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UNIQUENESS AND REGULARITY OF WEAK SOLUTIONS OF THE NONLINEAR FOKKER-PLANCK EQUATION

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Abstract. We study the Cauchy problem for the nonlinear parabolic equation $u_t + f(u)_x = K(u)_{xx}$, under the assumption of mild degeneracy – $Q(u) := K'(u) > 0$, for all $u \neq 0$. Such degenerate equations fail to admit classical solutions and, therefore, weak solutions are sought. It is known that weak solutions to the Cauchy problem exist; however, their uniqueness remained in question. We prove that uniqueness and address also the question of regularity by showing that, for smooth data, $K(u(\cdot,t))$ is uniformly Lipschitz continuous for all $t > 0$.

The case of one-signed weak solutions is of special interest. Uniqueness, in that case, has already been shown before. As for the regularity of the weak solutions in that case, the best result so far was the Lipschitz continuity of $K(u(\cdot,t))$. We improve that result, for a subclass of equations, by showing that $Q(u(\cdot,t))$ is Lipschitz continuous. This subclass includes the well known porous media equation for which this regularity result is sharp.

We also prove uniqueness for the Cauchy problem for a larger family of equations of the form $u_t + f_x = K_{xx} + g$, where f, K and g are smooth functions of (x,t,u).

1. Introduction. We study the nonlinear parabolic equation

$$u_t + f(u)_x = K(u)_{xx} \quad , \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ , \tag{1.1}$$

subject to the Cauchy data

$$u(x,0) = u_0(x) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) , \tag{1.2}$$

where f and K are smooth functions and K is strictly monotonic increasing. This equation is usually called the nonlinear Fokker-Planck equation due to its resemblance to the Fokker-Planck equation of statistical mechanics.

It is well known [9] that if (1.1) is uniformly parabolic, namely, $Q(u) = K'(u) \geq \varepsilon > 0$, the Cauchy problem (1.1)–(1.2) admits a unique classical solution. We, on the other hand, are interested here in the degenerate case, where $Q(u)$ may vanish for some value of u, say at $u = 0$:

$$Q(u) > 0 \qquad \forall u \neq 0 \quad \textit{and} \quad Q(0) = 0 . \tag{1.3}$$

Such degenerate equations arise in the study of several diffusion-advection processes and the simplest example is the porous media equation,

$$u_t = (|u|^{m-1}u)_{xx} \quad , \quad m > 1 . \tag{1.4}$$

In the degenerate case classical solutions usually do not exist and weak solutions, in the sense of distributions, are sought:

DEFINITION 1.1. *A bounded function $u(x,t)$ is a weak solution of (1.1)–(1.2) if it satisfies the following equality for every test function $\phi \in C_0^\infty(\mathbb{R}^2)$:*

$$\iint_{\mathbb{R}\times\mathbb{R}^+} [u\phi_t + f(u)\phi_x + K(u)\phi_{xx}]dxdt = -\int_{\mathbb{R}} u_0\phi(\cdot,0)dx \quad . \tag{1.5}$$

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The questions which arise are:

- *Existence.* Does the Cauchy problem (1.1)–(1.3) have always a weak solution?
- *Uniqueness.* Are the weak solutions uniquely determined by the initial data?
- *Regularity.* What can be said about the regularity of those solutions?

These questions were addressed in numerous manuscripts. We refer the reader to [5] and [6] where references to many of these manuscripts and a brief description of their results are provided. We describe below only those which are of importance to our discussion.

Since, in most practical applications, u is nonnegative, a large part of the study of equation (1.1) concentrates on that case. We may summarize the most recent uniqueness and regularity results for this case as follows, [6, Theorems 1, 4 & 7]:

THEOREM 1.2. *(Gilding). If $f, K \in C[0,\infty) \cap C^{2+\alpha}(0,\infty)$, $\alpha > 0$, and u_0 is nonnegative, bounded and continuous, the Cauchy problem (1.1)-(1.3) admits a unique weak solution. In addition, $K(u)_x$ exists and is uniformly bounded in $\mathbb{R} \times [\tau, T]$ for any $0 < \tau < T$. Furthermore, if $K(u_0)_x$ is uniformly bounded in $\mathbb{R}$, $K(u)_x$ is uniformly bounded in $\mathbb{R} \times [0, T]$ for any $T > 0$.*

We proceed by describing those results which concern weak solutions with no sign restriction.

Volpert and Hudjaev [12] dealt with a larger class of equations,

$$u_t + \frac{\partial}{\partial x} f(x,t,u) = \frac{\partial^2}{\partial x^2} K(x,t,u) + g(x,t,u) \quad , \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \quad , \tag{1.6}$$

where

$$Q(x,t,u) := \frac{\partial}{\partial u} K(x,t,u) \geq 0 \quad , \tag{1.7}$$

and proved the existence of weak solutions to the corresponding Cauchy problem:

DEFINITION 1.3. *A bounded function $u(x,t)$ is a weak solution of (1.6)+(1.2) if it satisfies the following equality for every test function $\phi \in C_0^\infty(\mathbb{R}^2)$:*

$$\iint_{\mathbb{R}\times\mathbb{R}^+} [u\phi_t + f(x,t,u)\phi_x + K(x,t,u)\phi_{xx} + g(x,t,u)\phi] dx dt = -\int_{\mathbb{R}} u_0 \phi(\cdot, 0) dx \quad . \tag{1.8}$$

However, their assumption on the nature of degeneracy, (1.7), is weaker than ours, (1.3), and includes also the case of first order hyperbolic equations ($Q \equiv 0$). Under this relaxed assumption, the weak solutions of (1.6) are not uniquely determined by their initial data (1.2), (e.g. [8]). Hence, they considered a subclass of weak solutions (which, in the context of hyperbolic conservation laws, $Q = g \equiv 0$, are known as *entropy solutions*) by imposing further restrictions on the solution, and proved uniqueness for (1.6)+(1.2) in that subclass.

Brézis and Crandall [2] considered equation (1.1) with $f \equiv 0$, under the assumption of the same kind of degeneracy as in (1.7), $Q(u) = K'(u) \geq 0$, and proved that weak solutions are uniquely determined by their initial data, (1.2). Since the first order term, $f(u)_x$, was taken to be zero, no further restrictions (entropy conditions) needed to be imposed on the weak solutions in order to guarantee uniqueness.

The question which remained open was whether weak solutions of (1.1) (where a first order nonlinear term is present) are uniquely determined by their initial data, (1.2), under the assumption of mild degeneracy, (1.3).

An important result which we shall use in our analysis is due to DiBenedetto [4]: he dealt with a class of equations larger than (1.1) and (1.6) and proved that any weak solution is continuous.

Our objectives in this paper are as follows:

1. Sharpening the regularity result of Theorem 1.2 for nonnegative solutions, by imposing further restrictions on f and K. This is done in §2.

