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Abstract

We prove long time diffusive behavior (homogenization) for convection-diffusion in a turbulent flow that it incompressible and has a stationary and square integrable stream matrix. Simple shear flow examples show that this result is sharp for flows that have stationary stream matrices.

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1 Introduction

Let $\mathbf{u}(\mathbf{x})$, with $\nabla\cdot\mathbf{u}(\mathbf{x}) = 0$, be an incompressible velocity field in R^d, $d \geq 2$, and let $\rho(t, \mathbf{x})$ be the density of an additive carried by the flow and dispersing diffusively. It satisfies the convection-diffusion equation

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \sigma \Delta \rho, \tag{1.1}$$

with $\rho(0, \mathbf{x}) = \rho_0(\mathbf{x})$ and where σ is the molecular diffusivity. The density ρ is non negative and we may assume that $\int \rho_0(\mathbf{x}) d\mathbf{x} = 1$ in which case (1.1) is the Fokker-Plank equation for a diffusing particle satisfying the stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t))dt + \sqrt{2\sigma} d\mathbf{w}(t), \tag{1.2}$$

with $\mathbf{x}(0) = \mathbf{x}_0$ and where $\mathbf{w}(t)$ is the d-dimensional Brownian motion process. A natural question to ask, and the one we consider here, is what happens to the density ρ, or the process $\mathbf{x}(t)$, after a long time. This is particularly interesting when the velocity field has a repetitive structure as, for example, when it is a periodic, an almost periodic or a stationary random function with mean zero. We expect then an overall diffusive behavior with an *effective* diffusion constant. In this paper we give sharp conditions on $\mathbf{u}$ for this to be the case.

To state our main result we introduce the stream matrix $\Psi(\mathbf{x})$ such that

$$\nabla \cdot \Psi(\mathbf{x}) = -\mathbf{u}(\mathbf{x}), \tag{1.3}$$

which is a skew symmetric matrix and always exists, because $\mathbf{u}$ is incompressible and has mean zero, but may not be stationary. We assume throughout this paper that the velocity field comes from a *stationary* stream matrix Ψ which is *square integrable* and (1.3) is meant in the weak sense. In the periodic case there always exists a periodic stream matrix. In the almost periodic or stationary random case the stream matrix exists but may not be almost periodic or stationary, respectively. In two dimensions the matrix Ψ has the form

$$\Psi(\mathbf{x}) = \begin{pmatrix} 0 & -\psi(\mathbf{x}) \\ \psi(\mathbf{x}) & 0 \end{pmatrix} \tag{1.4}$$

where $\psi(\mathbf{x})$ is the usual stream function. In three dimensions, Ψ has the form

$$\Psi(\mathbf{x}) = \begin{pmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix}. \tag{1.5}$$

where $\psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \psi_3(\mathbf{x}))$ is the vector potential of the flow $\mathbf{u}$ so that $\nabla \cdot \Psi = \nabla \times \psi = -\mathbf{u}$. In terms of the stream matrix Ψ, the convection diffusion equation (1.1) can be put into divergence form

$$\frac{\partial \rho(t,\mathbf{x})}{\partial t} = \nabla \cdot [(\sigma I + \Psi(\mathbf{x}))\nabla \rho(t,\mathbf{x})], \tag{1.6}$$

where I is the identity matrix. Note that the coefficient matrix $\sigma I + \Psi$ of this parabolic equation is not symmetric. Since we are interested in long time behavior we rescale space and time and let

$$\rho_n(t,\mathbf{x}) = \rho(n^2 t, n\mathbf{x})$$

with n a large parameter tending to infinity. The scaled density ρ_n satisfies the diffusion equation

$$\frac{\partial \rho_n(t,\mathbf{x})}{\partial t} = \nabla \cdot [(\sigma I + \Psi(n\mathbf{x}))\nabla \rho_n(t,\mathbf{x})], \tag{1.7}$$

whose coefficients are rapidly oscillating. The initial condition is $\rho_n(0,\mathbf{x}) = \rho_0(x) \in L^2(R^d)$, which is assumed to be independent of the parameter n.

The main result in homogenization (periodic, almost periodic or random) [18] tells us that if the stationary stream matrix $\Psi(\mathbf{x})$ is *uniformly bounded and ergodic* then there exists a constant effective diffusivity matrix σ^{eff} such that if $\bar{\rho}$ satisfies the effective diffusion equation

$$\frac{\partial \bar{\rho}(t,\mathbf{x})}{\partial t} = \sum_{i,j=1}^{d} \sigma_{ij}^{eff} \frac{\partial^2 \bar{\rho}(t,\mathbf{x})}{\partial x_i \partial x_j} \tag{1.8}$$

with $\bar{\rho}(0,\mathbf{x}) = \rho_0(\mathbf{x})$ then $\rho_n \to \bar{\rho}$ as n tends to infinity

$$\lim_{n\to\infty} \sup_{0\le t\le T} \int_{R^d} |\rho_n(t,\mathbf{x}) - \bar{\rho}(t,\mathbf{x})|^2 d\mathbf{x} = 0 \tag{1.9}$$

for any $T < \infty$ and with probability one. However, when the stream matrix $\Psi(\mathbf{x})$ is stationary and ergodic but unbounded then it is not clear that a diffusion approximation holds. The purpose of this paper is to prove the following theorem.

Theorem 1 *Suppose that the stream matrix $\Psi(\mathbf{x})$ is stationary and ergodic, that the diffusivity σ is positive and that ρ_0 is in $L^2(R^d)$. Then there exists a constant effective diffusivity matrix σ^{eff} and the random density ρ_n converges in the sense of (1.9) to $\bar{\rho}$ satisfying (1.8) if and only if*

$$\langle |\Psi(\mathbf{x})|^2 \rangle < \infty \tag{1.10}$$

where $\langle\cdot\rangle$ *denotes expectation.*

The effective diffusivity matrix σ^{eff} is determined from the solution of a *cell* problem, as in the case of periodic coefficients [4], which is described in detail in section 3.4. It is not symmetric in general but in the above theorem only its symmetric part enters.

