}
&= \tau \sum_{n=1}^{k} a\big((G-G_h)\partial_\tau A_h^n,\, (G_h-G)e_A^n\big) \\
&\le C\tau \sum_{n=1}^{k} \|(G-G_h)\partial_\tau A_h^n\|_1\, \|(G_h-G)e_A^n\|_1 \\
&\le C\tau\, h^2 \sum_{n=1}^{k} \|\partial_\tau A_h^n\|_0\, \|e_A^n\|_0 \le Ch^2 \sum_{n=1}^{k} 1 \\
&\le C\, h^2/\tau.
\end{aligned} \tag{4.13}
$$

One can estimate $(\mathrm{III})_2$ from (1.2), (2.7) and (3.2) that

$$
\begin{aligned}
(\mathrm{III})_2 &\le \sum_{n=1}^{k} \int_{I^n} \|f(t)\|_{-1}\, \|(G-G_h)e_A^n\|_1\, dt \\
&\le C\, h \sum_{n=1}^{k} \|e_A^n\|_0 \int_{I^n} \|f(t)\|_{-1}\, dt \\
&\le C\, h \sum_{n=1}^{k} \int_{I^n} \|f(t)\|_{-1}\, dt \le C\, h.
\end{aligned} \tag{4.14}
$$

Therefore we have obtained from (4.8), (4.12)–(4.14) that

$$
\begin{aligned}
\frac{1}{2}\|Ge_A^k\|_a^2 + \frac{1}{4}c_A\tau \sum_{n=1}^{k} \|e_\theta^n\|_0^2 \le& \frac{1}{2}\|Ge_A^0\|_a^2 + C(\tau + h + h^2/\tau) + C\tau \sum_{n=1}^{k} \|e_\chi^n\|_X^2 \\
&+ C \sum_{n=1}^{k} \big(\tau + \|M_D\|_{L^2(I^n)}^2\big)\, \|Ge_A^n\|_a^2.
\end{aligned} \tag{4.15}
$$

In the sequel, we turn to the estimation of e_χ^n. Firstly we integrate (1.9*b*) over I^n, multiply the resultant equation in both sides by $\tau^{-1}(\lambda - \chi^n)$ and use (1.9*c*) to obtain

$$
\begin{aligned}
&\big\langle \partial_\tau \chi^n,\, \lambda - \chi^n \big\rangle + I_K(\lambda) - \frac{1}{\tau}\int_{I^n} I_K\big(\chi(t)\big)\, dt \\
\ge& \frac{1}{\tau}\int_{I^n} \big\langle B(t,\theta(t),\chi(t)),\, \lambda - \chi^n \big\rangle\, dt + \frac{1}{\tau}\int_{I^n} \big\langle \eta(t),\, \chi^n - \chi(t) \big\rangle\, dt,\ \forall \lambda \in X.
\end{aligned} \tag{4.16}
$$

By means of the definition of I_K it is easy to check that (4.16) is equivalent to the following variational inequality: for *a.e.* $t \in I^n$, $\chi(t) \in K$ and for each $\lambda \in K$

$$
\big\langle \partial_\tau \chi^n,\, \lambda - \chi^n \big\rangle \ge \frac{1}{\tau}\int_{I^n} \big\langle B(t,\theta,\chi),\, \lambda - \chi^n \big\rangle\, dt + \frac{1}{\tau}\int_{I^n} \big\langle \eta(t),\, \chi^n - \chi(t) \big\rangle\, dt. \tag{4.17}
$$

But from (3.10) we know χ_h^n solves the following variational inequality: $\chi_h^n \in K_h$ such that for any $\lambda_h^n \in K_h$

$$\langle \partial_\tau \chi_h^n,\, \lambda_h^n - \chi_h^n \rangle \geq \frac{1}{\tau} \int_{I^n} \langle B(t, \theta_h^{n-1}, \chi_h^{n-1}),\, \lambda_h^n - \chi_h^n \rangle \, dt. \tag{4.18}$$

