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OVERLAPPING SCHWARZ METHODS ON UNSTRUCTURED MESHES USING NON-MATCHING COARSE GRIDS

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Abstract. We consider two level overlapping Schwarz domain decomposition methods for solving the finite element problems that arise from discretizations of elliptic problems on general unstructured meshes in two and three dimensions. Standard finite element interpolation from the coarse to the fine grid may be used. Our theory requires no assumption on the substructures which constitute the whole domain, so each substructure can be of arbitrary shape and of different size. The global coarse mesh is allowed to be non-nested to the fine grid on which the discrete problem is to be solved and both the coarse meshes and the fine meshes need not be quasi-uniform. In addition, the domains defined by the fine and coarse grid need not be identical. The one important constraint is that the closure of the coarse grid must cover any portion of the fine grid boundary for which Neumann boundary conditions are given. In this general setting, our algorithms have the same optimal convergence rate of the usual two level overlapping domain decomposition methods on structured meshes. The condition number of the preconditioned system depends only on the (possibly small) overlap of the substructures and the size of the coarse grid, but is independent of the sizes of the subdomains.

Key Words. Unstructured meshes, non-nested coarse meshes, Schwarz methods, optimal convergence rate.

AMS(MOS) subject classification. 65N30, 65F10

1. Introduction. Unstructured grids are very popular, and extra flexible to allow for complicated geometries and the resolution of fine scale structure in the solution [1], [18]. However, this flexibility may come with a price. Traditional solvers which exploit the regularity of the mesh may become less efficient on an unstructured mesh. Moreover, efficient vectorization and parallelization may require extra care. Thus, there is a need to adapt and develope current solution techniques for structured meshes so that they can run as efficiently on unstructured meshes.

In this paper, we will present Schwarz methods defined for overlapping subdomains, for solving elliptic problems on unstructured meshes in two and three space dimensions. These are extensions of existing domain decomposition methods, constructed in such a way so that, first, they can be applied to unstructured meshes, and second, they retain their optimal efficiency as for structured meshes. These methods are designed to possess inherent coarse grain parallelism in the sense that the subdomain problems can be solved independently on different processors.

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The theory and methodology of domain decomposition methods for elliptic problems on structured meshes are quite well developed, cf. [2], [3], [10], [12], etc. On a structured mesh, most of the existing theories and algorithms exploit the fact that the space of functions on the coarse mesh is a subspace of that on the fine mesh. Unfortunately, this property may no longer hold on an unstructured mesh. Both the theory and the algorithms need to be developed to accomodate this fact.

In this paper, we continue to develop the theory of overlapping Schwarz methods for elliptic problems in two and three dimensions on unstructured meshes begun by Cai [4] and Chan and Zou [6]. Our main new results are: (1) to prove convergence even when the domains defined by the fine grid, Ω^h, and coarse grid Ω^H are not identical, for instance the coarse grid only covers a (large) portion of the fine grid, and (2) to provide a simple proof of convergence when standard finite element interpolation from the coarse to fine grid is used that also holds for non-quasi-uniform triangulations. An important observation will be made that to obtain these strong results, in general, any Neumann boundary must be covered by the coarse grid. As in the earlier work, the subdomains are allowed to be of arbitrary shapes.

2. The finite element problem. We consider the following self-adjoint elliptic problem:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + b\,u = f, \qquad \text{in } \Omega \tag{1}$$

with Dirichlet boundary condition

$$u = 0, \qquad \text{on } \Gamma \tag{2}$$

and with mixed Neumann boundary condition

$$\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j} n_i + \alpha\, u = 0, \qquad \text{on } \partial\Omega \setminus \Gamma \tag{3}$$

where $\Omega \subset R^d\,(d = 2,3)$, $(a_{ij}(x))$ is symmetric, uniformly positive definite, and is allowed to be discontinuous but varies slowly on the domain, $b(x) \geq 0$ in Ω, $\alpha(x) \geq 0$ on $\partial\Omega$, and $n = (n_1, n_2, \cdots, n_d)$ is the unit outer normal of the boundary $\partial\Omega$.

By Green's formula, it is immediate to derive the variational problem corresponding to (1)–(3): Find $u \in H^1_\Gamma(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma\}$ such that

$$a(u, v) = f(v), \ \forall\ v \in H^1_\Gamma(\Omega) \tag{4}$$

with

$$\begin{aligned} a(u,v) &= \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + b\,uv\Big)dx + \int_{\partial\Omega\setminus\Gamma} \alpha\, uv\, ds, \\ f(v) &= \int_\Omega f\, v\, dx. \end{aligned}$$

We will solve the above variational problem (4) by the finite element method. Suppose we are given a family of triangulations $\{\mathcal{T}^h\}$ on Ω, consisting of simplices. We will not discuss the effects of approximating Ω but always assume in the paper that the triangulations $\{\mathcal{T}^h\}$ of Ω are exact. So we have $\Omega = \Omega^h \equiv \cup_{\tau\in\mathcal{T}^h}\tau$. Let $h = \overline{h} = \max_{\tau\in\mathcal{T}^h} h_\tau$, $h_\tau =$ diam τ, $\underline{h} = \min_{\tau\in\mathcal{T}^h} h_\tau$, $\rho_\tau =$ the radius of the largest ball inscribed in τ. Then we say $\mathcal{T}^h$ is *shape regular* if it satisfies

$$(5) \qquad \sup_h \max_{\tau\in\mathcal{T}^h} \frac{h_\tau}{\rho_\tau} \le \sigma_0\,,$$

and we say $\mathcal{T}^h$ is *quasi-uniform* if it is shape regular and satisfies

$$(6) \qquad \overline{h} \le \gamma\,\underline{h}\,,$$

with σ_0 and γ fixed positive constants, see Ciarlet [8]. In the paper, we will only assume that the elements are shape regular, but not necessarily quasi-uniform.

Let V^h be a piecewise linear finite element subspace of $H^1_\Gamma(\Omega)$ defined on $\mathcal{T}^h$ with its basis denoted by $\{\phi_i^h\}_{i=1}^n$, and O_i =supp ϕ_i^h. Later on we will use the following simple facts: if $\mathcal{T}^h$ is shape regular, there exist a positive constant C and an integer ν, both depending only on σ_0 appearing in (5) and independent of h, so that, for $i = 1, 2, \cdots, n$,

$$(7) \qquad \operatorname{diam} O_i \le C\,h_\tau,\ \forall\,\tau \subset O_i,$$

$$(8) \qquad \operatorname{card}\{\tau\in\mathcal{T}^h;\ \tau\subset O_i\} \le \nu.$$

Our finite element problem is: Find $u^h \in V^h$ such that

$$(9) \qquad a(u^h, v^h) = f(v^h),\ \forall\ v^h \in V^h.$$

The corresponding linear system is

$$(10) \qquad Au = f$$

with $a = \big(\,a(\phi_i^h, \phi_j^h)\,\big)_{i,j=1}^n$ being the corresponding stiffness matrix.