To put this theorem in its proper context and to explain its significance we provide several remarks. First, the diffusion equation (1.7) is not well defined when the stream function is unbounded so part of the theorem is to make sense of (1.7). We work entirely with time independent problems through the Laplace transform of (1.7)

$$\hat{\rho}_n(\mathbf{x},\lambda) = \int_0^\infty e^{-\lambda t}\rho_n(t,\mathbf{x})dt\ , \quad \lambda > 0 \tag{1.11}$$

which satisfies

$$-\nabla\cdot[(\sigma I + \Psi(n\mathbf{x}))\nabla\hat{\rho}_n(\mathbf{x},\lambda)] + \lambda\hat{\rho}_n(\mathbf{x},\lambda) = \rho_0(\mathbf{x}), \tag{1.12}$$

for $\mathbf{x} \in R^d$. Convergence of $\hat{\rho}_n$ in L^2, with probability one, to the Laplace transform of $\bar{\rho}$ for each $\lambda > 0$ implies (1.9). In this paper we will actually work with (1.12) over a bounded open set $\mathcal{O}$ in R^d with Dirichlet boundary conditions and $\lambda = 0$. All essential calculations are the same[1] for these two problems. Dropping the hat, the Dirichlet problem has the weak form

$$\int_{\mathcal{O}}(\sigma I + \Psi(n\mathbf{x}))\nabla\rho_n(\mathbf{x})\cdot\nabla\phi(\mathbf{x})d\mathbf{x} = \int_{\mathcal{O}}\rho_0(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \tag{1.13}$$

for every test function ϕ in $C_0^\infty(\mathcal{O})$. One of the first steps in our proof is to define (1.13) when Ψ is not in L^∞ but in L^2 in the sense of (1.10). The case of uniformly bounded coefficients that are also uniformly elliptic is covered by the usual homogenization results [18], whether they are symmetric or not. The case with bounded coefficients in the discrete setting(namely, random walks in random environments) was obtained by Kozlov [12] using martingale central limit theorems. In the discrete setting, the boundedness assumption is reflected in the uniform ellipticity of the transition probabilities.

Why is the L^2 condition (1.10) necessary and sufficient for diffusive behavior? There are shear-flow examples in two dimensions for which condition (1.10) is clearly necessary and

[1] They are more involved for the boundary value problem because of the singular boundary layers in the large n limit.

sufficient as can be seen from explicit computations. The examples are due to Matheron and De Marsily [14], who noted the significance of condition (1.10), and were studied extensively by Avellaneda and Majda [1]. This is all in the context of stationary stream matrices. In general, the stream matrix will have stationary increments (since the flow $\mathbf{u}$ is stationary) but will not be stationary. For nonstationary Ψ nondiffusive behavior is to be expected although there are no mathematical results to substantiate such behavior. Given the shear flow examples, and in the context of stationary stream matrices, it is therefore enough to show that (1.10) implies diffusive behavior. Previous attempts to extend the L^∞ homogenization results to unbounded coefficients required conditions like $\langle |\Psi|^p \rangle < \infty$ with $p = 2 + \epsilon$, $\epsilon > 0$ for $d = 2$ or $p = d$ for $d \geq 3$ which are not sharp, [2], [3], or certain additional regularity and growth conditions that are hard to verify [16]. The sharp result proven here relies essentially on the minimax variational principles used in [8] for the small σ (large Peclet number) analysis of the effective diffusivity. Similar variational principles were used to obtain bounds for complex dielectrics by Gibianski, Cherkaev and Milton [15]. A special form of the variational principles was also noted by Avellaneda and Majda [2] but it was not used.

Before reviewing the shear flow examples we note that along with the basic Theorem 1 we have a convergence theorem for the Dirichlet problem (1.13), as already mentioned, and the following.

Theorem 2 *Let $Q^{(n)}_{\mathbf{x}}$ be the probability measures on continuous paths starting at $\mathbf{x}$ for the process generated by the stochastic differential equation (1.2) with the scaling $\mathbf{x}(t) \to n\mathbf{x}(n^2 t)$. Under the hypotheses of Theorem 1 the measures $Q^{(n)}_{\mathbf{x}}$ converge weakly to the Brownian motion measure with infinitesimal covariance matrix $2\sigma^{eff}$, in measure relative to the law of the stationary flow field $\mathbf{u}$ and for each finite $\mathbf{x} \in R^d$.*

The convergence of the finite dimensional distributions follows immediately from Theorem 1. The tightness of the measures is proved in section 7 (cf. Theorem 7.1).

Let us briefly review the shear flow examples [14] which show that the L^2 condition (1.10) is sharp.

In two dimensions let $\mathbf{x} = (x, y)$ and $\mathbf{u}(\mathbf{x}) = (u(y), 0)$. Then the convection diffusion equation (1.1) becomes

$$\frac{\partial \rho}{\partial t} + u(y)\frac{\partial \rho}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right), \tag{1.14}$$

where we have set $\sigma = 1/2$ for simplicity, and the stochastic differential equation (1.2) becomes

$$\begin{aligned} dx(t) &= -u(y(t))dt + dw_1(t) \\ dy(t) &= dw_2(t) \end{aligned} \tag{1.15}$$

where $w_1(t)$ and $w_2(t)$ are independent Brownian motions on R^1, independent also of the random horizontal velocity $u(y)$. Assuming that $x(0) = 0$ and $y(0) = 0$ and letting

$$\langle u(y)u(0)\rangle = R(y) = \int_{-\infty}^{\infty} e^{iky}\hat{R}(k)dk \tag{1.16}$$

be the covariance R and power spectral density $\hat{R}$ of u we have

$$\begin{aligned} \langle E\{x^2(t)\}\rangle &= t + \int_0^t\int_0^t\int_{-\infty}^{\infty} E\{e^{ik(y(s_1)-y(s_2))}\}\hat{R}(k)dkds_1ds_2 \\ &= t + \int_0^t\int_0^t\int_{-\infty}^{\infty} e^{-\frac{k^2}{2}|s_1-s_2|}\hat{R}(k)dkds_1ds_2 \\ E\{y^2(t)\} &= t \end{aligned} \tag{1.17}$$