Letting $\lambda = \chi_h^n \in K_h \subset K$ in (4.17) and adding the resultant inequality to (4.18) implies that

$$\begin{aligned} \langle \partial_\tau e_\chi^n,\, e_\chi^n \rangle \leq \langle \partial_\tau \chi_h^n,\, \lambda_h^n - \chi^n \rangle + \frac{1}{\tau} \int_{I^n} \langle \eta(t),\, \chi(t) - \chi^n \rangle \, dt \\ + \frac{1}{\tau} \int_{I^n} \Big[\langle B(t, \theta, \chi),\, \chi^n - \chi_h^n \rangle + \langle B(t, \theta_h^{n-1}, \chi_h^{n-1}),\, \chi_h^n - \lambda_h^n \rangle \Big] \, dt, \end{aligned}$$

this leads to

$$\begin{aligned} \frac{1}{2}\|e_\chi^k\|_X^2 - \frac{1}{2}\|e_\chi^0\|_X^2 &\leq \tau \sum_{n=1}^k \langle \partial_\tau \chi_h^n,\, \lambda_h^n - \chi^n \rangle + \sum_{n=1}^k \int_{I^n} \langle \eta(t),\, \chi(t) - \chi^n \rangle \, dt \\ &\quad + \sum_{n=1}^k \int_{I^n} \langle B(t, \theta, \chi) - B(t, \theta_h^{n-1}, \chi_h^{n-1}),\, \chi^n - \chi_h^n \rangle \, dt \\ &\quad + \sum_{n=1}^k \int_{I^n} \langle B(t, \theta_h^{n-1}, \chi_h^{n-1}),\, \chi^n - \lambda_h^n \rangle \, dt \\ &=: (\mathrm{IV})_1 + (\mathrm{IV})_2 + (\mathrm{IV})_3 + (\mathrm{IV})_4. \end{aligned} \tag{4.19}$$

We evaluate now $(\mathrm{IV})_1 - (\mathrm{IV})_4$, one by one. By Schwarz's inequality and (3.1) we get

$$\begin{aligned} \big|(\mathrm{IV})_1\big| &\leq \Big(\sum_{n=1}^k \tau \|\partial_\tau \chi_h^n\|_X^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=1}^k \tau \|\chi^n - \lambda_h^n\|_X^2 \Big)^{\frac{1}{2}} \\ &\leq C\,\tau \|\chi_t\|_{L^2(Q_T)} + C \Big(\sum_{n=1}^k \tau \|\tilde{\chi}^n - \lambda_h^n\|_X^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.20}$$

It is standard to derive that

$$\begin{aligned} \big|(\mathrm{IV})_2\big| &\leq \sum_{n=1}^k \int_{I^n} \|\eta(t)\|_X \, \|\chi(t) - \chi^n\|_X \, dt \\ &\leq \tau \Big(\sum_{n=1}^k \int_{I^n} \|\eta(t)\|_X^2 \, dt \Big)^{\frac{1}{2}} \Big(\sum_{n=1}^k \int_{I^n} \|\chi_t\|_X^2 \, dt \Big)^{\frac{1}{2}} \leq C\,\tau. \end{aligned} \tag{4.21}$$

From (1.6d), Lemma 2.1 and Young's inequality we have

$$\begin{aligned}
\left|(\mathrm{IV})_3\right| &\le \sum_{n=1}^{k} \|e_\chi^n\|_X \int_{I^n} \left(\|\theta-\theta^{n-1}\|_0 + \|\chi-\chi^{n-1}\|_X\right) M_B(t)\,dt \\
&\quad + \sum_{n=1}^{k} \|e_\chi^n\|_X \left(\|e_\theta^{n-1}\|_0 + \|e_\chi^{n-1}\|_X\right) \int_{I^n} M_B(t)\,dt \\
&\le 2\tau \sum_{n=1}^{k} \|e_\chi^n\|_X \, \|M_B\|_{L^2(I^n)} \left(\|\theta_t\|_{L^2(I^n;L^2(\Omega))} + \|\chi_t\|_{L^2(I^n;X)}\right) \\
&\quad + \sqrt{\tau} \sum_{n=1}^{k} \|e_\chi^n\|_X \, \|M_B\|_{L^2(I^n)} \left(\|e_\theta^{n-1}\|_0 + \|e_\chi^{n-1}\|_X\right) \\
&\le C\tau^2 + C \sum_{n=1}^{k} \|M_B\|_{L^2(I^n)}^2 \|e_\chi^n\|_X^2 + C\tau \sum_{n=1}^{k} \|e_\chi^{n-1}\|_X^2 + \frac{1}{8} c_A \tau \sum_{n=1}^{k} \|e_\theta^{n-1}\|_0^2 \\
&\le C\,\tau + \frac{1}{8} c_A\,\tau \sum_{n=1}^{k-1} \|e_\theta^n\|_0^2 + C \sum_{n=1}^{k} \left(\tau + \|M_B\|_{L^2(I^n)}^2\right) \|e_\chi^n\|_X^2. \qquad (4.22)
\end{aligned}$$