Because of the ill-conditioning of the stiffness matrix A, our goal is to construct a good preconditioner M for A by domain decomposition methods to be used with the preconditioned conjugate gradient method.

As usual, we decompose the domain Ω into p nonoverlapping subdomains Ω_i such that $\overline{\Omega} = \cup_{i=1}^p \overline{\Omega}_i$, then extend each subdomain Ω_i to a larger one Ω_i' such that the distance between $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by $\delta_i > 0$. We denote the minimum of all δ_i by δ. We assume that $\partial\Omega_i'$ does not cut through any element $\tau\in\mathcal{T}^h$. For the subdomains meeting the boundary we cut off the part of Ω_i' which is outside of $\overline{\Omega}$. No other assumptions will be made on $\{\Omega_i\}$ in this paper except that any point $x\in\Omega$ belongs only to a finite number of subdomains $\{\Omega_i'\}$. This means that we allow each Ω_i to be of quite different size and of quite different shape from other subdomains. We define the subspaces of V^h corresponding to the subdomains $\{\Omega_i'\}$, $i = 1, 2, \cdots, p$ by

$$(11) \qquad V_i^h = \{v_h \in V^h;\, v_h = 0 \text{ on } (\Omega\setminus\Omega_i') \cup (\partial\Omega\setminus(\partial\Omega\cap\partial\Omega_i))\}$$

For interior subdomains, and those adjacent to only a Dirichlet boundary,

$$V_i^h = V^h \cap H_0^1(\Omega_i'). \tag{12}$$

To develop a two level method, we also introduce a coarse grid $\mathcal{T}^H$ which forms a shape regular triangulation of Ω, but has nothing to do with $\mathcal{T}^h$, i.e., none of the nodes of $\mathcal{T}^H$ need to be nodes of $\mathcal{T}^h$. In general, $\Omega^H \neq \Omega$. Let H be the maximum diameter of the elements of $\mathcal{T}^H$, and $\Omega^H = \cup_{\tau^H \in \mathcal{T}^H} \tau^H$, and more, let Γ^H denote the portion of the boundary $\partial\Omega^H$ to which we will apply Dirichlet boundary conditions. (If the original problem is not pure Neumann we require that the measure of Γ^H be at least the order of one coarse element size.)

By V^H we denote a subspace of $H^1_{\Gamma^H}(\Omega^H)$ consisting of piecewise polynomials defined on $\mathcal{T}^H$, by $\{\psi_i^H\}_{i=1}^m$ we denote its basis functions related to the nodes $\{q_i^H\}_{i=1}^m$. Let $O_i^H = \text{supp } \psi_i^H$. We note that V^H need not necessarily be piecewise linear; for example, it may be bilinear (2-D) and trilinear (3-D) elements or higher order elements. Thus we do not necessarily have the usual condition: $V^H \subset V^h$. We need to impose one important constraint on the coarse grid:

(A1): $\quad \partial\Omega \setminus \Gamma \subset \overline{\Omega^H}$.

That is, the coarse grid covers all of the Neumann boundary, See Fig. 1.

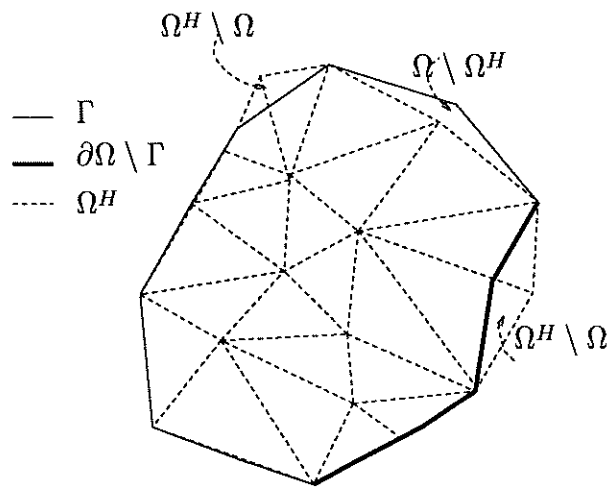

FIG. 1.

For technical reasons, we make two less restrictive assumptions further on the coarse grid:

(A2): $\quad \tau^H \cap \Omega \neq \emptyset$ for all $\tau^H \in \mathcal{T}^H$,

that is, no coarse grid element lies completely outside the fine grid. For the complement set $\Omega \setminus \Omega^H$, let S be the set of all vertices q_i^H of Ω^H which belongs to $\overline{\Omega \setminus \Omega^H}$, and $B_p(r)$ be a ball at the point p with radius r. We assume that

(A3): $\quad \Omega \setminus \Omega^H \subset \cup_{q_i^H \in S} B_{q_i^H}(\text{diam } O_i^H)$,

that is, the coarse grid must cover a significant part of the fine grid.

To overcome the difficulty that $V^H \not\subset V^h$, in both the theory and the algorithms, we need a way of mapping values from V^H to V^h. For the coarse space to be effective, this mapping must possess the properties of H^1-stability and L^2 optimal approximation, see Chan and Zou [6], Mandel [17]. In this paper we shall mainly consider two such mappings. The first is the standard finite element interpolation Π_h defined by the fine grid basis functions $\{\phi_i^h\}_{i=1}^n$. The second is the local, L^2-like projection, $\mathcal{R}_h$ used in Chan and Zou [6].

Throughout the paper, we use $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ to denote the norm and semi-norm of the usual Sobolev space $H^m(\Omega)$ for any integer $m \geq 0$. In addition, $||\cdot||_{m,r,\Omega}$ and $|\cdot|_{m,r,\Omega}$ will denote the norm and semi-norm of the spaces $W^{m,r}(\Omega)$ for any integer $m \geq 0$ and real number $r \geq 1$.

3. Two level overlapping Schwarz algorithms. Based on the finite element spaces V_i^h and V^H given in the last section, we derive the two level overlapping Schwarz methods for nonnested grids. Schwarz methods are preconditioning for the linear system $Au = f$ that are built using local and coarse grid solves. We first define these solves. From these we may write down the preconditioners using matrix notations.

The local solves are defined as in Dryja and Widlund [11], and Bramble, Pasciak, Wang, and Xu [3]. Define the H^1-projection operators $P_i : V^h \to V_i^h$, $i = 1, \cdots, p$ such that for any $u \in V^h$, $P_i u \in V_i^h$ satisfies

$$a(P_i u, v_i) = a(u, v_i), \qquad \forall\, v_i \in V_i^h. \tag{13}$$

The coarse grid projection-like operator must be defined slightly differently than in Dryja and Widlund [11], due to the non-nestedness of the coarse grid space. Let $\mathcal{I}_h$ be any linear operator which maps V^H into a subspace $\mathcal{I}_h V^H$ of V^h, so $\mathcal{I}_h$ may be chosen as the modified standard finite element interpolation operator Π_h or the locally defined operator $\mathcal{R}_h$, see Section 5 for more details.