Here $E\{\}$ denotes expectation with respect to the Brownian motions and we have assumed for simplicity that there exists a continuous power spectral density $\hat{R}$. From (1.17) we find easily that

$$\langle E\{x^2(t)\}\rangle = t + \int_{-\infty}^{\infty} 4\hat{R}(k)\left[\frac{t}{k^2} - \frac{2}{k^4}(1 - e^{-k^2t/2})\right]dk \tag{1.18}$$

so that

$$\frac{1}{t}\langle E\{x^2(t)\}\rangle \to 1 + \int_{-\infty}^{\infty}\frac{4\hat{R}(k)}{k^2}dk \tag{1.19}$$

at t tends to infinity, provided the integral is finite. It is also easy to see that the integral in (1.19) is finite if and only if the process $\int^y u(s)ds$ is stationary and square integrable. This is the shear flow version of condition (1.10). If on the other hand the integral in (1.19) is not finite, when typically $\hat{R}(0) \neq 0$, then after a simple computation we have

$$\frac{1}{t^{3/2}}\langle E\{x^2(t)\}\rangle \to \frac{8\sqrt{2\pi}}{3}\hat{R}(0) \tag{1.20}$$

This means that we do not have diffusive behavior in the horizontal direction since the mean square displacement behaves like $t^{3/2}$ for t large. Note that $\hat{R}(0) \neq 0$ means that there will be no stationary stream function for the shear flow. The large scale (k small) fluctuations in the horizontal velocity are strong enough to produce superdiffusive behavior in the mean square horizontal particle displacement.

In several dimensions the square integrability condition can be made more explicit by using the spectral representation of the flow $\mathbf{u}$, which is stationary, divergence free and square integrable. There exists a process $\hat{\mathbf{u}}(\boldsymbol{\kappa})$ with orthogonal increments such that with probability one

$$\mathbf{u}(\mathbf{x}) = \int_{R^d} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} d\hat{\mathbf{u}}(\boldsymbol{\kappa}) \tag{1.21}$$

where $\overline{d\hat{\mathbf{u}}(\boldsymbol{\kappa})} = d\hat{\mathbf{u}}(-\boldsymbol{\kappa})$, since $\mathbf{u}$ is real, and

$$\begin{aligned} \langle d\hat{u}_p(\boldsymbol{\kappa})\overline{d\hat{u}_q(\boldsymbol{\kappa})}\rangle &= \hat{R}_{pq}(\boldsymbol{\kappa})d\boldsymbol{\kappa} \\ \langle u_p(\mathbf{x}+\mathbf{y})u_q(\mathbf{y})\rangle &= R_{pq}(\mathbf{x}) = \int_{R^d} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}}\hat{R}_{pq}(\boldsymbol{\kappa})d\boldsymbol{\kappa} \\ \boldsymbol{\kappa}\cdot d\hat{\mathbf{u}}(\boldsymbol{\kappa}) &= 0 \quad \text{(by incompressibility)} \\ R_{pq}(\mathbf{x}) = R_{qp}(-\mathbf{x})\ , &\quad \hat{R}_{pq}(\boldsymbol{\kappa}) = \hat{R}_{qp}(-\boldsymbol{\kappa})\ , \quad p,q = 1,\dots,d. \end{aligned} \tag{1.22}$$

We assume here that the spectral measure of the covariance has a continuous density $\hat{R}_{pq}(\boldsymbol{\kappa})$ with respect to Lebesgue measure. The stream matrix Ψ satisfying (1.3) has the spectral representation

$$\Psi_{pq}(\mathbf{x}) = \int_{R^d} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \frac{1}{|\boldsymbol{\kappa}|^2}\left[-i\kappa_q d\hat{u}_p(\boldsymbol{\kappa}) + i\kappa_p d\hat{u}_q(\boldsymbol{\kappa})\right] \tag{1.23}$$

provided it is square integrable

$$\langle|\Psi(\mathbf{x})|^2\rangle = \sum_{pq}\langle\Psi_{pq}(\mathbf{x})\Psi_{pq}(\mathbf{x})\rangle = 2\sum_p \int_{R^d} \frac{\hat{R}_{pp}(\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|^2} d\boldsymbol{\kappa} < \infty. \tag{1.24}$$

Since the flow $\mathbf{u}$ is square integrable we have

$$\langle|\mathbf{u}(\mathbf{x})|^2\rangle = \sum_p \int_{R^d} \hat{R}_{pp}(\boldsymbol{\kappa})d\boldsymbol{\kappa} < \infty. \tag{1.25}$$

If, for example, the flow $\mathbf{u}$ has a spectral density that satisfies for some constant C

$$\sum_p \hat{R}_{pp}(\boldsymbol{\kappa}) \leq \frac{C}{|\boldsymbol{\kappa}|^\alpha} \tag{1.26}$$

in the neighborhood of the origin and $\alpha < d-2$ then Theorem 1 tells us that we have diffusive behavior. In three dimensions having a bounded power spectral density at the origin will suffice but in two and one dimensions it will not. Note that in dimensions three or more Theorem 1 is natural and what one wants physically. Note also that the L^∞ condition on the stream matrix Ψ that the usual homogenization results demand is quite unnatural. In two dimensions the power spectral density must vanish at the origin if we are to have diffusive behavior and for shear flows, as we saw above, nondiffusive behavior is more typical. This is reminiscent of wave localization which occurs typically in low dimensions, if the random fluctuations are not large.