Finally for $(\mathrm{IV})_4$ we obtain from (1.3), (1.4), (3.1) and (1.6d) that

$$\begin{aligned}
\left|(\mathrm{IV})_4\right| &\le \sum_{n=1}^{k} \|\chi^n - \lambda_h^n\|_X \int_{I^n} \|B(t, \theta_h^{n-1}, \chi_h^{n-1})\|_X \, dt \\
&\le \sum_{n=1}^{k} \sqrt{\tau}\, \|\chi^n - \lambda_h^n\|_X \left[\int_{I^n} \left(M_B^2(t) + \|B(t,\,0,\,0)\|_X^2\right) dt\right]^{\frac{1}{2}} \\
&\le C \left(\sum_{n=1}^{k} \tau \|\chi^n - \lambda_h^n\|_X^2\right)^{\frac{1}{2}} \le C\,\tau + C \left(\sum_{n=1}^{k} \tau \|\tilde{\chi}^n - \lambda_h^n\|_X^2\right)^{\frac{1}{2}}. \qquad (4.23)
\end{aligned}$$

Therefore from (4.19)–(4.23) we have deduced that

$$\begin{aligned}
\frac{1}{2}\|e_\chi^k\|_X^2 &\le C\,\tau + \frac{1}{2}\|e_\chi^0\|_X^2 + \frac{1}{8} c_A \tau \sum_{n=1}^{k} \|e_\theta^n\|_0^2 + C \left(\sum_{n=1}^{k} \tau \|\tilde{\chi}^n - \lambda_h^n\|_X^2\right)^{\frac{1}{2}} \\
&\quad + C \sum_{n=1}^{k} \left(\tau + \|M_B\|_{L^2(I^n)}^2\right) \|e_\chi^n\|_X^2. \qquad (4.24)
\end{aligned}$$

Adding (4.24) to (4.15) comes to

$$\frac{1}{2}\|e_\chi^k\|_X^2 + \frac{1}{2}\|G\,e_A^k\|_a^2 + \frac{1}{8} c_A \tau \sum_{n=1}^{k} \|e_\theta^n\|_0^2 \le \frac{1}{2}\left(\|G\,e_A^0\|_a^2 + \|e_\chi^0\|_X^2\right) + C\left(\tau + h + h^2/\tau\right)$$

$$+ C\Big(\sum_{n=1}^{M}\tau\|\tilde{\chi}^n-\lambda_h^n\|_X^2\Big)^{\frac{1}{2}} + C\sum_{n=1}^{k}\big(\tau+\|M_B\|_{L^2(I^n)}^2\big)\,\|e_\chi^n\|_X^2$$

$$+ C\sum_{n=1}^{k}\big(\tau+\|M_D\|_{L^2(I^n)}^2\big)\,\|G\,e_A^n\|_a^2. \tag{4.25}$$

which holds for any $1 \le k \le M$ and any $\lambda_h^n \in K_h$.