In **Method 1** we define $\tilde{P}_0$ by first defining $P_H u \in V^H$ on the original coarse grid space by

$$a(P_H u, v) = a(u, \mathcal{I}_h v), \qquad u \in V^h, \quad \forall v \in V^H \tag{14}$$

and then define $\tilde{P}_0 = \mathcal{I}_h P_H$: $V^h \to V_0^h$. The subspace $V_0^h \in V^h$ is defined by $\mathcal{I}_h V^H$.

In **Method 2** we define P_0 by calculating the projection directly onto the subspace V_0^h,

$$a(P_0 u, v) = a(u, v), \qquad u \in V^h, \quad \forall v \in V_0^h, \tag{15}$$

where $P_0 u \in V_0^h$.

REMARK 3.1. *We note here that for the left-hand side in (14), $a(u_H, v_H)$ for any $u_H, v_H \in V^H$, is not an integral over original domain Ω, but the one over the coarse domain Ω^H, i.e.*

$$a(u_H, v_H) = \int_{\Omega^H} \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + b\, uv \Big) dx + \int_{\partial\Omega^H \setminus \Gamma^H} \alpha\, uv\, ds\,. \tag{16}$$

Thus in the sequel we always assume that the coefficient functions a_{ij}, b, α are continuously extended onto Ω^H. And later on, we will use $\|\cdot\|_a$ and $\|\cdot\|_{a,\Omega^H}$ to denote the energy norms $a(\cdot,\cdot)$ over Ω and Ω^H, respectively.

We now derive the matrix representation of the operators P_i and $\tilde{P}_0$. Using these both the additive and multiplicative Schwarz preconditioners may be written down. For the rest of this section only, we will use u^h to denote finite element functions and u to denote the vector of coefficients of that finite element function, that is $u^h = \sum u_k \phi_k$.

Let $\{\phi^h_{i,j}\}_{j=1}^{n_i} \subset \{\phi^h_i\}_{i=1}^n$ be the set of nodal basis functions of V_i^h, $i = 1, 2, \cdots, p$. For each i, we define a matrix extension operator R_i^T as follows: For any $u_i^h \in V_i^h$, we denote by u_i the coefficient vector of u_i^h in the basis $\{\phi^h_{i,j}\}_{j=1}^{n_i}$, and we define that $R_i^T u_i$ to be the coefficient vector of u_i^h in the basis $\{\phi^h_i\}_{i=1}^n$.

It is immediate to check that

$$A_i = R_i A\, R_i^T \tag{17}$$

where A and A_i, $i = 1, 2, \cdots, p$ are the stiffness matrices corresponding to the fine subspace V^h and the subspaces V_i^h, $i = 1, 2, \cdots, p$. And from (13) it follows that for any $u^h \in V^h$, the coefficient vector of $P_i u^h$ in the basis $\{\phi^h_i\}_{i=1}^n$ is

$$R_i^T A_i^{-1} R_i A\, u \tag{18}$$

where u denotes the coefficient vector of u^h in the basis $\{\phi^h_i\}_{i=1}^n$.

Since $\{\psi_i^H\}_{i=1}^m$ is the set of basis functions of V^H, then $\{\mathcal{I}_h \psi_i^H\}_{i=1}^m$ is the set of basis functions of V_0^h. We define a matrix extension operator R_0^T as follows: For any $u_0^h \in V_0^h$, we denote by u_0 the coefficient vector of u_0^h in the basis $\{\mathcal{I}_h \psi_i^H\}_{i=1}^m$, define $R_0^T u_0$ to be the coefficient vector of u_0^h in the basis $\{\phi_j^h\}_{j=1}^n$. Then $R_{0_{ij}} = \mathcal{I}_h \psi_i^H(q_j)$ where q_j is the nodal vertex of ϕ_j^h. When $\mathcal{I}_h = \Pi_h$ then $R_{0_{ij}}$ is simply given by $\psi_i^H(q_j)$.

We first note that the coefficient vector of a function $v \in V^H$ in the basis $\{\psi_i^H\}_{i=1}^m$ is exactly the same as the one for the function $\mathcal{I}_h v$ in the basis $\{\mathcal{I}_h \psi_i^H\}_{i=1}^m$. So from (14) we find that the coefficient vector of $P_H u^h$ in the basis $\{\psi_i^H\}_{i=1}^m$ is

$$A_H^{-1} R_0 A\, u \tag{19}$$

where A_H is the stiffness matrix corresponding to the original coarse space V^H, that is $A_{H_{ij}} = a(\psi_j^H, \psi_i^H)$. Now using the previously given fact we know the coefficient vector of $\tilde{P}_0 u^h = \mathcal{I}_h P_H u^h \in V_0^h$ in the basis $\{\mathcal{I}_h \psi_i^H\}_{i=1}^m$ is also $A_H^{-1} R_0 A\, u$, therefore, by the definition of R_0^T, $R_0^T A_H^{-1} R_0 A\, u$ is the coefficient vector of $\tilde{P}_0 u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$.

For Method 2 it is straightforward to derive that

$$A_0 = R_0 A\, R_0^T \tag{20}$$

where A_0 is the stiffness matrix corresponding to the subspace V_0^h. It follows from (15) that the coefficient vector of $P_0 u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$ is

$$R_0^T A_0^{-1} R_0 A\, u. \tag{21}$$

From above, the additive Schwarz preconditioner may be written as

$$M_1 = R_0^T A_H^{-1} R_0 + \sum_{i=1}^{p} R_i^T A_i^{-1} R_i, \tag{22}$$

for Method 1 and

$$M_2 = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{p} R_i^T A_i^{-1} R_i, \tag{23}$$

for Method 2. These may be thought of as an overlapping block Jacobi method with the addition of a coarse grid correction. The multiplicative Schwarz method is the Gauss-Seidel version of the additive algorithm. We write down the symmetrized version, using Method 1 as,

$$\begin{aligned} M &= (I - (I - R_1^T A_1^{-1} R_1 A) \dots (I - R_p^T A_p^{-1} R_p A)(I - R_0^T A_H^{-1} R_0 A) \\ & \quad (I - R_p^T A_p^{-1} R_p A) \dots (I - R_1^T A_1^{-1} R_1 A)) A^{-1}. \end{aligned} \tag{24}$$

In practice the application of the multiplicative Schwarz preconditioner is done directly, not as given in (24).

REMARK 3.2. *From the above matrix representations M_1 and M_2 for* ***Method 1*** *and* ***Method 2****, we see that the only difference between them is in the global coarse problem solver. The latter coarse problem (with A_0^{-1}) is conducted on the newly constructed coarse subspace V_0^h, but the former (with A_H^{-1}) is conducted on the original coarse subspace V^H. Since V^H is not necessarily nested to V^h, A_H may not be expressed in terms of the stiffness matrix A as A_0 in (20).*

We give the convergence results for the additive algorithm, similar results may be obtained for the multiplicative algorithm using the techniques in Xu [25] or Dryja, Smith, and Widlund [10].