When the diffusion equation (1.1) is put into divergence form (1.6) the large time or homogenization asymptotic analysis is not sensitive to the dimension of the underlying space because we do not use Sobolev inequalities or other dimension-sensitive tools. In Theorems 1 and 2 dimension dependence enters only through the passage from the flow $\mathbf{u}$ to the stream matrix Ψ. The most natural way to relate these two quantities is the spectral representation (1.23) for which L^2 is the natural setting. That is another reason why it is important to have homogenization valid with just the L^2 condition (1.10). However, the main reason that we have looked at homogenization with unbounded coefficients so carefully is the minimax variational principles that we use and the mathematical technology around them. They are a powerful tool that may well be the key to unraveling multidimensional non-diffusive behavior (cf. [7]). They have already proven to be invaluable in the large Peclet number (small σ) analysis of the effective diffusivity for two dimensional periodic and random flows with bounded stream functions [8], [9]. It is important to note also that diffusion in random media is a big subject where many diverse issues arise. For example, we do not discuss here flows with nonzero mean or flows that are not incompressible. If the random fluctuations about the nonzero mean are small and in addition to stationarity we have some mixing then a few results are known [11], even with $\sigma = 0$. If the mean velocity is zero and the fluctuations are neither incompressible nor gradient fields then diffusive behavior has been proven for dimensions $d \geq 3$ and for small fluctuations (small Peclet number)[5]. The analytical methods for both of these cases differ substantially from those used in homogenization, and in this paper.

Since homogenization with unbounded coefficients is considered here for random incompressible flows it is natural to ask about problems with *symmetric* coefficient matrix in (1.7) that is unbounded. This is considered in detail in another paper. It illustrates nicely the use of variational principles, which in the symmetric case are well known.

2 Outline of the paper

Since the convection-diffusion equation (1.12) (or (1.13)) has a symmetric part $(\nabla \cdot \sigma \nabla)$ and a skew symmetric part $(\nabla \cdot \Psi_n \nabla)$, where $\Psi_n(\mathbf{x}) = \Psi(n\mathbf{x})$, it is natural to separate the symmetric part ((4.5) for the cell problem, (4.29) for the Dirichlet problem) and the skew symmetric part ((4.6) for the cell problem, (4.30) for the Dirichlet problem) of the solution by adding and subtracting to it the solution of the adjoint problem (4.27)-(4.28 or (4.2) for the cell problem). This is the motivation of the symmetrization procedure of Section 4 which applies to both the Dirichlet and the cell problem. This way, the equations can be written as a symmetric but non-definite system which are the Euler equations of a min-max variational principle (4.21), (4.39). Once the Euler equation corresponding to the min or the max is solved, the min-max principle is turned into a maximum (4.43), (4.49) or minimum principle (4.42), (4.46). This is done in Section 4, following a brief review in Section 3 of the analytical framework for stationary processes that was used in [18] and elsewhere.

The formulation of the minimum principle further motivates the definition of the Hilbert spaces $H_0(\Psi_n, \mathcal{O})$ (3.10) or $\mathcal{H}_g(\Psi)$ (3.28) in which the convection diffusion problems have a natural weak formulation. A key observation here is that the L^2-stationarity of Ψ is a necessary and sufficient condition for these spaces to contain all the functions with essentially bounded derivatives(Lemma 5.1). Therefore, under this assumption the existence and uniqueness questions become standard in the new spaces (Theorem 5.1, 5.2 and 5.4). This is the case for both the Dirichlet and the cell problems.

In Section 5 we address the $n \to \infty$ or homogenization limit. We would like to represent the exact solutions approximately in $H_0^1(\mathcal{O})$ by functions of the form (6.9) and (6.10) suggested by the multiple scales expansions (cf. Theorem 6.1, 6.2). The idea of the proof is to first show the attainability, within arbitrary error, of the min-max principle by trial functions of a specific form ((6.11) for the minimum principle, (6.12) for the maximum principle). The gap between the upper bound and the lower bound provided by the minimum and maximum principles respectively is closed by the density lemma 6.2. This is basically the content of Theorem 6.3 in Section 6.1.

Because of the local ellipticity ($\sigma = 1 > 0$), the approximation within arbitrarily small error of the exact solutions by (6.11)-(6.12) in $H_0^1(\mathcal{O})$ follows from the preceding convergence of functionals (Theorem 6.3). Since the limiting form of the approximations (6.11)-(6.12) is (6.9)-(6.10), again by the density lemma, the strong convergence Theorems 6.1- 6.2 are natural consequences of Theorem 6.3.

Since Ψ is unbounded there are no Nash estimates (cf. [17]) available. We have to obtain the tightness of the probability measures from sharp L^∞ resolvent estimates. This is done in Section 7 by noting that the L^2 estimates of Corollary 6.2 can be strengthened by averaging over the ensemble of fluid flows.

Let us also comment on some of the technical issues in this approach, which comprise much of Section 6 and 8.

To use the minimum and the maximum principles, which are nonlocal, we have to evaluate accurately the projection operator Γ_0 (3.9) acting on a fast oscillatory function. This amounts to solving in terms of approximate correctors in $H_0^1(\mathcal{O})$ the Poisson equations with large and rapidly oscillating source terms. This is the content of Lemmas 8.3 and 8.5. An additional difficulty which has to do with the boundary layers of the Dirichlet problem for large n enters and is handled by choosing the cut-off functions $\alpha_n(\mathbf{x})$ carefully. The resolvent estimates needed for Theorem 1, 2 and 7.1 are easier to obtain when the domain of interest is R^d.

It is natural to ask why we do not use (6.9)-(6.10) directly as trial functions? The answer is, as explained in Section 6.1, that they may not be admissible (that is, belong to $H_0(\Psi_n, \mathcal{O})$) unless Ψ is uniformly bounded. Therefore, it is essential to use the trial functions with essentially bounded derivatives and since only the minimal L^2 assumption is imposed on Ψ, some additional strong sublinear growth estimates (Lemma 8.2 and 6.1) for the trial functions are necessary for the proofs of Lemmas 8.3 and 8.5. Even when (6.9)-(6.10) do belong to $H_0(\Psi_n, \mathcal{O})$, the arguments of the proofs of Lemma 8.3 and 8.5 would not work because of the lack of strong sublinear growth estimates for the exact correctors. This illustrates the natural complementarity between the kind of estimates needed for the trial functions to make the variational framework work and the kind of assumptions imposed on Ψ: if the latter is uniformly bounded, then the former can be square integrable; if the latter is only square integrable, then the former has to be uniformly bounded.

