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UNIQUENESS OF WEAK SOLUTIONS OF THE CAUCHY PROBLEM FOR MULTIDIMENSIONAL DEGENERATE PARABOLIC EQUATIONS

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Abstract. We study the Cauchy problem for the multidimensional parabolic equation

$$\frac{\partial}{\partial t}u + \nabla \cdot f = \nabla \cdot (Q\nabla u) + g \quad , \quad (x,t) = (\vec{x},t) \in \mathbb{R}^N \times \mathbb{R}^+ \quad , \quad \nabla = \vec{\nabla} = \partial/\partial\vec{x} \; ,$$

$f = (f_i)_{i=1}^N$, $Q = (Q_{i,j})_{i,j=1}^N$ and g are smooth functions of (x,t,u), under the assumption of mild degeneracy: $Q > 0$ for all x, t and $u \neq 0$. Such degenerate equations, which arise in the study of several diffusion-advection processes (e.g., the porous media equation), fail to admit classical solutions and, therefore, weak solutions are sought. It is known that weak solutions to the Cauchy problem exist; however, their uniqueness remained in question. We prove that uniqueness by showing that the corresponding solution operator is L_1-stable.

1. Introduction. In a previous paper [4] we studied the nonlinear degenerate parabolic equation

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(x,t,u) = \frac{\partial^2}{\partial x^2}K(x,t,u) + g(x,t,u) \quad , \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \; , \tag{1.1}$$

where

$$Q(x,t,u) := \frac{\partial}{\partial u}K(x,t,u) > 0 \qquad \forall (x,t,u) \in \mathbb{R} \times \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \tag{1.2}$$

and proved the uniqueness of the weak solutions to the corresponding Cauchy problem, by showing that the solution operator of that equation is L_1-stable.

In the present paper we generalize these results to the multidimensional case. We are therefore studying weak solutions (to be defined later), $u = u(x,t)$, of the equation

$$\frac{\partial}{\partial t}u + \nabla \cdot f = \nabla \cdot (Q\nabla u) + g \quad , \quad (x,t) = (\vec{x},t) \in \mathbb{R}^N \times \mathbb{R}^+ \quad , \quad \nabla = \vec{\nabla} = \partial/\partial\vec{x} \; ; \tag{1.3}$$

here, f (the flux) denotes a vector field

$$f = \vec{f}(x,t,u) = (f_1(x,t,u), ..., f_N(x,t,u)) \; ,$$

$g = g(x,t,u)$ is a scalar source term and $Q = Q(x,t,u) = (Q_{i,j}(x,t,u))_{i,j=1}^N$ (the viscosity coefficient) is a symmetric matrix function which, in analogous to (1.2), is positive definite for all $u \neq 0$,

$$\xi^T Q(x,t,u)\xi > 0 \qquad \forall (x,t,u) \in \mathbb{R}^N \times \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \;\; and \;\; \xi \in \mathbb{R}^N \setminus \{0\} \; . \tag{1.4}$$

f_i $(1 \leq i \leq N)$, $Q_{i,j}$ $(1 \leq i,j \leq N)$ and g are assumed to be smooth functions of (x,t,u).

It is well known [2] that if equation (1.3) is uniformly parabolic, i.e., $Q(x,t,u)$ is uniformly positive definite for all (x,t,u), then the corresponding Cauchy problem admits a unique classical solution. We, on the other hand, are interested here in the degenerate case, (1.4), where $Q(x,t,u)$ may be singular for

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$u = 0$. Such degenerate equations arise in the study of several diffusion-advection processes and the simplest example is the porous media equation,

$$u_t = \Delta(|u|^{m-1}u) \quad , \quad m > 1 \ .$$

In the degenerate case classical solutions usually do not exist and weak solutions, in the sense of distributions, are sought:

DEFINITION 1.1. *A bounded function $u(x,t)$ is a weak solution of (1.3), subject to the Cauchy data*

$$u(x,0) = u_0(x) \in L_\infty(\mathbb{R}^N) \ , \tag{1.5}$$

if $Q\nabla u$ exists in the sense of distributions and

$$\iint_{\mathbb{R}^N \times \mathbb{R}^+} [u\phi_t + f \cdot \nabla\phi - (Q\nabla u)\cdot\nabla\phi + g\phi]dxdt = -\int_{\mathbb{R}^N} u_0\phi(\cdot,0)dx \qquad \forall\phi \in C_0^\infty(\mathbb{R}^{N+1}) \ . \tag{1.6}$$