It is easy to check that

$$\kappa(M_1 A) = \kappa(\tilde{P}_0 + \sum_{i=1}^{p} P_i), \quad \kappa(M_2 A) = \kappa(\sum_{i=0}^{p} P_i). \tag{25}$$

For these condition numbers, we have the following bounds:

THEOREM 3.1. *Suppose that both triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular (not necessarily quasi-uniform), and satisfy Assumptions* ***(A1) - (A3)****. Then we have*

$$\kappa(M_1 A),\ \kappa(M_2 A) \le C\,(1 + \frac{H}{\delta})^2. \tag{26}$$

Theorem (3.1) will be proved at the end of Section 5.

4. Boundedness properties of the operator $\mathcal{I}_h$. Let W^h and W^H be any two finite element subspaces related to the triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$, respectively. Since $W^H \not\subset W^h$ for our interest, the convergence proof for the overlapping two level Schwarz methods requires the operator $\mathcal{I}_h : W^H \to W^h$ to possess the following H^1 stability and L^2 optimal approximation properties:

$$(27) \qquad |\mathcal{I}_h u|_{1,\Omega} \le C|u|_{1,\Omega^H}, \qquad \forall u \in W^H,$$

and

$$(28) \qquad ||\mathcal{I}_h u - u||_{0,\Omega} \le Ch|u|_{1,\Omega^H}, \qquad \forall u \in W^H.$$

There exist many options for the operator $\mathcal{I}_h$, for example, L^2 and quasi-L^2 projection operators Q_h and $\tilde{Q}_h$, or Clément's local L^2 projection operator $\mathcal{R}_h$. For these discussions, we refer to Chan and Zou [6]. In this paper, we are mainly interested in the most natural option for $\mathcal{I}_h$, i.e., the standard finite element interpolation operator Π_h and the Clément's local L^2 projection operator $\mathcal{R}_h$.

Generally, for $\mathcal{I}_h = \Pi_h$, in three dimensions (27) and (28) are not true for all $u \in H^1(\Omega)$. Fortunately, they are true in the general finite element spaces. We state this fact in the following lemma. Several alternative proofs for this result exist, see, for instance, Cai [4], Zhang [26] and Widlund [23].

LEMMA 4.1. *Assume that $\mathcal{T}^h$ and $\mathcal{T}^H$ are both shape regular, not necessarily quasi-uniform, and W^h and W^H are any two corresponding finite element spaces consisting of continuous piecewise polynomials defined on Ω and Ω^H, respectively. Furthermore, we assume that $\Omega \subset \Omega^H$. Then (27) and (28) hold in both two and three dimensions for $\mathcal{I}_h = \Pi_h$.*

Proof. Let $\tau^h \in \mathcal{T}^h$, then (see, for example, Ciarlet [8], Theorem 3.1.5), for $r > 3$ and $s = 0, 1$, we know for any $u \in W^H$

$$(29) \qquad |u - \Pi_h u|^2_{s,\tau^h} \le Ch^{2(1-s)} h_\tau^{2d(1/2-1/r)} |u|^2_{1,r,\tau^h},$$

this implies

$$(30) \qquad \sum_{\tau^h \cap \tau^H \neq \emptyset} |u - \Pi_h u|^2_{s,\tau^h} \le C\, h^{2(1-s)} \sum_{\tau^h \cap \tau^H \neq \emptyset} h_\tau^{2d(1/2-1/r)} |u|^2_{1,r,\tau^h}.$$

Now apply the Cauchy inequality

$$\sum_i a_i b_i \le (\sum_i a_i^q)^{1/q} (\sum_i b_i^p)^{1/p}$$

to the right hand side with $p = r/2 > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}
(31) \quad \sum_{\tau^h \cap \tau^H \neq \emptyset} |u - \Pi_h u|^2_{s,\tau^h} &\le C\, h^{2(1-s)} (\sum_{\tau^h \cap \tau^H \neq \emptyset} h_\tau^{2d(1/2-1/r)q})^{1/q} (\sum_{\tau^h \cap \tau^H \neq \emptyset} |u|^{2p}_{1,r,\tau^h})^{1/p} \\
(32) \qquad\qquad &\le C\, h^{2(1-s)} (\sum_{\tau^h \cap \tau^H \neq \emptyset} h_\tau^d)^{1-2/r} (\sum_{\tau^h \cap \tau^H \neq \emptyset} |u|^r_{1,r,\tau^h})^{2/r} \\
(33) \qquad\qquad &\le C\, h^{2(1-s)} H_\tau^{d(1-2/r)} (\sum_{\bar{\tau}^H \cap \bar{\tau}^{H'} \neq \emptyset} |u|^r_{1,r,\tau^{H'}})^{2/r} \\
(34) \qquad\qquad &\le C\, h^{2(1-s)} H_\tau^{d(1-2/r)} \left(H_\tau^{d(2/r-1)} \sum_{\bar{\tau}^H \cap \bar{\tau}^{H'} \neq \emptyset} |u|^2_{1,\tau^{H'}} \right).
\end{aligned}$$

The last line follows from (7) and (8), and a standard local inverse inequality, see, for instance, Ciarlet [8], Theorem 3.2.6. Note that since the sum is only over those elements that are neighbors to τ^H we do not need the quasi-uniform assumption, called the inverse assumption by Ciarlet [8], only the shape regular assumption.

Taking the sum over τ^H, we obtain

$$\sum_{\tau^H} \sum_{\tau^h \cap \tau^H \neq \emptyset} |u - \Pi_h u|^2_{s,\tau^h} \leq C h^{2(1-s)} |u|^2_{1,\Omega^H}, \tag{35}$$

which implies (27) and (28). □

For our later use, we introduce a special, locally defined projection operator $\mathcal{R}_h$ which was used in the domain decomposition context in [7] and [6]. Operators with similar properties to $\mathcal{R}_h$ can be also found in Scott and Zhang [20].

We denote the set of basis functions of V^h by $\{\phi_i^h\}_{i=1}^n$ corresponding to the vertices $\{q_i^h\}_{i=1}^n$. Let $O_i =$supp ϕ_i^h, $i = 1, 2, \cdots, n$.

DEFINITION 4.1. *The mapping $\mathcal{R}_h^0 : L^2(\Omega) \to V^h$ is defined by*

$$\mathcal{R}_h^0 u = \sum_{i=1}^{n} Q_i\, u(q_i^h)\, \phi_i^h, \ \forall\ u \in L^2(\Omega), \tag{36}$$

where $Q_i u \in \mathcal{P}_1(O_i)$ satisfies

$$\int_{O_i} Q_i u\, p\, dx = \int_{O_i} u\, p\, dx, \ \forall\ p \in \mathcal{P}_1(O_i) \tag{37}$$

where $\mathcal{P}_1(O_i)$ is the space of linear functions defined on O_i.

