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COMPUTATIONAL AND APPLIED MATHEMATICS

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July 1994

CAM Report 94-22

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July 8, 1994

ABSTRACT

Domain decomposition preconditioners are constructed for general two and three dimensional non-symmetric parabolic problems on unstructured meshes. Our convergence theory needs fairly weak assumptions concerning the smoothness of the given data and the regularity of the continuous solutions, say only H^1-regularity in space, and does not require that the coarse mesh and the subdomain size be small enough. In each preconditioning step, all local subdomain problems are symmetric positive definite, only the small-scale coarse mesh problem is non-symmetric. The coarse mesh need not be nested to the fine mesh and the coarse domain Ω^H need not match the original domain Ω, and neither of the coarse and fine meshes are required to be quasi-uniform. The subdomains are allowed to be of arbitrary shape.

1 Introduction

This paper is to develop efficient domain decomposition algorithms for solving the linear systems of equations which arise from the finite element discretizations of second-order parabolic initial and boundary value problems on unstructured meshes.

The domain decomposition algorithms and theory for second-order parabolic problems have been well developed for structured meshes, see [7], [8], [9], [15], [17], and

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the references therein for symmetric parabolic problems in 2D; and [2], [13], [14] for non-symmetric parabolic problems in 2D and 3D.

In this paper, we will consider both two and three dimensional non-symmetric parabolic problems of more general form than those mentioned above. The domain decomposition algorithms and the convergence theory will be developed for these parabolic problems on unstructured meshes. Our convergence theory does not require that the coarse mesh and the subdomain size be small enough, and we need fairly weak assumptions concerning the smoothness of the given data and the regularity of the continuous solutions, say only H^1-regularity in space. The coarse mesh need not be nested to the fine mesh, and neither need to be quasi-uniform. The subdomains are allowed to be of arbitrary shape. Moreover, in each preconditioning step, all local subdomain problems are symmetric positive definite, only the small-scale coarse mesh problem is non-symmetric, and this idea is initially motivated by the work [21].

Let Ω be a bounded domain in R^d and $n = (n_1, \cdots, n_d)$ the unit outer normal of the boundary $\partial\Omega$. We shall deal with parabolic problems of the following general form:

$$\frac{\partial u}{\partial t} + Lu = f \quad in\ \Omega \times (0,T), \tag{1.1}$$

where

$$Lu = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}[\sum_{j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j} + b_i u] + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + du \tag{1.2}$$

with $(a_{ij}(x,t))$ symmetric, uniformly positive definite on $\overline{\Omega} \times [0,T]$, and the functions $b_i(x,t)$, $c_i(x,t)$ and $d(x,t)$ continuous on $\bar{\Omega} \times [0,T]$. The initial and boundary conditions are

$$u(x,0) = u_0(x) \quad on\ \Omega \tag{1.3}$$

and

$$\sum_{i=1}^{d}[\sum_{j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j} + b_i u] \cdot n_i + \alpha u = g \quad on\ \partial\Omega \times (0,T). \tag{1.4}$$

We always assume $\alpha(x,t) \geq 0$ on $\overline{\Omega} \times [0,T]$. Thus (1.4) includes Neumann boundary condition as a special case. The results of this paper can be generalized to the Dirichlet and mixed boundary conditions by using the techniques introduced in our previous work [3] [4]. There for the elliptic problems, we have developed the theory and algorithms for domain decompositions on unstructured meshes.

Throughout the paper, we use $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ to denote the norm and semi-norm of the usual Sobolev space $H^m(\Omega)$ for any integer $m \geq 0$. In addition, $||\cdot||_{m,r,\Omega}$ and $|\cdot|_{m,r,\Omega}$ will denote the norm and semi-norm of the spaces $W^{m,r}(\Omega)$ for any integer $m \geq 0$ and real number $r \geq 1$. The capital C denotes generic positive constants which are independent of the mesh parameters h, H and the time step τ.

2 Finite elements and domain decompositions

We will use finite element methods to discretize the parabolic problem (1.1)–(1.4). Suppose we are given a family of triangulations $\{\mathcal{T}^h\}$ on Ω, consisting of simplices. We will not discuss the effects of approximating Ω but always assume in the paper that the triangulations $\{\mathcal{T}^h\}$ of Ω are exact. So we have $\Omega = \Omega^h \equiv \cup_{\tau\in\mathcal{T}^h}\tau$. Let $h = \overline{h} = \max_{\tau\in\mathcal{T}^h} h_\tau$, $h_\tau =$ diam τ, $\underline{h} = \min_{\tau\in\mathcal{T}^h} h_\tau$, $\rho_\tau =$ the radius of the largest ball inscribed in τ. Then we say $\mathcal{T}^h$ is *shape regular* if it satisfies

$$\sup_h \max_{\tau\in\mathcal{T}^h} \frac{h_\tau}{\rho_\tau} \le \sigma_0\,, \tag{2.1}$$

and we say $\mathcal{T}^h$ is *quasi-uniform* if it is shape regular and satisfies

$$\overline{h} \le \gamma\,\underline{h}\,, \tag{2.2}$$

with σ_0 and γ fixed positive constants, see Ciarlet [5]. In the paper, we will only assume that the elements are shape regular, but not necessarily quasi-uniform.

Let V^h be a piecewise linear finite element subspace of $H^1(\Omega)$ defined on $\mathcal{T}^h$ with its basis denoted by $\{\phi_i^h\}_{i=1}^n$, and $O_i = \operatorname{supp}\phi_i^h$. Later on we will use the following simple facts: if $\mathcal{T}^h$ is shape regular, there exist a positive constant C and an integer ν, both depending only on σ_0 appearing in (2.1) and independent of h, so that, for $i = 1, 2, \cdots, m$,

$$\operatorname{diam} O_i \le C\, h_\tau, \;\forall\, \tau \subset O_i, \tag{2.3}$$

$$\operatorname{card}\{\tau\in\mathcal{T}^h;\, \tau\subset O_i\} \le \nu. \tag{2.4}$$

Because of the ill-conditioning of the finite element stiffness matrix A obtained from the discretization of the parabolic problems, our goal is to construct a good preconditioner M for A by domain decomposition methods to be used with *Krylov subspace* methods, cf. [11].