It is well known that the Cauchy problem (1.3)–(1.5), denoted *(CP)* henceforth, has weak solutions [5] which are continuous [1]. Furthermore, these solutions satisfy the equation in the classical sense in the neighborhood of points where $u \neq 0$. However, their uniqueness remained so far in question.

Volpert and Hudjaev [5] have studied equation (1.3) where the viscosity coefficient, $Q(x,t,u)$, is assumed only to satisfy

$$\xi^T Q(x,t,u)\xi \geq 0 \qquad \forall (x,t,u) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R} \ \ \textit{and} \ \ \xi \in \mathbb{R}^N \setminus \{0\} \tag{1.7}$$

(as oppose to our stricter assumption, (1.4)). Under this relaxed assumption, weak solutions of (1.3) are not uniquely determined by the initial data, (1.5). Therefore, Volpert and Hudjaev dealt with a subclass of weak solutions (which are usually referred to as *entropy solutions* in the context of hyperbolic conservation laws, $Q \equiv 0$, $g \equiv 0$) and proved uniqueness in that subclass.

We show here that under the assumption of mild degeneracy, (1.4), L_1-weak solutions of (1.3) are uniquely determined by their initial value. We first prove that weak solutions of *(CP)* are regular in some sense (Proposition 2.1) and then we use this regularity in proving that the solution operator of (1.3) is L_1-stable (Theorem 2.2). This L_1-stability implies the uniqueness of L_1-weak solutions of the Cauchy problem *(CP)*.

2. Let u be a weak solution of *(CP)* and let Ω denote the set of points where it vanishes,

$$\Omega = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^+ : u(x,t) = 0\} \ .$$

Since u is continuous, the complement set, Ω^c, is open and u satisfies equation (1.3) in the classical sense there. Clearly, u satisfies the equation in the classical sense also in the interior[1] of Ω, Ω^o. Hence, u fails to be smooth only on the interface, $\partial\Omega$. However, it is still regular there in some weak sense:

PROPOSITION 2.1. *Define, for all $t \geq 0$,*

$$\partial\Omega(t) \equiv \partial\Omega \cap (\mathbb{R}^N \times \{t\}) \ .$$

Let $n = \vec{n}(x,t) \in \mathbb{R}^N$ be a normal vector to $\partial\Omega(t)$. Then

$$\langle Q\nabla u\rangle(x,t)\cdot n = 0 \qquad \forall (x,t) \in \partial\Omega(t) \ , \tag{2.1}$$

where $\langle \cdot \rangle$ denotes the jump along the normal direction, n:

$$\langle v\rangle(x,t) = v(x + 0\cdot n, t) - v(x - 0\cdot n, t) \quad , \quad (x,t) \in \partial\Omega(t) \ .$$

[1] For any domain D, D^o denotes its interior.

Therefore, $Q\nabla u$ may not be continuous (as it is in the one-dimensional case, $N=1$, consult [4, Proposition 4.1]) but for any fixed t, $(Q\nabla u)\cdot n$ is continuous along normal directions, n, to the interface $\partial\Omega(t)$.

Remark. Since u is continuous, $\partial\Omega$ consists of an at most countable, nowhere dense, collection of smooth manifolds in $\mathbb{R}^N\times\mathbb{R}^+$, with dimensions less than or equal to N. Therefore, for all $t>0$, $\partial\Omega(t)$ consists of an at most countable, nowhere dense, collection of smooth manifolds in $\mathbb{R}^N$, with dimensions less than or equal to $N-1$. On manifolds in $\mathbb{R}^N$ with dimension $N-1$, the normal vector $n=\vec{n}(x)\in\mathbb{R}^N$ is determined uniquely modulo scalar multiplications. However, in any point on a manifold in $\mathbb{R}^N$ with dimension $k<N-1$, there is an infinite $(N-1-k)$-parametered family of normal vectors. In that case, equality (2.1) holds for any choice of a normal vector.

