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ABSTRACT

Instead of solving original problems defined on irregular domains, we solve extended parabolic problems on regular domains which contain actual ones to get required solutions of original problems. Optimal error estimates between continuous solutions of original problems and finite element solutions of extended problems are obtained both in L^2-norm and in energy-norm.

1 Introduction

With the development of domain decomposition methods, fictitious domain methods for partial differential equations have also attracted much attention in the

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past years, see Glowinski et al. [8], Glowinski and Pan [9] and Young et al. [16]. The methods of the kind (they are sometimes called domain embedding methods) have been regarded as a most interesting potential method for solving complicated problems from practical applications. Although there exist many different ways for developing and constructing fictitious domain methods, see Astrakhantsev [2], Börgers and Widlund [3], Buzbee et al. [4], Finogenov and Kuznetsov [6], Glowinski et al [8], Hoffmann and Tiba [10], Männikko et al. [12], Matsokin [13], they have one common property. That is, they all introduce so-called fictitious domains– quite regular, simply shaped– which contain the actual irregular or curved domains and then solve newly constructured problems defined on fictitious domains to get approximate solutions of original problems defined on actual irregular domains.

Since fictitious domains are very regular, for example, rectangular (2-D) or cubic domains (3-D), they may have some advantages: (1) approximations of curved boundaries of original domains are not necessary any more; (2) one may use some efficient solvers suitable for fairly structured domains; (3) one may introduce simple or more desirable boundary conditions, on fictitious domains, for the considered problems. For example, periodical boundary conditions for using spectral methods for Navier-Stokes equations.

In this paper, we consider the fictitious domain/penalty method for solving parabolic problems with Neumann boundary conditions defined on domains with curved boundary. The fictitious domain/penalty method was proposed by Glowinski et al. [8] for elliptic problems. Our main work is to analyse the convegence and approximation of fictitious domain problems discretized by finite element methods in space and by Crank-Nicolson scheme in time. And we obtain the optimal error estimates both in L^2-norm and energy norm with respect to time step t and mesh size h. Section 2 will introduce the parabolic problems which we solve. Section 3 will be devoted to the formulation of fictitious domain/penalty solution method and its discretization. Finally, error estimates between finite element solution of fictitious domain problem and continuous solution of original parabolic problem will be conducted in Section 4.

Throughout the paper we utilize $|\cdot|_{m,\Omega}$ and $||\cdot||_{m,\Omega}$ to denote the semi-norm and norm of the usual Sobolev space $H^m(\Omega)$. Constants C denote always generic constants which are independent of mesh size h and time step τ.

2 Parabolic problems with Neumann boundary conditons

In this paper, we take the following parabolic problem with Neumann boundary condition as example for the analysis:

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right) + b(x)\,u = f \text{ in } \Omega\,, \tag{2.1}$$

$$u(x,0) = u_0(x), \tag{2.2}$$

$$\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j} n_i = g(x,t), \quad (x,t) \in \partial\Omega \times (0,T) \tag{2.3}$$

with $n = (n_1, n_2, \cdots, n_d)$ being the unit outward normal of the boundary $\partial\Omega$, Ω being a open domain in R^d with appropriate smooth boundary, $b(x) \geq 0$, $f \in L^2(0,T; L^2(\Omega))$ and $\left(a_{ij}\right)$ is symmetric and satisfies

$$\alpha_1|\eta|^2 \leq \sum_{i,j=1}^{d} a_{ij}\eta_i\,\eta_j \leq \alpha_2|\eta|^2, \ \forall\, \eta \in R^d. \tag{2.4}$$

By Green's formulae it is immediate to derive the variatinal formulation of the problem (2.1)–(2.3)

(VP): Find $u \in L^2\ (0,T; H^1(\Omega)) \cap H^1\ (0,T; (H^1(\Omega))')$ such that

$$u(\cdot,0) = u_0 \tag{2.5}$$

and for almost every $t \in (0,T)$ the following equation holds

$$(u_t, v)_\Omega + a_\Omega(u,v) = (f,\, v)_\Omega + (g,\, v)_{\partial\Omega}, \ \forall\ v \in H^1(\Omega) \tag{2.6}$$

where

$$(u,v)_\Omega = \int_\Omega u\,v\,dx, \quad (g,v)_{\partial\Omega} = \int_{\partial\Omega} g\,v ds, \tag{2.7}$$

$$a_\Omega(u,v) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + b\,u\,v\right)\,dx. \tag{2.8}$$

Under appropriate smoothness assumptions on the given data, one can show that Problem (VP) is equivalent to Problem (2.1)–(2.3).

