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Tony F. Chan
Jun Zou

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

ADDITIVE SCHWARZ DOMAIN DECOMPOSITION METHODS FOR ELLIPTIC PROBLEMS ON UNSTRUCTURED MESHES *

TONY F. CHAN † AND JUN ZOU ‡

Abstract. We give several additive Schwarz domain decomposition methods for solving finite element problems which arise from the discretizations of elliptic problems on general unstructured meshes in two and three dimensions. Our theory requires no assumption (for the main results) on the substructures which constitute the whole domain, so each substructure can be of arbitrary shape and of different size. The global coarse mesh is allowed to be non-nested to the fine grid on which the discrete problem is to be solved and both the coarse meshes and the fine meshes need not be quasi-uniform. In this general setting, our algorithms have the same optimal convergence rate of the usual domain decomposition methods on structured meshes. The condition numbers of the preconditoned systems depend only on the (possibly small) overlap of the substructures and the size of the coarse grid, but is independent of the sizes of the subdomains.

Key Words. Unstructured meshes, non-nested coarse meshes, additive Schwarz algorithm, optimal convergence rate.

AMS(MOS) subject classification. 65N30, 65F10

1. Introduction. Unstructured meshes have become quite popular recently in large scale scientific computing [1] [17]. One of the main advantages over structured meshes is the extra flexibility in adapting efficiently to complicated geometries and to regions with large variations in the solution. However, this flexibility may come with a price. Traditional solvers which exploit the regularity of the mesh may become less efficient on an unstructured mesh. Moreover, efficient vectorization and parallelization may require extra care. Thus, there is a need to adapt and develope current solution techniques for structured meshes so that they can run as efficiently on unstructured meshes.

In this paper, we will present some domain decomposition methods, in particular additive Schwarz methods defined for overlapping subdomains, for solving elliptic problems on unstructured meshes in two and three space dimensions. These are extensions of existing domain decomposition methods, constructed in such a way so that, first, they can be applied to unstructured meshes, and second, they retain their optimal efficiency

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† Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024-1555. E-mail: chan@math.ucla.edu. The work of this author was partially supported by the National Science Foundation under contract ASC 92-01266, the Army Research Office under contract DAAL03-91-G-0150, and ONR under contract ONR-N00014-92-J-1890.

‡ Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024-1555 and Computing Center, the Chinese Academy of Sciences, Beijing 100080, P. R. China. The work of this author was partially supported by the National Science Foundation under contract ASC 92-01266, the Army Research Office under contract DAAL03-91-G-0150, and subcontract DAAL03-91-C-0047.

as for structured meshes. These methods are designed to possess inherent coarse grain parallelism in the sense that the subdomain problems can be solved independently on different processors.

The theory and methodology of domain decomposition methods for elliptic problems on structured meshes are quite well developed [7] [14] [12]. It is known, for example, that to achieve an *optimal* rate of convergence, a coarse grid solver must be employed in addition to individual subdomain solves. On a structured mesh, most of the existing theories and algorithms exploit the fact that the space of functions on the coarse mesh is a subspace of that on the fine mesh. Unfortunately, this property may no longer hold on an unstructured mesh. Both the theory and the algorithms need to be developed to accomodate this fact.

A natural approach is to introduce an appropriate mapping of the coarse space so that the image space is a proper subspace of the fine space. In this paper, we consider several possibilities, including interpolation, projection and localized projection. The use of interpolation is most natural and has been used by Cai and Saad [5], Cai [4] and Chan and Smith [8]. We believe the use of the other two projection operators in this context is new. The key step in the theory is in establishing a basic decomposition lemma corresponding to the stability of a representation of finite element functions on the fine mesh as a combination of functions defined on the newly constructed coarse space and the subspaces corresponding to the subdomains. After a brief formulation of the problem in Section 2, the new coarse spaces and the corresponding decomposition lemmas are established in Sections 3-5. The additive Schwarz algorithms and the establishment of their optimal convergence rate are then given in Section 6. Finally, in Section 7, we apply the local projection coarse subspace method to *structured* meshes, thereby improving the algorithm by allowing the amount of overlap of a subdomain with its neighbors to vary proportionally to its size, while still retaining the optimal convergence rate. This feature may improve the efficiency of the algorithm in cases where the subdomains can have large variations in size.

2. The formulation of the problem. In this paper, we consider the following self-adjoint elliptic problem:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + b\,u = f, \qquad \text{in } \Omega \tag{1}$$

$$\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j} n_i + \alpha\, u = g, \qquad \text{on } \partial\Omega \tag{2}$$

where $\Omega \subset R^d\,(d = 2,3)$ is a polygon or polyhedron, $(a_{ij}(x))$ is symmetric, uniformly elliptic, and is allowed to be discontinuous, $b(x) \geq 0$ on Ω, $\alpha(x) \geq 0$ on $\partial\Omega$, and $n = (n_1, n_2, \cdots, n_d)$ is the unit outer normal of the boundary $\partial\Omega$.

By Green's formula, it is immediate to derive the variational problem corresponding to (1) and (2): Find $u \in H^1(\Omega)$ such that

$$a(u,v) = f(v), \;\forall\; v \in H^1(\Omega) \tag{3}$$

with

$$
\begin{aligned}
a(u,v) &= \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + b\, uv \Big) dx + \int_{\partial\Omega} \alpha\, uv\, ds, \\
f(v) &= \int_\Omega f\, v\, dx + \int_{\partial\Omega} g\, v\, ds.
\end{aligned}
$$