2. Proving uniqueness for (1.1)-(1.3) in the class of weak solutions, (1.5), with no sign restriction and obtaining for these solutions a similar regularity result to that of Theorem 1.2. This is the content of §3. We also give an example, due to Kamin and Vazquez [7], which demonstrates the sharpness of our regularity estimates, as well as the difference between the cases of one-signed and two-signed weak solutions.

In §4 we generalize the uniqueness and regularity results of the previous section to the more general class of equations (1.6), under the assumption of mild degeneracy (as in (1.3)),

$$Q(x,t,u) > 0 \qquad \forall (x,t,u) \in \mathbb{R} \times \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \ . \tag{1.9}$$

2. Improved regularity for one-signed weak solutions. Consider the porous media equation (1.4): according to Theorem 1.2, if u is a nonnegative solution of that equation then $u^m(\cdot,t)$ is Lipschitz continuous for $t > 0$. This implies $\frac{1}{m}$-Hölder continuity of $u(\cdot,t)$. However, we recall that, as Aronson showed in [1], $u(\cdot,t)$ is in fact Hölder continuous with (optimal) exponent $\min\{\frac{1}{m-1}, 1\}$. Therefore, the regularity result of Theorem 1.2 is not always sharp.

In this section we show that, by assuming several assumptions on f and K, also $Q(u(\cdot,t))$, like $K(u(\cdot,t))$, is Lipschitz continuous. In the case of the porous media equation, which satisfies those assumptions, it means that $u^{m-1}(\cdot,t)$ is Lipschitz continuous, which implies the Hölder continuity of $u(\cdot,t)$ with exponent $\min\{\frac{1}{m-1}, 1\}$, in agreement with the sharp results of [1].

We start with the following Lemma which we prove by using a well known technique due to Bernstein (e.g. [9]). We assume throughout the section that $f \in C^2$ and $Q \in C^3$ for $u > 0$. These assumptions may be somewhat relaxed by means of regularization.

LEMMA 2.1. *Let $u = u(x,t)$ be a smooth positive classical solution of (1.1) in $R = (a,b) \times (0,T]$. Assume that*

$$\lim_{u \to 0} \left| \frac{f''(u)}{Q'(u)} \right| < \infty \tag{2.1}$$

and that for all $u \in (0, M]$, $M = \max_{\overline{R}} u$,

$$Q'(u) > 0 \ , \tag{2.2}$$

$$-1 < \alpha \le G(u) := \left(\frac{Q(u)}{Q'(u)} \right)' \le \beta \ , \tag{2.3}$$

$$G'(u) \le 0 \ , \tag{2.4}$$

where α and β are some constants. Then for any proper subrectangle of R, $R^* = (a_1, b_1) \times (\tau, T]$,

$$|Q(u)_x| \le C \qquad in \ \ \overline{R^*} \ , \tag{2.5}$$

where the constant C depends on $f(\cdot), Q(\cdot), M, a_1 - a, b - b_1, \tau$ and is independent of the lower bound of u.

If, in addition, $M_0 = \max_{[a,b]} |Q(u_0)_x| < \infty$, then (2.5) holds for $R^* = (a_1, b_1) \times (0, T]$, where C depends on M_0 instead of τ.

Proof. We first make the change of variables $u \mapsto v = Q(u)$. Due to assumption (2.2), this transformation is invertible and $u = q(v)$, $q = Q^{-1}$. Equation (1.1) therefore translates to

$$v_t + f'(q(v)) v_x = \left(\frac{q''(v)}{q'(v)} v + 1 \right) v_x^2 + v v_{xx} \ . \tag{2.6}$$

For $0 \le r \le 1$ let $\psi(r) = Nr(3-r)/2$ where $N = Q(M)$. Note that when r increases from 0 to 1, ψ strictly increases from 0 to N. Hence, since $0 < v \le N$, the equation $v = \psi(w)$ defines a function $w = w(x,t)$ which takes values in the interval $(0,1]$ and is as smooth as v. Substituting $v = \psi(w)$ in (2.6) yields the following equation for w:

$$w_t + f'(q)w_x = \left(\frac{q''}{q'}\psi + 1\right)\psi' w_x^2 + \psi\frac{\psi''}{\psi'}w_x^2 + \psi w_{xx} \; . \tag{2.7}$$

Here, $q^{(i)} = q^{(i)}(\psi(w))$ and $\psi^{(i)} = \psi^{(i)}(w)$, $0 \le i \le 2$. Differentiating (2.7) with respect to x and multiplying by $p = w_x$, we arrive at

$$\frac{1}{2}(p^2)_t - \psi p p_{xx} = F_1 \cdot p^4 + F_2 \cdot p^2 p_x - (f''(q)q'\psi' p^3 + f'(q)pp_x) \; , \tag{2.8}$$

where

$$F_1 = (\psi')^2 \cdot \left[\left(\frac{q''}{q'}\right)'\psi + \frac{q''}{q'}\right] + \psi'' \cdot \left[2 + \frac{q''}{q'}\psi\right] + \psi \cdot \left(\frac{\psi''}{\psi'}\right)' \; , \tag{2.9}$$

and

$$F_2 = \psi' \cdot \left[3 + 2\frac{q''}{q'}\psi\right] + 2\psi\frac{\psi''}{\psi'} \; . \tag{2.10}$$

Let $\eta = \eta(x,t)$ be a $C^2(\overline{R})$ function such that $\eta = 1$ on $\overline{R^*}$, $\eta = 0$ in a neighborhood of $x = a$, $x = b$ and $t = 0$, and $0 \le \eta \le 1$. Set $z = \eta^2 p^2$ and let $(x_0, t_0) \in R$ be the point in R where z attains its maximal value. Since $z_x = 0$, $z_{xx} \le 0$ and $z_t \ge 0$ in that point, we conclude that

$$\eta p_x = -\eta_x p\Big|_{(x_0,t_0)} \; , \tag{2.11}$$

and (recall that $\psi \ge 0$)

$$\psi z_{xx} - z_t \le 0\Big|_{(x_0,t_0)} \; . \tag{2.12}$$

Substituting $z = \eta^2 p^2$ into (2.12) and rearranging, we get that

$$\eta^2\left\{\frac{1}{2}(p^2)_t - \psi p p_{xx}\right\} \ge \psi\eta^2 p_x^2 + 4\psi\eta\eta_x p p_x + \psi\eta_x^2 p^2 + \psi\eta\eta_{xx}p^2 - \eta\eta_t p^2 \; . \tag{2.13}$$

Since $|4\psi\eta\eta_x p p_x| \le \psi\eta^2 p_x^2 + 4\psi\eta_x^2 p^2$, (2.13) implies that

$$\eta^2\left\{\frac{1}{2}(p^2)_t - \psi p p_{xx}\right\} \ge -3\psi\eta_x^2 p^2 + \psi\eta\eta_{xx}p^2 - \eta\eta_t p^2 \; . \tag{2.14}$$

We may now conclude, in view of (2.8), (2.11) and (2.14), that the following inequality holds at (x_0, t_0):

$$-F_1\eta^2 p^4 \le \left\{3\psi\eta_x^2 - \psi\eta\eta_{xx} + \eta\eta_t\right\}p^2 - F_2 p^3 \eta\eta_x - f''(q)q'\psi' p^3\eta^2 + f'(q)\eta\eta_x p^2 \; . \tag{2.15}$$

We shall now estimate the coefficients of p in this inequality. We start with F_1 and F_2.