By using Poincaré inequality, the definitions of $\mathcal{R}_h^0$ and the relations (7) and (8), we can show the following properties of $\mathcal{R}_h^0$, see also Clément [9].

LEMMA 4.2. *The operator $\mathcal{R}_h^0$ defined by (36) and (37) has the properties*

$$||\mathcal{R}_h^0 u||_{r,\Omega} \leq C\, ||u||_{r,\Omega}, \ \forall\ u \in H_\Gamma^1(\Omega), r = 0, 1, \tag{38}$$

$$||u - \mathcal{R}_h^0 u||_{0,\Omega} \leq C\, h |u|_{1,\Omega}, \ \forall\ u \in H_\Gamma^1(\Omega) \tag{39}$$

where the constant C is independent of h.

REMARK 4.1. *In Lemma 4.2, we assume only that T^h is shape regular, not necessarily quasi-uniform, unlike the usual L^2 projection.*

REMARK 4.2. *The definition $\mathcal{R}_h^0$ can be generalized to more general finite element spaces V^h, not restricted to the subspaces of V^h, e.g., to bilinear element (2-D), trilinear element (3-D), and higher order elements. In these cases, one needs only to replace $\mathcal{P}_1(O_i)$ in the relations (36) and (37) by $\tilde{P}(O_i)$ which are determined by the types of elements used in V^h, and Lemma 4.2 will still hold.*

5. Partition lemma. In this section, we give a partition lemma for the finite element space V^h which is very essential to the convergence proof of Theorem 3.1. As denoted previously, let $\{\psi_i^H\}_{i \in N_H}$ be the set of basis functions of V^H with $N_H = \{1, 2, \cdots, m\}$, and let $\{q_i^H\}_{i \in N_H}$ be the corresponding nodes and $O_i =$ supp ψ_i^H.

We introduce an auxiliary subspace $\tilde{V}^H$ of V^H:

$$\tilde{V}^H = \text{span } \{\psi_i^H; i \in N_H^0\} \tag{40}$$

with $N_H^0 = \{i \in N_H;\ \psi_i^H = 0 \text{ on } \Gamma\}$. We only need $\tilde{V}^H$ for the proof of our main theorem, we do not require its explicit computation for our algorithms. It is easy to check that $\tilde{V}^H|_\Omega \subset H^1_\Gamma(\Omega)$.

By $\tilde{\Omega}^H$ and Ω_N^H we denote

$$\tilde{\Omega}^H = \cup_{i \in N_H^0} \text{supp } \psi_i^H, \quad \Omega_N^H = \tilde{\Omega}^H \cup (\Omega \setminus \tilde{\Omega}^H). \tag{41}$$

Let $\mathcal{R}_H$ be defined for V^H similarly to $\mathcal{R}_h$ defined for V^h in (36) and (37), with natural modifications, see Remark 4.2. Now we define a modified operator $\tilde{\mathcal{R}}_H : L^2(\tilde{\Omega}^H) \to \tilde{V}^H$ as follows

$$\tilde{\mathcal{R}}_H u = \sum_{i \in N_H^0} Q_i\, u(q_i^H)\, \psi_i^H, \ \forall\ u \in L^2(\tilde{\Omega}^H), \tag{42}$$

where $Q_i\, u \in \mathcal{P}(O_i)$ satisfies

$$\int_{O_i} Q_i u\, p\, dx = \int_{O_i} u\, p\, dx, \ \forall\ p \in \mathcal{P}(O_i). \tag{43}$$

Here $\mathcal{P}(O_i)$ is determined by the type of elements used in V^H. If V^H consists of piecewise polynomials of degree $\leq q$, then $\mathcal{P}(O_i) = \mathcal{P}_q(O_i)$. We note that $\tilde{\mathcal{R}}_H u$ is well-defined on Ω by extending by 0.

For the operator $\tilde{\mathcal{R}}_H$, we have

LEMMA 5.1. *The operator $\tilde{\mathcal{R}}_H$ defined by (42) and (43) has the properties*

$$\|u - \tilde{\mathcal{R}}_H u\|_{r,\Omega_N^H} \leq C\, \|u\|_{r,\Omega_N^H}, \ \forall\ u \in H^1_\Gamma(\Omega_N^H), r = 0, 1, \tag{44}$$

$$\|u - \tilde{\mathcal{R}}_H u\|_{0,\Omega_N^H} \leq C\, H |u|_{1,\Omega_N^H}, \ \forall\ u \in H^1_\Gamma(\Omega_N^H) \tag{45}$$

where the constant C is independent of h and H.

Proof. Analogous to Lemma 4.2, we can prove that

$$\|u - \mathcal{R}_H u\|_{r,\Omega^H} \leq C\, \|u\|_{r,\Omega^H} \ \forall\ u \in H^1(\Omega^H), r = 0, 1, \tag{46}$$

$$\|u - \mathcal{R}_H u\|_{0,\Omega^H} \leq C\, H |u|_{1,\Omega^H}, \ \forall\ u \in H^1(\Omega^H). \tag{47}$$

Let $\partial N_H = \{i; i \in N_H \setminus N_H^0\}$. We see that for any $u \in H^1_\Gamma(\Omega_N^H)$

$$u - \tilde{\mathcal{R}}_H u = u - \mathcal{R}_H u + \sum_{i \in \partial N_H} Q_i u(q_i^H) \psi_i^H. \tag{48}$$

Using (46) and (47), we need only to estimate the final term of (48).

For any $i \in \partial N_H$, we have by a local finite element inverse inequality, Poincaré inequality, (cf. Ladyzhenskaya et al. [15]), and the previous assumption **(A3)** on

$\Omega \setminus \Omega^H$, that

$$\begin{aligned} \|Q_i u(q_i^H)\psi_i^H\|_{0,O_i}^2 &\leq \sum_{\tau^H \subset O_i} \|\psi_i^H\|_{0,\tau^H}^2 \, \|Q_i u\|_{L^\infty(\tau^H)}^2 \\ &\leq C \sum_{\tau^H \subset O_i} H_\tau^d \left(H_\tau^{-d} \|Q_i u\|_{0,\tau^H}^2\right) \leq C \, \|Q_i u\|_{0,O_i}^2 \leq C \, \|u\|_{0,O_i}^2 \quad (49) \\ &\leq C \, \|u\|_{0,\tilde{O}_i}^2 \leq C \, (\mathrm{diam} \ O_i)^2 |u|_{1,\tilde{O}_i}^2 \leq C \, H^2 \, |u|_{1,\tilde{O}_i}^2, \quad (50) \end{aligned}$$

where $\tilde{O}_i =$ the union of O_i with the part of Ω which is outside O_i, cf. Fig. 1.