We will solve the above variational problem (3) by finite element methods. Suppose we are given a family of triangulations $\{\mathcal{T}^h\}$ on Ω. Let $\overline{h} = \max_{\tau\in\mathcal{T}^h} h_\tau$, $h_\tau =$ diam τ, $\underline{h} = \min_{\tau\in\mathcal{T}^h} h_\tau$, $\rho_\tau =$ the radius of the ball inscribed in τ. Then we say $\mathcal{T}^h$ is shape regular if it satisfies

$$
\sup_h \max_{\tau\in\mathcal{T}^h} \frac{h_\tau}{\rho_\tau} \le \sigma_0\,, \tag{4}
$$

and we say $\mathcal{T}^h$ is quasi-uniform if it is shape regular and satisfies

$$
\overline{h} \le \gamma\, \underline{h}, \tag{5}
$$

with σ_0 and γ fixed positive constants, see Ciarlet [10], Xu [20]. Let V^h be the piecewise linear subspace of $H^1(\Omega)$ defined on $\mathcal{T}^h$ with its basis denoted by $\{\phi_i^h\}_{i=1}^n$, and $O_i = \operatorname{supp} \phi_i^h$. Later on we will use the following simple facts: if $\mathcal{T}^h$ is shape regular, there exist a positive constant C and an integer ν, both depending only on σ_0 appearing in (4) and independent of h such that for $i = 1, 2, \cdots, n$,

$$
\operatorname{diam} O_i \le C\, h_\tau, \ \forall\, \tau \subset O_i, \tag{6}
$$

$$
\operatorname{card} \{\tau \in \mathcal{T}^h;\, \tau \subset O_i\} \le \nu. \tag{7}
$$

Our finite element problem is: Find $u^h \in V^h$ such that

$$
a(u^h, v^h) = f(v^h), \ \forall\, v^h \in V^h. \tag{8}
$$

Because of the ill-conditioning of the stiffness matrix A induced by the bilinear form $a(\cdot,\cdot)$ in (8), our goal is to construct a good preconditioner M for A by domain decomposition methods to be used with the preconditioned conjugate gradient method.

As usual, we decompose the domain Ω into p nonoverlapping subdomains Ω_i such that $\overline{\Omega} = \cup_{i=1}^p \overline{\Omega}_i$, then extend each subdomain Ω_i to a larger one Ω_i' such that the distance between $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by $\delta_i > 0$. We denote the minimum of all δ_i by δ. We assume that $\partial\Omega_i'$ does not cut through any element $\tau \in \mathcal{T}^h$. For the subdomains meeting the boundary we cut off the part of Ω_i' which is outside of $\overline{\Omega}$. No other assumptions will be made on $\{\Omega_i\}$ in this paper except that any point $x \in \Omega$ belongs only to a finite number of subdomains $\{\Omega_i'\}$. This means that we allow each Ω_i to be of quite different size and of quite different shape from other subdomains. Throughout the paper, we define the subspaces of V^h corresponding to the subdomains $\{\Omega_i'\}$, $i = 1, 2, \cdots, p$ by

$$
V_i^h = \{v_h \in V^h;\, v_h = 0 \text{ on } \partial\Omega_i' \cap \Omega\}. \tag{9}
$$

It is well-known that we have to add some global coarse problems in order to get an optimal or almost optimal preconditioner, see Widlund [19]. Therefore, we also introduce a coarse grid $\mathcal{T}^H$ which form a shape regular triangulation of Ω. For the simplicity of our exposition, though our algorithms and theory are applicable for more general cases, cf. Chan, Smith and Zou [6], we assume here that the boundary $\partial\Omega^H$ of $\Omega^H = \cup_{\tau^H \in \mathcal{T}^H} \tau^H$ coincides with the boundary $\partial\Omega^h$ of $\Omega^h = \cup_{\tau^h \in \mathcal{T}^h} \tau^h$, but otherwise has nothing to do with $\mathcal{T}^h$, i.e., none of the interior nodes of $\mathcal{T}^H$ need to be nodes of $\mathcal{T}^h$. Thus, $\mathcal{T}^H$ is in general non-nested to $\mathcal{T}^h$. Let H be the maximum diameter of the elements of $\mathcal{T}^H$, and V^H be a subspace of $H^1(\Omega)$ consisting of piecewise polynomials defined on $\mathcal{T}^H$. We note that V^H is not necessarily piecewise linear as V^h; for example, it may be bilinear (2-D) and trilinear (3-D) elements or higher order elements. Thus we do not necessarily have the usual condition: $V^H \subset V^h$.

To overcome the difficulty that $V^H \not\subset V^h$, we introduce an operator $I_h : V^H \to V^h$ so that $I_h V^H$ is a subspace of V^h. We will use I_h to define the global coarse space. Certainly, for the coarse space to be effective, I_h must possess the properties of H^1-stability and L^2 optimal approximation, see Mandel [16] and Lemma 4.1 and Lemma 5.1 below. For this purpose we will introduce three such options which are all effective in both two and three dimensions. The simplest and most natural one among them is $I_h = \Pi_h$, the piecewise linear interpolation operator related to V^h, as in Cai and Saad [5], Cai [4], Chan and Smith [8]. We will discuss this algorithm in Section 3. The second option is $I_h = Q_h$, the L^2 projection onto V^h, see Bramble and Xu [2]. It meets all the requirements mentioned above, but requires the triangulation $\mathcal{T}^h$ be quasi-uniform. Another disadvantage of Q_h is its global nature, but this can be overcome by using its numerical counterpart, i.e., the so-called L^2 quasi-projection, see Xu [20]. We will discuss this second option in Section 4.

In order to consider general unstructured meshes, we would like to relax as many as possible the restrictions on the coarse subspaces, on the fine subspaces, and in particular, on the substructures $\{\Omega_i\}$. In Section 5, we try to achieve these aims by introducing a special locally defined projection operator $\mathcal{R}_h$ which meets all our requirments mentioned previously and makes the forming of the new coarse subspace quite parallellizable. Although this operator $\mathcal{R}_h$ is slightly more complicated to calculate than Π_h, it may be more stable and behavior better than Π_h in practical computations since $\mathcal{R}_h$ is a kind of averaging. Another reason for us to introduce this local operator $\mathcal{R}_h$ is that its coarse subspace form $\mathcal{R}_H$ can be used to remove the requirement of the quasi-uniformity on the coarse triangulation $\mathcal{T}^H$. This is very important for the development of domain decomosition methods on unstructured meshes because they often are highly non-quasi-uniform, especially for modelling complex geometries and solution behaviour. Our theory is quite similar to that developed independently by Cai [4]. A major difference is that, by making use of the coarse space local projection operator $\mathcal{R}_H$, our results cover the case of non-quasi-uniform meshes.