It can be easily verified that $\psi(w)$ satisfies

$$\frac{N}{2} \le \psi' \le \frac{3N}{2} \quad , \quad \psi'' = -N \quad , \quad \left|\frac{\psi''}{\psi'}\right| \le 2 \quad \textit{and} \quad \left(\frac{\psi''}{\psi'}\right)' < 0 \tag{2.16}$$

for $w \in [0,1]$. Therefore

$$\psi\left(\frac{\psi''}{\psi'}\right)' \le 0 \qquad \forall w \in [0,1] \; . \tag{2.17}$$

Furthermore, since $q'(v) = 1/Q'(u)$ and $q''(v)/q'(v) = -Q''(u)/Q'(u)^2$, we have that

$$2 + \frac{q''}{q'}\psi = 2 + \frac{q''(v)}{q'(v)}v = 2 - \frac{Q''(u)}{Q'(u)^2}Q(u) = 1 + G(u) \tag{2.18}$$

and

$$\left(\frac{q''}{q'}\right)'\psi + \frac{q''}{q'} = \left(\frac{q''(v)}{q'(v)}v\right)' = \frac{d}{dv}\left(-\frac{Q''(u)Q(u)}{Q'(u)^2}\right) = \frac{G'(u)}{Q'(u)} . \tag{2.19}$$

Hence, in view of (2.9), (2.16)–(2.19) and (2.2)–(2.4),

$$F_1 = (\psi')^2\frac{G'(u)}{Q'(u)} + \psi'' \cdot (1 + G(u)) + \psi\left(\frac{\psi''}{\psi'}\right)' \leq -N\hat{\alpha} \quad , \quad \hat{\alpha} = 1 + \alpha > 0 . \tag{2.20}$$

As for F_2, using (2.10), (2.3), (2.16) and (2.18), we get that

$$|F_2| \leq \psi' \cdot (1 + 2|G(u)|) + 2\psi\left|\frac{\psi''}{\psi'}\right| \leq \frac{3N}{2}(1 + 2\gamma) + 4N , \tag{2.21}$$

where $\gamma = \max|G(u)| = \max(|\alpha|, |\beta|)$.

In addition,

$$|f'(q)| = |f'(u)| \leq Const_{f,M} \quad , \quad 0 < u \leq M , \tag{2.22}$$

and, in view of (2.1) and (2.2),

$$|f''(q)q'| = \left|\frac{f''(u)}{Q'(u)}\right| \leq Const_{f,Q,M} \quad , \quad 0 < u \leq M . \tag{2.23}$$

Finally, we return to (2.15) and conclude by (2.20)–(2.23) that

$$\hat{\alpha}N\eta^2p^4 \leq C_1p^2 + \eta C_2|p|^3 \tag{2.24}$$

at (x_0, t_0), where C_1 and C_2 depend on $f, Q, M, a_1 - a, b - b_1$ and τ. Using the simple quadratic inequality

$$\frac{2\eta C_2}{\hat{\alpha}N}|p|^3 \leq \eta^2p^4 + \frac{C_2^2p^2}{\hat{\alpha}^2N^2} \quad ,$$

we conclude by (2.24) that

$$\eta^2p^4 \leq \frac{2C_1}{\hat{\alpha}N}p^2 + \frac{C_2^2p^2}{\hat{\alpha}^2N^2}$$

at (x_0, t_0), or

$$\max_{\overline{R}} z(x,t) \leq C_3 := \frac{2C_1}{\hat{\alpha}N} + \frac{C_2^2}{\hat{\alpha}^2N^2} .$$

Hence, $\max_{\overline{R^*}}|w_x| \leq C_3^{\frac{1}{2}}$, and since $v_x = \psi'(w)w_x$ and $|\psi'| \leq 3N/2$, we arrive at (2.5) with $C = 3NC_3^{\frac{1}{2}}/2$.

This proves the first assertion of the Lemma. In order to prove the second assertion we take $\eta = \eta(x)$ to be a $C_0^2[a,b]$-function such that $\eta = 1$ on $[a_1, b_1]$ and $0 \leq \eta \leq 1$, and proceed in the same manner. □

Since the local bound on $|Q(u)_x|$, given in Lemma 2.1, is independent of the lower bound of u, we may conclude the same for nonnegative weak solutions of (1.1)–(1.2) as well.

To this end, we first state and prove the following:

LEMMA 2.2. *Assume that Q satisfies assumption (1.3) and vanishes at least algebraicly fast at $u = 0$. Then there exists $\varepsilon > 0$ such that (2.3) holds for $u \in (0, \varepsilon]$, for some constants α, β.*

Proof. Assume that $Q(u)$ vanishes algebraicly fast at $u = 0$, i.e., $Q(u) = a_p u^p + r(u)$, where $a_p > 0$, $p > 0$ and $r(u) = o(u^p)$. Then, for $u \neq 0$,

$$G(u) = \left(\frac{Q(u)}{Q'(u)}\right)' = 1 - \frac{Q(u)Q''(u)}{Q'(u)^2} = 1 - \frac{p(p-1)u^{2p-2} + o(u^{2p-2})}{p^2u^{2p-2} + o(u^{2p-2})} .$$

Hence, $\lim_{u\to 0} G(u) = 1/p$ and, therefore, (2.3) holds near $u = 0$ with $\alpha = 0$ and any $\beta > 1/p$. If, on the other hand, $Q(u)$ vanishes faster than any algebraic power at $u = 0$, we may consider the sequence $Q_p(u) = u^p + Q(u)$. Q_p, Q'_p, Q''_p converge uniformly on $[0, \frac{1}{2}]$ to Q, Q', Q'', respectively, as $p \to \infty$. Since, according to our previous arguments, $G_p(u) = (Q_p(u)/Q'_p(u))' \to 1/p$ as $u \to 0$, we conclude that $\lim_{u\to 0} G(u) = 0$ and, therefore, (2.3) holds near $u = 0$ with any $\alpha \in (-1, 0)$ and $\beta > 0$. □

THEOREM 2.3. *Let $u = u(x,t)$ be the weak solution of (1.1)–(1.3), where $u_0(x)$ is bounded and nonnegative. Assume that condition (2.1) holds, Q vanishes at least algebraicly fast at $u = 0$ and there exists $\varepsilon > 0$ such that (2.2) and (2.4) are satisfied for all $u \in (0, \varepsilon]$. Then $Q(u(\cdot, t))$ is locally Lipschitz continuous for all $t > 0$.*

Proof. Since weak solutions are continuous, each point (x,t) in which $u(x,t) \neq 0$ has a neighborhood where $u \neq 0$ and hence $Q > 0$. In this neighborhood, equation (1.1) becomes uniformly parabolic and therefore satisfied by u in the classical sense. Hence, whenever $u \neq 0$, the classical derivative $Q(u)_x$ exists and $Q(u)$ is locally Lipschitz continuous.

