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Abstract

We summarize some recent work of ours on the design and analysis of multigrid and overlapping domain decomposition algorithms for solving elliptic problems on unstructured meshes in two and three dimensions. We describe a class of algorithms which are based on the construction of a coarse grid hierarchy, and the associated standard finite element interpolation operators, from the given fine grid. We develop a theory which shows that our domain decomposition algorithms have the same optimal convergence rate as the usual algorithms for structured meshes. Very general meshes and subdomains are allowed: neither the fine mesh nor the coarse mesh need to be quasi-uniform, the subdomains can be of arbitrary shapes and sizes, and the coarse mesh need not be nested to, or cover the same physical domain as, the fine mesh. Some numerical results will be presented to demonstrate both the theory and the algorithms.

Key Words. Unstructured meshes, multigrid, domain decomposition.

1 Introduction

Unstructured meshes have become increasing popular in scientific computing [1, 12]. Compared to structured meshes, they have increased flexibility in adapting to complicated geometries as well as to large changes in the solution. However, traditional solvers have to be modified so that their efficiency will not be adversely affected by the lack of structure in the mesh.

In this short paper, we summarize some recent work of ours on the design and analysis of multigrid and overlapping domain decomposition algorithms for solving elliptic problems on unstructured meshes in two and three dimensions. For this class

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of algorithms, unstructured meshes present several additional practical and theoretical difficulties: the absence of a natural coarse grid hierarchy, the need to construct well-balanced subdomains, the typically highly non-uniformity of the mesh, the non-matching of the fine grid boundary and the coarse grid boundary and the non-nestedness of the finite element spaces on the fine mesh and the coarse mesh. We shall describe some approaches we have taken to overcome these problems.

We describe a class of algorithms which are based on the construction of a coarse grid hierarchy, and the associated standard finite element interpolation operators, from the given fine grid. We use recursive spectral graph partitioning techniques to construct the subdomains. We develop a convergence theory for overlapping domain decomposition algorithms which allows very general meshes and subdomains: neither the fine mesh nor the coarse mesh need to be quasi-uniform, the subdomains can be of arbitrary shapes and sizes, and the coarse mesh need not be nested to, or cover the same physical domain as, the fine mesh. The theory shows that our algorithms have the same optimal convergence rate as the usual algorithms for structured meshes.

There has been relatively little work on domain decomposition and multigrid algorithms for unstructured meshes. Some references are: [12, 11, 5, 4, 14].

The present paper is essentially a summary of our results in the papers [8, 9, 7]. More details can be found in those papers.

2 Construction of coarse grid hierarchy

In both multigrid and domain decomposition algorithms, a coarse grid hierarchy is need. Unlike for a structured mesh, this is not naturally given as part of the refinement procedure. Our approach is to construct the coarse grid hierarchy, as well as the associated interpolation and restriction operators, directly from the given unstructured fine mesh. It suffices to describe this for one coarse level because the procedure can be recursively applied to obtain all the coarse meshes. Typically, the whole coarse grid hierarchy is used by a multigrid algorithm but only one of the coarser grids is used by a standard two-level domain decomposition algorithm.

We shall need the notion of a *maximal independent set* of the vertices of a graph. A subset of vertices V of a graph G is said to be *independent* if no two vertices of V are connected by an edge. V is said to be *maximally independent* if adding any additional vertex to it makes it dependent. Note that maximal independent sets of vertices of a graph are generally not unique.

Our procedure has four steps:

1. Form a maximally independent set of the boundary vertices and from these construct a set of coarse boundary edges,

2. Form a maximally independent set of the interior vertices,

3. Apply a Cavendish type algorithm [6] to triangulate the resulting collection of coarse boundary edges and coarse interior vertices,

4. Construct the interpolation and restriction operators.

Step (i) is fairly straightforward. For each disjoint boundary segment, the boundary vertices are ordered say in a clockwise direction, starting with a random vertex. Then every other vertex is thrown out and the remaining ones are connected with new coarse boundary edges. This forms a coarse representation of the boundary segment. After several coarsenings, one may find that the boundary is no longer qualitatively similar to the original boundary. This may be prevented by simply retaining some of the vertices in the coarse grid boundary that would normally be dropped.