Throughout the paper, we use $||\cdot||_m$ and $|\cdot|_m$ to denote the norm and semi-norm of the usual Sobolev space $H^m(\Omega)$ for any integer $m \geq 0$. We denote the scalar product in $L^2(\Omega)$ by $(\cdot,\cdot)$. C will denote the generic constant independent of all mesh parameters.

3. Global coarse subspace based on interpolation. In this section we discuss briefly the interpolated coarse subspace used by Cai [4], Chan and Smith [8]. We use the following decomposition for the finite element space V^h:

$$V^h = V_0^h + V_1^h + \cdots + V_p^h \tag{10}$$

where V_i^h, $i = 1, 2, \cdots, p$ are defined by (9), but the global coarse subspace V_0^h is defined as the range of the interpolation operator Π_h on the non-nested coarse subspace V^H, that is,

$$V_0^h = \Pi_h V^H \equiv \{v \in V^h;\ \exists\, w \in V^H \text{ such that } v = \Pi_h w\}. \tag{11}$$

Then we have the following decomposition lemma

LEMMA 3.1. *Suppose $\Omega \subset R^d (d = 2, 3)$, $\mathcal{T}^h$ is shape regular, and the triangulation $\mathcal{T}^H$ related to V^H is quasi-uniform. Then for any $v \in V^h$, there exist elements $v_i \in V_i^h$, $i = 1, 2, \cdots, p$, $v_H \in V^H$ such that*

$$v = \Pi_h v_H + v_1 + \cdots + v_p \tag{12}$$

and

$$||\Pi_h v_H||_1^2 + \sum_{i=1}^{p} ||v_i||_1^2 \le C\left(1 + \frac{H}{\delta}\right)^2 ||v||_1^2, \ \forall\, v \in V^h. \tag{13}$$

Proof. For any $v \in V^h$, choose $v_H = Q_H v \in V^H$, $v_0 = \Pi_h v_H \in V_0^h$ and $v_i = \Pi_h(\theta_i v - \theta_i v_0) \in V_i^h$, where Q_H is the L^2 projection onto V^H and $\{\theta_i\}$ the partition of unity for Ω related to the covering $\{\Omega_i'\}$. Then Lemma 3.1 can be proved in the standard way by using the H^1-stability of Π_h and its L^2 optimal approximation on the coarse space V^H, see Section 5 below for the more general case. □

With the decomposition defined in (10) it is straightforward to derive that the corresponding additive Schwarz algorithm has a condition number of the order $(1 + H/\delta)^2$.

Lemma 3.1 was first established for the 2D case by Cai [4]. The key point in the proof of Lemma 3.1 is that the interpolation operator Π_h has the H^1-stability and L^2 optimal approximation in both two and three dimensions:

$$||u - \Pi_h u||_s \le C\, \bar{h}^{1-s} |u|_1, \ \forall\, u \in V^H, s = 0, 1. \tag{14}$$

One way to prove (14) is to use the standard finite element interpolation results and the inverse inequality as well as the fact that $V^H \subset H^{1+\beta}(\Omega)$ for some $\beta \in [0, 1/2)$(see Xu [20]) and $H^{1+\beta}(\Omega)$ is imbedded in $C(\overline{\Omega})$ continuously in two dimensions. However, this last imbedding relation is not true in three dimensions and therefore we cannot use this approach to prove (14) in that case. Moreover, the non-nestedness of the coarse mesh $\mathcal{T}^H$ to the fine mesh $\mathcal{T}^h$ and the non-quasi-uniformity of $\mathcal{T}^h$ make it not so straightforward to prove (14). The 3D result of (14) was recently proved independently by Cai in his revision of [4] in 1994 and by us in Chan, Smith and Zou [6].

One limitation of Lemma 3.1 is the necessity to assume the quasi-uniformity of the coarse finite element space V^H due to the use of Q_H in the proof which requires the quasi-uniformity of $\mathcal{T}^H$ for its H^1-stability and L^2 optimal approximation. It would be desirable to remove this restriction because one of the main advantages of unstructured meshes is that it can be highly non-quasi-uniform. Actually, one can find many ways to remove this limitation by using another local operator $\mathcal{R}_H$ with the appropriate stability and approximation properties to replace Q_H in the proof. In the later section, we will introduce one of such operators.

4. Global coarse subspace based on the L^2 quasi-projection. In this and the next sections, we construct the global coarse space V_0^h by using some special projection operators. First, we define the subspaces V_i^h, $i = 1, 2, \cdots, p$ as in (9), but the coarse grid subspace is now defined by:

$$V_0^h = \tilde{Q}_h V^H \tag{15}$$

where $\tilde{Q}_h$ is the L^2 quasi-projection onto V^h defined by (see also Xu [20]):

$$(\tilde{Q}_h w, \phi)_h = (w, \phi), \ \forall \ w \in L^2(\Omega), \phi \in V^h \tag{16}$$

with $(\cdot, \cdot)_h$ defined as follows

$$(u, v)_h = \frac{1}{d+1} \sum_{\tau \in \mathcal{T}^h} |\tau| \sum_{q_i \in N_h \cap \tau} (u\, v)(q_i). \tag{17}$$

Here $\{N_h\}$ is the set of nodal points of $\mathcal{T}^h$.

REMARK 4.1. *The forming of the coarse subspace V_0^h defined by (15) is simple, since V_0^h =span$\{\tilde{Q}_h \phi_i^H\}$ with $\{\phi_i^H\}$ being the basis functions of V^H. Each $\{\tilde{Q}_h \phi_i^H\}$ can be calculated directly, as the corresponding coefficient matrix for $\tilde{Q}_h$ is diagonal.*

For the L^2 quasi-projection operator $\tilde{Q}_h$ we have the following H^1-stability and L^2 optimal approximation properties, see Lemma 3.6 in Xu [20] for $H_0^1(\Omega)$ whose proof is also applicable to the present case.