3 A fictitious domain method and its finite element scheme

Now we introduce a fictitious domain method to reduce the solution of Problem (VP) into a parabolic problem defined on a larger regular domain $\mathcal{O}$ which contains the original domain Ω, see Figure 1.

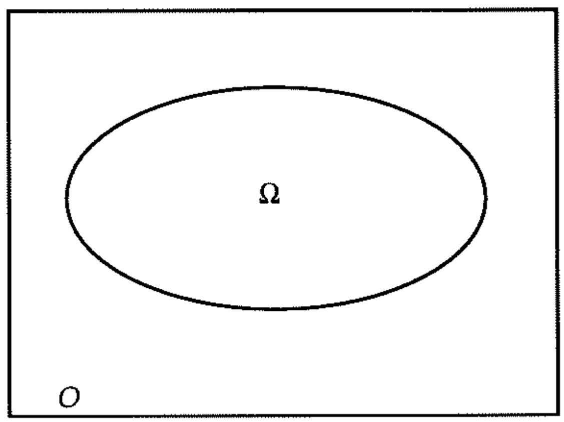

Figure 1. Original domain Ω and its fictitious domain $\mathcal{O}$

Let $\varepsilon > 0$ be a arbitrarily given parameter. Consider the following fictitious domain formulation:

(FDP): Find $u^\varepsilon \in L^2(0,T;H_0^1(\mathcal{O})) \cap H^1(0,T;H^{-1}(\mathcal{O}))$ such that

$$u(\cdot,0) = u_0 \tag{3.1}$$

and for almost every $t \in (0,T)$ the following equation holds

$$\begin{aligned}(u_t^\varepsilon, v)_\Omega + a_\Omega(u^\varepsilon, v) + \varepsilon\,(u_t^\varepsilon, v)_{\mathcal{O}} + \varepsilon\, a_{\mathcal{O}}(u^\varepsilon, v) = (f,\, v)_\Omega + (g,\, v)_{\partial\Omega} \\ +\varepsilon\,(f_0,\, v)_{\mathcal{O}}, \quad \forall\, v \in H_0^1(\Omega).\end{aligned} \tag{3.2}$$

where f_0 is a given function in $L^2(0,T;L^2(\mathcal{O}))$, $a_{\mathcal{O}}(\cdot,\,\cdot)$ and $(\cdot,\,\cdot)_{\mathcal{O}}$ are defined similarly as in (2.7) and (2.8). Later on, we will also use norms $||\cdot||_{a,\Omega}$ and $||\cdot||_{a,\mathcal{O}}$ to denote norms $(a_\Omega(\cdot,\cdot))^{1/2}$ and $(a_{\mathcal{O}}(\cdot,\cdot))^{1/2}$. By Lions' theorem, see Wang [15], we know that Problem (FDP) has a unique solution $u^\varepsilon \in L^2(0,T;H_0^1(\mathcal{O}))$.

In this paper we take Crank-Nicolson's scheme to discretize Problem (FDP) in time. For other discretization scheme for time, we can get the similar results. Let $\tau = T/M$ be time step size with M a positive integer. For any $n = 1, 2, \cdots, M$, we denote $t_n = n\tau$ and $I^n = (t_{n-1}, t_n]$. For a given sequence $\{u^n\}_{n=0}^M \subset L^2(\Omega)$, we define

$$\partial_\tau u^n = \frac{u^n - u^{n-1}}{\tau}, \quad \bar{u}^{n-\frac{1}{2}} = \frac{1}{2}(u^n + u^{n-1}) \tag{3.3}$$

For a continuous mapping $u : [0, T] \to L^2(\Omega)$, we define $u^n = u(\cdot, n\tau)$, $0 \le n \le M$.