Now (44) with $r = 0$ and (45) follow from (46)–(50). (44) with $r = 1$ can be proved analogously to the case of $r = 0$ above. □

Now we choose

$$V_0^h = \mathcal{I}_h V^H, \tag{51}$$

where $\mathcal{I}_h$ can be any linear operator which maps V^H onto the subspace $\mathcal{I}_h V^H$ of V^h and keeps the H^1 stability and L^2- optimal approximation *in any subspace (not necessarily in the whole V^H) of V^H of functions that vanish on* Γ. This essential observation will become very clear when going through the following proof of Lemma 5.2. Therefore, $\mathcal{I}_h$ may be chosen as the standard finite element interpolation operator, or local L^2 projection operator $\mathcal{R}_h$ after simple and natural modifications for meeting the Dirichlet boundary condition on Γ. For example, $\mathcal{I}_h$ may be chosen as $\mathcal{R}_h^0$ defined in Definition 4.1, or as Π_h^0 which is defined as follows

$$\Pi_h^0 u = \sum_{i=1}^{n} u(q_i^h)\phi_i^h. \tag{52}$$

From (51), we note that we require that the coarse grid covers the fine grid Neumann boundary, cf. **(A1)**. If not so, $\mathcal{I}_h$ makes no sense for the part $\Omega \setminus \Omega^H$. But the coarse grid does not need to cover the fine grid Dirichlet boundary, since we impose also homogeneous Dirichlet boundary conditions on the corresponding coarse grid boundary, so $\mathcal{I}_h$ still makes sense by naturally extending coarse grid functions by 0 for the part $\Omega \setminus \Omega^H$. Our numerical experiments will show that this is important for practical computations.

Then we have the following partition lemma for the fine space V^h:

$$V^h = V_0^h + V_1^h + \cdots + V_p^h. \tag{53}$$

LEMMA 5.2. *Let $\Omega \subset R^d \, (d = 2, 3)$. We assume that both triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular, but not necessarily quasi-uniform. Then for any $u \in V^h$, there exists a constant C independent of h, p, H, δ, and $u_i \in V_i^h$, $i = 1, \cdots, p$ and $u_0 = \mathcal{I}_h u_H \in V_0^h$ with $u_H \in V^H$ such that*

$$u = u_0 + u_1 + \cdots + u_p \tag{54}$$

and

$$\sum_{i=1}^{p}||u_i||_{1,\Omega}^2 \le C\,(1+\frac{H}{\delta})^2||u||_{1,\Omega}^2, \tag{55}$$

$$||u_0||_{1,\Omega} \le C\,||u||_{1,\Omega},\ \ ||u_H||_{1,\Omega^H} \le C\,||u||_{1,\Omega}. \tag{56}$$

Proof. Let $\hat{\Omega}$ be an open domain in R^d large enough such that $\Omega \subset \Omega_N^H \subset\subset \hat{\Omega}$. Then we know, cf. Stein [22], that there exists a linear extension operator $E: H^1(\Omega) \to H^1(\hat{\Omega})$ such that $Eu|_\Omega = u$ and

$$||Eu||_{1,\hat{\Omega}} \le C\,||u||_{1,\Omega}. \tag{57}$$

We note that we do not require Eu for $u \in V^h$ be a finite element function. For any $u \in V^h$, we choose $u_0 = \mathcal{I}_h u_H$ with $u_H = \tilde{\mathcal{R}}_H \tilde{u}$ and $\tilde{u} = Eu\Big|_{\Omega_N^H}$. Then from Lemma 5.1 and Lemma 4.1 we obtain

$$\begin{aligned}||u_0||_{1,\Omega} &= ||\mathcal{I}_h\tilde{\mathcal{R}}_H\tilde{u}||_{1,\Omega} \le C\,||\tilde{\mathcal{R}}_H\tilde{u}||_{1,\Omega^H} = C\,||\tilde{\mathcal{R}}_H\tilde{u}||_{1,\Omega_N^H}\\ &\le C\,||\tilde{u}||_{1,\Omega_N^H} \le C\,||\tilde{u}||_{1,\hat{\Omega}} \le C\,||u||_{1,\Omega}\end{aligned} \tag{58}$$

which implies (56), and

$$\begin{aligned}||u-u_0||_{0,\Omega} &\le ||u-\tilde{\mathcal{R}}_H\tilde{u}||_{0,\Omega} + ||\tilde{\mathcal{R}}_H\tilde{u} - \mathcal{I}_h\tilde{\mathcal{R}}_H\tilde{u}||_{0,\Omega}\\ &\le ||\tilde{u}-\tilde{\mathcal{R}}_H\tilde{u}||_{0,\Omega_N^H} + C\,h\,|\tilde{\mathcal{R}}_H\tilde{u}|_{1,\Omega^H}\\ &\le CH|\tilde{u}|_{1,\Omega_N^H} + C\,h|\tilde{u}|_{1,\Omega_N^H} \le C\,H|\tilde{u}|_{1,\hat{\Omega}} \le C\,H|u|_{1,\Omega}.\end{aligned} \tag{59}$$

It is well-known, see Dryja and Widlund [11], Bramble et al. [3] that there exists a partition $\{\theta_i\}_{i=1}^p$ of unity for Ω related to the subdomains $\{\Omega_i'\}$ such that $\sum_{i=1}^p \theta_i(x) = 1$ on Ω and for $i = 1, 2, \cdots, p$,

$$\operatorname{supp}\theta_i \subset \Omega_i' \cup \partial\Omega,\ \ 0 \le \theta_i \le 1 \text{ and } \ ||\nabla\theta_i||_{L^\infty(\Omega_i)} \le C\,\delta_i^{-1}. \tag{60}$$

Now for any $u \in V^h$, let $u_0 = \mathcal{I}_h\tilde{\mathcal{R}}_H\,\tilde{u} \in V^h$ be chosen as above, and $u_i = \Pi_h\theta_i(u-u_0)$ with Π_h being the standard interpolation of V^h. Obviously, $u_i \in V_i^h$ and

$$u = u_0 + u_1 + \cdots + u_p. \tag{61}$$

Then (55) follows in the standard way, see Dryja and Widlund [11] and Smith [21]. But we still give a complete proof here so that one can see clearly that no quasi-uniformity assumption on $\mathcal{T}^h$ and the subdomains $\{\Omega_i\}$ are required in the present case. Let τ be any element belonging to Ω_k' with h_τ being its diameter and $\overline{\theta}_k$ the average of θ_k on element τ. Then from (60) and the fact that $u - u_0 \in V^h$, we get:

$$\begin{aligned}|u_k|_{1,\tau}^2 &\le 2|\overline{\theta}_k\Pi_h(u-u_0)|_{1,\tau}^2 + 2|\Pi_h(\theta_k-\overline{\theta}_k)(u-u_0)|_{1,\tau}^2\\ &\le 2|u-u_0|_{1,\tau}^2 + 2|\Pi_h(\theta_k-\overline{\theta}_k)(u-u_0)|_{1,\tau}^2.\end{aligned}$$