3 Abstract framework

We begin with a brief review of the framework of stationary processes that is used in homogenization [18].

3.1 Random stationary stream matrix

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\Psi(\mathbf{x}, \omega)$ be a *strictly stationary* random skew-symmetric matrix of $\mathbf{x} \in R^d$ such that each element Ψ_{ij} is a L^2 function

$$\langle |\Psi_{ij}(\mathbf{x}, \cdot)|^2 \rangle < \infty, \quad \forall i, j, \tag{3.1}$$

where $\langle \cdot \rangle$ denotes the average or integral with respect to the measure P. By strict stationarity we mean that the joint distribution of $\Psi_{ij}(\mathbf{x}_1, \omega), \Psi_{ij}(\mathbf{x}_2, \omega), \ldots, \Psi_{ij}(\mathbf{x}_n, \omega)$ for any points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in R^d$ and that of $\Psi_{ij}(\mathbf{x}_1 + \ell, \omega), \Psi_{ij}(\mathbf{x}_2 + \ell, \omega), \ldots, \Psi_{ij}(\mathbf{x}_n + \ell, \omega)$ for any $\ell \in R^d$ is the same, so the averages in (3.1) are independent of $\mathbf{x}$. Without loss of generality (see Doob, [6]), we may assume that there is a group of transformations $\tau_{\mathbf{x}}$, $\mathbf{x} \in R^d$ from Ω into Ω that is one to one and preserves the measure P. That is, $\tau_{\mathbf{x}}\tau_{\mathbf{y}} = \tau_{\mathbf{x}+\mathbf{y}}$ and $P(\tau_{\mathbf{x}} A) = P(A)$ for any $A \in \mathcal{F}$. We may also suppose that there is a measurable matrix function $\widetilde{\Psi}(\omega)$ on Ω satisfying (0.1) such that

$$\Psi(\mathbf{x}, \omega) = \widetilde{\Psi}(\tau_{-\mathbf{x}} \omega) , \qquad \mathbf{x} \in R^d , \qquad \omega \in \Omega .$$

We assume the group of transformations $\tau_{\mathbf{x}}$ is ergodic with respect to the probability measure P.

The random stationary divergence free velocity $\mathbf{u}$ which we consider in this paper is given by

$$-\mathbf{u}(\mathbf{x}, \omega) = \nabla \cdot \Psi(\mathbf{x}, \omega). \tag{3.2}$$

In dimension two and three, the stream matrix Ψ has the familiar form such as (1.4) and (1.5) respectively

3.2 Hilbert spaces of stationary functions

The group of transformations $\tau_{\mathbf{x}}$ acting on Ω induces a group of operators on the Hilbert space of real-valued functions $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$ with inner product

$$(\tilde{f}, \tilde{g}) \equiv \langle \tilde{f}\tilde{g} \rangle \equiv \int_\Omega P(d\omega)\, \tilde{f}(\omega)\, \tilde{g}(\omega) \,, \qquad \tilde{f}, \tilde{g} \in \mathcal{H}$$

Here $\langle\cdot\rangle$ stands for integration over Ω with respect to P, $\int_\Omega P(d\omega)\cdot$. The group of operators $T_{\mathbf{x}}$ on $\mathcal{H}$ is given by

$$(T_{\mathbf{x}}\tilde{f})(\omega) = \tilde{f}(\tau_{-\mathbf{x}}\omega) \,, \qquad \mathbf{x} \in R^d$$

Since $\tau_{\mathbf{x}}$ is measure preserving, the operators $T_{\mathbf{x}}$ form a unitary group. Therefore they have closed densely defined infinitesimal generators $\tilde{\nabla}_i$ in each direction $i = 1, 2, \ldots, d$ with domain $\mathcal{D}_i \subset \mathcal{H}$. Then,

$$\tilde{\nabla}_i = \frac{\partial}{\partial x_i} T_{\mathbf{x}} \Big|_{\mathbf{x}=0} \,, \qquad i = 1, \ldots, d \,,$$

with differentiation defined in the sense of convergence in $\mathcal{H}$ for elements of $\mathcal{D}_i$. The closed subset of $\mathcal{H}$

$$\mathcal{H}^1 = \bigcap_{i=1}^{d} \mathcal{D}_i$$

becomes a Hilbert space with the inner product

$$\begin{aligned}(\tilde{f}, \tilde{g})_1 &\equiv \langle \tilde{f}\tilde{g} \rangle + \langle \tilde{\nabla}\tilde{f} \cdot \tilde{\nabla}\tilde{g} \rangle \\ &\equiv \int_\Omega P(d\omega)\, \tilde{f}(\omega)\, \tilde{g}(\omega) + \sum_{i=1}^{d} \int_\Omega P(d\omega)\, \tilde{\nabla}_i \tilde{f}(\omega)\, \tilde{\nabla}_i \tilde{g}(\omega)\end{aligned}$$

The hypothesis that the action of the translation group $\tau_{\mathbf{x}}$ is ergodic on Ω takes the following form in $\mathcal{H}$: the only functions in $\mathcal{H}$ that are invariant under $T_{\mathbf{x}}$ are the constant functions.