Step (ii) uses a greedy wavefront type algorithm. A random interior vertex is selected for inclusion in the maximally independent set. Then every interior vertex connected to it is eliminated from consideration for inclusion in the maximally independent set. Next, one of the interior vertices connected to the newly eliminated vertices is selected for inclusion and the procedure repeats until all interior vertices have been considered. An algorithm similar to this has been used by Barnard and Simon [2] in designing graph partitioning algorithms, also see Guillard [11]. This procedure can be implemented in linear time, i.e. proportional to the total number of interior vertices. An alternative to step (ii) is to construct a maximal independent set of the *dual* graph of the mesh and use the center of the remaining elements as the coarse grid vertices.

The input to Step (iii) is thus a collection of coarse boundary edges and coarse interior vertices. A version of Cavendish's algorithm [6] is then applied to triangulate this collection. This algorithm is an advancing front technique and "grows" new triangles from those already built by selecting an interior vertex to be "mated" to an existing edge. In doing so, it tries to optimize the aspect ratio of the new triangle formed, preferring those that are close to being equilateral. It is possible to implement this algorithm in linear time, i.e. proportional to the number of interior vertices, but our current implementation is not optimal.

Finally, in Step (iv), the interpolation operator is constructed in the form of a sparse matrix and stored. To determine the entries of this interpolation matrix, the coarse triangles are taken in sequence and the entries corresponding to all the fine grid vertices within the coarse triangle are then computed using the standard piecewise linear interpolation. This procedure can also be implemented in linear time because the fine grid triangles close to the vertices of the coarse triangle (which are also fine grid vertices as well) can be found by a local search. We emphasize that this is not possible if the coarser grids are generated completely independently. The restriction matrix is then just the transpose of the interpolation matrix.

3 Convergence analysis of Two Level Schwarz Methods

We will develop a theory to make the two level Schwarz overlapping domain decomposition methods applicable to unstructured meshes and keep the same optimal convergence. We give only the results for the model problem with Dirichlet boundary condition. For more general problems and boundary conditions, we refer to Chan, Smith and Zou [9] [7].

3.1 Uniformly elliptic problems with non-matching boundary

We consider the following model problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) \equiv \int_\Omega \rho(x)\,\nabla u\,\nabla v\,dx = f(v),\ \forall\, v \in H_0^1(\Omega) \tag{1}$$

where $\Omega \subset R^d$ (d=2, 3) and $\rho(x) \geq \alpha > 0$, in Ω.

Suppose we are given a family of triangulations $\{\mathcal{T}^\langle\}$ on Ω, consisting of simplices. We will not discuss the effects of approximating Ω but always assume in the paper that the triangulations $\{\mathcal{T}^\langle\}$ of Ω are exact. So we have $\Omega = \Omega^h \equiv \cup_{\tau\in\mathcal{T}^\langle}\tau$. Let $h_\tau =$ diam τ and $h = \overline{h} = \max_{\tau\in\mathcal{T}^\langle} h_\tau$. Let V^h be a piecewise linear finite element subspace of $H_0^1(\Omega)$ defined on $\mathcal{T}^\langle$ with its basis denoted by $\{\phi_i^h\}_{i=1}^n$, and $O_i =$supp ϕ_i^h. As usual, we decompose the domain Ω into p nonoverlapping subdomains Ω_i such that $\overline{\Omega} = \cup_{i=1}^p \overline{\Omega}_i$, then extend each subdomain Ω_i to a larger one Ω_i' such that the distance between $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by $\delta_i > 0$. We denote the minimum of all δ_i by δ. We assume that $\partial\Omega_i'$ does not cut through any element $\tau \in \mathcal{T}^\langle$. For the subdomains meeting the boundary we cut off the part of Ω_i' which is outside of $\overline{\Omega}$. No other assumptions will be made on $\{\Omega_i\}$ except that any point $x \in \Omega$ belongs only to a finite number of subdomains $\{\Omega_i'\}$. Therefore, we allow each Ω_i to be of quite different size and of quite different shape from other subdomains. We define the subspaces of V^h corresponding to the subdomains $\{\Omega_i'\}$, $i = 1, 2, \cdots, p$ by