Proof. Let Γ be a N-dimensional manifold in $\partial\Omega$ and P be a point in the interior of Γ. Assume that there exists a closed ball, $B\subset\mathbb{R}^N\times\{t:\ t>0\}$, centered at P, such that $B\cap\partial\Omega=B\cap\Gamma$. Therefore, Γ splits B into two components, B_1 and B_2, in the interior of which u is smooth.

Let ϕ be a test function in $C_0^\infty(B^o)$. Then, by (1.6),

$$0=\iint_B[u\phi_t+f\cdot\nabla\phi-(Q\nabla u)\cdot\nabla\phi+g\phi]dxdt=\sum_{j=1}^{2}I_j\ , \tag{2.2}$$

where

$$I_j=\iint_{B_j}[u\phi_t+f\cdot\nabla\phi-(Q\nabla u)\cdot\nabla\phi+g\phi]dxdt\ .$$

Since u satisfies equation (1.3) in the strong sense in B_j^o, we get that

$$I_j=\iint_{B_j}[(u\phi)_t+\nabla\cdot((f-Q\nabla u)\phi)]dxdt\ , \tag{2.3}$$

where j stands henceforth for $j=1,2$.

We introduce the following notations:

- $\Gamma_B=\Gamma\cap B$ is the inner boundary between B_1 and B_2.
- $\vec{\nu}_j=\vec{\nu}_j(x,t)\in\mathbb{R}^{N+1}$ is the outer unit normal to B_j in $(x,t)\in\partial B_j$.
- $n_j=\vec{n}_j\in\mathbb{R}^N$ and $m_j\in\mathbb{R}$ are, respectively, the spatial and time components of $\vec{\nu}_j$:

$$\vec{\nu}_j=\begin{pmatrix} n_j \\ m_j \end{pmatrix}\ . \tag{2.4}$$

Let us now define

$$\Gamma_B^j(\varepsilon_j):=\Gamma_B-\varepsilon_j\begin{pmatrix} n_j(P) \\ 0 \end{pmatrix}\ .$$

$\Gamma_B^j(\varepsilon_j)$ is therefore a translation of Γ_B along the normal direction to $\Gamma_B\cap(\mathbb{R}^N\times\{t\})$ in P, towards the interior of B_j. Γ_B is the internal part of ∂B_j (the external part of ∂B_j is $\partial B_j\cap\partial B$); by replacing Γ_B with $\Gamma_B^j(\varepsilon_j)$, B_j shrinks into a new domain, denoted $B_j(\varepsilon_j)$.

We now consider the integrals

$$I_j(\varepsilon_j)=\iint_{B_j(\varepsilon_j)}[(u\phi)_t+\nabla\cdot((f-Q\nabla u)\phi)]dxdt\ . \tag{2.5}$$

Clearly, since

$$B_j(\varepsilon_j)\underset{\varepsilon_j\to 0}{\longrightarrow}B_j\ ,$$

we have that

$$I_j(\varepsilon_j) \underset{\varepsilon_j\to 0}{\longrightarrow} I_j \ . \tag{2.6}$$

Applying The Divergence Theorem in (2.5), we get that

$$I_j(\varepsilon_j) = \int_{\partial B_j(\varepsilon_j)} \left\{ \left[\begin{pmatrix} f - Q\nabla u \\ u \end{pmatrix} \cdot \vec{\nu}_j^{\varepsilon_j} \right] \phi \right\} (x,t) dx \ , \tag{2.7}$$

where $\vec{\nu}_j^{\varepsilon_j}(x,t) \in \mathbb{R}^{N+1}$ is the outer unit normal to $B_j(\varepsilon_j)$ in $(x,t) \in \partial B_j(\varepsilon_j)$. Since ϕ vanishes on ∂B, we get that

$$I_j(\varepsilon_j) = \int_{\Gamma_B(\varepsilon_j)} \left\{ \left[\begin{pmatrix} f - Q\nabla u \\ u \end{pmatrix} \cdot \vec{\nu}_j^{\varepsilon_j} \right] \phi \right\} (x,t) dx \ , \tag{2.8}$$

or, after the changes of variables $x \mapsto x - \varepsilon_j n_j(P)$,

$$I_j(\varepsilon_j) = \int_{\Gamma_B} \left\{ \left[\begin{pmatrix} f - Q\nabla u \\ u \end{pmatrix} \cdot \vec{\nu}_j^{\varepsilon_j} \right] \phi \right\} (x - \varepsilon_j n_j(P), t) dx \ . \tag{2.9}$$