In space we will approximate the problem (FDP) by finite element method. Suppose we are given a family of triangulations $\{\mathcal{T}^h\}$ consisting of d-simplices on $\mathcal{O}$. Let $V^h \subset H_0^1(\mathcal{O})$ be a given finite element subspace, and Π_h be the corresponding finite element interpolation operator. For the sake of simplicity, we assume that V^h is a piecewise linear finite element subspace. But all our results in the paper can be directly generalized to other finite element subspaces and non-simplex elements without any difficulty. Our finite element problem is then formulated as follows:

(FEP): Find $u_h^n \in V^h$ such that $u_h^0 = \Pi_h u^0$ and for $n = 1, 2, \cdots, M$,

$$\begin{aligned}(\partial_\tau u_h^n, v)_\Omega + a_\Omega(\bar{u}_h^{n-1/2}, v) + \varepsilon\,(\partial_\tau u_h^n, v)_\mathcal{O} + \varepsilon\, a_\mathcal{O}(\bar{u}_h^{n-1/2}, v) \\ = (f^{n-1/2}, v)_\Omega + (g^{n-1/2}, v)_{\partial\Omega} + \varepsilon\,(f_0^{n-1/2}, v)_\mathcal{O}, \ \forall\, v \in V^h.\end{aligned} \tag{3.4}$$

where $\bar{u}_h^{n-1/2} = (u_h^n + u_h^{n-1})/2$.

By Lax-Milgram lemma, see Ciarlet [5], we know that Problem (FEP) has a unique solution u_h^n, for any $n : n = 1, 2, \cdots, M$.

4 Error estimates between (FEP) and (VP)

This section is devoted to the error estimates between the solutions of the finite element problem (FEP) and of the original variational problem (VP), or (2.1)–(2.3). For the purpose, we first extend functions defined on Ω into $\mathcal{O}$. There exist various extensions, cf. Agmon [1], Gilbarg and Trudinger [7], Stein [14], etc. Here we cite a result from Stein [14]

Lemma 4.1 *Let $\Omega \subset R^d (d \ge 2)$ be an open, bounded domain with a piecewise smooth, uniformly Lipschitz continuous boundary Γ. Then there exists a linear operator E extending functions on Ω to functions on $\mathcal{O}$ with $\Omega \subset\subset \mathcal{O}$ such that $Eu \in H_0^m(\mathcal{O})$, $Eu\,|_\Omega = u$ and*

$$||Eu||_{H^m(\mathcal{O})} \le C(\Omega)||u||_{H^m(\Omega)}, \ \forall\, v \in H^m(\Omega) \tag{4.1}$$

i.e., E maps $H^m(\Omega)$ continuously into $H^m(\mathcal{O})$, m is a nonnegative integer. $C(\Omega)$ depends on d, m and the Lipschitz constant of the region only.

Our main results of this paper are stated in the following lemma

Theorem 4.1 *Let Ω is a bounded open domain in R^d with a $C^{1,1}$ boundary and $\Omega \subset\subset \mathcal{O}$ with $\mathcal{O}$ being a polygonal domain. Then we have*

$$\max_{1\le n\le M} ||u_h^n - u^n||_{1,\Omega} \le C\,(h + \tau + h^2\,\tau^{-1/2} + \sqrt{\varepsilon}), \tag{4.2}$$

$$\max_{1\le n\le M} ||u_h^n - u^n||_{0,\Omega} \le C\,(h^2 + \tau^2 + \sqrt{\varepsilon}). \tag{4.3}$$

We first cite the well-known standard finite element interpolation results, cf. Ciarlet [5]:

Lemma 4.2 *For any $u \in H^2(\mathcal{O})$, we have*

$$||u - \Pi_h u||_{s,\mathcal{O}} \le C\,h^{2-s}|u|_{2,\mathcal{O}}, s = 0, 1. \tag{4.4}$$

Proof of Theorem 4.1. Taking $t = t^{n-1/2}$ and $v \in V^h$ in (2.6) gives

$$(u_t^{n-1/2}, v)_\Omega + a_\Omega(u^{n-1/2}, v) = (f^{n-1/2}, v)_\Omega + (g^{n-1/2}, v)_{\partial\Omega}, \tag{4.5}$$

which can be rewritten as

$$\begin{aligned}(\partial_\tau u^n, v)_\Omega + a_\Omega(\bar{u}^{n-1/2}, v) \;&=\; (f^{n-1/2}, v)_\Omega + (g^{n-1/2}, v)_{\partial\Omega} \\ &+(\partial_\tau u^n - u_t^{n-1/2}, v)_\Omega + a_\Omega(\bar{u}^{n-1/2} - u^{n-1/2}, v).\end{aligned} \tag{4.6}$$

Let $\rho_h^n = u_h^n - \Pi_h E u^n$ with u_h^n and u^n being the solutions of Problem (FEP) and (4.6), respectively. To prove Theorem 4, we first estimate ρ_h^n in the following and then our required results follow from the triangle's inequality.