As usual, we decompose the domain Ω into p nonoverlapping subdomains Ω_i such that $\overline{\Omega} = \cup_{i=1}^p \overline{\Omega}_i$, then extend each subdomain Ω_i to a larger Ω_i' such that the distance between $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by $\delta_i > 0$. We denote the minimum of all δ_i by δ. We assume that $\partial\Omega_i'$ does not cut through any element $\tau\in\mathcal{T}^h$. For the subdomains meeting the boundary, we cut off the part of Ω_i' which is outside of $\overline{\Omega}$. No other assumptions will be made on $\{\Omega_i\}$ in this paper except that any point $x\in\Omega$ belongs only to a finite number of subdomains $\{\Omega_i'\}$. This means that we allow each Ω_i to be of quite different size and of quite different shape from other subdomains. We define the subspaces of V^h corresponding to the subdomains $\{\Omega_i'\}$, $i = 1, 2, \cdots, p$ by

$$V_i^h = \{v_h \in V^h;\, v_h = 0 \text{ on } (\Omega\setminus\Omega_i')\cup(\partial\Omega\setminus(\partial\Omega\cap\partial\Omega_i))\} \tag{2.5}$$

For interior subdomains,

$$V_i^h = V^h \cap H_0^1(\Omega_i'). \tag{2.6}$$

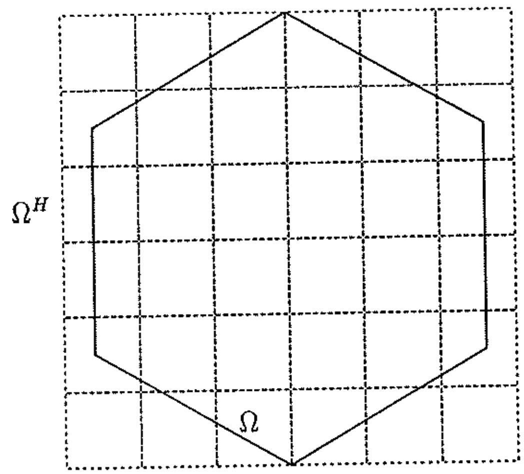

Figure 1: Fine grid Ω and coarse gird Ω^H

To construct optimal preconditioners, we introduce a coarse grid $\mathcal{T}^H$ which forms only a shape regular triangulation of Ω and covers Ω completely, that is, $\Omega \subset \Omega^H$, but otherwise has nothing to do with $\mathcal{T}^h$, i.e., none of the nodes of $\mathcal{T}^H$ need to be nodes of $\mathcal{T}^h$. Therefore, we can also use quite regular coarse grids, cf. Figure 1, which in some cases allow very efficient solvers for the coarse mesh problems.

Let H be the maximum diameter of the elements of $\mathcal{T}^H$, and $\Omega^H = \cup_{\tau^H \in \mathcal{T}^H} \tau^H$.

By V^H we denote a subspace of $H^1(\Omega^H)$ consisting of piecewise polynomials defined on $\mathcal{T}^H$. We note that V^H need not necessarily be piecewise linear; for example, it may be bilinear (2-D) and trilinear (3-D) elements or higher order elements. Thus we do not necessarily have the usual condition: $V^H \subset V^h$. See Figure 2.1.

In order to overcome the difficulty that $V^H \not\subset V^h$, in both the theory and the algorithms, we need a mapping $\mathcal{I}_h$ from V^H to V^h. For the coarse space to be effective, this mapping must possess the properties of H^1-stability and L^2 optimal approximation, see Chan-Zou [4], Mandel [16]. So we make the following assumptions on the mapping $\mathcal{I}_h$: for any $u \in V^H$,

(A1): $|\mathcal{I}_h u|_{1,\Omega} \leq C\, |u|_{1,\Omega^H}$,

(A2): $||u - \mathcal{I}_h u||_{0,\Omega} \leq C\, h\, |u|_{1,\Omega^H}$.

In our previous work [3] [4], we introduced several operators $\mathcal{I}_h$ which fulfill assumptions **(A1)** and **(A2)**. In particular, the standard finite element interpolation operator Π_h and the Clément's local L^2-like projection operator $\mathcal{R}_h$ are two such choices.

Further, we assume that there exists an operator $\mathcal{R}_H : H^1(\Omega^H) \to V^H$ which satisfies that for any $u \in H^1(\Omega^H)$,

(A3): $|\mathcal{R}_H u|_{1,\Omega^H} \leq C\, |u|_{1,\Omega^H}$,

(A4): $||u - \mathcal{R}_H u||_{0,\Omega^H} \leq C\, H\, |u|_{1,\Omega^H}$.

There are a few options for such operators $\mathcal{R}_H$, e.g., Clément's local L^2-like projection operator and Scott and Zhang's local operator, cf. [3] [4] for details.

3 Self-adjoint cases

To deal with the non-selfadjoint parabolic problems, we need some results from the selfadjoint case. So in this section we first consider self-adjoint parabolic problems, that is, we assume $b_i = c_i$ for $i = 1, \cdots, d$ in (1.2). For any $u, v \in H^1(\Omega)$, define

$$\begin{aligned} b(u,v) &= \sum_{i,j=1}^{d} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^{d} \int_\Omega b_i(x) \left(u \frac{\partial v}{\partial x_i} + \frac{\partial u}{\partial x_i} v \right) dx \\ &+ \int_\Omega d(x) uv\, dx + \int_{\partial\Omega} \alpha(x) uv\, ds, \end{aligned} \tag{3.1}$$

where and in the sequel we drop the dependence of the functions a_{ij}, b_i, d and α on the time t, since we are mainly interested in solving the corresponding elliptic equations at each fixed time step. We see that there are positive constants C_1, C_2, C_3 depending only on the coefficients such that

$$C_1 ||u||_{1,\Omega}^2 - C_2 ||u||_{0,\Omega}^2 \leq b(u,u) \leq C_3 ||u||_{1,\Omega}^2, \ \forall\ u \in H^1(\Omega). \tag{3.2}$$

After discretizing the variational problem corresponding to the system (1.1)–(1.4) by using some finite difference schemes in time and the finite element space V^h in space, the resulting linear system may be formulated as follows

$$a(u^h, v^h) = (f, v^h)\,,\ \forall\ v^h \in V^h. \tag{3.3}$$

The stiffness matrix of (3.3) is denoted by $A = \left(a(\phi_i^h, \phi_j^h) \right)_{i,j=1}^N$ with $a(\cdot,\cdot)$ defined below

$$a(u, v) = (u, v) + \tau\, b(u, v). \tag{3.4}$$

Here τ is the time step. It is known that (3.3) allows usual time discretizations, for example, implicit Euler scheme and Crank-Nicolson scheme.

In order to find an optimal preconditioner M for the matrix A, we employ the overlapping domain decomposition method and then use the PCG method to solve the problem (3.3) with the preconditioner M. Let $(\cdot,\cdot)$ be the usual l^2-inner product in R^n and the corresponding norm $||\cdot|| = (\cdot,\cdot)^{1/2}$. We denote $(A\cdot,\cdot)$, which defines an inner product on R^n, by $(\cdot,\cdot)_A$ and its induced-norm by $||\cdot||_A$. Let u^k be the kth PCG iteration with the initial guess u^0. It is well known that [12]

$$||u - u^k||_A \leq 2 \left(\frac{\sqrt{\kappa(MA)} - 1}{\sqrt{\kappa(MA)} + 1} \right)^k ||u - u^0||_A,$$

where $\kappa(MA)$ is the condition number of MA.

Based on the finite element spaces V_i^h and V^H given in the last section, we derive the two level overlapping Schwarz methods for nonnested grids.

For each subspace V_i^h, we define the H^1-projection operators $P_i : V^h \to V_i^h$, $1 \leq i \leq p$ such that for any $u^h \in V^h$, $P_i u^h \in V_i^h$ satisfies

$$a(P_i u^h, v_i) = a(u^h, v_i), \qquad \forall\ v_i \in V_i^h. \tag{3.5}$$

Next we define the coarse grid projection-like operator. Let $\mathcal{I}_h$ be any linear operator which maps V^H into a subspace $\mathcal{I}_h V^H$ of V^h, e.g. $\mathcal{I}_h$ may be chosen as the standard finite element interpolation operator Π_h or the locally defined operator $\mathcal{R}_h$, see Section 5 for more details.