LEMMA 4.1. *Suppose $\mathcal{T}^h$ is quasi-uniform, then for any $v \in H^1(\Omega)$ and $s \in [0, 1]$*

$$||\tilde{Q}_h v||_s \leq C\, ||v||_s, \tag{18}$$

$$||v - \tilde{Q}_h v||_s \leq C\, h^{1-s} ||v||_1. \tag{19}$$

Now we have the following partition of V^h from the next lemma:

$$V^h = V_0^h + V_1^h + \cdots + V_p^h. \tag{20}$$

LEMMA 4.2. *Suppose $\Omega \subset R^d\, (d = 2, 3)$, $\mathcal{T}^h$ is quasi-uniform, and the triangulation $\mathcal{T}^H$ related to V^H is shape regular. Then for any $v \in V^h$, there exist elements $v_i \in V_i^h$, $i = 1, 2, \cdots, p$, $v_0 \in V_0^h$ such that*

$$v = v_0 + v_1 + \cdots + v_p \tag{21}$$

and

$$\|v_0\|_1^2+\sum_{i=1}^{p}\|v_i\|_1^2 \le C\left(1+\frac{H}{\delta}\right)^2\|v\|_1^2,\ \forall\ v\in V^h. \tag{22}$$

Proof. For any $v\in V^h$, choose $v_H=\mathcal{R}_H v\in V^H$, $v_0=\tilde{Q}_h v_H\in V_0^h$ and $v_i=\Pi_h(\theta_i v-\theta_i v_0)\in V_i^h$, where $\mathcal{R}_H$ is a locally defined projection operator onto V^H that will be introduced in next section and $\{\theta_i\}$ is the partition of unity for Ω related to the covering $\{\Omega_i'\}$. Then Lemma 4.2 can be proved in the standard way by using Lemma 4.1, see Dryja and Widlund [14] or Section 5 below for more general case. □

With the decomposition defined in (20), it is immediate to derive that the corresponding additive Schwarz method has a condition number of the order $(1+H/\delta))^2$, see Section 6 for more details.

5. Global coarse subspace based on locally defined projection operator. Now we introduce a special locally defined Clément projection operator $\mathcal{R}_h$ [11] which, to our knowledge, was first used in the domain decomposition context by Chen and Zou [9]. This operator $\mathcal{R}_h$ can be used not only in the algorithms, but more importantly, its coarse space form $\mathcal{R}_H$ can also be used in the convergence proof of the algorithms to remove the requirement of the quasi-uniformity on the coarse triangulation $\mathcal{T}^H$. Although the removement seems quite straightforward by replacing the usual L^2 projection operator Q_H in the convergence proof of the additive Schwarz methods, we think the recognition of this point is a very important step in the development of domain decomposition methods on unstructured meshes. Of course, one can find many other local operators which have the same properties as $\mathcal{R}_h$ or $\mathcal{R}_H$.

Let $\{q_i^h\}_{i=1}^n$ and $\{q_i^H\}_{i=1}^m$ be the nodal points of the triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$, respectively; and $\{\phi_i^h\}_{i=1}^n$ and $\{\phi_i^H\}_{i=1}^m$ be the sets of nodal basis functions of V^h and V^H, respectively. Denote $O_i^h=\text{supp}\,\phi_i^h$, $1\le i\le n$ and $O_i^H=\text{supp}\,\phi_i^H$, $1\le i\le m$.

DEFINITION 5.1. *The mapping $\mathcal{R}_h: L^2(\Omega)\to V^h$ is defined by*

$$\mathcal{R}_h u=\sum_{i=1}^{N} Q_i\, u(q_i)\,\phi_i^h,\ \forall\ u\in L^2(\Omega), \tag{23}$$

where $Q_i u\in\mathcal{P}_1(O_i)$ satisfies

$$\int_{O_i} Q_i u\, p\, dx=\int_{O_i} u\, p\, dx,\ \forall\ p\in\mathcal{P}_1(O_i) \tag{24}$$

where $\mathcal{P}_1(O_i)$ is the space of linear functions defined on O_i.

Analogously, we define $\mathcal{R}_H: L^2(\Omega)\to V^H$ the same as $\mathcal{R}_h$ by replacing V^h by V^H, and the nodal points and basis functions of V^h by the corresponding ones of V^H.

By using Poincaré's inequality, the definitions of $\mathcal{R}_h$ and $\mathcal{R}_h$, the relations (6) and (7), we can show the following results, see also Clément [11].

LEMMA 5.1. *Assume that the triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular, then the operator $\mathcal{R}_h$ and $\mathcal{R}_H$ defined above have the properties*

$$\|\mathcal{R}_h u\|_r\le C\|u\|_r,\quad \|\mathcal{R}_H u\|_r\le C\|u\|_r,\ \forall\ u\in H^r(\Omega), r=0,1, \tag{25}$$

$$\|u-\mathcal{R}_h u\|_0\le C\,h|u|_1,\quad \|u-\mathcal{R}_H u\|_0\le C\,H|u|_1,\ \forall\ u\in H^1(\Omega). \tag{26}$$

REMARK 5.1. *In Lemma 5.1, we assume only that the triangulations are shape regular, not necessarily quasi-uniform, unlike the usual L^2 projection. The main difference between $\mathcal{R}_H$ and $\tilde{Q}_H$ of the last section is that the former is defined locally, but the latter globally.*

In the remainder of this section we construct a special coarse subspace which plays an important role in our unstructured domain decomposition theory.

We define, the same as in (9), the subspaces V_i^h, $i = 1, 2, \cdots, p$, but the global coarse subspace is defined by

$$V_0^h = \mathcal{R}_h V^H. \tag{27}$$

Now we have the following lemma for the decomposition of the fine subspace V^h:

$$V^h = V_0^h + V_1^h + \cdots + V_p^h. \tag{28}$$

LEMMA 5.2. *Let $\Omega \subset R^d (d = 2, 3)$. We assume that both triangulations $\mathcal{T}^h$ corresponding to the fine space V^h and $\mathcal{T}^H$ corresponding to the coarse space V^H are shape regular, but not necessarily quasi-uniform. Then for any $u \in V^h$, there exist $u_i \in V_i^h$, $i = 0, 1, \cdots, p$ such that*

$$u = u_0 + u_1 + \cdots + u_p \tag{29}$$

and

$$\sum_{i=0}^{p} ||u_i||_1^2 \le C\,(1 + \frac{H}{\delta})^2 ||u||_1^2, \ \forall\ u \in V^h. \tag{30}$$