We therefore restrict our attention to points (x,t), $t > 0$, where the parabolic equation degenerates, i.e. $u(x,t) = 0$.

The key idea in the proof is the observation that if we let $u_\delta(x,t)$ denote the (classical) solution of (1.1)–(1.3) with the uniformly positive Cauchy data $u_\delta(x,0) = u_0(x) + \delta$, $\delta > 0$, then $u_\delta(x,t)$ form a decreasing and lower bounded sequence of smooth functions,

$$u_{\delta_1}(x,t) > u_{\delta_2}(x,t) > 0 \qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+ \quad , \quad \delta_1 > \delta_2 > 0 .$$

This sequence therefore converges pointwise to a function $u(x,t)$ which is the weak solution of (1.1)–(1.3). Since this classical argument is a straightforward consequence of the maximum principle, we omit further details (consult [10]).

According to our assumption, there exists an $\varepsilon > 0$ such that (2.2) and (2.4) hold in the interval $(0, \varepsilon]$. In view of Lemma 2.2, we may assume that also (2.3) holds in that interval. Now, let (x_0, t_0), $t_0 > 0$, be a point where $u = 0$. Since u is continuous, there exists a rectangle $R \subset \mathbb{R} \times \mathbb{R}^+$, such that $(x_0, t_0) \in R$ and $\max_{\overline{R}} u \le \varepsilon/2$. Hence, thanks to the locally uniform convergence of u_δ to u, we conclude that $0 < u_\delta(x,t) \le \varepsilon$ in $\overline{R}$ for sufficiently small δ, say $\delta \le \delta_0$. Applying Lemma 2.1 to u_δ, we conclude that there exists a constant $C = C(\varepsilon)$, independent of δ, such that

$$\max_{\overline{R^*}} |Q(u_\delta)_x| \le C \qquad \forall \delta \le \delta_0$$

for any proper subrectangle $\overline{R^*} \subset R$. Letting $\delta \to 0$, we find that $Q(u(\cdot, t))$ is Lipschitz continuous in (x_0, t_0) with a local Lipschitz constant less than or equal to C. This concludes the proof. □

Remarks.

1. In practice, the coefficients $Q(u)$ are non-oscillatory. For such coefficients, condition (2.2) clearly holds for small $u > 0$, as a result of assumption (1.3).

2. Condition (2.4) narrows down the class of degenerate coefficients, (1.3), for which Theorem 2.3 applies. A wide subclass of degenerate coefficients which satisfy this condition is the following:

$$Q(u) = a_p u^p + a_q u^q + o(u^q) \quad , \quad a_p > 0 \; , \; a_q \ge 0 \; , \; q > p > 0 . \tag{2.25}$$

Every degenerate coefficient, (1.3), which vanishes algebraicly fast at $u = 0$, has a leading term as in (2.25). The restriction here is on the sign of the second most significant term, which has to be nonnegative. The coefficients associated with the porous media equation, (1.4), $Q(u) = mu^{m-1}$, belong to this subclass. We also note that since $f \equiv 0$ for that equation, condition (2.1) holds as well. Therefore, if u is a nonnegative solution of (1.4), $u(\cdot, t)^{m-1}$ is locally Lipschitz continuous.

3. Consider the subclass of degenerate coefficients which take the form

$$Q(u) = a_p u^p + r(u) \quad , \quad a_p > 0 \ , \ p > 0 \ ,$$

where

$$r(u) = o(u^p) \quad and \quad |r(u)| \leq O(u^{2p}) \ .$$

Although the coefficients in this subclass do not necessarily satisfy condition (2.4) near $u = 0$, a simple computation shows that $G'(u)/Q'(u)$ remains bounded for these coefficients when $u \to 0$. This proves sufficient in order to obtain the Lipschitz continuity of $Q(u(\cdot,t))$, as in Theorem 2.3. Note that the porous media equation belongs to this subclass as well.

4. Theorem 2.3 applies also for nonpositive weak solutions if we replace condition (2.4) by $G'(u) \geq 0$ for $u < 0$, $|u|$ sufficiently small.

In Theorem 2.3 we established *local* Lipschitz continuity for $Q(u(\cdot,t))$. In order to obtain a uniform estimate, $Q(u)$ must satisfy the conditions of Theorem 2.3 for all values of u and not only for small ones:

THEOREM 2.4. *Let $u = u(x,t)$ be the weak solution of (1.1)–(1.3), where $u_0(x)$ is bounded and nonnegative. Assume that condition (2.1) holds and that there exists $M_+ > M := \max u_0$ such that (2.2)–(2.4) are satisfied for all $u \in (0, M_+]$. Then $Q(u(\cdot,t))$ is uniformly Lipschitz continuous in any domain $\mathbb{R} \times [\tau, T]$, $0 < \tau < T$.*

If, in addition, $Q(u_0)$ is uniformly Lipschitz continuous, then $Q(u(\cdot,t))$ is uniformly Lipschitz continuous in $\mathbb{R} \times [0,T]$.

Remark. Since for the porous media equation $Q(u) = mu^{m-1}$ and $f \equiv 0$, it satisfies the conditions of Theorem 2.4 (note that $G(u) \equiv (m-1)^{-1}$ and $G'(u) \equiv 0$ in this case).

Proof. We consider the sequence of classical solutions u_δ, defined in the proof of Theorem 2.3, which converges to the weak solution, u, as δ tends to 0. The maximum principle implies that for $\delta < M_+ - M$, $\delta \leq u_\delta \leq M_+$. Therefore, according to Lemma 2.1, for these values of δ, $Q(u_\delta(\cdot,t))$ are uniformly Lipschitz continuous in $\mathbb{R} \times [\tau, T]$, $0 < \tau < T$ (or in $\mathbb{R} \times [0,T]$, under the further assumption), with a Lipschitz constant independent of δ. By letting δ go to 0 we obtain the uniform Lipschitz continuity of $Q(u(\cdot,t))$. □

3. Uniqueness and regularity for weak solutions with changing sign. Our first aim is to establish uniqueness of weak solutions to the Cauchy problem (1.1)–(1.3). We would like to note that thanks to the weak nature of the degeneracy, (1.3), the weak solutions are continuous [4] and therefore there is no need for entropy conditions (as in [12]) in order to guarantee uniqueness.

We begin with the following proposition.

PROPOSITION 3.1. *Let u be any weak solution of (1.1)–(1.3). Then $K(u)_x$ is continuous for $t > 0$.*

Proof. Let Ω denote the zero set of u, i.e.,

$$\Omega = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : u(x,t) = 0\} \ .$$

Thanks to the continuity of u, the complement set, Ω^c, is open and u satisfies equation (1.1) in the classical sense there. Hence, $K(u)_x$ is obviously continuous in Ω^c. Since $K(u)_x \equiv 0$ in the interior[1] of Ω, Ω^o, it remains to prove the continuity of $K(u)_x$ along the interface, $\partial\Omega$, where the equation changes its type from parabolic to hyperbolic.