We define $\tilde{P}_0 = \mathcal{I}_h P_H$: $V^h \to V_0^h$ with $P_H : V^h \to V^H$ defined on the original coarse space by

$$a(P_H u^h, v^H) = a(u^h, \mathcal{I}_h v^H), \qquad u^h \in V^h, \quad \forall v^H \in V^H. \tag{3.6}$$

Here the subspace $V_0^h = \mathcal{I}_h V^H \subset V^h$.

Remark 3.1 *We note here that for the left-hand side in (3.6), $a(u_H, v_H)$ for any $u_H, v_H \in V^H$, is not an integral over original domain Ω, but the one over the coarse domain Ω^H, i.e, replacing Ω in the definition of $a(\cdot,\cdot)$ by Ω^H. Thus in the sequel we always assume that the coefficient functions a_{ij}, b_i, c_i, α are continuously extended onto Ω^H. And later on, we will use $\|\cdot\|_{a,\Omega}$ and $\|\cdot\|_{a,\Omega^H}$ to denote the energy norms $a(\cdot,\cdot)$ over Ω and Ω^H, respectively.*

The additive Schwarz algorithm is to use the *Conjugate Gradient Iteration* to solve the following operator equation

$$\tilde{P} u^h \equiv \tilde{P}_0 u^h + \sum_{i=1}^{p} P_i u^h = \tilde{g}_h \equiv \tilde{g}_0^h + \sum_{i=1}^{p} g_i^h \tag{3.7}$$

where g_i^h, $i = 1, \cdots, p$ satisfies

$$a(g_i^h, v_i) = (f, v_i), \ \forall\ v_i \in V_i^h, \tag{3.8}$$

but $\tilde{g}_0^h$ is defined as $\mathcal{I}_h g_H$ with g_H fulfilling

$$a(g_H, v) = (f, \mathcal{I}_h v), \ \forall\ v \in V^H. \tag{3.9}$$

We now derive the matrix representation of the operator $\tilde{P}$. Using this both the additive and multiplicative Schwarz preconditioners may be written down. We will often use u^h to denote the finite element function and u to denote the vector of coefficients of that finite element function, that is $u^h = \sum u_k \phi_k$.

Let $\{\phi_i^h\}_{i=1}^n$ be the standard nodal basis functions of V^h, and $\{\phi_{i,j}^h\}_{j=1}^{n_i} \subset \{\phi_i^h\}_{i=1}^n$ be the nodal basis functions of V_i^h, $i = 1, 2, \cdots, p$. For each i, we define a matrix extension operator R_i^T as follows: For any $u_i^h \in V_i^h$, we denote by u_i the coefficient vector of u_i^h

in the basis $\{\phi^h_{i,j}\}_{j=1}^{n_i}$, and we define that $R_i^T u_i$ to be the coefficient vector of u_i^h in the basis $\{\phi_i^h\}_{i=1}^n$.

Let $\{\psi_i^H\}_{i=1}^m$ be the basis functions of V^H, then $\{\mathcal{I}_h\psi_i^H\}_{i=1}^m$ are the basis functions of V_0^h. We define a matrix extension operator R_0^T as follows: For any $u_0^h \in V_0^h$, we denote by u_0 the coefficient vector of u_0^h in the basis $\{\mathcal{I}_h\psi_i^H\}_{i=1}^m$, define $R_0^T u_0$ to be the coefficient vector of u_0^h in the basis $\{\phi_j^h\}_{j=1}^n$. Then $R_{0_{ij}} = \mathcal{I}_h\psi_i^H(q_j)$ where q_j is the nodal vertex of ϕ_j^h. When $\mathcal{I}_h = \Pi_h$ then $R_{0_{ij}}$ is simply given by $\psi_i^H(q_j)$.

It is straightforward to derive, cf. [3] [4] that for any $u^h \in V^h$, the matrix representation of $\tilde{P}u^h$ is

$$MAu \equiv R_0^T A_H^{-1} R_0 Au + \sum_{i=1}^{p} R_i^T A_i^{-1} R_i Au, \tag{3.10}$$

where $A_i = R_i A R_i^T$ for $1 \le i \le p$ are the stiffness matrices corresponding to the subspaces V_i^h, but A_H the stiffness matrix corresponding to the original coarse space V^H.

These may be thought of as an overlapping block Jacobi method with the addition of a coarse grid correction. The multiplicative Schwarz method is the Gauss-Seidel version of the additive algorithm. We write down the symmetrized version:

$$\begin{aligned} M_1 \;\; = \;\; & (I - (I - R_1^T A_1^{-1} R_1 A)\ldots(I - R_p^T A_p^{-1} R_p A)(I - R_0^T A_H^{-1} R_0 A) \\ & (I - R_p^T A_p^{-1} R_p A)\ldots(I - R_1^T A_1^{-1} R_1 A))A^{-1}. \end{aligned} \tag{3.11}$$

In practice the application of the multiplicative Schwarz preconditioner is done directly as a block Gauss-Seidel iteration, not as is given in (3.11).

We give the convergence results for the additive algorithm, similar results may be obtained for the multiplicative algorithm using the techniques in [10] or [20].

It is easy to check that

$$\kappa(MA) = \kappa(\tilde{P}).$$

More exactly, the minimal and maximal eigenvalues of MA are given by

$$\lambda_{\min}(MA) = \min_{u\neq 0} \frac{(AMAu, u)}{(Au, u)} = \min_{u^h \neq 0} \frac{a(\tilde{P}u^h, u^h)}{a(u^h, u^h)} \tag{3.12}$$

and

$$\lambda_{\max}(MA) = \max_{u\neq 0} \frac{(AMAu, u)}{(Au, u)} = \max_{u^h \neq 0} \frac{a(\tilde{P}u^h, u^h)}{a(u^h, u^h)}. \tag{3.13}$$

We have the following bounds for the condition number $\kappa(MA)$ which will be improved greatly and localized in Section 5:

Theorem 3.1 *Suppose that both triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular, but not necessarily quasi-uniform. Then we have*

$$\kappa(MA) \le C\,(1 + H/\delta)^2. \tag{3.14}$$

First we show

Lemma 3.1 *We have*

$$\lambda_{\max}(MA) \le C. \qquad (3.15)$$

Proof. For any $u^h \in V^h$, it is quite routine to prove that

$$\sum_{i=1}^{p} a(P_i u^h,\, u^h) \le C\, a(u^h, u^h), \qquad (3.16)$$

while from (3.6) we see that

$$a(P_H u^h, P_H u^h) = a(\mathcal{I}_h P_H u^h, u^h), \qquad (3.17)$$

thus by Cauchy-Schwarz's inequality and the assumption **(A1)**- **(A2)** and (3.2),

$$||P_H u^h||^2_{a,\Omega^H} \le ||u^h||_{a,\Omega}\, ||\mathcal{I}_h P_H u^h||_{a,\Omega} \le C\, ||u^h||_{a,\Omega}\, ||P_H u^h||_{a,\Omega^H},$$

i.e., $||P_H u^h||_{a,\Omega^H} \le C\, ||u^h||_{a,\Omega}$, which leads to the following

$$a(\tilde{P}_0 u^h, u^h) = a(P_H u^h, P_H u^h) \le C\, a(u^h, u^h),$$

this, together with (3.16) and (3.13) implies Lemma 3.1. □

To estimate the lower bound of $\kappa(MA)$, we first give a partition lemma for the finite element space V^h:

$$V^h = V_0^h + V_1^h + \cdots + V_p^h. \qquad (3.18)$$

with V_0^h defined as previously by $V_0^h = \mathcal{I}_h V^H$.