Proof. It is well-known, see Dryja and Widlund [13], Bramble et al. [3] that there exists a partition $\{\theta_i\}_{i=1}^p$ of unity for Ω related to the subdomains $\{\Omega_i'\}$ such that $\sum_{i=1}^p \theta_i(x) = 1$ on Ω and for $i = 1, 2, \cdots, p$,

$$\operatorname{supp} \theta_i \subset \Omega_i' \cup \partial\Omega,\ 0 \le \theta_i \le 1 \text{ and } ||\nabla \theta_i||_{L^\infty(\Omega_i)} \le C\,\delta_i^{-1}. \tag{31}$$

Now for any $u \in V^h$, let $u_0 = \mathcal{R}_h \mathcal{R}_H\, u \in V^h$ and $u_i = \Pi_h(\theta_i\, u - \theta_i\, u_0)$ with Π_h being the standard interpolation of V^h. Obviously, $u_i \in V_i^h$ and

$$u = u_0 + u_1 + \cdots + u_p. \tag{32}$$

Now we prove (30). By Lemma 5.1 we see that

$$||u_0||_1 = ||\mathcal{R}_h \mathcal{R}_H u||_1 \le C\,||\mathcal{R}_H u||_1 \le C\,||u||_1\,, \tag{33}$$

and

$$\begin{aligned} ||u - u_0||_0 &\le ||u - \mathcal{R}_H u||_0 + ||\mathcal{R}_H u - \mathcal{R}_h \mathcal{R}_H u||_0 \\ &\le C\,H|u|_1 + C\,h|\mathcal{R}_H u|_1 \le C\,H|u|_1, \qquad (34)\\ |u - u_0|_1 &\le |u|_1 + |\mathcal{R}_h \mathcal{R}_H u|_1 \le C\,|u|_1. \qquad (35)\end{aligned}$$

Then (30) follows in the standard way, see Dryja and Widlund [13] and Smith [18]. But we still give a complete proof here so that one can see clearly that no quasi-uniformity assumption on $\mathcal{T}^h$ and the subdomains $\{\Omega_i\}$ are required in the present case. Let τ be any element belonging to Ω'_k with h_τ being its diameter and $\overline{\theta}_k$ the average of θ_k on element τ. Then from (31) and the fact that $u - u_0 \in V^h$, we get:

$$\begin{aligned} |u_k|^2_{1,\tau} &\leq 2|\overline{\theta}_k \Pi_h (u-u_0)|^2_{1,\tau} + 2|\Pi_h(\theta_k - \overline{\theta}_k)(u-u_0)|^2_{1,\tau} \\ &\leq 2|u-u_0|^2_{1,\tau} + 2|\Pi_h(\theta_k - \overline{\theta}_k)(u-u_0)|^2_{1,\tau}. \end{aligned}$$

By using the local inverse inequality which requires only the shape regularity of $\mathcal{T}^h$ (see Proposition 3.2 in Xu [20]), we obtain:

$$\begin{aligned} |u_k|^2_{1,\tau} &\leq 2|u-u_0|^2_{1,\tau} + C\, h_\tau^{-2} ||\Pi_h(\theta_k - \overline{\theta}_k)(u-u_0)||^2_{0,\tau} \\ &\leq 2|u-u_0|^2_{1,\tau} + C\, h_\tau^{-2} \frac{h_\tau^2}{\delta_k^2} ||u-u_0||^2_{0,\tau} \\ &\leq 2|u-u_0|^2_{1,\tau} + C\, \frac{1}{\delta_k^2} ||u-u_0||^2_{0,\tau}. \end{aligned}$$

By taking the sum over $\tau \in \Omega'_k$ we have

$$|u_k|^2_{1,\Omega'_k} \leq 2|u-u_0|^2_{1,\Omega'_k} + C\, \frac{1}{\delta_k^2} ||u-u_0||^2_{0,\Omega'_k}. \tag{36}$$

Noticing the assumption made previously that any point $x \in \Omega$ belongs only to a finite number of subdomains $\{\Omega'_i\}$, it follows from (33)–(36) that

$$\sum_{k=1}^{p} |u_k|^2_{1,\Omega'_k} \leq C\,(|u-u_0|^2_1 + \frac{1}{\delta^2}||u-u_0||^2_0) \tag{37}$$

$$\leq C\,\Big(1 + \frac{H}{\delta}\Big)^2 |u|^2_1. \tag{38}$$

Analogously, we derive that

$$\sum_{k=1}^{p} ||u_k||^2_{0,\Omega'_k} \leq C\,\Big(1 + \frac{h^2}{\delta^2}\Big)||u||^2_0,$$

which completes the proof of (30). □

Note that the bound H^2/δ^2 appearing in Lemma 5.2 is not sharp compared to the results in the structured case, see Dryja and Widlund [15]. In fact, we can also improve this bound to make it as optimal as in the structured case by simply using one estimation by Dryja and Widlund [15]. To reach this aim, we suppose that $\delta_i = \delta$, $i = 1, 2, \cdots, p$, for simplicity, and let $\Gamma_{\delta,i} \subset \Omega_i$ be the set of points which are within a distance δ of $\partial\Omega_i$, H_i be the diameter of Ω_i.