$$V_i^h = V^h \cap H_0^1(\Omega_i'). \tag{2}$$

In order to get an optimal preconditioner even when the number of subdomains increases greatly, we introduce a coarse grid $\mathcal{T}^H$ which can be completely independent of $\mathcal{T}^\langle$, i.e., none of the nodes of $\mathcal{T}^H$ need to be nodes of $\mathcal{T}^\langle$. In general, $\Omega^H \neq \Omega$. Let H be the maximum diameter of the elements of $\mathcal{T}^H$, and $\Omega^H = \cup_{\tau^H\in\mathcal{T}^H}\tau^H$. By V^H we denote a subspace of $H_0^1(\Omega^H)$ consisting of piecewise polynomials defined on $\mathcal{T}^H$, by $\{\psi_i^H\}_{i=1}^m$ we denote its basis functions related to the nodes $\{q_i^H\}_{i=1}^m$. Let $O_i^H =$ supp ψ_i^H. We note that V^H need not necessarily be piecewise linear as V^h. Thus we do not necessarily have the usual condition: $V^H \subset V^h$.

For technical reasons, we make two additional reasonable assumptions on the coarse grid:

(A1): $\tau^H \cap \Omega \neq \emptyset$ for all $\tau^H \in \mathcal{T}^H$,

that is, no coarse grid element lies completely outside the fine grid. For the complement set $\Omega \setminus \Omega^H$, let S be the set of all vertices q_i^H of Ω^H which belongs to $\overline{\Omega \setminus \Omega^H}$, and $B_p(r)$ be a ball at the point p with radius r. We assume that

(A2): $\Omega \setminus \Omega^H \subset \cup_{q_i^H \in S} B_{q_i^H}(\text{diam } O_i^H)$,

that is, the coarse grid must cover a significant part of the fine grid.

To overcome the difficulty that $V^H \not\subset V^h$, in both the theory and the algorithms, we need a way of mapping values from V^H to V^h. For the coarse space to be effective, this mapping must possess the properties of H^1-stability and L^2 optimal approximation. We will show that the standard finite element interpolant Π_h is one of such choices.

3.2 Two level overlapping Schwarz algorithms

Based on the finite element spaces V_i^h and V^H given previously, we formulate the two level overlapping Schwarz methods for nonnested grids. Schwarz methods are techniques to construct preconditioners for the stiffness matrix $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} = a(\phi_i^h, \phi_j^h)$. These preconditioners are built using local and coarse grid solves.

The local solves are defined as in Dryja and Widlund [10], and Bramble, Pasciak, Wang, and Xu [3]. Define the H^1-projection operators $P_i : V^h \to V_i^h$, $i = 1, \cdots, p$ such that for any $u \in V^h$, $P_i u \in V_i^h$ satisfies

$$a(P_i u, v_i) = a(u, v_i), \qquad \forall\, v_i \in V_i^h. \tag{3}$$

There are two ways to define the coarse grid projection, in order to handel the non-nestedness of the coarse grid space. Let $\mathcal{I}_l$ be any linear operator which maps V^H into a subspace $\mathcal{I}_h V^H$ of V^h. But in the rest of the paper, we only discuss the choice $\mathcal{I}_h = \Pi_h$-the standard finite element interpolant.

In **Method 1** we define $\tilde{P}_0$ by first defining $P_H u \in V^H$ on the original coarse grid space by

$$a(P_H u, v) = a(u, \mathcal{I}_h v), \qquad u \in V^h, \quad \forall v \in V^H \tag{4}$$

and then define $\tilde{P}_0 = \mathcal{I}_h P_H$: $V^h \to V_0^h$. Here the subspace $V_0^h = \mathcal{I}_h V^H \subset V^h$.