By using the local inverse inequality which requires only the shape regularity of $\mathcal{T}^h$ (see Proposition 3.2 in Xu [24]), we obtain:

$$\begin{aligned} |u_k|_{1,\tau}^2 &\le 2|u-u_0|_{1,\tau}^2 + C\,h_\tau^{-2}||\Pi_h(\theta_k-\overline{\theta}_k)(u-u_0)||_{0,\tau}^2 \\ &\le 2|u-u_0|_{1,\tau}^2 + C\,h_\tau^{-2}\frac{h_\tau^2}{\delta_k^2}||u-u_0||_{0,\tau}^2 \\ &\le 2|u-u_0|_{1,\tau}^2 + C\,\frac{1}{\delta_k^2}||u-u_0||_{0,\tau}^2. \end{aligned}$$

By taking the sum over $\tau \in \Omega_k'$ we have

$$|u_k|_{1,\Omega_k'}^2 \le 2|u-u_0|_{1,\Omega_k'}^2 + C\,\frac{1}{\delta_k^2}||u-u_0||_{0,\Omega_k'}^2. \tag{62}$$

Noticing the assumption made previously that any point $x \in \Omega$ belongs only to a finite number of subdomains $\{\Omega_i'\}$, it follows from (58), (59) and (62) that

$$\sum_{k=1}^{p} |u_k|_{1,\Omega_k'}^2 \le C\,(|u-u_0|_1^2 + \frac{1}{\delta^2}||u-u_0||_0^2) \tag{63}$$

$$\le C\left(1+\frac{H}{\delta}\right)^2 |u|_1^2. \tag{64}$$

Analogously, we derive that

$$\sum_{k=1}^{p} ||u_k||_{0,\Omega_k'}^2 \le C\left(1+\frac{h^2}{\delta^2}\right)||u||_0^2,$$

which completes the proof of (55). □

In the rest of this section, we prove Theorem 3.1. To do so, we first state a general abstract lemma which is a natural extension of that due to Lions [16], Nepomnyaschikh [19], Dryja and Widlund [11], Zhang [27], and Griebel and Oswald [14]. The proof of it is straightforward, similar to the one of Theorem A in [14].

Given a Hilbert space V and a symmetric, positive definite bilinear form $a(\cdot,\cdot)$ and a set of auxiliary spaces V_i for which the bilinear form is also defined, but which are not necessarily the subspaces of V. Suppose there exist "interpolation" operators $I_i : V_i \to V$ and define $T_i : V \to V_i$ by

$$a(T_i u, v) = a(u, I_i v),\ \forall\ v \in V_i. \tag{65}$$

Then $T = \sum_{i=0}^{p} I_i T_i$ satisfies

LEMMA 5.3.

$$a(T^{-1}u, u) = \min_{\substack{u_i \in V_i \\ u=\sum_{i=0}^{p} I_i u_i}} \sum_{i=0}^{p} a(u_i, u_i). \tag{66}$$

Proof of Theorem 3.1. The estimate of $\kappa(M_2A)$ is quite routine by using Lemma 5.2 and (25). To get the bound of $\kappa(M_1A)$, it suffices to show that there exist two constants C_0 and C_1 independent of H, δ, h such that for any $u^h \in V^h$,

$$C_0\, a(\tilde{P}u^h, u^h) \leq a(u^h, u^h) \leq C_1 \left(1 + \frac{H}{\delta}\right)^2 a(\tilde{P}u, u). \tag{67}$$

We first estimate the upper bound. First, from (14) we see that

$$a(P_H u^h, P_H u^h) = a(\mathcal{I}_h P_H u^h, u^h), \tag{68}$$

thus by Cauchy-Schwarz's inequality and stability of $\mathcal{I}_h$,

$$||P_H u^h||^2_{a,\Omega^H} \leq ||u^h||_a\, ||\mathcal{I}_h P_H u^h||_a \leq C\, ||u^h||_a\, ||P_H u^h||_{a,\Omega^H}, \tag{69}$$

i.e., $||P_H u^h||_{a,\Omega^H} \leq C\, ||u^h||_a$, which leads to the following

$$a(\tilde{P}_0 u^h, u^h) = a(P_H u^h, P_H u^h) \leq C\, a(u^h, u^h). \tag{70}$$

But the standard coloring arguments and the fact that the norm of a projection operator equals one gives that, cf. Zhang [27]

$$\sum_{i=1}^{p} a(P_i u^h, u^h) \leq C\, a(u^h, u^h), \tag{71}$$

therefore we have proved the first inequality in (67). For the second inequality, we choose $I_0 = \mathcal{I}_h$, I_i for $i > 0$ to be the identity operator, and $V_0 = V^H$, then applying Lemma 5.3 and Lemma 5.2 gives our results.

REMARK 5.1. *We can improve the bound of Theorem 3.1 by replacing* $(1 + H/\delta)^2$ *by* $(1 + H_{sub}/\delta)$ *if the subdomains* $\{\Omega_i\}_{i=1}^p$ *forms a quasi-uniform triangulation of* Ω *and* $H \leq \beta H_{sub}$ *for some fixed constant* β. *Here* H_{sub} *is the maximum of all diameters of subdomains. This can be done by using a result by Dryja and Widlund [13], cf. Chan and Zou [6].*

6. Numerical Experiments. In this section, we give two numerical experiments for the case $\mathcal{I}_h = \Pi_h$. In our first numerical experiment, we demonstrate that the assumption **(A1)** is necessary in practice, i.e., it is very important to cover the Neumann boundary. When the coarse grid does not completely cover the fine grid Neumann boundary, one obtains rather poor convergence.

We consider the Possion problem on the unit square with either pure homogeneous Dirichlet or mixed boundary conditions. In the case of mixed boundary conditions, we prescribe homogeneous Dirichlet boundary condition for $x \leq 0.2$ and homogeneous Neumann boundary condition for $x > 0.2$. A uniform triangulation using linear finite elements is used. The coarse grid is defined on the square $[0, 1 + \beta] \times [0, 1]$. If β is less than zero we are not covering the right edge of the fine grid, see Fig. 2. Note that only when $\beta = 0$ do we have a nested coarse grid space.

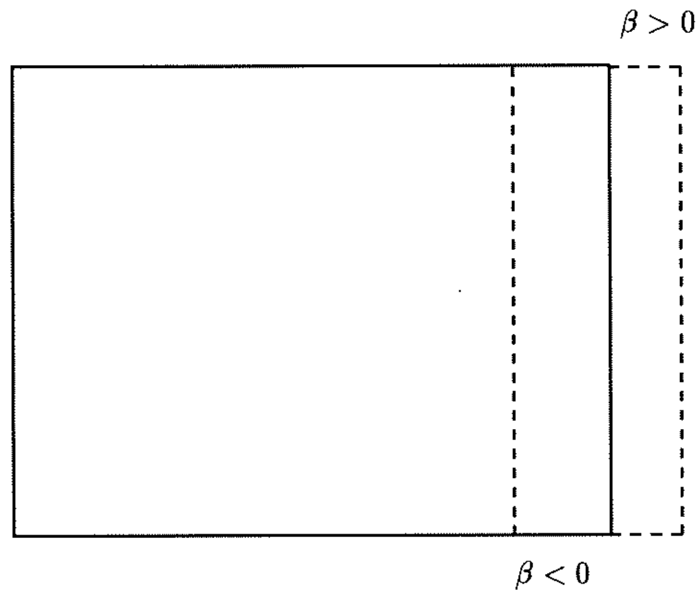

FIG. 2.