LEMMA 5.3. *[15] Suppose that the substructures $\{\Omega_i\}$ is shape regular, let H_i be the diameter of Ω_i. Then for any $u \in H^1(\Omega_i)$, we have*

$$||u||^2_{0,\Gamma_{\delta,i}} \leq C\,\delta^2((1 + \frac{H_i}{\delta})|u|^2_{1,\Omega_i} + \frac{1}{H_i\delta}||u||^2_{0,\Omega_i}). \tag{39}$$

LEMMA 5.4. *In addition to the assumptions of Lemma 5.2, we assume that the subdomains* $\{\Omega_i\}_{i=1}^p$ *are quasi-uniform. Then for any* $u \in V^h$, *there exist* $u_i \in V_i^h$, $i = 0, 1, \cdots, p$ *such that*

$$u = u_0 + u_1 + \cdots + u_p \tag{40}$$

and

$$\sum_{i=1}^{p} \|u_i\|_1^2 \le C\left(1 + \frac{H_{sub}}{\delta} + \frac{H^2}{\delta\, H_{sub}}\right)\|u\|_1^2, \ \forall\, u \in V^h. \tag{41}$$

Here H_{sub} *is the maximum diameter of all subdomains.*

Proof. We choose $u_i \in V_i^h$ and $\{\theta_i\}$ exactly the same as in the proof of Lemma 5.2, but in addition, for $\{\theta_i\}$, we make them satisfy (31) and $\theta_i \equiv 1$ in the interior part of Ω_i which does not belong to $\Gamma_{\delta,i}$. Then (41) follows by using Lemma 5.3, and the techniques used in the proofs of Lemma 5.2 above and of Theorem 3 in Dryja and Widlund [15]. □

REMARK 5.2. *We see* V_0^h *=span*$\{\mathcal{R}_h\phi_i^H\}$, *with* $\{\phi_i^H\}$ *being the basis functions of* V^H. *To get* $\mathcal{R}_h\phi_i^H$ *one needs to solve a* 3×3 *algebraic system of equations in two dimensions and a* 4×4 *algebraic system of equations in three dimensions at each node of* $\mathcal{T}^h$ *which belongs to the closure of the support of the basis function* ϕ_i^H. *One can easily find that there are many repeated calculations for getting* $\{\mathcal{R}_h\phi_i^H\}$ *which may be utilized to save a lot of the calculations. In particular, it can be shown that if* V^H *is a piecewise linear space defined on* $\mathcal{T}^H$, *then for all nodes the support sets of whose basis functions are in the interior part of one element of* $\mathcal{T}^H$, *the values of* $\mathcal{R}_h u, u \in V^H$ *at these nodes are the same as that of* u.

REMARK 5.3. *We can remove the need for solving the local* 3×3 *or* 4×4 *linear systems mentioned in the above remark. To do this, we replace* $\mathcal{P}_1(O_i)$ *in the defintion of* R_h *in (23) and (24) by* $V^h(O_i)$, *the restriction of* V^h *on* O_i, *and replace the left-hand side integral by its numerical counterpart like that used in (17). Then Lemma 5.1 still holds for this modified* R_h *and the coefficient matrices of all these corresponding small algebraic systems are diagonal ones, see Remark 4.1.*

6. Additive Schwarz algorithms. Based on the decomposition of the finite element space V^h given in Sections 3–5, now we derive the condition numbers of the corresponding additive Schwarz algorithms. Because of the similarity, we only consider the most general case from Section 5. The main theorem below states that the additive Schwarz algorithm corresponding to the decomposition of the finite element space V^h given in Section 5 has a condition number which is bounded by $(1 + H_{sub} + H^2/(\delta H_{sub}))$ for quasi-uniform substructures and by $(1 + H/\delta)^2$ for the arbitrary substructures.

From (28) we see that

$$V^h = V_0^h + V_1^h + \cdots + V_p^h \tag{42}$$

where $V_0^h = \mathcal{R}_h V^H$, and V_i^h, $i = 1, 2, \cdots, p$, are defined by (9). Now we define the H^1-projection operators $P_i : V^h \to V_i^h$, $i = 0, 1, \cdots, p$ such that for any $u \in V^h$, $P_i u \in V_i^h$ satisfies

$$a(P_i u,\, v_i) = a(u,\, v_i), \;\forall\; v_i \in V_i^h. \tag{43}$$

Then it is easy to check that the solution u^h of (8) is also the unique solution of the operator equation

$$P\,u \equiv \sum_{i=0}^{p} P_i\,u = g_h \equiv \sum_{i=0}^{p} g_i^h \tag{44}$$

where g_i^h, $i = 0, 1, \cdots, p$ satisfy

$$a(g_i^h,\, v_i) = (f,\, v_i), \;\forall\; v_i \in V_i^h. \tag{45}$$

The additive Schwarz algorithm is to use the conjugate gradient method to solve the operator equation (44).

Notice that for the above algorithm we have to form the stiffness matrix for the newly constructed coarse space $V_0^h = \mathcal{R}_h V^H$. However, if we would like to use the stiffness matrix corresponding to the original coarse space V^H, we can define the coarse operator P_0 in (43) in another way: First we define a projection operator P_H on the original coarse space V^H by

$$a(P_H u, v) = a(u, \mathcal{R}_h v), \;\forall\; u \in V^h, \quad v \in V^H \tag{46}$$

and then define $\tilde{P}_0 = \mathcal{R}_h P_H$: $V^h \to V_0^h$. Now it is the same as above to check that the solution of u^h of (27) is also the unique solution of the operator equation

$$\tilde{P}u \equiv (\tilde{P}_0 + P_1 + \cdots + P_p)u = \tilde{g}_h \tag{47}$$

where $\tilde{g}_h = \tilde{g}_0^h + g_1^h + \cdots + g_p^h$ with g_i^h, $i = 1, 2, \cdots, p$ defined by (45) but $\tilde{g}_0^h = \mathcal{R}_h g_H$ and g_H is defined as follows:

$$a(g_H, v) = (f, \mathcal{R}_h v), \;\forall\; v \in V^H. \tag{48}$$

For the condition numbers of the operator P and $\tilde{P}$, we have the following bounds

THEOREM 6.1. *Under the same assumptions as in Lemma 5.2, one has*

$$\kappa(P),\; \kappa(\tilde{P}) \le C\,(1 + \frac{H}{\delta})^2, \tag{49}$$

if the subdomains $\{\Omega_i\}_{i=1}^p$ are neither quasi-uniform nor shape regular; and

$$\kappa(P),\; \kappa(\tilde{P}) \le C\,(1 + \frac{H_{sub}}{\delta} + \frac{H^2}{\delta\, H_{sub}}), \tag{50}$$

if the subdomains $\{\Omega_i\}_{i=1}^p$ are quasi-uniform.