From (3.4) and (4.6), we derive for any $v \in V^h$ that

$$\begin{aligned}(\partial_\tau \rho_h^n, v)_\Omega \;&+\; a_\Omega(\bar{\rho}_h^{n-1/2}, v) + \varepsilon\,(\partial_\tau u_h^n, v)_\mathcal{O} + \varepsilon\, a_\mathcal{O}(\bar{u}_h^{n-1/2}, v) \\ &= (\partial_\tau(Eu^n - \Pi_h Eu^n), v)_\Omega + a_\Omega(E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2}, v) + \varepsilon(f_0^{n-1/2}, v)_\mathcal{O} \\ &+(Eu_t^{n-1/2} - \partial_\tau Eu^n, v)_\Omega + a_\Omega(Eu^{n-1/2} - E\bar{u}^{n-1/2}, v).\end{aligned} \tag{4.7}$$

We rewrite (4.7) into

$$\begin{aligned}(\partial_\tau \rho_h^n, v)_\Omega \;&+\; a_\Omega(\bar{\rho}_h^{n-1/2}, v) + \varepsilon\,(\partial_\tau \rho_h^n, v)_\mathcal{O} + \varepsilon\, a_\mathcal{O}(\bar{\rho}_h^{n-1/2}, v) \\ &= (\partial_\tau(Eu^n - \Pi_h Eu^n), v)_\Omega + a_\Omega(E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2}, v) + \varepsilon\big(f_0^{n-1/2}, v\big)_\mathcal{O} \\ &+(u_t^{n-1/2} - \partial_\tau u^n, v)_\Omega + a_\Omega(u^{n-1/2} - \bar{u}^{n-1/2}, v) \\ &-\varepsilon\,(\partial_\tau \Pi_h E u^n, v)_\mathcal{O} - \varepsilon a_\mathcal{O}(\Pi_h E\bar{u}^{n-1/2}, v).\end{aligned} \tag{4.8}$$

Substituting $v = \bar{\rho}_h^{n-1/2}$ into (4.8) implies that

$$\begin{aligned}
&\frac{1}{2}||\rho_h^n||_{0,\Omega}^2 - \frac{1}{2}||\rho_h^{n-1}||_{0,\Omega}^2 + \tau\,||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 \\
&+\frac{\varepsilon}{2}||\rho_h^n||_{0,\mathcal{O}}^2 - \frac{\varepsilon}{2}||\rho_h^{n-1}||_{0,\mathcal{O}}^2 + \tau\varepsilon||\bar{\rho}_h^{n-1/2}||_{a,\mathcal{O}}^2 \\
&\le \tau\left[(\partial_\tau(Eu^n - \Pi_h Eu^n),\, \bar{\rho}_h^{n-1/2})_\Omega + a_\Omega(E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2},\, \bar{\rho}_h^{n-1/2})\right] \\
&+\tau\left[(u_t^{n-1/2} - \partial_\tau u^n,\, \bar{\rho}_h^{n-1/2})_\Omega + a_\Omega(u^{n-1/2} - \bar{u}^{n-1/2},\, \bar{\rho}_h^{n-1/2})\right] \\
&+\tau\varepsilon\left[(f_0^{n-1/2},\, \bar{\rho}_h^{n-1/2})_\mathcal{O} - (\partial_\tau \Pi_h Eu^n,\, \bar{\rho}_h^{n-1/2})_\mathcal{O} - a_\mathcal{O}(\Pi_h E\bar{u}^{n-1/2},\, \bar{\rho}_h^{n-1/2})\right] \\
&=: r^1 + r^2 + r^3. \qquad (4.9)
\end{aligned}$$

Now we estimate r^1, r^2, r^3, one by one.