Let $H_s(R^d; \mathcal{H})$ be the space of all stationary random processes $f(\mathbf{x}, \omega)$ on R^d, such that $\int_\Omega P(d\omega) f^2(\mathbf{x}, \omega) = \text{const.} < \infty$. Clearly $H_s(R^d; \mathcal{H})$ is in one-to-one correspondence with $\mathcal{H}$ since it is simply the space of all translates of $\mathcal{H}$, that is, $f(\mathbf{x}, \omega) \in H(R^d; \mathcal{H})$ iff

$f(\mathbf{x},\omega) = T_{\mathbf{x}}\tilde{f}(\omega)$, $\tilde{f}(\omega) \in \mathcal{H}$. Similarly, we may identify $\mathcal{H}^1$ with the set of mean square differentiable, stationary processes $H^1_s(R^d;\mathcal{H})$. In particular, if $f \in H^1_s$, then its derivatives are also a stationary processes and

$$\nabla_i f(\mathbf{x},\omega) = \frac{\partial f(\mathbf{x},\omega)}{\partial x_i} = \widetilde{\nabla}_i f(\mathbf{x},\omega)$$

with equality holding $d\mathbf{x} \times P$ almost everywhere. Thus, we have $H^1_s(R^d;\mathcal{H}) = H_s(R^d;\mathcal{H}^1)$.

We define also the Hilbert spaces $\mathcal{H}_g$ and $\mathcal{H}_c$ which correspond to gradient fields and curl fields, respectively,

$$\begin{aligned} \mathcal{H}_g &= \Big\{ \widetilde{F}_i(\omega) \in \mathcal{H}\ ,\ i = 1,\ldots,d \mid \widetilde{\nabla}_i \widetilde{F}_j = \widetilde{\nabla}_j \widetilde{F}_i, \forall i,j = 1,...,d \\ &\qquad \text{weakly and } \textstyle\int_\Omega \widetilde{F}_i(\omega)\, P(d\omega) = 0 \Big\} \end{aligned}$$

$$\begin{aligned} \mathcal{H}_c &= \Big\{ \widetilde{G}_i(\omega) \in \mathcal{H}\ ,\ i = 1,\ldots,d \mid \textstyle\sum_i \widetilde{\nabla}_i \widetilde{G}_i = 0 \\ &\qquad \text{weakly and } \textstyle\int_\Omega \widetilde{G}_i(\omega)\, P(d\omega) = 0 \Big\} \end{aligned}$$

3.3 Weak formulation of boundary value problem

Consider the inhomogeneous boundary value problem (1.13) with the fast oscillatory stream matrix $\Psi_n(\mathbf{x},\omega) = \Psi(n\mathbf{x},\omega)$:

$$\nabla \cdot (I + \Psi_n)\nabla \rho_n = s + \nabla \cdot \mathbf{S}, \quad \text{in} \quad \mathcal{O}, \tag{3.3}$$

$$\rho_n = 0, \qquad \text{on} \quad \partial\mathcal{O}, \tag{3.4}$$

for inhomogeneous terms $s \in L^2(\mathcal{O}), \mathbf{S} \in (L^2(\mathcal{O}))^d$, where $\mathcal{O}$ is a bounded, smooth domain in R^d. This is a little more general than (1.13) and, as before, n is a large parameter that eventually we let tend to infinity.

If the stream matrix is bounded

$$\text{ess-sup}_{\omega\in\Omega}|\widetilde{\Psi}| < \infty, \tag{3.5}$$

then there exists a unique $\rho_n \in H^1_0(\mathcal{O})$ such that

$$\int_{\mathcal{O}} d\mathbf{x}\ (I + \Psi_n)\nabla\rho_n \cdot \nabla\phi + \int_{\mathcal{O}} d\mathbf{x}\ (s\phi - \mathbf{S}\cdot\nabla\phi) = 0 \tag{3.6}$$

for all $\phi \in H_0^1(\mathcal{O})$. The proof follows from the the Lax-Milgram Lemma on the space $H_0^1(\mathcal{O})$. If we let $\phi = \rho_n$ and integrate by parts, we obtain

$$\int_{\mathcal{O}} dx\ \nabla\rho_n \cdot \nabla\rho_n \leq |s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})} \tag{3.7}$$

using the Schwarz inequality.

For unbounded stream matrices, the marix $I + \Psi_n$ defines an unbounded bilinear form, so the Lax-Milgram Lemma does not work on $H_0^1(\mathcal{O})$ right away. To motivate the introduction of the right spaces for this problem we first write (3.3) in the integral form

$$\nabla\rho_n + \Gamma_0\Psi_n\nabla\rho_n = \nabla(\Delta_0)^{-1}s + \Gamma_0\mathbf{S} \tag{3.8}$$

where the projection operator

$$\Gamma_0 = \nabla(\Delta_0)^{-1}\nabla\cdot \tag{3.9}$$

projects square integrable fields to square integrable fields that are gradients of functions with zero Dirichlet data on $\partial\mathcal{O}$ and $(\Delta_0)^{-1}$ is the inverse of the Laplacian with zero Dirichlet data on $\partial\mathcal{O}$. By the classical Calderon-Zygmond estimate, Γ_0 is a bounded operator from L^p to L^p, for all $p > 0$.

The natural space associated with (3.8) is

$$H_0(\Psi_n, \mathcal{O}) = \{g \in H_0^1(\mathcal{O}); \Gamma_0\Psi_n\nabla g \in (L^2(\mathcal{O}))^d\} \tag{3.10}$$

endowed with the norm

$$||g||^2_{\Psi_n} = |\nabla g|^2_{L^2(\mathcal{O})} + |\Gamma_0\Psi_n\nabla g|^2_{L^2(\mathcal{O})} \tag{3.11}$$

Note that the definition of $H_0(\Psi_n, \mathcal{O})$ incorporates only partial information of $\Psi_n\nabla g$. For example, we have no knowledge about the square integrability of $\Psi_n\nabla g$ for $g \in H_0(\Psi_n, \mathcal{O})$. As will be further explained in Section 6, this poses a severe difficulty for the standard homogenization methods such as [18] or Tartar's argument [19], but can be handled nicely by the variational methods.

Clearly, $H_0(\Psi_n, \mathcal{O}) \subseteq H_0^1(\mathcal{O})$ and $H_0(\Psi_n, \mathcal{O}) = H_0^1(\mathcal{O})$, if $\tilde{\Psi}$ is bounded. We show in Section 5 that

$$C_0^\infty(\mathcal{O}) \subset H_0(\Psi_n, \mathcal{O}), \forall n > 0, \tag{3.12}$$

for almost all ω, if the stream matrix is square integrable. The square integrability condition is the minimum required for $H_0(\Psi_n, \mathcal{O})$ to contain $C_0^\infty(\mathcal{O})$.