Since u is continuous, this interface consists of an at most countable nowhere dense collection of points and continuous curves. We show below that $K(u)_x$ is continuous along each of the curves in $\partial\Omega$. As for the

[1] For any domain D, D^o denotes its interior.

single points which $\partial\Omega$ may contain – by imbedding each of them on a small curve and applying the same line of proof, it follows that $K(u)_x$ is continuous in those points as well.

Let Γ be a curve in $\partial\Omega$ and P be a point in the interior of Γ. Assume that there exists a disc, B, centered at P, such that $B \cap (\mathbb{R} \times \{0\}) = \emptyset$ and $B \cap \partial\Omega = B \cap \Gamma$. Therefore, Γ splits B into two components, B_1 and B_2, and u is smooth in their interior.

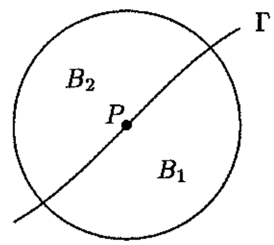

Figure 1

Let $\phi \in C_0^\infty(B^o)$, i.e.

$$\overline{supp\phi} \subset B^o \quad . \tag{3.1}$$

In light of (1.5),

$$0 = \iint_B (u\phi_t + f(u)\phi_x + K(u)\phi_{xx})dxdt = \tag{3.2}$$

$$\iint_{B_1} (u\phi_t + f(u)\phi_x + K(u)\phi_{xx})dxdt + \iint_{B_2} (u\phi_t + f(u)\phi_x + K(u)\phi_{xx})dxdt \quad .$$

Integration by parts yields, using (3.1), that

$$\iint_{B_i} K(u)\phi_{xx}dxdt = (-1)^i \int_\Gamma K(u)\phi_x dt - \iint_{B_i} K(u)_x\phi_x dxdt \qquad i = 1,2 \quad . \tag{3.3}$$

Equalities (3.2) and (3.3) imply that

$$\iint_{B_1} (u\phi_t + f(u)\phi_x - K(u)_x\phi_x)dxdt + \iint_{B_2} (u\phi_t + f(u)\phi_x - K(u)_x\phi_x)dxdt = 0 \quad . \tag{3.4}$$

Since in B_i^o u satisfies equation (1.1) in the classical sense, we get that

$$\iint_{B_i} (u\phi_t + f(u)\phi_x - K(u)_x\phi_x)dxdt = \iint_{B_i} [(u\phi)_t + (f(u)\phi)_x - (K(u)_x\phi)_x]dxdt \qquad i = 1,2 \quad . \tag{3.5}$$

Applying Green's Theorem we get, in view of (3.1), that

$$\iint_{B_i} [(u\phi)_t + (f(u)\phi)_x - (K(u)_x\phi)_x]dxdt = (-1)^i \int_\Gamma \phi \cdot \{-udx + (f(u) - K(u)_x)dt\} \qquad i = 1,2 \quad . \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we arrive at

$$\int_\Gamma \phi \cdot \{-[u]dx + [f(u)]dt - [K(u)_x]dt\} = 0 \quad ,$$

where $[\cdot]$ denotes the jump across Γ. But, as u is continuous, $[u] = [f(u)] = 0$ and we are left with

$$\int_\Gamma \phi \cdot [K(u)_x] dt = 0 \qquad \forall \phi \in C_0^\infty(B^o) \ .$$

By letting $supp\phi$ shrink to P we conclude that the $K(u)_x$ is continuous in P.

If P is an end point of Γ we may extend Γ beyond P (note that along any such extension $K(u)_x$ is continuous) and repeat the same arguments.

The proof in case that P lays on the intersection of two (or more) curves is modified slightly by decomposing B into four (or more) components.

It remains to deal with the case of accumulation points. Suppose that $P \in \Gamma \subset \partial\Omega$ is such that each of its neighborhoods intersects infinitely many degeneracy curves from $\partial\Omega$. Let B be a disc centered at P and $B \cap (\mathbb{R} \times \{0\}) = \emptyset$. Let B_1 and B_2 be its two components from either side of Γ and $\{\Gamma_n\}_{n\in\mathbb{N}}$ be the curves of $\partial\Omega$ which intersect B, where $\lim_{n\to\infty} \text{dist}(P, \Gamma_n) = 0$. For the sake of simplicity we assume that Γ_n lay on one side of Γ, say on the right. We then denote by B_1^n the component of B_1 which lays to the right of Γ_n so that $B_1 = \cup_{n\in\mathbb{N}} B_1^n$, see Figure 2.

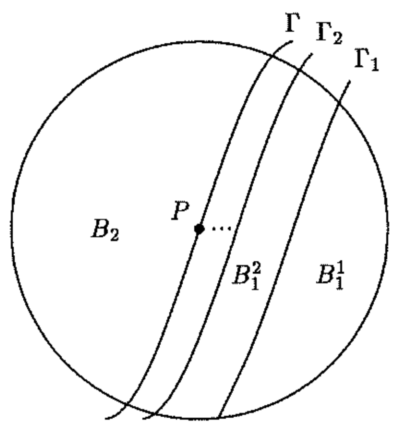

Figure 2

As in (3.4), we get here that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \iint_{B_1^n} (u\phi_t + f(u)\phi_x - K(u)_x\phi_x) dx dt + \iint_{B_2} (u\phi_t + f(u)\phi_x - K(u)_x\phi_x) dx dt = 0 \ .$$

We proceed, as before, by applying Green's Theorem in each component. Since along each isolated curve Γ_n we already showed that $K(u)_x$ is continuous, we conclude that

$$\int_\Gamma \phi \cdot K(u)_x^- dt - \lim_{N\to\infty} \int_{\Gamma_N} \phi \cdot K(u)_x^+ dt = 0 \qquad \forall \phi \in C_0^\infty(B^o) \ ,$$

where $K(u)_x^\pm$ denote the left and right limits. By letting $supp\phi$ shrink to P we get the that $K(u)_x$ is continuous in P. □

A consequence of Proposition 3.1 is that the solution operator of (1.1) is contractive in L_1. This is the content of the following theorem.

THEOREM 3.2. *(L_1-Contraction). Let u and v be two weak solutions of (1.1), (1.3). Then*

$$\|u(\cdot,t)-v(\cdot,t)\|_{L_1} \le \|u(\cdot,0)-v(\cdot,0)\|_{L_1} \quad , \quad t>0 \ . \tag{3.7}$$

Proof. As in [8], we divide the real line into intervals, $\mathbb{R} = \cup_n I_n(t)$, $I_n(t) = [x_n(t), x_{n+1}(t))$, so that

$$(-1)^n[u(\cdot,t)-v(\cdot,t)]\Big|_{I_n(t)} \ge 0 \tag{3.8}$$

and consequently, thanks to the continuity of u and v,

$$u(x_n(t),t) = v(x_n(t),t) \ . \tag{3.9}$$

Using (3.8) and (3.9) we conclude that

$$\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_{L_1(\mathbb{R})} = \tag{3.10}$$

$$= \frac{d}{dt}\sum_n (-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u(x,t)-v(x,t)]dx = \sum_n (-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u_t(x,t)-v_t(x,t)]dx \ .$$

We show below that all the terms in the last sum in (3.10) are nonpositive.