Proof. The estimate of the condition number $\kappa(P)$ is quite routine by using the estimates of (30) and (41) for the decomposition of the finite element space V^h, and the equivalence between the norm $||\cdot||_1$ and $||\cdot||_a = (a(\cdot,\cdot))^{1/2}$; see Dryja and Widlund [13], [15] and Xu [21].

Next we estimate the condition number of $\kappa(\tilde{P})$, our proof is similar to that used in Cai [4]. We prove only (49). It suffices to show that there exist two constants C_0 and C_1 independent of H, δ, h such that for any $u^h \in V^h$,

$$C_0\, a(\tilde{P}u^h, u^h) \leq a(u^h, u^h) \leq C_1 \left(1 + \frac{H}{\delta}\right)^2 a(\tilde{P}u, u). \tag{51}$$

First, from (46) we see that

$$a(P_H u^h, P_H u^h) = a(R_h P_H u^h, u^h), \tag{52}$$

thus by Cauchy-Schwarz's inequality and Lemma 5.1,

$$||P_H u^h||_a^2 \leq ||u^h||_a\, ||R_h P_H u^h||_a \leq C\, ||u^h||_a\, ||P_H u^h||_a, \tag{53}$$

i.e., $||P_H u^h||_a \leq C\, ||u^h||_a$, which leads to the following

$$a(\tilde{P}_0 u^h, u^h) = a(P_H u^h, P_H u^h) \leq C\, a(u^h, u^h), \tag{54}$$

But the standard arguments gives that

$$\sum_{i=1}^{p} a(P_i u^h, u^h) \leq C\, a(u^h, u^h), \tag{55}$$

therefore we have proved the first inequality in (51). For the second inequality, it follows from (52) that

$$\begin{aligned} a(\tilde{P}u^h, u^h) &= a(R_h P_H u^h, u^h) + \sum_{i=1}^{p} a(P_i u^h, u^h) \\ &= ||P_H u^h||_a^2 + \sum_{i=1}^{p} ||P_i u^h||_a^2. \end{aligned} \tag{56}$$

Using Lemma 5.2 for the partition of $u^h = \sum_{i=0}^{p} u_i^h$ with $u_i^h \in V_i^h$ and $u_0^h = R_h R_H u^h$, and the Cauchy-Schwarz's inequality and (56), we have:

$$\begin{aligned} a(u^h, u^h) &= a(u^h, \sum_{i=0}^{p} u_i^h) = a(u^h, R_h R_H u^h) + \sum_{i=1}^{p} a(u^h, u_i^h) \\ &= a(P_H u^h, R_H u^h) + \sum_{i=1}^{p} a(P_i u^h, u_i^h) \\ &\leq \left(\sum_{i=1}^{p} ||P_i u^h||_a^2 + ||P_H u^h||_a^2\right)^{1/2} \left(\sum_{i=1}^{p} ||u_i^h||_a^2 + ||R_H u^h||_a^2\right)^{1/2} \\ &\leq C \left(1 + \frac{H}{\delta}\right) a(\tilde{P}u^h, u^h)^{1/2}\, ||u^h||_a, \end{aligned}$$

which completes the proof of the second inequality in (51).

□

In the rest of this section, we shall give the matrix representations of the preconditioners induced by the two additive Schwarz algorithms defined above.

Let $\{\phi_i^h\}_{i=1}^n$ be the standard nodal basis functions of V^h, and $\{\phi_{i,j}^h\}_{j=1}^{n_i} \subset \{\phi_i^h\}_{i=1}^n$ be the nodal basis functions of V_i^h, $i = 1, 2, \cdots, p$. For each i, we define a matrix extension operator $R_i^\top$ as follows: For any $u_i^h \in V_i^h$, we denote by u_i the coefficient vector of u_i^h in the basis $\{\phi_{i,j}^h\}_{j=1}^{n_i}$, and we define that $R_i^\top u_i$ to be the coefficient vector of u_i^h in the basis $\{\phi_i^h\}_{i=1}^n$.

It is immediate to check that

$$A_i = R_i A\, R_i^\top \tag{57}$$

where A and A_i, $i = 1, 2, \cdots, p$ are the stiffness matrices corresponding to the fine subspace V^h and the subspaces V_i^h, $i = 1, 2, \cdots, p$. And from (43) it follows that for any $u^h \in V^h$, the coefficient vector of $P_i u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$ is

$$R_i^\top A_i^{-1} R_i A\, u \tag{58}$$

where u denotes the coefficient vector of u^h in the basis $\{\phi_i^h\}_{i=1}^n$.

Now let $\{\phi_i^H\}_{i=1}^m$ be the basis functions of V^H, then $\{\mathcal{R}_h \phi_i^H\}_{i=1}^m$ are the basis functions of $V_0^h = \mathcal{R}_h V^H$. We define a matrix extension operator $R_0^\top$ as follows: For any $u_0^h \in V_0^h$, we denote by u_0 the coefficient vector of u_0^h in the basis $\{\mathcal{R}_h \phi_i^H\}_{i=1}^m$, and we define $R_0^\top u_0$ to be the coefficient vector of u_0^h in the basis $\{\phi_i^h\}_{i=1}^n$.

It is straightforward to derive that

$$A_0 = R_0 A\, R_0^\top \tag{59}$$

where A_0 is the stiffness matrix corresponding to the subspace V_0^h. And it follows from (43) (with $i = 0$) that the coefficient vector of $P_0 u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$ is

$$R_0^\top A_0^{-1} R_0 A\, u. \tag{60}$$

Thus from (58) and (60) we deduce that the preconditioner M for the stiffness matrix A induced by the sum operator $P = \sum_{i=0}^p P_i$ is

$$M = R_0^\top A_0^{-1} R_0 + \sum_{i=1}^{p} R_i^\top A_i^{-1} R_i. \tag{61}$$

Next we derive the preconditioner $\tilde{M}$ for the stiffness matrix A induced by the sum operator $\tilde{P} = \tilde{P}_0 + P_1 + \cdots + P_p$. We first note that the coefficient vector of a function $v \in V^H$ in the basis $\{\phi_i^H\}_{i=1}^m$ is exactly the same as the one for the function $\mathcal{R}_h v$ in the basis $\{\mathcal{R}_h \phi_i^H\}_{i=1}^m$. So from (46) we find that the coefficient vector of $P_H u^h$ in the basis $\{\phi_i^H\}_{i=1}^m$ is

$$A_H^{-1} R_0 A\, u \tag{62}$$

where A_H is the stiffness matrix corresponding to the original coarse space V^H. Now using the previously given fact we know the coefficient vector of $\tilde{P}_0 u^h = \mathcal{R}_h P_H u^h \in V_0^h$ in the basis $\{\mathcal{R}_h \phi_i^H\}_{i=1}^m$ is also $A_H^{-1} R_0 A u$, therefore, by the definition of $R_0^\top$, $R_0^\top A_H^{-1} R_0 A u$ is the coefficient vector of $\tilde{P}_0 u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$.