First, using Lemma 4.1 and Lemma 4.2 and the standard arguments, cf. Hoffmann and Zou [11], Zou [17], we get

$$\begin{aligned}
&\left|\tau(\partial_\tau(Eu^n - \Pi_h Eu^n),\, \bar{\rho}_h^{n-1/2})_\Omega\right| \le \tau^{1/2}||\bar{\rho}_h^{n-1/2}||_{0,\Omega}\left(\int_{t_{n-1}}^{t_n} ||Eu_t - \Pi_h Eu_t||_{0,\Omega}^2 dt\right)^{1/2} \\
&\le C\,\tau^{1/2}\,h^2||\bar{\rho}_h^{n-1/2}||_{0,\Omega}\left(\int_{t_{n-1}}^{t_n} |u_t|_{2,\Omega}^2 dt\right)^{1/2} \\
&\le C\,\tau\,||\bar{\rho}_h^{n-1/2}||_{0,\Omega}^2 + C\,h^4\int_{t_{n-1}}^{t_n} |u_t|_{2,\Omega}^2\,dt, \qquad (4.10)
\end{aligned}$$

while by (2.4) and Lemma 4.2 we have

$$\begin{aligned}
\tau\left|a_\Omega(E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2},\, \bar{\rho}_h^{n-1/2})\right| &\le \tau\,||\bar{\rho}_h^{n-1/2}||_{a,\Omega}\,||E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2}||_{1,\Omega} \\
&\le \frac{1}{4}\tau\,||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 + C\,\tau\,h^4|\bar{u}^{n-1/2}|_{3,\Omega}^2. \quad (4.11)
\end{aligned}$$

Thus from (4.10) and (4.11) we derive

$$|r^1| \le \frac{1}{4}\tau\,||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 + C\,\tau\,||\bar{\rho}_h^{n-1/2}||_{0,\Omega}^2 + C\,h^4\left(\int_{t_{n-1}}^{t_n} |u_t|_{2,\Omega}^2\,dt + \tau\,|\bar{u}^{n-1/2}|_{3,\Omega}^2\right). \qquad (4.12)$$

Secondly, we analyse r^2. By using dual arguments, Taylor's formulae and the equivalence between the norms $||\cdot||_{a,\Omega}$ and $||\cdot||_{1,\Omega}$, we obtain by direct computations that

$$\left|\tau\,(u_t^{n-1/2} - \partial_\tau u^n,\, \bar{\rho}_h^{n-1/2})_\Omega\right| \le \frac{1}{4}\tau\,||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 + 2\tau^4\int_{t_{n-1}}^{t_n} ||u_{ttt}||_{(H^1(\Omega))'}^2\,dt, \quad (4.13)$$

and

$$\left|\tau\, a_\Omega(u^{n-1/2} - \bar{u}^{n-1/2},\, \bar{\rho}_h^{n-1/2})\right| \le \frac{1}{4}\tau\, ||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 + \tau^4 \int_{t_{n-1}}^{t_n} ||u_{tt}||_{1,\Omega}^2\, dt. \quad (4.14)$$

Therefore we deduce from (4.13) and (4.14) that

$$|r^2| \le \frac{1}{2}\tau\, ||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 + \tau^4 \int_{t_{n-1}}^{t_n} \left(||u_{tt}||_{1,\Omega}^2 + ||u_{ttt}||_{(H^1(\Omega))'}^2\right) dt\,. \quad (4.15)$$

Finally we turn to the estimation of r^3. It follows by Cauchy-Schwarz's inequality and standard arguments that

$$\tau\,\varepsilon\left|(f_0^{n-1/2},\, \bar{\rho}_h^{n-1/2})_{\mathcal{O}}\right| \le \frac{\varepsilon}{2}\tau\, ||\bar{\rho}_h^{n-1/2}||_{0,\mathcal{O}}^2 + \frac{1}{2}\tau\,\varepsilon\, ||f_0^{n-1/2}||_{0,\mathcal{O}}^2, \quad (4.16)$$

$$\tau\,\varepsilon\left|(\partial_\tau \Pi_h E u^n,\, \bar{\rho}_h^{n-1/2})_{\mathcal{O}}\right| \le \frac{1}{2}\tau\,\varepsilon\, ||\bar{\rho}_h^{n-1/2}||_{0,\mathcal{O}}^2 + \frac{1}{2}\varepsilon \int_{t_{n-1}}^{t_n} ||u_t||_{1,\Omega}^2\, dt, \quad (4.17)$$

$$\tau\,\varepsilon\left|a_{\mathcal{O}}(\Pi_h E\bar{u}^{n-1/2},\, \bar{\rho}_h^{n-1/2})\right| \le \frac{1}{16}\tau\,\varepsilon\, ||\bar{\rho}_h^{n-1/2}||_{a,\mathcal{O}}^2 + \tau\,\varepsilon\, ||\bar{u}^{n-1/2}||_{1,\Omega}^2. \quad (4.18)$$

where we have also used Lemma 4.1 and the stability of Π_h in $H^1(\mathcal{O})$-norm.