By applying discrete Gronwall's inequality for (4.25), using (1.4), (2.8) and taking $\lambda_h^n = P_h\tilde{\chi}^n \in K_h$ we have demonstrated

Theorem 4.1. *Let (χ, θ) be the solution of Problem (P) and (χ_h^n, θ_h^n) be the solution of Problem (FEP), $n = 1, 2, \cdots, M$. Then we have the error estimates*

$$\max_{1\le n\le M}\|\chi^n-\chi_h^n\|_X^2 + \max_{1\le n\le M}\|A(t^n,\theta^n,\chi^n)-A(t^n,\theta_h^n,\chi_h^n)\|_{-1}^2 + \tau\sum_{n=1}^{M}\|\theta^n-\theta_h^n\|_0^2$$

$$\le C\,(\tau+h+h^2/\tau) + C\,\|A(0,\,\theta^0,\,\chi^0)-A(0,\,\theta_h^0,\,\chi_h^0)\|_{-1}^2 + C\|\chi^0-\chi_h^0\|_X^2$$

$$+ C\Big(\sum_{n=1}^{M}\tau\|\tilde{\chi}^n-P_h\tilde{\chi}^n\|_X^2\Big)^{1/2}. \tag{4.26}$$

Remark 4.1: For practical applications, we know that the convex subset K is often the following, see [4, 5, 13]

$$K = \{\chi \in \big(L^2(\Omega)\big)^m; \chi_i \ge 0, i = 1, \cdots, m,\ \sum_{i=1}^{m}\chi_i \le 1 \ a.e. \text{ on } \Omega\}.$$

Suppose χ^0 and θ^0 are appropriately smooth (depending on different concrete problems), and for the solution $\chi = (\chi_i, \cdots, \chi_m)$ of Problem (P), we have $\chi_i \in L^2\big(0,T;H^1(\Omega)\big)$, $i = 1, 2, \cdots, m$. For concrete applications, e.g., for the three problems described in [4, Section 2], it is not difficult by using Lemma 2.1 to check that

$$\|\chi^0-\chi_h^0\|_X^2 = O(h^2),\ \|A(0,\,\theta^0,\,\chi^0)-A(0,\,\theta_h^0,\,\chi_h^0)\|_{-1}^2 = O(h^2),$$

$$\Big(\sum_{n=1}^{M}\tau\|\tilde{\chi}^n-P_h\tilde{\chi}^n\|_X^2\Big)^{1/2} = O(h),$$

The last error relation above is derived by using the following Lemma 4.2 which can be obtained in the same way as in [10, 14]

Lemma 4.2. *For each $h > 0$ there exists a linear operator $S_h : H^1(\Omega) \to H^2(\Omega) \subset C(\overline{\Omega})(d \le 3)$ such that*

(i) $\|S_hv\|_2 \le C\,h^{-1}\|v\|_1$, $\|v-S_hv\|_0 \le C\,h\|v\|_1$,

(ii) $0 \le S_h v \le 1$ on Ω if $0 \le v \le 1$ *a.e.* on Ω

since $\tilde{\chi}^n \in K$, we see $(\Pi_h S_h \tilde{\chi}_1^n, \cdots, \Pi_h S_h \tilde{\chi}_m^n) \in K_h$ with here Π_h the finite element interpolate related to V_h, therefore from the standard finite element interpolation theory [2], Lemma 2.1 and Lemma 4.2 we get

$$\|\tilde{\chi}^n - P_h \tilde{\chi}^n\|_X \le C \sum_{i=1}^m \|\tilde{\chi}_i^n - \Pi_h S_h \tilde{\chi}_i^n\|_0$$
$$\le C \sum_{i=1}^m \left(\|\tilde{\chi}_i^n - S_h \tilde{\chi}_i^n\|_0 + \|S_h \tilde{\chi}_i^n - \Pi_h S_h \tilde{\chi}_i^n\|_0\right)$$
$$\le C \sum_{i=1}^m \left(h\,\|\tilde{\chi}_i^n\|_1 + h^2 \|S_h \tilde{\chi}_i^n\|_2\right) \le C \sum_{i=1}^m h\,\|\tilde{\chi}_i^n\|_1,$$

that is,

$$\Big(\sum_{n=1}^M \tau \|\tilde{\chi}^n - P_h \tilde{\chi}^n\|_X^2\Big)^{1/2} \le C\,\Big(\sum_{i=1}^m \tau\, h^2 \sum_{n=1}^M \|\tilde{\chi}_i^n\|_1^2\Big)^{1/2} \le C\,h.$$

From above, we see that the right-hand side of (4.26) has the error order of $O(\tau + h + h^2/\tau)$.

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