Lemma 3.2 *Let $\Omega \subset R^d\,(d = 2, 3)$. We assume that both triangulations T^h and T^H are shape regular, but not necessarily quasi-uniform. Then for any $u \in V^h$, there exist $u_i \in V_i^h$, $i = 0, 1, \cdots, p$ such that $u = \sum_{i=0}^{p} u_i$ and*

$$\sum_{i=0}^{p} ||u_i||^2_{1,\Omega} \le C\,(1 + H/\delta)^2 ||u||^2_{1,\Omega},\ \forall\, u \in V^h. \qquad (3.19)$$

Proof. The proof follows [3] [4]. Since we will need some of the details in Section 5, we now give a complete proof here.

Let $\hat{\Omega}$ is an open domain in R^d large enough such that $\Omega \subset\subset \hat{\Omega}$. Then we know, cf. Stein [19], that there exists a linear extension operator $E : H^1(\Omega) \to H^1(\hat{\Omega})$ such that $Eu|_\Omega = u$ and

$$||Eu||_{1,\hat{\Omega}} \le C\, ||u||_{1,\Omega}. \qquad (3.20)$$

For any $u \in V^h$, we choose $u_0 = \mathcal{I}_h \mathcal{R}_H Eu$. Then from the assumptions **(A1)**–**(A4)** we obtain

$$\begin{aligned} \|u_0\|_{1,\Omega} &= \|\mathcal{I}_h \mathcal{R}_H Eu\|_{1,\Omega} \le C\,\|\mathcal{R}_H Eu\|_{1,\Omega^H} \\ &\le C\,\|Eu\|_{1,\Omega^H} \le C\,\|Eu\|_{1,\hat{\Omega}} \le C\,\|u\|_{1,\Omega} \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \|u - u_0\|_{0,\Omega} &\le \|Eu - \mathcal{R}_H Eu\|_{0,\Omega} + \|\mathcal{R}_H Eu - \mathcal{I}_h \mathcal{R}_H Eu\|_{0,\Omega} \\ &\le \|Eu - \mathcal{R}_H Eu\|_{0,\Omega^H} + C\,h\,|\mathcal{R}_H Eu|_{1,\Omega^H} \\ &\le CH\,|Eu|_{1,\Omega^H} + C\,h\,|Eu|_{1,\Omega^H} \le C\,H\,|Eu|_{1,\hat{\Omega}} \le C\,H\,|u|_{1,\Omega}. \end{aligned} \tag{3.22}$$

It is known, cf. [1], that there exists a partition $\{\theta_i\}_{i=1}^p$ of unity for Ω related to the subdomains $\{\Omega_i'\}$ such that $\sum_{i=1}^p \theta_i(x) = 1$ on Ω and for $i = 1, 2, \cdots, p$,

$$\operatorname{supp}\theta_i \subset \Omega_i' \cup \partial\Omega, \; 0 \le \theta_i \le 1 \text{ and } \|\nabla\theta_i\|_{L^\infty(\Omega_i)} \le C\,\delta_i^{-1}. \tag{3.23}$$

Now for any $u \in V^h$, let $u_0 = \mathcal{I}_h \mathcal{R}_H\, Eu \in V^h$ be chosen as above, and $u_i = \Pi_h \theta_i (u - u_0)$ with Π_h being the standard interpolation of V^h. Obviously, $u_i \in V_i^h$ and

$$u = u_0 + u_1 + \cdots + u_p. \tag{3.24}$$

Let τ be any element belonging to Ω_k' with h_τ being its diameter and $\overline{\theta}_k$ the average of θ_k on element τ. Then from (3.23) and the fact that $u - u_0 \in V^h$, we get:

$$\begin{aligned} |u_k|_{1,\tau}^2 &\le 2|\overline{\theta}_k \Pi_h (u - u_0)|_{1,\tau}^2 + 2|\Pi_h(\theta_k - \overline{\theta}_k)(u - u_0)|_{1,\tau}^2 \\ &\le 2|u - u_0|_{1,\tau}^2 + 2|\Pi_h(\theta_k - \overline{\theta}_k)(u - u_0)|_{1,\tau}^2. \end{aligned}$$

By using the local inverse inequality which requires only the shape regularity of $\mathcal{T}^h$, we obtain:

$$\begin{aligned} |u_k|_{1,\tau}^2 &\le 2|u - u_0|_{1,\tau}^2 + C\,h_\tau^{-2}\|\Pi_h(\theta_k - \overline{\theta}_k)(u - u_0)\|_{0,\tau}^2 \\ &\le 2|u - u_0|_{1,\tau}^2 + C\,h_\tau^{-2}\frac{h_\tau^2}{\delta_k^2}\|u - u_0\|_{0,\tau}^2 \\ &\le 2|u - u_0|_{1,\tau}^2 + C\,\frac{1}{\delta_k^2}\|u - u_0\|_{0,\tau}^2. \end{aligned}$$

By taking the sum over $\tau \in \Omega_k'$ we have

$$|u_k|_{1,\Omega_k'}^2 \le 2|u - u_0|_{1,\Omega_k'}^2 + C\,\frac{1}{\delta_k^2}\|u - u_0\|_{0,\Omega_k'}^2. \tag{3.25}$$

Noticing the assumption made previously that any point $x \in \Omega$ belongs only to a finite number of subdomains $\{\Omega_i'\}$, it follows from (3.21), (3.22) and (3.25) that

$$\sum_{k=1}^p |u_k|_{1,\Omega_k'}^2 \le C\,(|u - u_0|_1^2 + \frac{1}{\delta^2}\|u - u_0\|_0^2) \le C\,(1 + H/\delta)^2 |u|_1^2. \tag{3.26}$$

Analogously, we derive that

$$\sum_{k=1}^{p} ||u_k||^2_{0,\Omega'_k} \le C\left(1+\frac{h^2}{\delta^2}\right)||u||^2_0,$$

which completes the proof of (3.19). □

By making use of above Lemma 3.2, we can get the lower bound estimate for the condition number $\kappa(MA)$:

Lemma 3.3 *We have*

$$\lambda_{\min}(MA) \ge C(1+H/\delta)^{-2}. \tag{3.27}$$

Proof. For any $u_h \in V_h$, by Lemma 3.2 there exist elements $u_i \in V_i^h$, $i = 0, 1, \cdots, p$ such that $u = \sum_{i=0}^{p} u_i$ and

$$\sum_{i=0}^{p} ||u_i||^2_{1,\Omega} \le C\left(1+\frac{H}{\delta}\right)^2 ||u||^2_{1,\Omega},\ \forall\ u \in V^h. \tag{3.28}$$