We ran with four size grids, a 20 by 20, a 40 by 40, an 80 by 80 and a 160 by 160 grid. With each refinement the number of subdomains was increased by a factor of four from 16 to 64 to 256 to 1024. A constant overlap of one element was used. The coarse grid was refined from 5 by 5 to 10 by 10 to 20 by 20, to 40 by 40. The value of the "missing" overlap, $|\beta|$, was changed from 0.1 to 0.05 to 0.025 to 0.0125. Note that we keep $|\beta| \asymp H$. For all our calculations, we always choose the initial iterative guess of zero and stop iterations when a relative decrease in the discrete norm of residual of 10^{-5} is obtained.

As one can see in Tables 1 and 2, the number of iterations was essentially unaffected by the "missing" overlap for the Dirichlet boundary conditions. However for the case of Neumann boundary conditions the number of iterations required to achieve the same tolerance increased greatly. This agrees very well with our theory.

TABLE 1
Convergence for Multiplicative Schwarz

Boundary Conditions	β	Fine Mesh			
		20x20	40x40	80x80	160x160
Dirichlet	0	10	10	9	9
	+	10	10	11	12
	-	9	10	10	10
Mixed	0	10	10	10	10
	+	10	10	10	10
	-	15	22	30	43

TABLE 2

Convergence for Additive Schwarz

Boundary Conditions	β	Fine Mesh			
		20x20	40x40	80x80	160x160
Dirichlet	0	30	28	26	25
	+	29	30	28	30
	-	27	28	28	29
Mixed	0	23	28	29	29
	+	23	28	29	28
	-	33	50	77	110

TABLE 3

Multiplicative DD iterations for the Airfoil mesh. 32 Subdomains

Overlap (no. elements)	coarse grid	
0	None	55
0	G_2	30
0	G_1	20
1	None	31
1	G_2	17
1	G_1	11
2	None	24
2	G_2	13
2	G_1	9

In our second experiment, we solve a mildly varying coefficient problem:

$$\frac{\partial}{\partial x}\left((1+xy)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left((1+\sin(4x+4y))\frac{\partial u}{\partial y}\right)=f$$

discretized by standard piecewise linear finite element method on the unstructured airfoil grid shown in Fig. 3. The airfoil is imbedded in the unit square. We use nonhomogeneous Dirichlet boundary conditions for $x \leq 0.2$ and homogenous Neumann boundary conditions for $x > 0.2$. The right hand side f is chosen to be $x^2 \sin(3y)$. Note that since the present software we use for the calculations can only generate coarse grids which are interior to the fine grid, we do violate Assumption **(A1)** here. The subdomains are shown in Fig. 3 and two sets of coarse grids are given in Fig. 4. Since the theoretical convergence behavior of additive and multiplicative overlapping Schwarz is very similar, we have chosen to only include the results for the multiplicative case. Other numerical studies may be found in Chan and Smith [5]

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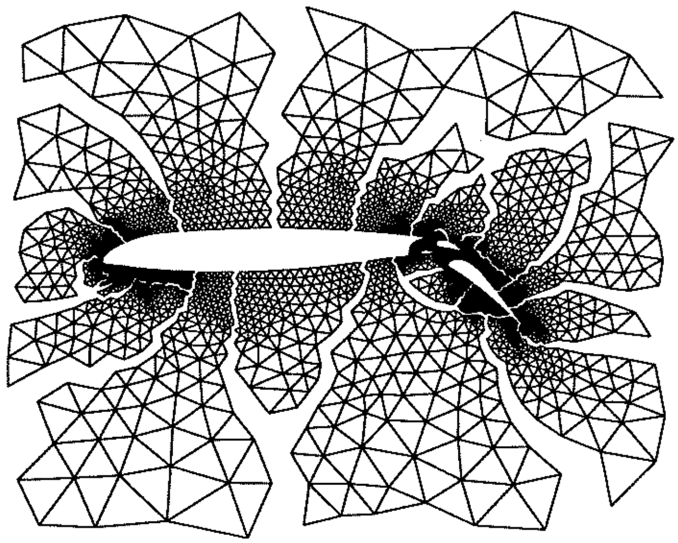

FIG. 3. *Airfoil grid partitioned into 32 subdomains*

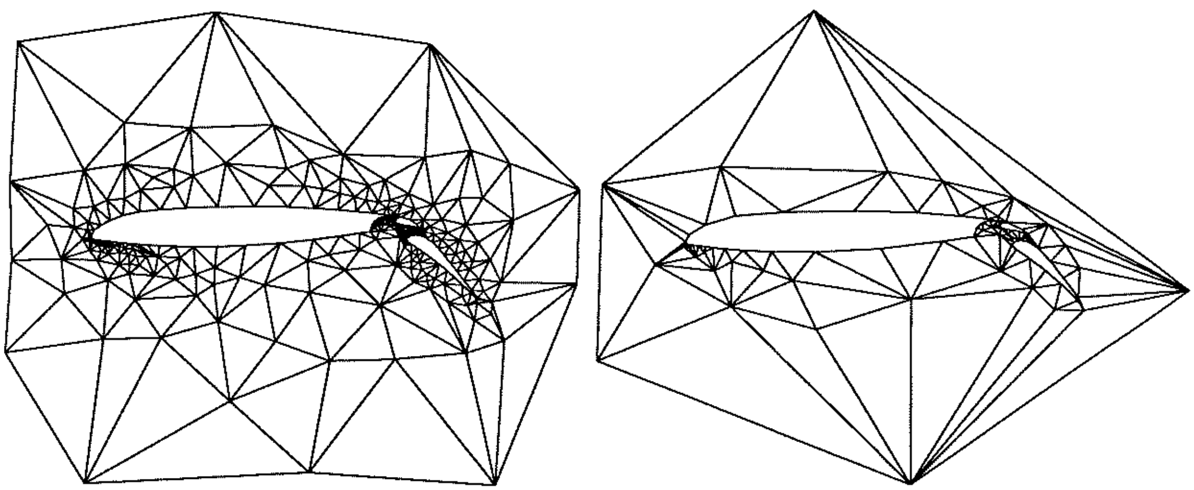

FIG. 4. *Two coarse grids: G_1 (left), G_2 (right)*

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