First, let us assume that neither $u(\cdot,t)$, nor $v(\cdot,t)$, vanish in the interior of the interval $I_n(t)$. Therefore, both u and v satisfy equation (1.1) there and we conclude that

$$(-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u_t(x,t)-v_t(x,t)]dx = \tag{3.11}$$

$$= (-1)^n\Big[-f(u)+f(v)\Big]_{x_n(t)}^{x_{n+1}(t)} + \Big[(-1)^n(K(u)-K(v))_x\Big]_{x_n(t)}^{x_{n+1}(t)} \ .$$

The first term on the right hand side of (3.11) vanishes in view of (3.9). Since, in light of (3.8), (3.9) and the monotonicity of $K(\cdot)$, $(-1)^n(K(u)-K(v))$ is nonnegative in $I_n(t)$ and vanishes in its end points, $x_n(t)$ and $x_{n+1}(t)$, we conclude that the second term on the right hand side of (3.11) is nonpositive. Therefore,

$$(-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u_t(x,t)-v_t(x,t)]dx \le 0 \tag{3.12}$$

in this case.

Next, we handle those intervals $I_n(t)$ where either u or v vanish. Let us assume, without loss of generality, that $v \ne 0$ along $I_n(t)$ while u vanish in one point $(\xi,t) \in I_n(t)$. Then

$$(-1)^n \int_{x_n(t)}^{x_{n+1}(t)} [u_t(x,t)-v_t(x,t)]dx = \tag{3.13}$$

$$= (-1)^n \left(\int_{x_n(t)}^{\xi} [u_t(x,t)-v_t(x,t)]dx + \int_{\xi}^{x_{n+1}(t)} [u_t(x,t)-v_t(x,t)]dx \right) =$$

$$= (-1)^n\Big[-f(u)+f(v)\Big]_{x_n(t)}^{\xi} + \Big[(-1)^n(K(u)-K(v))_x\Big]_{x_n(t)}^{\xi} +$$

$$+(-1)^n\Big[-f(u)+f(v)\Big]_{\xi}^{x_{n+1}(t)}+\Big[(-1)^n(K(u)-K(v))_x\Big]_{\xi}^{x_{n+1}(t)} .$$

In light of Proposition 3.1, $K(u)_x$ and $K(v)_x$ are continuous. Therefore, since also u and v are continuous, the contributions of the end point ξ in (3.13) add up to zero. Hence, (3.11) holds for such intervals as well. This implies that (3.12) holds always and, by (3.10),

$$\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_{L_1(\mathbb{R})}\leq 0 .$$

Namely, the solution operator of (1.1) is L_1-contractive. □

An immediate consequence of Theorem 3.2 is uniqueness:

COROLLARY 3.3. *(Uniqueness). The Cauchy problem (1.1)–(1.3) admits a unique weak solution.*

Next, we address the question of regularity. Proposition 3.1 implies that $K(u(\cdot,t))$, $t>0$, is locally Lipschitz continuous. We improve that local regularity result by showing that, whenever the initial data are smooth, $\{K(u(\cdot,t))\}_{t\geq 0}$ are uniformly Lipschitz equicontinuous in $\mathbb{R}$ (Theorem 3.5).

The main ingredient in proving the uniform Lipschitz continuity of $K(u(\cdot,t))$ is the following lemma, due to E. Tadmor [11].

LEMMA 3.4. *(Tadmor). Consider the uniformly parabolic equation,*

$$u_t+f(u)_x=K(u)_{xx} \quad , \quad Q(u)=K'(u)\geq\varepsilon>0 , \tag{3.14}$$

subject to the bounded initial data

$$u(x,0)=u_0(x) . \tag{3.15}$$

Then if $K(u_0)_x$ is uniformly bounded, there exists a constant C, independent of ε, such that

$$\|K(u)_x\|_{L_\infty(\mathbb{R}\times\mathbb{R}+)}\leq C . \tag{3.16}$$

Proof. We first recall that, thanks to the uniform parabolicity, (3.14)–(3.15) admits a unique classical solution. After differentiation of (3.14) with respect to t and integration with respect to x, we find that $w(x,t):=\int_{-\infty}^{x}u_t(\xi,t)d\xi$ satisfies

$$w_t+f'(u)w_x=(Q(u)w_x)_x \quad . \tag{3.17}$$

This is a uniformly parabolic linear equation in w and, therefore, by the maximum principle,

$$\|w\|_{L_\infty(\mathbb{R}\times\mathbb{R}+)}\leq\|w(\cdot,0)\|_{L_\infty(\mathbb{R})} . \tag{3.18}$$

But, since equation (3.14) and the definition of w imply that $w=K(u)_x-f(u)$, we conclude by (3.18) and the maximum principle for (3.14) that (3.16) holds with

$$C=2\max_{|u|\leq\|u_0\|_{L_\infty}}|f(u)|+\|K(u_0)_x\|_{L_\infty} . \tag{3.19}$$

□

Since estimate (3.16) is independent of ε, a similar estimate may be obtained in the degenerate case as well:

THEOREM 3.5. *(Regularity). Let u be the unique weak solution of (1.1)–(1.3), where $u_0\in W^{1,\infty}\cap BV$. Then there exists a constant C such that*

$$\|K(u)_x\|_{L_\infty(\mathbb{R}\times\mathbb{R}+)}\leq C . \tag{3.20}$$

Remarks.

1. This theorem generalizes Theorem 1.2 by removing the restriction on the sign of the solution and, furthermore, by obtaining a uniform bound, independent of t, for $|K(u)_x|$.

2. In the course of the proof of this theorem we also prove the existence of weak solutions to (1.1)–(1.2) and their L_1-stability with respect to uniformly parabolic perturbations. These proofs of existence and stability, which employ some classical arguments, are different from the ones provided in [12] and are interesting for their own sake.

Proof. Let u^ε be defined as the unique classical solution of the uniformly parabolic equation

$$u_t^\varepsilon + f(u^\varepsilon)_x = K^\varepsilon(u^\varepsilon)_{xx} \quad , \quad K^\varepsilon(u^\varepsilon) := K(u^\varepsilon) + \varepsilon u^\varepsilon \ , \tag{3.21}$$

subject to the same Cauchy data as u, i.e.,

$$u^\varepsilon(x,0) = u_0(x) \ . \tag{3.22}$$

The proof consists of three steps: In Step 1 we show that the family $\{u^\varepsilon(x,t)\}_{\varepsilon>0}$ is compact and has a subsequence which converges, as ε tends to 0, to a limit function $u(x,t)$. Afterwards, in Step 2, we show that $u(x,t)$ is a weak solution of (1.1)–(1.2) and therefore, in light of Corollary 3.3, the unique one. Finally, in Step 3, we prove that u satisfies (3.20) and conclude the proof.