Thus, from (3.28), (3.2) and (3.4) we have

$$\begin{aligned}\sum_{i=0}^{p} b(u_i,u_i) &\le C(1+H/\delta)^2||u_h||^2_{1,\Omega}\\ &\le C(1+H/\delta)^2\left(b(u_h,u_h)+||u_h||^2_{0,\Omega}\right),\end{aligned} \tag{3.29}$$

this combining the definition of $a(\cdot,\cdot)$ in (3.4) gives that

$$\begin{aligned}\sum_{i=0}^{p} a(u_i,u_i) &= \sum_{i=0}^{p}||u_i||^2_{0,\Omega} + \tau\sum_{i=0}^{p} b(u_i,u_i)\\ &\le C(1+H/\delta)^2\left(||u_h||^2_{0,\Omega}+\tau\, b(u_h,u_h)\right)\\ &= C(1+H/\delta)^2 a(u_h,u_h).\end{aligned} \tag{3.30}$$

It follows from (3.17) that

$$\begin{aligned}a(\tilde{P}u^h,u^h) &= a(\mathcal{I}_h P_H u^h, u^h) + \sum_{i=1}^{p} a(P_i u^h, u^h)\\ &= ||P_H u^h||^2_{a,\Omega^H} + \sum_{i=1}^{p}||P_i u^h||^2_{a,\Omega}.\end{aligned} \tag{3.31}$$

Now by Cauchy-Schwarz inequality, the definition of P_i, (3.30) and (3.31), we obtain

$$a(u^h,u^h) = a(u^h,\sum_{i=0}^{p} u_i) = a(u^h,\mathcal{I}_h\mathcal{R}_H u^h) + \sum_{i=1}^{p} a(u^h,u_i)$$

$$
\begin{aligned}
&= a(P_H u^h, \mathcal{R}_H u^h) + \sum_{i=1}^{p} a(P_i u^h, u_i) \\
&\le \left(\sum_{i=1}^{p} ||P_i u^h||_{a,\Omega}^2 + ||P_H u^h||_{a,\Omega^H}^2 \right)^{1/2} \left(\sum_{i=1}^{p} ||u_i||_{a,\Omega}^2 + ||\mathcal{R}_H u^h||_{a,\Omega^H}^2 \right)^{1/2} \\
&\le C \left(1 + \frac{H}{\delta}\right) a(\tilde{P} u^h, u^h)^{1/2} ||u^h||_{a,\Omega},
\end{aligned}
$$

which, together with (3.12) completes the proof of Lemma 3.3. □

4 Non-selfadjoint cases

Now we consider the general non-selfadjoint parabolic problem (1.1)-(1.4), that is, we remove the restriction $b_i = c_i$ $(i = 1, 2)$ made in last section for selfadjoint cases. For any $u, v \in H^1(\Omega)$, we define

$$
\begin{aligned}
a(u, v) &= \int_\Omega uv\,dx + \tau \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \\
&\quad + \tau \left(\int_\Omega d(x) uv\,dx + \int_{\partial\Omega} \alpha(x) uv\,ds \right)
\end{aligned} \tag{4.1}
$$

and

$$
b(u, v) = \int_\Omega \sum_{i=1}^{d} \left(b_i(x) u \frac{\partial v}{\partial x_i} + c_i(x) \frac{\partial u}{\partial x_i} v \right) dx. \tag{4.2}
$$

It is easy to see that if $\tau \min_{x\in\bar\Omega} d(x) \ge -\frac{1}{2}$, then there exist positive constants C_1, C_2 depending only on the coefficients a_{ij}, b_i, c_i, d and α such that

$$
\frac{1}{2} ||u||_{0,\Omega}^2 + C_1 \tau\, ||\nabla u||_{0,\Omega}^2 \le ||u||_{a,\Omega}^2 \le C_2 \left(||u||_{0,\Omega}^2 + \tau ||\nabla u||_{0,\Omega}^2 \right), \forall u \in H^1(\Omega) \tag{4.3}
$$

where $||u||_{a,\Omega}$ and the following $||u||_{a,\Omega^H}$ are energy norms defined on Ω and Ω^H, respectively, refer to *Remark* 3.1.

Further, by using (4.3) and Cauchy-Schwarz inequality, we see that there exists a positive constant C_3 depending only on the coefficients such that for any $u, v \in H^1(\Omega)$,

$$
\tau\,|b(u, v)| = \tau^{\frac{1}{2}} \int_\Omega \sum_{i=1}^{d} \left(|b_i(x) u|\, |\tau^{\frac{1}{2}} \frac{\partial v}{\partial x_i}| + |c_i(x)\, \tau^{\frac{1}{2}} \frac{\partial u}{\partial x_i}|\, |v| \right) dx \le C_3\, \tau^{\frac{1}{2}}\, ||u||_{a,\Omega} ||v||_{a,\Omega}. \tag{4.4}
$$

We denote

$$
c(u, v) = a(u, v) + \tau\, b(u, v),\ u, v \in H^1(\Omega) \tag{4.5}
$$

which is the resultant bilinear form by discretizing the parabolic system (1.1)–(1.4) by usual difference schemes in time and finite element methods in space.

The finite element problem we solve is: Find $u^h \in V^h$ such that

$$e(u^h, v^h) = (f, v^h), \ \forall \ v^h \in V^h. \tag{4.6}$$

We next constrcut a preconditioner for the stiffness matrix E. Let E, A and B denote the stiffness matrices related to the bilinear forms $e(\cdot,\cdot), a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, respectively, that is,

$$E = \left(e(\phi_i^h, \phi_j^h)\right)_{i,j=1}^n, \quad A = \left(a(\phi_i^h, \phi_j^h)\right)_{i,j=1}^n, \quad B = \left(b(\phi_i^h, \phi_j^h)\right)_{i,j=1}^n, \tag{4.7}$$

and E_0 and A_0 denote the non-symmetric and symmetric coarse stiffness matrices over subspace V^H, i.e.

$$E_0 = \left(e(\phi_i^H, \phi_j^H)\right)_{i,j=1}^m, \quad A_0 = \left(a(\phi_i^H, \phi_j^H)\right)_{i,j=1}^m \tag{4.8}$$

Obviously, matrices A, A_0 are both symmetric and positive definite. As in last section, we shall denote by A_i the stiffness matrix over the subspaces V_i^h with respect to the symmetric bilinear form $a(\cdot,\cdot)$.

With these preparations, we define the preconditioner for the stiffness matrix E by

$$M = R_0^T E_0^{-1} R_0 + \sum_{i=1}^{p} R_i^T A_i^{-1} R_i, \tag{4.9}$$

where R_i $(i = 0, 1, \cdots, p)$ are defined as in the previous section.