We now let $\varepsilon_j \to 0$. Since u, $f(u)$ and ϕ are continuous and

$$\vec{\nu}_j^{\varepsilon_j}(x - \varepsilon_j n_j(P), t) \underset{\varepsilon_j\to 0}{\longrightarrow} \vec{\nu}_j(x,t) \qquad \forall (x,t) \in \Gamma_B \ ,$$

we conclude that

$$\lim_{\varepsilon_j\to 0} I_j(\varepsilon_j) = \int_{\Gamma_B} \left[\begin{pmatrix} f(u(x,t)) - (Q\nabla u)(x - 0\cdot n_j(P), t) \\ u(x,t) \end{pmatrix} \cdot \vec{\nu}_j(x,t) \right] \phi(x,t) dx \ . \tag{2.10}$$

Since $\vec{\nu}_1 = -\vec{\nu}_2$ on Γ_B, we get, using (2.10), (2.6), (2.2) and (2.4), that

$$\int_{\Gamma_B} \left\{ \left[(Q\nabla u)(x + 0\cdot n_1(P), t) - (Q\nabla u)(x - 0\cdot n_1(P), t) \right] \cdot n_1(x,t) \right\} \phi(x,t) dx = 0 \ . \tag{2.11}$$

Note that since $\vec{\nu}_1$ is a normal vector to $\partial\Omega$ in $\mathbb{R}^{N+1}$, n_1 is a normal vector to $\partial\Omega(t)$ in $\mathbb{R}^N$. Finally, by letting *supp*ϕ shrink to P we conclude that (2.1) holds in P.

Next, we handle the more delicate possible cases:

Assume that $\Gamma \subset \partial\Omega$ is a manifold of dimension $k < N$. Let P be a point in $\Gamma(t) = \Gamma \cap (\mathbb{R}^N \times \{t\})$ and $\vec{n} \in \mathbb{R}^N$ be any normal vector to $\Gamma(t)$ in P. Then, there exists a N-dimensional manifold, $\tilde{\Gamma} \subset \mathbb{R}^{N+1}$, such that $\Gamma \subset \tilde{\Gamma}$ and $\vec{n}$ is the normal vector to $\tilde{\Gamma}(t) = \tilde{\Gamma} \cap (\mathbb{R}^N \times \{t\})$ in P. By repeating our arguments, as before, for $\tilde{\Gamma}$, we conclude that (2.1) holds in this case as well.

If P lies on the boundary of a manifold $\Gamma \subset \Omega$, we may extend Γ to a manifold $\tilde{\Gamma}$ such that $P \in (\tilde{\Gamma})^o$, in order to apply our previous line of proof.

The proof in case that P lies on the intersection of two (or more) manifolds is slightly modified by decomposing B into four (or more) components.

Finally, the case of an accumulation point (i.e., each neighborhood of P is intersected by infinitely many manifolds of $\partial\Omega$), is treated similarly by applying a limit process. □

A consequence of Proposition 2.1 is that the solution operator of (1.3) is stable in $L_1 = L_1(\mathbb{R}^N)$:

THEOREM 2.2. *(L_1-Stability). Let u and v be two weak solutions of (1.3)–(1.4). Let $M_T^{\pm}$ be such that*

$$M_T^- \le u(x,t), v(x,t) \le M_T^+ \qquad \forall (x,t) \in \mathbb{R}^N \times [0,T] \ , \tag{2.12}$$

and assume that

$$u(\cdot,t) - v(\cdot,t) \in L_1 \qquad \forall t \in [0,T] \ . \tag{2.13}$$

Then

$$\|u(\cdot,t)-v(\cdot,t)\|_{L_1} \le e^{\gamma t}\|u(\cdot,0)-v(\cdot,0)\|_{L_1} \qquad \forall t\in[0,T]\ , \tag{2.14}$$

where

$$\gamma=\gamma(T):=\sup_{\mathbb{R}^N\times[0,T]\times[M_T^-,M_T^+]} g_u(x,t,u)\ . \tag{2.15}$$