Step 1. The solution operator of (3.21) is, in view of Theorem 3.2, L_1-contractive. Since it is also translation invariant, it follows that

$$\|u^\varepsilon(\cdot,t)\|_{BV} \le \|u_0\|_{BV} \qquad \forall t>0 \ .$$

Furthermore, the maximum principle implies that

$$\|u^\varepsilon(\cdot,t)\|_{L_\infty} \le \|u_0\|_{L_\infty} \qquad \forall t>0 \ . \tag{3.23}$$

Hence, by Helly's theorem, we may extract for each $t>0$, a subsequence, $u^{\varepsilon_i}(\cdot,t)$, which converges in L_1 on compact intervals, $I_R=[-R,R]$, to some function $u(\cdot,t)\in L_1(I_R)\cap BV(I_R)$.

Assume that $\{t^k\}_{k\in\mathbb{N}}$ is an ordering of the rational numbers in $[0,T]$. Then, by a successive application of the above argument for each t^k, we can find subsequences $\{u^{\varepsilon_i^k}\}_{i\in\mathbb{N}}$ and functions $u(\cdot,t^k)\in L_1(I_R)\cap BV(I_R)$ such that

$$\{\varepsilon_i^k\}_{i\in\mathbb{N}} \subset \{\varepsilon_i^{k-1}\}_{i\in\mathbb{N}} \qquad \forall k>1$$

and

$$\lim_{i\to\infty} \|u^{\varepsilon_i^k}(\cdot,t^k) - u(\cdot,t^k)\|_{L_1(I_R)} = 0 \qquad \forall k \ . \tag{3.24}$$

We claim that the diagonal sequence, $\{u^{\varepsilon_i^i}(x,t)\}_{i\in\mathbb{N}}$, is an $L_1(I_R)$-Cauchy sequence for all $t\in[0,T]$. In order to show that, we decompose the L_1-distance between two functions in this sequence as follows (t^k denotes some rational number in $[0,T]$):

$$\|u^{\varepsilon_i^i}(\cdot,t) - u^{\varepsilon_j^j}(\cdot,t)\|_{L_1(I_R)} \le$$

$$\le \|u^{\varepsilon_i^i}(\cdot,t) - u^{\varepsilon_i^i}(\cdot,t^k)\|_{L_1(I_R)} + \|u^{\varepsilon_i^i}(\cdot,t^k) - u^{\varepsilon_j^j}(\cdot,t^k)\|_{L_1(I_R)} + \|u^{\varepsilon_j^j}(\cdot,t^k) - u^{\varepsilon_j^j}(\cdot,t)\|_{L_1(I_R)} = T_1+T_2+T_3 \ .$$

Since the rational numbers are dense in $\mathbb{R}$ and since the solution operator of (3.21) is continuous in L_1, we can make T_1 and T_3 as small as we please by choosing t^k sufficiently close to t. Then, by (3.24), if we choose i and j sufficiently large, T_2 will become as small as we please. Therefore, $\{u^{\varepsilon_i^i}(\cdot,t)\}_{i\in\mathbb{N}}$ converges in $L_1(I_R)$ to a function $u(\cdot,t)$, for all $t\in[0,T]$. By letting R and T go to ∞ we conclude the following:

The sequence of classical solutions of (3.21)–(3.22), $u^\varepsilon(x,t)$, has a subsequence, still denoted $u^\varepsilon(x,t)$, which converges in $L_{1,loc}(\mathbb{R}_x)$, as $\varepsilon\downarrow 0$, to a function $u(x,t)$, for all $t>0$.

Step 2. Next, we show that the constructed limit, $u(x,t)$, is the weak solution of (1.1)–(1.2).

Since u^ε is the classical solution of (3.21)–(3.22), it is also a solution in the weak sense, (1.5). Therefore, for all $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}\times\mathbb{R}^+} [u^\varepsilon \phi_t + f(u^\varepsilon)\phi_x + (K(u^\varepsilon) + \varepsilon u^\varepsilon)\phi_{xx}]dxdt = -\int_{\mathbb{R}} u_0 \phi(\cdot, 0)dx \quad . \tag{3.25}$$

Since, by (3.23),

$$\left| \iint_{\mathbb{R}\times\mathbb{R}^+} \varepsilon u^\varepsilon \phi_{xx} dxdt \right| \leq \varepsilon \|u_0\|_{L_\infty} \|\phi_{xx}\|_{L_1(\mathbb{R}\times\mathbb{R}^+)} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad ,$$

and $u^\varepsilon(\cdot,t)$ converges in $L_{1,loc}$ to $u(\cdot,t)$, we obtain, letting ε go to zero in (3.25), that u satisfies (1.5) for all $\phi \in C_0^\infty(\mathbb{R}^2)$. Hence, $u(x,t)$ is the (unique) weak solution of (1.1)–(1.2).

Step 3. According to Lemma 3.4,

$$\|K^\varepsilon(u^\varepsilon)_x\|_{L_\infty(\mathbb{R}\times\mathbb{R}^+)} \leq C^\varepsilon \ ,$$

where

$$C^\varepsilon = 2 \max_{|u| \leq \|u_0\|_{L_\infty}} |f(u)| + \|K^\varepsilon(u_0)_x\|_{L_\infty} \ .$$

Since $u_0 \in W^{1,\infty}$,

$$\|K^\varepsilon(u_0)_x\|_{L_\infty} \leq \|K(u_0)_x\|_{L_\infty} + \varepsilon\|(u_0)_x\|_{L_\infty} \leq \|K(u_0)_x\|_{L_\infty} + |u_0|_{W^{1,\infty}} \qquad \forall \varepsilon \in (0,1] \ .$$

Therefore, for $\varepsilon \in (0,1]$,

$$\|K^\varepsilon(u^\varepsilon(\cdot,t))_x\|_{L_\infty(\mathbb{R})} \leq C \qquad \forall t \geq 0 \ , \tag{3.26}$$

where C is independent of ε and is given by

$$C = 2 \max_{|u| \leq \|u_0\|_{L_\infty}} |f(u)| + \|K(u_0)_x\|_{L_\infty} + |u_0|_{W^{1,\infty}} \ .$$

Inequality (3.26) may be written, equivalently, as follows:

$$\sup_{\phi\in\Phi} \left| \int_{\mathbb{R}} K^\varepsilon(u^\varepsilon)\phi_x dx \right| \leq C \qquad \forall t \geq 0 \ ,$$

where

$$\Phi = \{\phi \in C_0^\infty(\mathbb{R}) : \ \|\phi\|_{L_1} = 1\} \ .$$

But since

$$\left| \int_{\mathbb{R}} K^\varepsilon(u^\varepsilon)\phi_x dx - \int_{\mathbb{R}} K(u)\phi_x dx \right| \leq \left| \int_{\mathbb{R}} (K(u^\varepsilon) - K(u))\phi_x dx \right| + \left| \int_{\mathbb{R}} \varepsilon u^\varepsilon \phi_x dx \right| \leq$$

$$\leq \|\phi_x\|_{L_\infty} \cdot \max_{|u| \leq \|u_0\|_{L_\infty}} |Q(u)| \cdot \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L_1(Supp\phi)} + \varepsilon\|u_0\|_{L_\infty}\|\phi_x\|_{L_1} \underset{\varepsilon\to 0}{\longrightarrow} 0 \ ,$$

we conclude that

$$\sup_{\phi\in\Phi} \left| \int K(u)\phi_x dx \right| \leq C \qquad \forall t \geq 0 \ .$$