From above we see the preconditioner $\tilde{M}$ for the stiffness matrix A induced by the sum operator $\tilde{P} = \tilde{P}_0 + P_1 + \cdots + P_p$ is

$$\tilde{M} = R_0^\top A_H^{-1} R_0 + \sum_{i=1}^{p} R_i^\top A_i^{-1} R_i. \tag{63}$$

REMARK 6.1. *From the matrix representations (61) and (63) for the preconditioners M and $\tilde{M}$, we see the major difference between algorithms (44) and (47) is in the global coarse problem solver. The former coarse problem (with A_0^{-1}) is conducted on the newly constructed coarse subspace V_0^h, but the latter (with A_H^{-1}) is conducted on the original coarse subspace V^H. Since V^H is not necessarily nested to V^h, A_H may not be expressed in terms of the stiffness matrix A as A_0 in (59).*

7. An improved result for structured mesh with shape regular subdomains. We add this section to show an improved result for structured mesh, by using the locally defined projection operator $\mathcal{R}_H$ introduced in Section 5, for the additive Schwarz algorithms with the global coarse subspace constructed directly by the shape regular subdomains. We assume now V^H is also a piecewise linear finite element space corresponding to the triangulation $\mathcal{T}^H = \{\Omega_i\}_{i=1}^p$ and that $\mathcal{T}^h$ is refined from $\mathcal{T}^H =$, thus $V^H \subset V^h$. Let H_i be the diameter of Ω_i, $i = 1, 2, \cdots, p$. The notations $\{\Omega_i'\}$, δ, δ_i and the operators $\mathcal{R}_H$ and the subspaces V_i^h $(i = 0, 1, \cdots, p)$ are defined as in Section 5. Here $V_0^h = V^H$. We note the best results for additive Schwarz methods in this case are that the condition numbers are bounded by $O(1 + H/\delta)$, see Dryja and Widlund[15]. Here H is the maximum diameter of all Ω_i, $i = 1, 2, \cdots, p$ and δ is the minimum overlap of all overlaps δ_i, $i = 1, 2, \cdots, p$. In this section, we will show a more refined result which indicates that the condition numbers of the additive Schwarz methods can be bounded by $O(1 + \max_{1 \le i \le p} H_i/\delta_i)$. That means the larger subdomains may have a larger overlap with its neighbors, and the smaller subdomains may have a smaller overlap. Certainly this property is very useful in practical applications. Our main result is stated in the following theorem:

THEOREM 7.1. *Suppose $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular, then for any $u \in V^h$, there exist elements $u_i \in V_i^h$, $i = 0, 1, \cdots, p$ such that*

$$u = u_0 + u_1 + \cdots + u_p \tag{64}$$

and

$$\sum_{i=0}^{p} \|u_i\|_1^2 \le C\left(1 + \max_{1 \le i \le p} \frac{H_i}{\delta_i}\right)\|u\|_1^2. \tag{65}$$

Thus the corresponding additive Schwarz algorithm (see Section 6) has a condition number bounded by $O(1 + \max_{1 \le i \le p} H_i/\delta_i)$.

Proof. For any $u \in V^h$, let $u_0 = \mathcal{R}_H u \in V^H \subset V^h$, where $\mathcal{R}_H$ is defined in Section 5. Let $u_i = \Pi_h(\theta_i u - \theta_i u_0)$ with $\{\theta_i\}$ the same as in the proof of Lemma 5.2. Thus it is easy to check that (64) holds true. To prove (65), we give an estimate for the error $u - \mathcal{R}_H u$: there exists a constant C, independent of $h, H_i, i = 1, 2, \cdots, p$ such that

$$||u - \mathcal{R}_H u||_{0,\Omega_i'} \leq C H_i |u|_{1,S_i} \tag{66}$$

$$|u - \mathcal{R}_H u|_{1,\Omega_i'} \leq C |u|_{1,S_i}. \tag{67}$$

Here $S_i = \cup_{q_j^H \in \overline{\Omega}_i} \operatorname{supp} \phi_j^H$ and $\{\phi_j^H\}$ are the basis of V^H related to the vertices $\{q_j^H\}$. The estimates (66) and (67) can be proved by using Poincaré's inequality, the regularity of $\mathcal{T}^H$, the relations (6) and (7), and in particular, the local definition of $\mathcal{R}_H$. Now (65) follows by combining (66) and (67) with the procedures used in proving Lemma 5.2 and Lemma 5.4, with only a small modification arising from (66) and (67). $\square$

8. Concluding remarks. In this paper, we proposed three additive Schwarz methods for elliptic problems in two and three dimensions on unstructured meshes. Our main results are: (1) to introduce a locally defined operator $\mathcal{R}_H$ to construct a new special global coarse problem by using it to transform the original coarse space and (2) to show that the same optimal condition numbers for structured meshes can be obtained for unstructured meshes as well. For our main results, we assume only that the fine mesh $\mathcal{T}^h$ and the coarse mesh $\mathcal{T}^H$ are both shape regular, but not necessarily quasi-uniform and $\mathcal{T}^H$ is allowed to be non-nested to $\mathcal{T}^h$. The subdomains may be quite arbitrarily shaped. The global coarse problems of the corresponding additive Schwarz algorithms involve only solving the original coarse problems, although the newly constructed global coarse spaces are changed. The same principle can be applied for obtaining similar results for multiplicative Schwarz methods and also for non-selfadjoint elliptic problems.

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