Before proving Theorem 2.2, we state and prove the following lemma:

LEMMA 2.3. *Let D be a bounded domain in $\mathbb{R}^N$ and $w=w(x)$ be such that*

$$w\Big|_D \ge 0 \quad and \quad w\Big|_{\partial D}=0\ . \tag{2.16}$$

Let $Q=Q(x)$ be a $N\times N$ nonnegative definite matrix function. Then if n is the outer unit normal to ∂D,

$$(Q(x)\nabla w)\cdot n\Big|_{\partial D}\le 0\ . \tag{2.17}$$

Proof. Let x_0 be a point in ∂D. Since $Q(x_0)$ is a nonnegative definite matrix, there exists an orthogonal matrix P, such that

$$PQ(x_0)P^T=\Lambda=diag\{\lambda_1,...,\lambda_N\}\quad,\quad \lambda_i\ge 0\ ,\ \ 1\le i\le N\ . \tag{2.18}$$

We make the following change of variables, $x\mapsto\tilde{x}=Px$. Denoting the gradient with respect to the new variables by $\tilde{\nabla}=\partial/\partial\tilde{x}$ and the new outer unit normal vector to ∂D by $\tilde{n}$, we have that

$$\tilde{\nabla}=P\nabla \quad and \quad \tilde{n}=Pn\ . \tag{2.19}$$

Using (2.18)–(2.19) we get that

$$(Q\nabla w)\cdot n\Big|_{x=x_0}=(P^T\Lambda P\nabla w)\cdot n\Big|_{x=x_0}=(\Lambda\tilde{\nabla}w)\cdot\tilde{n}\Big|_{\tilde{x}=Px_0}=\sum_{i=1}^N\lambda_i\frac{\partial w}{\partial\tilde{x}_i}\tilde{n}_i\Big|_{\tilde{x}=Px_0}\ . \tag{2.20}$$

However, assumption (2.16) implies that

$$\frac{\partial w}{\partial\tilde{x}_i}\tilde{n}_i\ge 0 \qquad 1\le i\le N\ , \tag{2.21}$$

in $\tilde{x}=Px_0$. Hence, since the eigenvalues λ_i are nonnegative, (2.18), we get by (2.20)–(2.21) that inequality (2.17) holds in $x=x_0$. That concludes the proof. □

Proof of Theorem 2.2. For every $t\ge 0$, we divide the space $\mathbb{R}^N$ to sub-domains, $\mathbb{R}^N=\cup_k D_k(t)$, so that

$$(-1)^k[u(\cdot,t)-v(\cdot,t)]\Big|_{D_k(t)}\ge 0 \qquad \forall k \tag{2.22}$$

and

$$u(\cdot,t)=v(\cdot,t)\Big|_{\partial D_k(t)} \qquad \forall k\ . \tag{2.23}$$

Using (2.22) and (2.23) we conclude that

$$\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_{L_1}= \tag{2.24}$$

$$= \frac{d}{dt}\sum_k (-1)^k \int_{D_k(t)} [u(x,t) - v(x,t)]dx = \sum_k (-1)^k \int_{D_k(t)} [u_t(x,t) - v_t(x,t)]dx = \sum_k I_k \ .$$

We show below that all the terms in the last sum in (2.24) are nonpositive. We concentrate on terms I_k which correspond to bounded sub-domains $D_k(t)$. Later on we comment about the treatment of unbounded sub-domains.