Therefore,

$$\|K(u(\cdot,t))_x\|_{L_\infty(\mathbb{R})} \le C \qquad \forall t \ge 0 \ ,$$

which proves (3.20). This concludes the proof. □

Example. Consider the porous media equation, (1.4), subject to a compactly supported initial data, $u(x,0) = u_0(x)$. Assume that

$$\int_{\mathbb{R}} u_0(x)dx = 0 \quad \textit{and} \quad P := -\int_{\mathbb{R}} xu_0(x)dx > 0 \quad .$$

In [7] it is shown that

$$t^{\frac{1}{m}}\|u(\cdot,t) - z(\cdot,t)\|_{L_\infty} \underset{t\to\infty}{\longrightarrow} 0 \quad ,$$

where $z(x,t)$ is the solution of (1.4) which takes a dipole as initial data, $z(x,0) = \delta'(x)$. This solution, which was published by Barenblatt and Zeldovich [3], is given by

$$z(x,t) = -dt^{-\frac{1}{m}}\xi^{\frac{1}{m}}(C - q|\xi|^{\frac{m+1}{m}})_+^{\frac{1}{m-1}} \quad , \quad (\cdot)_+ = \max(\cdot,0) \ ,$$

where $\xi = xt^{-\frac{1}{2m}}$ and d, C, q are some constants which depend on m and P.

Equation (1.4) degenerates, for this solution, at $x = 0$ (where z changes its sign) and at the tips of the compact support, $x = x_\pm(t) = \pm(C/q)^{\frac{m}{m+1}}t^{\frac{1}{2m}}$. Along $x = 0$, $z(x,t)$ is Hölder continuous with exponent $\frac{1}{m}$. This demonstrates the sharpness of our estimate that $K(z)_x = (|z|^{m-1}z)_x$ is bounded. Note that, on the other hand, along the interfaces $x_\pm(t)$ the solution is Hölder continuous with exponent $\min\{\frac{1}{m-1}, 1\}$. Hence, in the neighborhood of these interfaces, where the solution is one-signed, our estimate from §2 holds, namely, $Q(z)_x = (m|z|^{m-1})_x$ is locally bounded.

4. Uniqueness and regularity for a larger class of degenerate equations. In this section we would like to show how the results of §3, concerning equation (1.1), may be extended to the more general class of equations (1.6), where a source term is included and the coefficients depend explicitly on x and t.

Since the proofs are very similar to those of §3, we will only point out the modifications which are necessary for the treatment of this larger class of equations.

We begin with a proposition which is the analogous of Proposition 3.1. Since the proof of the later applies also here, with the proper straightforward modifications, we do not include a detailed proof of the present one.

PROPOSITION 4.1. *Let u be any weak solution of (1.6), (1.9). Then $\frac{\partial}{\partial x}K(x,t,u)$ is continuous for* $t > 0$.

Remark. We use the subscripts x and t to denote partial differentiation with respect to those variables and the notations $\partial/\partial x$ and $\partial/\partial t$ to denote complete differentiation, i.e.,

$$\frac{\partial}{\partial x}K(x,t,u) = K_x(x,t,u) + K_u(x,t,u)\cdot u_x \quad , \quad \frac{\partial}{\partial t}K(x,t,u) = K_t(x,t,u) + K_u(x,t,u)\cdot u_t \quad .$$

Proposition 4.1 implies the L_1-stability of the solution operator:

THEOREM 4.2. *(L_1-Stability). Let u and v be two weak solutions of (1.6), (1.9). Let $M_T^\pm$ be such that*

$$M_T^- \le u(x,t), v(x,t) \le M_T^+ \qquad \forall (x,t) \in \mathbb{R}\times[0,T] \ . \tag{4.1}$$

Then

$$(4.2) \qquad \|u(\cdot,t)-v(\cdot,t)\|_{L_1} \le e^{\gamma t}\|u(\cdot,0)-v(\cdot,0)\|_{L_1} \qquad \forall t\in[0,T] \ ,$$

where

$$(4.3) \qquad \gamma=\gamma(T):=\sup_{\mathbb{R}\times[0,T]\times[M_T^-,M_T^+]} g_u(x,t,u) \ .$$

Remark. The maximum principle does not hold in the presence of source terms. However, if the initial data are bounded, the solution will be uniformly bounded in any strip $\mathbb{R}\times[0,T]$ (with a uniform bound which depends on T).

Proof. The proof is similar to the one of Theorem 3.2. Here, instead of (3.11), the following equality holds:

$$(4.4) \qquad (-1)^n\int_{x_n(t)}^{x_{n+1}(t)}[u_t(x,t)-v_t(x,t)]dx=(-1)^n\Big[-f(x,t,u)+f(x,t,v)\Big]_{x_n(t)}^{x_{n+1}(t)}+$$

$$+\Big[(-1)^n\frac{\partial}{\partial x}\left(K(x,t,u)-K(x,t,v)\right)\Big]_{x_n(t)}^{x_{n+1}(t)}+(-1)^n\int_{x_n(t)}^{x_{n+1}(t)}[g(x,t,u)-g(x,t,v)]dx \ .$$

As before, the first term on the right hand side of (4.4) vanishes and the second one is nonpositive, due to the increasing monotonicity of $K(x,t,\cdot)$. As for the last term,

$$(4.5) \qquad (-1)^n\int_{x_n(t)}^{x_{n+1}(t)}[g(x,t,u)-g(x,t,v)]dx=\int_{x_n(t)}^{x_{n+1}(t)}g_u(x,t,w)\cdot|u-v|dx \ ,$$

where $w=w(x,t)$ is a mid-value between $u(x,t)$ and $v(x,t)$. Therefore, (4.4)–(4.5), together with (4.1) and (4.3), imply that

$$(4.6) \qquad (-1)^n\int_{x_n(t)}^{x_{n+1}(t)}[u_t(x,t)-v_t(x,t)]dx\le\gamma\int_{x_n(t)}^{x_{n+1}(t)}|u-v|dx \qquad \forall t\in[0,T] \ .$$

Finally, from (4.6) and (3.10) we get that

$$\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_{L_1(\mathbb{R})}\le\gamma\cdot\|u(\cdot,t)-v(\cdot,t)\|_{L_1(\mathbb{R})} \qquad \forall t\in[0,T] \ ,$$

which implies (4.2). □

COROLLARY 4.3. *(Uniqueness). The Cauchy problem for equation (1.6), (1.9) admits a unique weak solution.*

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