The problem now is to seek $\rho_n \in H_0(\Psi_n, \mathcal{O})$, rather than $H_0^1(\mathcal{O})$, such that

$$\int_{\mathcal{O}} d\mathbf{x}\ (\nabla\rho_n \cdot \nabla\phi + \Gamma_0\Psi_n\nabla\rho_n \cdot \nabla\phi) + \int_{\mathcal{O}} d\mathbf{x}\ (s\phi - \mathbf{S}\cdot\nabla\phi) = 0 \tag{3.13}$$

for all $\phi \in H_0^1(\mathcal{O})$. At this stage, the integrals in (3.13) at least make sense for $\phi \in H_0^1(\mathcal{O})$ and $\rho_n \in H_0(\Psi_n, \mathcal{O})$. But there is no energy estimate that puts ρ_n in $H_0(\Psi_n, \mathcal{O})$ since the term with Ψ_n drops out of the energy identity. We will address the questions of existence and uniqueness in Section 4 using the variational methods developed in [8].

3.4 Abstract cell problem and the effective diffusivity

Assuming that $\Psi(\mathbf{x}, \omega)$ is uniformly bounded and stochastically continuous, Papanicolaou and Varadhan [18] showed that

$$\langle\rho_n(\mathbf{x}, \cdot)\rangle \to \rho, \quad H_0^1(\mathcal{O}) \quad \text{weakly} \tag{3.14}$$

in the limit $n \to \infty$. Here ρ is the solution of a deterministic variational problem with constant coefficients σ^{eff}

$$\int_{\mathcal{O}} d\mathbf{x}\ \sigma^{eff}\nabla\rho\cdot\nabla\phi + \int_{\mathcal{O}} d\mathbf{x}\ (s\phi - \mathbf{S}\cdot\nabla\phi) = 0 \tag{3.15}$$

for $\forall\phi \in H_0^1(\mathcal{O})$. The matrix $\sigma^{eff} = [\sigma_{ij}^{eff}]$ is called the effective diffusivity and is determined by solving the abstract cell problem: Find two stationary random fields $\mathbf{E}_i(\mathbf{x}, \omega)$ and $\mathbf{D}_i(\mathbf{x}, \omega) \in (H_s^1(R^d; \mathcal{H}))^d, i = 1, ..., d$, such that

$$\mathbf{D}_i(\mathbf{x}, \omega) = (I + \Psi(\mathbf{x}, \omega))(\mathbf{E}_i(\mathbf{x}, \omega) + \mathbf{e}^i) \tag{3.16}$$

$$\nabla \times \mathbf{E}_i(\mathbf{x}, \omega) = 0 \tag{3.17}$$

$$\nabla \cdot \mathbf{D}_i(\mathbf{x}, \omega) = 0 \tag{3.18}$$

$$\langle\mathbf{E}_i(\mathbf{x}, \cdot)\rangle = 0 \tag{3.19}$$

where $\{\mathbf{e}^i\}$ is an orthonormal set of vectors in R^d and

$$\sigma_{ij}^{eff} = \langle\mathbf{D}_i(\mathbf{x}, \cdot)\cdot\mathbf{e}^j\rangle, \quad i, j = 1, ..., d. \tag{3.20}$$

The connection between this cell problem and homogenization as in Theorem 1 comes about by the usual multiple scale arguments and is formally the same in the random as in the periodic case [4, 18].

When $\Psi(\mathbf{x},\omega)$ is strictly stationary as defined in Section 3.1, the abstract cell problem (3.16)-(3.19) becomes

$$\widetilde{\mathbf{D}}_i(\omega) = (I + \widetilde{\Psi}(\omega))(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i) \tag{3.21}$$

$$\widetilde{\nabla} \times \widetilde{\mathbf{E}}_i(\omega) = 0 \tag{3.22}$$

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{D}}_i(\omega) = 0 \tag{3.23}$$

$$\langle \widetilde{\mathbf{E}}_i(\cdot) \rangle = 0 \tag{3.24}$$

whose variational form is to find $\widetilde{\mathbf{E}}_i \in \mathcal{H}_g, i = 1, ..., d$, such that

$$\int_\Omega P(d\omega) \left(I + \widetilde{\Psi}(\omega)\right) \left(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i\right) \cdot \widetilde{\mathbf{F}}(\omega) = 0 \tag{3.25}$$

for all $\widetilde{\mathbf{F}}(\omega) \in \mathcal{H}_g$, and

$$\sigma_{ij}^{eff} = \int_\Omega P(d\omega) \left(I + \widetilde{\Psi}(\omega)\right) \left(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i\right) \cdot \mathbf{e}^j, \quad i,j = 1, ..., d. \tag{3.26}$$

By Lax-Milgram lemma (3.25) has a unique solution for bounded $\widetilde{\Psi}$.

For unbounded stream matrices $\widetilde{\Psi}$, the abstract cell problem can be put into a form parallel to (3.13), namely, to find $\widetilde{\mathbf{E}}_i \in \mathcal{H}_g(\widetilde{\Psi})$, $i = 1, ..., d$, such that

$$\int_\Omega P(d\omega) \left(I + \widetilde{\Gamma}\widetilde{\Psi}(\omega)\right) \left(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i\right) \cdot \widetilde{\mathbf{F}}(\omega) = 0 \tag{3.27}$$

for all $\widetilde{\mathbf{F}}(\omega) \in \mathcal{H}_g$, where the space $\mathcal{H}_g(\widetilde{\Psi})$ is defined as

$$\mathcal{H}_g(\widetilde{\Psi}) = \{\mathbf{G} \in \mathcal{H}_g; \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{G}} \in (\mathcal{H})^d\}. \tag{3.28}$$

Here

$$\widetilde{\Gamma} = \widetilde{\nabla}\widetilde{\Delta}^{-1}\widetilde{\nabla}\cdot$$

with $\widetilde{\nabla} = (\widetilde{\nabla}_1, ..., \widetilde{\nabla}_d)$ and $\widetilde{\Delta} = \widetilde{\nabla}\cdot\widetilde{\nabla}$ is the projection operator that takes vector fields in $\mathcal{H}^d$ to curl free fields in $\mathcal{H}_g$. Since $\mathcal{H}_g(\widetilde{\Psi})$ is defined through the projection $\widetilde{\Gamma}$, the same remark following the definition (3.10) applies, namely, we do not know if $\widetilde{\Psi}\widetilde{\mathbf{G}} \in \mathcal{H}^d, \forall \widetilde{\mathbf{G}} \in \mathcal{H}_g(\widetilde{\Psi})$.