First, let us assume that neither $u(\cdot,t)$ nor $v(\cdot,t)$ vanish in $D_k(t)^o$. Therefore, both u and v satisfy equation (1.3) in the strong sense there and we conclude that

$$I_k = (-1)^k \int_{D_k(t)} \nabla \cdot [f(x,t,v) - f(x,t,u)]dx + \tag{2.25}$$

$$+(-1)^k \int_{D_k(t)} \nabla \cdot [Q(x,t,u)\nabla u - Q(x,t,v)\nabla v]dx + (-1)^k \int_{D_k(t)} [g(x,t,u) - g(x,t,v)]dx = I_k^1 + I_k^2 + I_k^3 \ .$$

The first term on the right hand side of (2.25) is zero, due to The Divergence Theorem and equality (2.23):

$$I_k^1 = \int_{\partial D_k(t)} [f(s,t,v) - f(s,t,u)] \cdot n ds = 0 \ ; \tag{2.26}$$

$n \in \mathbb{R}^N$ denotes here and henceforth the outer unit normal to $D_k(t)$.

As for the second term, it equals, by The Divergence Theorem, to

$$I_k^2 = (-1)^k \int_{\partial D_k(t)} [Q(s,t,u)\nabla u - Q(s,t,v)\nabla v] \cdot n ds \ .$$

Since $u = v$ on $\partial D_k(t)$, (2.23), it may be written as follows:

$$I_k^2 = \int_{\partial D_k(t)} [\tilde{Q}(s,t)\nabla w] \cdot n ds \ , \tag{2.27}$$

where $\tilde{Q}(s,t) = Q(s,t,u = u(s,t))$ and $w = (-1)^k(u - v)$. Since, by (2.22)–(2.23), w is nonnegative in $D_k(t)$ and vanishes on $\partial D_k(t)$, and $\tilde{Q}(s,t) \geq 0$, Lemma 2.3 implies that

$$I_k^2 \leq 0 \ . \tag{2.28}$$

Using The Mid-value Theorem, (2.22) and (2.15) for the last term on the right hand side of (2.25), we get that

$$I_k^3 \leq \gamma \int_{D_k(t)} |u - v| dx \qquad \forall t \in [0,T] \ . \tag{2.29}$$

Combining (2.25)–(2.29) we conclude that

$$I_k \leq \gamma \int_{D_k(t)} |u - v| dx \qquad \forall t \in [0,T] \ . \tag{2.30}$$

Next, we handle those sub-domains, $D_k(t)$, in the interior of which either u or v vanish. We denote, as in (2.1), the zero sets of u and v by Ω_u and Ω_v, respectively.

Assume that $D_k(t)^o$ is intersected by one of the manifolds of Ω_u, Γ,

$$D_k(t)^o \cap \Omega_u = D_k(t)^o \cap \Gamma \neq \emptyset \tag{2.31}$$

and that

$$D_k(t)^o \cap \Omega_v = \emptyset \ . \tag{2.32}$$

The case where $D_k(t)^o$ is intersected by more than one manifold of either of the two zero sets, is treated in a similar manner, as we explain later on.

If the dimension of Γ is less than N, we imbed it in a N-dimensional manifold, still denoted by Γ. Therefore, $S := \Gamma \cap D_k(t)$, splits $D_k(t)$ into two components, $D_k^1(t)$ and $D_k^2(t)$, and in view of (2.31)–(2.32) u and v satisfy equation (1.3) in the strong sense in $D_k^j(t)^o$, $j = 1, 2$. Therefore

$$I_k = (-1)^k \int_{D_k(t)} [u_t(x,t) - v_t(x,t)]dx = \sum_{j=1}^{2} \left\{ (-1)^k \int_{D_k^j(t)} \nabla \cdot [f(x,t,v) - f(x,t,u)]dx + \right. \tag{2.33}$$

$$\left. +(-1)^k \int_{D_k^j(t)} \nabla \cdot [Q(x,t,u)\nabla u - Q(x,t,v)\nabla v]dx + (-1)^k \int_{D_k^j(t)} [g(x,t,u) - g(x,t,v)]dx \right\} .$$

Let n_j denote the outer unit normal to $D_k^j(t)$. Note that on S, the interface between $D_k^1(t)$ and $D_k^2(t)$, $n_1 = -n_2$, and that on $\partial D_k^j(t) \setminus S$, n_j coincides with n, the outer unit normal to $D_k(t)$.