Remark 4.1 *With the preconditioner M defined by (4.9) for the non-symmetric stiffness matrix E, only the coarse problem (E_0^{-1}) is still a non-symmetric system, but all subdomain solvers (A_i^{-1}) are symmetric positive definite. This is important for practical applications.*

Instead of solving the finite element problem (4.6), we solve the following preconditioned system by the generalized minimal residual (GMRES) method, cf. Saad and Schultz [18]:

$$M E u = g, \tag{4.10}$$

with g a given vector in R^n. GMRES starts with an initial approximation solution $u^0 \in R^n$ and an initial residual $r^0 = g - M E u^0$. At the mth iteration, a correction vector w^m is computed in the Krylov subspace

$$\mathcal{K}_m(r^0) = span\{r^0, M E r^0, \cdots, (M E)^{m-1} r^0\} \tag{4.11}$$

which minimizes the residual, $\min_{w \in \mathcal{K}_m(r^0)} ||g - M E(u^0 + w)||_A$ with $||\cdot||_A$ being the norm of R^n induced by the inner product $(A\cdot,\cdot)$. The mth iteration is then given by $u^m = u^0 + w^m$. Let

$$\beta_1 = \min_{u \neq 0} \frac{(u, M E u)_A}{(u, u)_A} \text{ and } \beta_2 = \max_{u \neq 0} \frac{||M E u||_A}{||u||_A}. \tag{4.12}$$

It is known that if $\beta_1 > 0$, GRMES method converges, and at the mth iteration the residual is bounded as

$$||g - MEu^m||_A \leq \left(1 - \frac{\beta_1^2}{\beta_2^2}\right)^{m/2} ||r^0||_A, \tag{4.13}$$

Now we state our main theorem of this section which gives the bounds of the convergence rate of the preconditioned GMRES method:

Theorem 4.1 *There exists a constant $C_0 > 0$ independent of h, H and τ such that for any $\tau \leq C_0$, we have*

$$\min_{u \neq 0} \frac{(u, MEu)_A}{(u,u)_A} \geq C\,(1 + H/\delta)^{-2}, \tag{4.14}$$

$$\max_{u \neq 0} \frac{||MEu||_A}{||u||_A} \leq C. \tag{4.15}$$

For proving the theorem, we first introduce a symmetric projection operator $P_0 = \mathcal{I}_h P_H : V^h \rightarrow V_0^h = \mathcal{I}_h V^H$ with P_H defined by

$$a(P_H u^h, v_H) = a(u^h, \mathcal{I}_h v_H), \; u^h \in V^h, \; \forall \; v_H \in V^H \tag{4.16}$$

and a nonsymmetric projection operator $Q_0 = \mathcal{I}_h Q_H : V^h \rightarrow V_0^h$ with Q_H defined by

$$c(Q_H u^h, v_H) = c(u^h, \mathcal{I}_h v_H), \; u^h \in V^h, \; \forall \; v_H \in V^H. \tag{4.17}$$

In the sequel, we always associate any vector $v \in R^n$ with a finite element function $v^h \in V^h$ by the realtion $v^h = \sum_{i=1}^n v_i \phi_i^h$. And for any matrix $D \in R^{n\times n}$, we use $||D||_A$ to denote the matrix norm :

$$||D||_A = \max_{u \neq 0} \frac{||Du||_A}{||u||_A}. \tag{4.18}$$

With the above notations, we first give a lemma used later for proving Theorem 4.1

Lemma 4.1 *If $\tau \leq 1/(16C_3^2)$ with C_3 appearing in (4.4), then for any $u^h \in V^h$,*

$$||Q_H u^h||_{a,\Omega^H} \leq C\,||u^h||_{a,\Omega} \tag{4.19}$$

and for any $u \in R^n$, let $w = A^{-1}Bu$, we have

$$||w||_A \leq C_3\,\tau^{-1/2}\,||u||_A. \tag{4.20}$$

Proof. Taking $v_H = Q_H u^h$ in (4.17) gives

$$\|Q_H u^h\|^2_{a,\Omega^H} = a(u^h, \mathcal{I}_h Q_H u^h) - \tau\, b(Q_H u^h, Q_H u^h) + \tau\, b(u^h, \mathcal{I}_h Q_H u^h). \tag{4.21}$$

Using the definition (4.16) and the same way as in proving Lemma 3.1, we get

$$\|P_H u^h\|_{a,\Omega^H} \le C\, \|u^h\|_{a,\Omega}. \tag{4.22}$$

Thus from (4.16), (4.22) and Cauchy-Schwarz inequality it follows that

$$\begin{aligned} a(u^h, \mathcal{I}_h Q_H u^h) &= a(P_H u^h, Q_H u^h) \le \|P_H u^h\|_{a,\Omega^H}\, \|Q_H u^h\|_{a,\Omega^H} \\ &\le \frac{1}{4}\|Q_H u^h\|^2_{a,\Omega^H} + C\, \|u^h\|^2_{a,\Omega}, \end{aligned} \tag{4.23}$$

while from (4.4) and assumptions **(A1)** and **(A2)** we deduce that

$$\tau\, \left|b(Q_H u^h, Q_H u^h)\right| \le C_3\, \tau^{1/2}\, \|Q_H u^h\|^2_{a,\Omega^H}, \tag{4.24}$$

$$\begin{aligned} \tau\, \left|b(u^h, \mathcal{I}_h Q_H u^h)\right| &\le C_3\, \tau^{1/2}\, \|u^h\|_{a,\Omega}\, \|Q_H u^h\|_{a,\Omega^H} \\ &\le \frac{1}{4}\|Q_H u^h\|^2_{a,\Omega^H} + C_3^2\, \tau\, \|u^h\|^2_{a,\Omega}. \end{aligned} \tag{4.25}$$

Now (4.19) follows from (4.23)–(4.25) and $\tau \le 1/(16C_3^2)$.

For the proof of (4.20), we see by the choice of w that

$$(Aw, w) = (Bu, w) = b(u^h, w^h),$$

thus combining (4.4) shows

$$\|w\|^2_A \le C_3\, \tau^{-1/2}\, \|u\|_A\, \|w\|_A$$

this implies (4.20). □

Proof of Theorem 4.1. We first prove (4.15). The preconditioner M may be rewritten as the sum

$$M = \tilde{M} + M_0$$

with

$$\tilde{M} = \sum_{i=0}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i, \quad M_0 = R_0^{\mathrm{T}} E_0^{-1} R_0 - R_0^{\mathrm{T}} A_0^{-1} R_0.$$

By Lemma 3.1 and Lemma 3.3 of Section 3, we know that

$$\lambda_{\min}(\tilde{M} A) \ge C\,(1 + H/\delta)^{-2}, \quad \|\tilde{M} A\|_A = \lambda_{\max}(\tilde{M} A) \le C \tag{4.26}$$

We see

$$M\, Eu = \tilde{M}\, Au + \tau\, \tilde{M}\, Bu + M_0 Eu. \tag{4.27}$$

The first two terms can be bounded easily by (4.26) and (4.20) that

$$||\tilde{M} A u||_A \leq ||\tilde{M} A||_A \, ||u||_A \leq C \, ||u||_A, \tag{4.28}$$

and

$$\tau \, ||\tilde{M} B u||_A \leq \tau \, ||\tilde{M} A||_A \, ||A^{-1} B u||_A \leq C \, \tau^{-1/2} \, ||u||_A. \tag{4.29}$$