In **Method 2** we define P_0 by calculating the projection directly onto the subspace V_0^h: For any $u \in V^h$, $P_0 u \in V_0^h$ such that

$$a(P_0 u, v) = a(u, v), \quad \forall v \in V_0^h. \tag{5}$$

We now derive the matrix representation of the operators P_i and $\tilde{P}_0$. Next, we use u^h to denote finite element functions and u its coefficient vector, that is $u^h = \sum u_k \phi_k$.

Let $\{\phi_{i,j}^h\}_{j=1}^{n_i} \subset \{\phi_i^h\}_{i=1}^n$ be the set of nodal basis functions of V_i^h, $i = 1, 2, \cdots, p$. For each i, we define a matrix extension operator R_i^T as follows: For any $u_i^h \in V_i^h$, we denote by u_i the coefficient vector of u_i^h in the basis $\{\phi_{i,j}^h\}_{j=1}^{n_i}$, and we define that $R_i^T u_i$ to be the coefficient vector of u_i^h in the basis $\{\phi_i^h\}_{i=1}^n$.

Since $\{\psi_i^H\}_{i=1}^m$ is the set of basis functions of V^H, then $\{\mathcal{I}_h \psi_i^H\}_{i=1}^m$ is the set of basis functions of V_0^h. We define a matrix extension operator R_0^T as follows: For any $u_0^h \in V_0^h$, we denote by u_0 the coefficient vector of u_0^h in the basis $\{\mathcal{I}_h \psi_i^H\}_{i=1}^m$, define $R_0^T u_0$ to be the coefficient vector of u_0^h in the basis $\{\phi_j^h\}_{j=1}^n$. Let A_H be the stiffness matrix corresponding to the original coarse space V^H.

Then it is straightforward to derive that for any $u^h \in V^h$, the matrix representation of $\tilde{P}u^h$ with $\tilde{P} = \tilde{P}_0 + \sum_{i=1}^p P_i$ is

$$M_1 Au = R_0^T A_H^{-1} R_0 Au + \sum_{i=1}^{p} R_i^T A_i^{-1} R_i Au, \tag{6}$$

and the matrix representation of Pu^h with $P = \sum_{i=0}^p P_i$ is

$$M_2 Au = \sum_{i=0}^{p} R_i^T A_i^{-1} R_i Au \tag{7}$$

where $A_i = R_i A R_i^{\mathrm{T}}$ for $i = 0, 1, \cdots, p$ are the stiffness matrices corresponding to the subspaces V_i^h. The preconditioners M_1 and M_2 may be also thought of as an overlapping block Jacobi method with the addition of a coarse grid correction. The multiplicative Schwarz method is the Gauss-Seidel version of the additive algorithm. We write down the symmetrized version, using Method 1 as,

$$\begin{aligned} M \;=\; & (I - (I - R_1^T A_1^{-1} R_1 A) \ldots (I - R_p^T A_p^{-1} R_p A)(I - R_0^T A_H^{-1} R_0 A) \\ & (I - R_p^T A_p^{-1} R_p A) \ldots (I - R_1^T A_1^{-1} R_1 A)) A^{-1}. \end{aligned} \tag{8}$$

In practice the application of the multiplicative Schwarz preconditioner is done directly, not as given in (8).

For **Method 1** and **Method 2**, we have the following estimates for their condition numbers, see Chan, Smith and Zou [7] for details:

Theorem 3.1 *Suppose that both triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular (not necessarily quasi-uniform), and satisfy Assumptions* **(A1)** *and* **(A2)**. *Then, with $\mathcal{I}_h = \Pi_h$, we have*

$$\kappa(M_1 A),\ \kappa(M_2 A) \leq C\,(1 + \frac{H}{\delta})^2. \tag{9}$$

We remark that the ratio H/δ in (9) can be replaced by a completely localized ratio.

4 Numerical results

In this section we present convergence results for two basic algorithms: two level overlapping multiplicative Schwarz methods and standard V-cycle multigrid. The fine grid is as depicted in Figure 1. In Figure 2 we depict the three coarser grids which have been generated automatically.