Therefore, using The Divergence Theorem and equality (2.23), the first term on the right hand side of (2.33) vanishes:

$$\sum_{j=1}^{2} (-1)^k \int_{D_k^j(t)} \nabla \cdot [f(x,t,v) - f(x,t,u)]dx = \sum_{j=1}^{2} (-1)^k \int_{\partial D_k^j(t)} [f(s,t,v) - f(s,t,u)] \cdot n_j ds = \tag{2.34}$$

$$= (-1)^k \left\{ \int_S [f(s,t,v) - f(s,t,u)] \cdot (n_1 + n_2) ds + \int_{\partial D_k(t)} [f(s,t,v) - f(s,t,u)] \cdot n ds \right\} = 0 \ .$$

As for the second term, it is nonpositive:

$$\sum_{j=1}^{2} (-1)^k \int_{D_k^j(t)} \nabla \cdot [Q(x,t,u)\nabla u - Q(x,t,v)\nabla v]dx = \tag{2.35}$$

$$= \sum_{j=1}^{2} (-1)^k \int_{\partial D_k^j(t)} [Q(s,t,u)\nabla u - Q(s,t,v)\nabla v] \cdot n_j ds =$$

$$= \int_{\partial D_k(t)} [\tilde{Q}(s,t)\nabla w] \cdot n ds + (-1)^k \int_S \langle Q(s,t,u)\nabla u \rangle \cdot n_1 ds - (-1)^k \int_S \langle Q(s,t,v)\nabla v \rangle \cdot n_1 ds \ ,$$

where, as before, $\tilde{Q}(s,t) = Q(s,t,u = u(s,t))$, $w = (-1)^k(u - v)$ and $\langle \cdot \rangle$ denotes the jump across S in the normal direction, n_1. The first term on the right hand side of (2.35) is nonpositive, in light of Lemma 2.3, while the other two terms vanish in view of Proposition 2.1.

Since the last term on the right hand side of (2.33) may be bounded as in (2.29), we conclude, by (2.33)–(2.35), that inequality (2.30) holds in this case as well.

If $D_k(t)$ is intersected by any number (finite or infinite) of manifolds from either Ω_u or Ω_v, it may be decomposed into $D_k(t) = \cup_{j \in J} D_k^j(t)$, so that both u and v are smooth in $D_k^j(t)^o$, $j \in J$, and the proof goes along the same lines as above.

Finally, if $D_k(t)$ is an unbounded sub-domain, we may consider an increasing sequence of bounded domains, $\{D_k^m(t)\}_{m=1}^{\infty}$, such that $\cup_{m=1}^{\infty} D_k^m(t) = D_k(t)$. The boundary of $D_k^m(t)$ may be decomposed to

$$\partial D_k^m(t) = S_1^m \cup S_2^m \quad , \quad S_1^m = \partial D_k^m(t) \cap \partial D_k(t) \quad , \quad S_2^m = \partial D_k^m(t) \setminus \partial D_k(t) \ .$$

In light of (2.23), $u = v$ on S_1^m. On the other hand, in view of (2.13), $\{D_k^m(t)\}_{m=1}^{\infty}$ may be chosen so that

(2.36) $$\lim_{m\to\infty} \|u - v\|_{L_1(S_2^m)} = 0 ,$$

since S_2^m is the part of the boundary "near the infinity". Therefore, when $m \to \infty$, the contribution of the integral on S_2^m to I_k^1 in (2.26) tends to zero while the contribution to I_k^2 in (2.27) tends to a nonpositive value.

To summarize all of the above, inequality (2.30) holds for all k. Hence, we get from (2.24) that

$$\frac{d}{dt}\|u(\cdot,t) - v(\cdot,t)\|_{L_1} \leq \gamma \cdot \|u(\cdot,t) - v(\cdot,t)\|_{L_1} \qquad \forall t \in [0,T] ,$$

which implies (2.14). □

An immediate consequence of Theorem 2.2 is uniqueness:

COROLLARY 2.4. *(Uniqueness). The Cauchy problem for equation (1.3)-(1.4) admits a unique* $L_1(\mathbb{R}^N)$*-weak solution.*

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