For the third term, $M_0 E = R_0^\top E_0^{-1} R_0 E - R_0^\top A_0^{-1} R_0 E$. By direct computations, the matrix expressions for $\mathcal{I}_h P_H u^h$ and $\mathcal{I}_h Q_H u^h$ are:

$$R_0^\top A_0^{-1} R_0 A u, \quad R_0^\top E_0^{-1} R_0 E u, \tag{4.30}$$

respectively, cf. Section 3. Using these expressions, (4.19), (4.22), and assumptions **(A1)** and **(A2)** leads to

$$\begin{aligned} ||R_0^\top E_0^{-1} R_0 E u||_A^2 &= (A R_0^\top E_0^{-1} R_0 E u, R_0^\top E_0^{-1} R_0 E u) \\ &= a(\mathcal{I}_h Q_H u^h, \mathcal{I}_h Q_H u^h) \leq C \, ||Q_H u^h||_{a,\Omega^H}^2 \leq C \, ||u||_A^2, \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} ||R_0^\top A_0^{-1} R_0 E u||_A &\leq ||R_0^\top A_0^{-1} R_0 A u||_A + \tau \, ||R_0^\top A_0^{-1} R_0 A \, (A^{-1} B u)||_A \\ &= a(\mathcal{I}_h P_H u^h, \mathcal{I}_h P_H u^h)^{1/2} + \tau \, a(\mathcal{I}_h P_H w^h, \mathcal{I}_h P_H w^h)^{1/2} \\ &\leq C \, ||P_H u^h||_{a,\Omega^H} + C \, \tau \, ||P_H w^h||_{a,\Omega^H} \leq C \, ||u||_A, \end{aligned} \tag{4.32}$$

thus (4.15) follows from (4.27)–(4.32).

To prove (4.14), we first know by (4.27)

$$(M E u, u)_A = (\tilde{M} A u, u)_A + \tau \, (\tilde{M} B u, u)_A + (M_0 E u, u)_A \, . \tag{4.33}$$

The first two terms can be estimated by using (4.26) and (4.20) :

$$(\tilde{M} A u, u)_A \geq C \, (1 + H/\delta)^{-2} \, (u, u)_A \tag{4.34}$$

and

$$\tau \, \left|(\tilde{M} B u, u)_A\right| \;\leq\; \tau \, ||\tilde{M} A||_A \, ||A^{-1} B u||_A \, ||u||_A \leq C \, \tau^{1/2} \, (u, u)_A. \tag{4.35}$$

Now we need only to bound the term $(M_0 E u, u)_A$. We obtain by using (4.30), (4.16) and (4.17) that

$$\begin{aligned} (M_0 E u, u)_A &= (A R_0^\top E_0^{-1} R_0 E u, u) - (A R_0^\top A_0^{-1} R_0 A u, u) - \tau \, (A R_0^\top A_0^{-1} R_0 B u, u) \\ &= a(\mathcal{I}_h Q_H u^h, u^h) - a(\mathcal{I}_h P_H u^h, u^h) - \tau \, (A R_0^\top A_0^{-1} R_0 A u, A^{-1} B u) \\ &= a(\mathcal{I}_h Q_H u^h, u^h) - a(\mathcal{I}_h P_H u^h, u^h) - \tau \, a(\mathcal{I}_h P_H u^h, w^h). \end{aligned} \tag{4.36}$$

From the definitions of Q_H and P_H, we get

$$\begin{aligned} a(\mathcal{I}_h Q_H u^h, u^h) &= a(Q_H u^h, P_H u^h) = e(Q_H u^h, P_H u^h) - \tau\, b(Q_H u^h, P_H u^h) \\ &= e(u^h, \mathcal{I}_h P_H u^h) - \tau\, b(Q_H u^h, P_H u^h) \\ &= a(u^h, \mathcal{I}_h P_H u^h) + \tau\, b(u^h, \mathcal{I}_h P_H u^h) - \tau\, b(Q_H u^h, P_H u^h) \end{aligned} \tag{4.37}$$

therefore, we see from (4.36) and (4.37) that

$$(M_0 Eu, u)_A = \tau\, b(u^h, \mathcal{I}_h P_H u^h) - \tau\, b(Q_H u^h, P_H u^h) - \tau\, a(\mathcal{I}_h P_H u^h, w^h). \tag{4.38}$$

By means of (4.4), (4.19), (4.20), (4.22) and assumptions **(A1)** and **(A2)** we derive

$$\tau \left| b(u^h, \mathcal{I}_h P_H u^h) \right| \le C_3 \tau^{1/2} \|u^h\|_{a,\Omega} \|\mathcal{I}_h P_H u^h\|_{a,\Omega} \le C\, \tau^{1/2} (u, u)_A, \tag{4.39}$$

$$\tau \left| b(Q_H u^h, P_H u^h) \right| \le C_3\, \tau\, \|Q_H u^h\|_{a,\Omega^H} \|P_H u^h\|_{a,\Omega^H} \le C\, \tau\, (u, u)_A, \tag{4.40}$$

and

$$\begin{aligned} \tau \left| a(\mathcal{I}_h P_H u^h, w^h) \right| &\le \tau\, \|w^h\|_{a,\Omega} \|\mathcal{I}_h P_H u^h\|_{a,\Omega} \\ &\le C\, \tau\, \|w\|_A \|u^h\|_{a,\Omega} \le C\, \tau^{1/2} (u, u)_A. \end{aligned} \tag{4.41}$$

So we obtain by (4.38)–(4.41) that

$$|(M_0 Eu, u)_A| \le C\, \tau^{1/2} (u, u)_A\,. \tag{4.42}$$

Evidently, (4.14) follows now from (4.33)–(4.35) and (4.42), this completes the proof of Theorem 4.1.

5 Improvement for bounds of condition numbers

In this section, we shall briefly discuss how to improve the bounds of condition numbers of the Schwarz algorithms addressed in Sections 3 and 4. We consider only the case $\mathcal{I}_h = \Pi_h$, similar results hold true for the case $\mathcal{I}_h = \mathcal{R}_h$. All the notations here are the same as in previous sections.

For simplicity, let V^H be any Lagrangian finite element space consisting of piecewise polynomials of degree $\le q$. Let $\{\psi_i^H\}_{i=1}^m$ be the basis functions of V^H, $\{q_i^H\}_{i=1}^m$ be the corresponding nodes and $O_i = \text{supp}\ \psi_i^H$ the supporting sets. We define a local L^2-like projection operator $\mathcal{R}_H : L^2(\Omega^H) \to V^H$ as follows

$$\mathcal{R}_H u = \sum_{i=1}^m Q_i\, u(q_i^H)\, \psi_i^H, \ \forall\, u \in L^2(\Omega^H), \tag{5.1}$$

where $Q_i u \in \mathcal{P}_q(O_i)$ satisfies

$$\int_{O_i} Q_i u\, p\, dx = \int_{O_i} u\, p\, dx, \ \forall\, p \in \mathcal{P}_q(O_i). \tag{5.2}$$

We introduce three more notations:

$$h_k = \max_{\tau^h \subset \Omega'_k} h_\tau, \quad B_k = \cup_{\tau^H \cap \Omega'_k \neq \emptyset} \tau^H, \quad H_k = \max_{\tau^H \subset B_k} H_\tau. \tag{5.3}$$

We assume

(A5): Any point $x \in \Omega$ belongs only to a finite number of subdomains $\{\Omega'_i\}$, and also to a finite number of $\{B_k\}$.