For our computations we have used Laplace's equation with homogeneous boundary conditions. This problem was discretized with the usual piecewise linear finite elements. In our calculations the discrete right hand side was chosen to be the vector of all 1's. Other numerical examples may be found in Chan and Smith [8] and Chan, Smith and Zou [7].

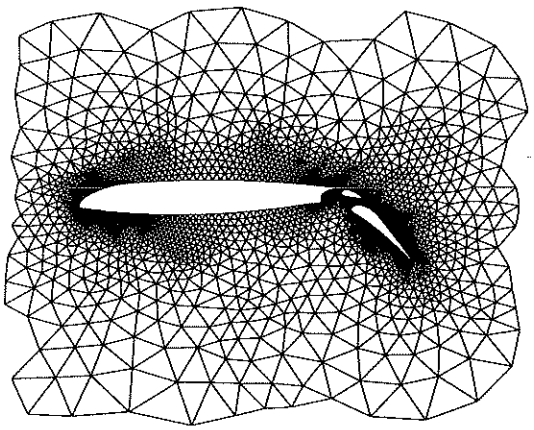

Figure 1: The *Airfoil* mesh: 4253 nodes

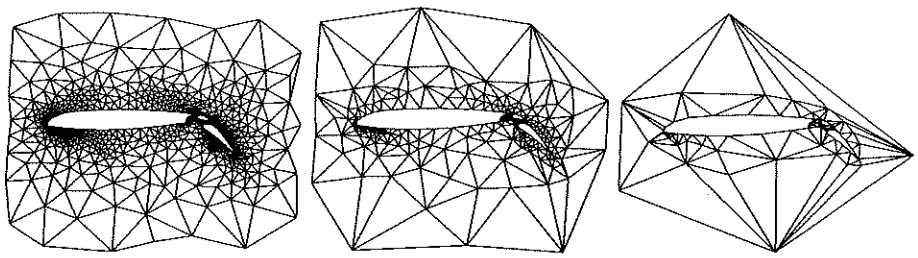

Figure 2: The coarser *Airfoil* meshes

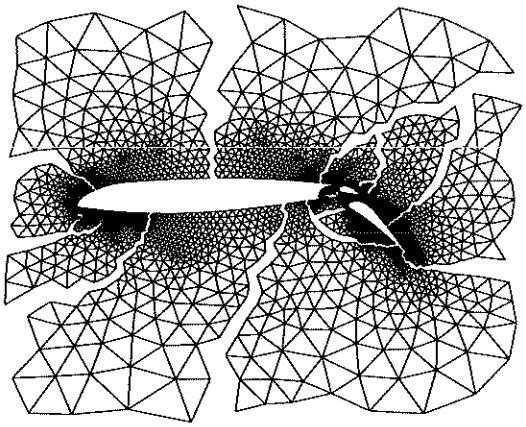

Figure 3: The *Airfoil* mesh: 16 subdomains computed by RSB

4.1 Multiplicative Schwarz results

We have partitioned the original fine grid into 16 pieces using the *recursive spectral bisection* method, [13]. These are given in Figure 3. In Table 1 we give the number of iterations required to achieve a relative decrease in the discrete two norm of 10^{-6}. Exact solvers (sparse LU factorization) were used to solve both the local problems and the coarse grid problem.

Table 1: Iterations for the *Airfoil* mesh. Multiplicative Schwarz

Overlap (no. elements)	Level of coarse grid	Regular coarsening	Dual graph coarsening
0	None	56	56
0	4	21	22
0	3	15	12
1	None	16	16
1	4	10	10
1	3	7	7
2	None	14	14
2	4	8	8
2	3	5	5

4.2 Multigrid results

In Table 2 we give the convergence results for standard V-cycle multigrid. We have used 2 pre and 2 post smoothing steps of symmetric pointwise Gauss-Seidel on each level. These results are comparable to those obtained on a uniformly refined mesh.

Table 2: Multigrid iterations for the *Airfoil* mesh

	Regular coarsening		Dual graph coarsening	
MG Levels	Nodes	Dir. B.C. .	Nodes	Dir. B.C.
2	1180	4	1507	4
3	518	4	328	4
4	89	4	171	5

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