Thus from (4.16)–(4.18) it follows that

$$\begin{aligned} |r^3| \;&\le\; \varepsilon\,\tau\, ||\bar{\rho}_h^{n-1/2}||_{0,\mathcal{O}}^2 + \tau\,\varepsilon\, ||f_0^{n-1/2}||_{0,\mathcal{O}}^2 + \varepsilon \int_{t_{n-1}}^{t_n} ||u_t||_{1,\Omega}^2\, dt \\ &\quad + \frac{1}{16}\tau\,\varepsilon\, ||\bar{\rho}_h^{n-1/2}||_{a,\mathcal{O}}^2 + \tau\,\varepsilon\, ||\bar{u}^{n-1/2}||_{1,\mathcal{O}}^2. \end{aligned} \quad (4.19)$$

Thus taking the sum from $n = 1$ to $n = k \le M$ in (4.9) and using (4.12), (4.15) and (4.19) comes to

$$\begin{aligned} &||\rho_h^k||_{0,\Omega}^2 + \frac{1}{2}\tau \sum_{n=1}^{k} ||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 + \varepsilon\, ||\rho_h^k||_{0,\mathcal{O}}^2 + \frac{1}{2}\tau\,\varepsilon \sum_{n=1}^{k} ||\bar{\rho}_h^{n-1/2}||_{a,\mathcal{O}}^2 \\ &\le \tau \sum_{n=1}^{k} ||\bar{\rho}_h^{n-1/2}||_{0,\Omega}^2 + \tau\,\varepsilon \sum_{n=1}^{k} ||\bar{\rho}_h^{n-1/2}||_{0,\mathcal{O}}^2 + \varepsilon\, ||f_0||_{C(0,T;L^2(\mathcal{O}))}^2 + \varepsilon\, ||u||_{C(0,T;H^2(\Omega))}^2 \\ &\qquad + C\,(h^4 + \tau^4 + \varepsilon) \int_0^T \left(|u_t|_{2,\Omega}^2 + ||u_{tt}||_{1,\Omega}^2 + ||u_{ttt}||_{(H^1(\Omega))'}^2 + ||u_t||_{2,\mathcal{O}}^2\right) dt \\ &\le \tau \sum_{n=1}^{k} ||\rho_h^n||_{0,\Omega}^2 + \tau\,\varepsilon \sum_{n=1}^{k} ||\rho_h^n||_{0,\mathcal{O}}^2 + C\,(h^4 + \tau^4 + \varepsilon). \end{aligned} \quad (4.20)$$

where we have used the estimates of $||\rho_h^0||_{0,\mathcal{O}}$.

Now by Gronwall's inequality, we obtain that

$$\max_{1\le n\le M} ||\rho_h^n||_{0,\Omega}^2 + \varepsilon \max_{1\le n\le M} ||\rho_h^n||_{0,\mathcal{O}}^2 + \tau \sum_{n=1}^{M} \left(||\bar{\rho}_h^{n-1/2}||_{a,\Omega}^2 + \varepsilon \,||\bar{\rho}_h^{n-1/2}||_{a,\mathcal{O}}^2\right)$$
$$\le C\,(h^4 + \tau^4 + \varepsilon). \tag{4.21}$$

Then (4.2) follows from above and the triangle's inequality for $u^n - u_h^n = u^n - \Pi_h u^n - \rho_h^n$.

To prove (4.3), substituting $v = \tau\,\partial_\tau \rho_h^n$ into (4.8) and using Young's inequality implies that

$$\frac{1}{2}\tau\left(||\partial_\tau\rho_h^n||_{0,\Omega}^2 + \varepsilon\,||\partial_\tau\rho_h^n||_{0,\mathcal{O}}^2\right)^2 + \frac{1}{2}\left(||\rho_h^n||_{a,\Omega}^2 - ||\rho_h^{n-1}||_{a,\Omega}^2\right) + \frac{\varepsilon}{2}\left(||\rho_h^n||_{a,\mathcal{O}}^2 - ||\rho_h^{n-1}||_{a,\mathcal{O}}^2\right)$$
$$\le \tau\,||\partial_\tau(Eu^n - \Pi_h Eu^n)||_{0,\Omega}^2 + \tau\,\varepsilon\,||f_0^{n-1/2}||_{0,\mathcal{O}}^2 + \tau\,||u_t^{n-1/2} - \partial_\tau u^n||_{0,\Omega}^2$$
$$+\tau\, a_\Omega(E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2},\, \partial_\tau\rho_h^n) + \tau\, a_\Omega(u^{n-1/2} - \bar{u}^{n-1/2},\, \partial_\tau\rho_h^n)$$
$$+\tau\,\varepsilon\,||\partial_\tau\Pi_h Eu^n||_{0,\mathcal{O}}^2 + \tau\,\varepsilon\,||\Pi_h E\bar{u}^{n-1/2}||_{a,\mathcal{O}}^2. \tag{4.22}$$