(A6): $h_k \leq \mu\, H_k$, for $k = 1, 2, \cdots, p$, and μ is a mesh independent positive constant.

With above preparations, we get

Lemma 5.1 *For $k = 1, 2, \cdots, p$ and $s = 0, 1$, the following estimates holds:*

$$||u - \mathcal{R}_H u||_{s,B_k} \leq C\, H_k^{1-s}\, |u|_{1,S_k}, \ \forall\, u \in H^1(\Omega^H) \tag{5.4}$$

with $S_k = \cup_{q_i^H \in \bar{B}_k} O_i$.

Proof. By combining a known result, cf. Clément [6],

$$||u - \mathcal{R}_H u|_{s,\tau^H} \leq C\, H_\tau^{1-s} \sum_{i: q_i^H \in \tau^H} |u|_{1,O_i}, \ \forall\, \tau^H \subset \mathcal{T}^H \tag{5.5}$$

with the same properties as (2.3) and (2.4) for $\mathcal{T}^H$, and the definition of B_k, we obtain (5.4).

For the standard finite element interpolation operator Π_h, we have

Lemma 5.2 *For $k = 1, 2, \cdots, p$ and $s = 0, 1$,*

$$|u - \Pi_h u|_{s,\Omega'_k} \leq C\, h_k^{1-s}\, |u|_{1,B_k}, \ \forall\, u \in V^H. \tag{5.6}$$

Proof. Let $\tau^h \in \mathcal{T}^h$ and $u \in V^h$, we know, cf. Ciarlet [5], for $r > 3$ and $s = 0, 1$ that

$$|u - \Pi_h u|^2_{s,\tau^h} \leq C h_\tau^{2(1-s)} h_\tau^{2d(1/2-1/r)} |u|^2_{1,r,\tau^h}, \tag{5.7}$$

this implies

$$\sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \Omega'_k}} |u - \Pi_h u|^2_{s,\tau^h} \leq C\, h_k^{2(1-s)} \sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \Omega'_k}} h_\tau^{2d(1/2-1/r)} |u|^2_{1,r,\tau^h}. \tag{5.8}$$

Now apply the Cauchy inequality

$$\sum_i a_i b_i \leq (\sum_i a_i^q)^{1/q} (\sum_i b_i^p)^{1/p}$$

to the right hand side with $p = r/2 > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the local finite element inverse inequality, we get

$$\begin{aligned}
\sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \Omega'_k}} |u - \Pi_h u|^2_{s,\tau^h} &\leq C\, h_k^{2(1-s)} \Big(\sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \Omega'_k}} h_\tau^{2d(1/2-1/r)q} \Big)^{1/q} \Big(\sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \Omega'_k}} |u|^{2p}_{1,r,\tau^h} \Big)^{1/p} \\
&\leq C\, h_k^{2(1-s)} \Big(\sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \Omega'_k}} h_\tau^{d} \Big)^{1-2/r} \Big(\sum_{\substack{\tau^h \cap \tau^H \neq \emptyset \\ \tau^h \subset \Omega'_k}} |u|^{r}_{1,r,\tau^h} \Big)^{2/r} \\
&\leq C\, h_k^{2(1-s)} H_\tau^{d(1-2/r)} \Big(\sum_{\substack{\tau^H \cap \tau^{H'} \neq \emptyset \\ \tau^{H'} \subset B_k}} |u|^{r}_{1,r,\tau^{H'}} \Big)^{2/r} \\
&\leq C\, h_k^{2(1-s)} H_\tau^{d(1-2/r)} \Big(H_\tau^{d(2/r-1)} \sum_{\substack{\tau^H \cap \tau^{H'} \neq \emptyset \\ \tau^{H'} \subset B_k}} |u|^{2}_{1,\tau^{H'}} \Big). \qquad (5.9)
\end{aligned}$$

Taking the sum for $\tau^h \subset \Omega'_k$ implies (5.6). □

We state now our main results:

Theorem 5.1 *Suppose that both triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular, not necessarily quasi-uniform. Let A be the stiffness matrix of Section 3 and M be its additive Schwarz preconditioner defined in (3.10). Then we have*

$$\kappa(MA) \leq C \left(1 + \max_{1 \leq i \leq p} \frac{H_k}{\delta_k}\right)^2 . \qquad (5.10)$$

Proof. By checking the procedures of proving Theorem 3.1, it suffices to derive a new bound in (3.19) for the decomposition $u = \sum_{i=0}^{p} u_i$ for any $u \in V^h$. Let $u_0 = \Pi_h \mathcal{R}_H E u$ be chosen the same as there. We obtain for $s = 0, 1$ by using Lemmas 5.1 and 5.2 that

$$\begin{aligned}
||u - u_0||_{s,\Omega'_k} &\leq ||Eu - \mathcal{R}_H Eu||_{s,\Omega'_k} + ||\mathcal{R}_H Eu - \Pi_h \mathcal{R}_H Eu||_{s,\Omega'_k} \\
&\leq ||Eu - \mathcal{R}_H Eu||_{s,B_k} + C\, h_k^{1-s}\, |\mathcal{R}_H Eu|_{1,B_k} \\
&\leq C\, H_k^{1-s}\, |Eu|_{1,S_k} + C\, h_k^{1-s}\, |\mathcal{R}_H Eu|_{1,B_k}. \qquad (5.11)
\end{aligned}$$

In the same way as proving (3.19) by using (5.11) and the properties of $\mathcal{R}_H$ and E, we derive that for any $u \in V^h$ there exist elements $u_k \in V_k^h$, $k = 0, 1, \cdots, p$ such that $u = \sum_{k=0}^{p} u_i$ and

$$\sum_{k=1}^{p} ||u_k||^2_{1,\Omega} \leq C \sum_{k=1}^{p} ||u_k||^2_{1,\Omega'_k} \leq C \sum_{k=1}^{p} \Big(|u - u_0|^2_{1,\Omega'_k} + \frac{1}{\delta_k^2} ||u - u_0||^2_{0,\Omega'_k} \Big)$$

$$\leq C \sum_{k=1}^{p} \left((1 + \frac{H_k^2}{\delta_k^2}) |Eu|_{1,S_k}^2 + (1 + \frac{h_k^2}{\delta_k^2}) |\mathcal{R}_H Eu|_{1,B_k}^2 \right)$$

$$\leq C \left(1 + \max_{1 \leq i \leq p} \frac{H_k}{\delta_k} \right)^2 ||u||_{1,\Omega}^2. \quad (5.12)$$

Then (5.10) follows immediately from above.

Obviously, by means of Therorem 4.1 we can obtain the same improvement for the results of Section 4 for non-selfadjoint parabolic problems.

Remark 5.1 *From (5.10) we can see that if the overlapping δ_k of the subdomain Ω_k is a fraction of the size H_k (note: this is a completely local requirement), then the condition number is completely independent of the fine mesh size h, coarse mesh size H and time step τ.*

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