By means of the techniques used in (4.10), (4.13), (4.11), (4.14) and (4.17) and (4.18), we get

$$\tau\,||\partial_\tau(Eu^n - \Pi_h Eu^n)||_{0,\Omega}^2 \le C\,h^4 \int_{t_{n-1}}^{t_n} |u_t|_{2,\Omega}^2\,dt\,, \tag{4.23}$$
$$\tau\,||u_t^{n-1/2} - \partial_\tau u^n||_{0,\Omega}^2 \le C\,\tau^4 \int_{t_{n-1}}^{t_n} ||u_{tt}||_{1,\Omega}^2 dt, \tag{4.24}$$
$$\tau\,||E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2}||_{a,\Omega}^2 \le C\,\tau\, h^4 |\bar{u}^{n-1/2}|_{3,\Omega}^2, \tag{4.25}$$
$$\tau\,||u^{n-1/2} - \bar{u}^{n-1/2}||_{a,\Omega}^2 \le \tau^4 \int_{t_{n-1}}^{t_n} ||u_{tt}||_{1,\Omega}^2 dt, \tag{4.26}$$
$$\tau\,\varepsilon\,||\partial_\tau\Pi_h Eu^n||_{0,\mathcal{O}}^2 \le \frac{1}{2}\varepsilon \int_{t_{n-1}}^{t_n} ||u_t||_{2,\mathcal{O}}^2\,dt, \tag{4.27}$$
$$\tau\,\varepsilon\,||\Pi_h E\bar{u}^{n-1/2}||_{a,\mathcal{O}}^2 \le \tau\varepsilon\,||\bar{u}^{n-1/2}||_{2,\mathcal{O}}^2. \tag{4.28}$$

From (4.25) and (4.26) we deduce that

$$a_\Omega(E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2},\, \tau\partial_\tau\rho_h^n) \le ||E\bar{u}^{n-1/2} - \Pi_h E\bar{u}^{n-1/2}||_{a,\Omega}(||\rho_h^n||_{a,\Omega} + ||\rho_h^{n-1}||_{a,\Omega})$$
$$\le C\tau(||\rho_h^n||_{a,\Omega}^2 + ||\rho_h^{n-1}||_{a,\Omega}^2) + \tau^{-1}\,h^4\,||u||_{C(0,T;H^3(\Omega))}^2 \tag{4.29}$$

and

$$a_\Omega(u^{n-1/2} - \bar{u}^{n-1/2},\, \partial_\tau\rho_h^n) \le ||u^{n-1/2} - \bar{u}^{n-1/2}||_{a,\Omega}(||\rho_h^n||_{a,\Omega} + ||\rho_h^{n-1}||_{a,\Omega})$$
$$\le \tau\,(||\rho_h^n||_{a,\Omega}^2 + ||\rho_h^{n-1}||_{a,\Omega}^2) + \tau^2 \int_{t_{n-1}}^{t_n} ||u_{tt}||_{1,\Omega}^2 dt. \tag{4.30}$$

Therefore, taking the sum from $n = 1$ to $n = k \le M$ in (4.22), using (4.23)–(4.30) and Gronwall's inequality, we have

$$\max_{1 \le n \le M} \left(||\rho_h^n||_{a,\Omega}^2 + \varepsilon \, ||\rho_h^n||_{a,\mathcal{O}}^2 \right) + \tau \sum_{n=1}^{M} \left(||\partial_\tau \rho_h^n||_{0,\Omega}^2 + \varepsilon \, ||\partial_\tau \rho_h^n||_{0,\mathcal{O}}^2 \right)$$
$$\le C \, (h^2 + \tau^2 + \varepsilon + h^4 \, \tau^{-1}). \tag{4.31}$$

Thus we proved (4.3), this completes the proof of Theorem 4.1.

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