The existence and uniqueness of (3.27), as well as the existence of effective diffusivity, will be addressed in Section 4.

4 Variational principles

The main step in the derivation of variational principles is the symmetrization procedure that transforms the original problem and its adjoint into a symmetric, but nondefinite system which are the Euler equations of a min-max principle. For the derivations in this section we assume that the stream matrix Ψ is uniformly bounded so all the calculations make sense in the usual way. For unbounded but square integrable Ψ we take the symmetrized system as starting point of the analysis and establish existence and uniqueness in appropriate spaces, then work our way back to the original problems. This is done in Section 5.

4.1 Symmetrization and min-max variational principle

4.1.1 Symmetrization of abstract cell problem

Following closely [8], we denote the intensity and flux fields of the abstract cell problem (3.21)-(3.24) with the superscript + and those of the adjoint problem with the superscript −. Thus

$$\widetilde{\mathbf{D}}^{+}_{\mathbf{e}^i} = (I + \tilde{\Psi}(\omega))\widetilde{\mathbf{E}}^{+}_{\mathbf{e}^i} \tag{4.1}$$

$$\widetilde{\mathbf{D}}^{-}_{\mathbf{e}^j} = (I - \tilde{\Psi}(\omega))\widetilde{\mathbf{E}}^{-}_{\mathbf{e}^j} \tag{4.2}$$

$$\widetilde{\nabla} \times \widetilde{\mathbf{E}}^{+}_{\mathbf{e}^i} = \widetilde{\nabla} \times \widetilde{\mathbf{E}}^{-}_{\mathbf{e}^j} = 0 \tag{4.3}$$

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{D}}^{+}_{\mathbf{e}^i} = \widetilde{\nabla} \cdot \widetilde{\mathbf{D}}^{-}_{\mathbf{e}^j} = 0. \tag{4.4}$$

Define now the sum and difference fields

$$\widetilde{\mathbf{E}}_{ij} = \frac{1}{2}(\widetilde{\mathbf{E}}^{+}_{\mathbf{e}^i} + \widetilde{\mathbf{E}}^{-}_{\mathbf{e}^j}), \tag{4.5}$$

$$\widetilde{\mathbf{E}}'_{ij} = \frac{1}{2}(\widetilde{\mathbf{E}}^{+}_{\mathbf{e}^i} - \widetilde{\mathbf{E}}^{-}_{\mathbf{e}^j}), \tag{4.6}$$

$$\widetilde{\mathbf{D}}_{ij} = \frac{1}{2}(\widetilde{\mathbf{D}}^{+}_{\mathbf{e}^i} + \widetilde{\mathbf{D}}^{-}_{\mathbf{e}^j}), \tag{4.7}$$

$$\widetilde{\mathbf{D}}'_{ij} = \frac{1}{2}(\widetilde{\mathbf{D}}^{+}_{\mathbf{e}^i} - \widetilde{\mathbf{D}}^{-}_{\mathbf{e}^j}), \tag{4.8}$$

which are related to each other by

$$\widetilde{\mathbf{D}}_{ij} = \widetilde{\mathbf{E}}_{ij} + \widetilde{\Psi}\widetilde{\mathbf{E}}'_{ij} \tag{4.9}$$

$$\widetilde{\mathbf{D}}'_{ij} = \widetilde{\mathbf{E}}'_{ij} + \widetilde{\Psi}\widetilde{\mathbf{E}}_{ij} \tag{4.10}$$

$$\widetilde{\nabla} \times \widetilde{\mathbf{E}}_{ij} = \widetilde{\nabla} \times \widetilde{\mathbf{E}}'_{ij} = 0 \tag{4.11}$$

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{D}}_{ij} = \widetilde{\nabla} \cdot \widetilde{\mathbf{D}}'_{ij} = 0 \tag{4.12}$$

The effective diffusivity is defined by

$$\sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \int_\Omega P(d\omega)\, \widetilde{\mathbf{D}}^+_{\mathbf{e}^i} \cdot \mathbf{e}^j \tag{4.13}$$

and we define also

$$\sigma_-^{eff}(\mathbf{e}^j, \mathbf{e}^i) = \int_\Omega P(d\omega)\, \widetilde{\mathbf{D}}^-_{\mathbf{e}^j} \cdot \mathbf{e}^i \tag{4.14}$$

with the mean field conditions

$$\int_\Omega P(d\omega)\, \widetilde{\mathbf{E}}^+_{\mathbf{e}^i} = \mathbf{e}^i \ , \quad \int_\Omega P(d\omega)\, \widetilde{\mathbf{E}}^-_{\mathbf{e}^j} = \mathbf{e}^j \ . \tag{4.15}$$

It is easy to see that

$$\sigma_-^{eff}(\mathbf{e}^j, \mathbf{e}^i) = \sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) \tag{4.16}$$

because

$$\begin{aligned} \sigma_-^{eff}(\mathbf{e}^j, \mathbf{e}^i) &= \int_\Omega P(d\omega)\, \left(I - \tilde{\Psi}\right) \widetilde{\mathbf{E}}^-_{\mathbf{e}^j} \cdot \widetilde{\mathbf{E}}^+_{\mathbf{e}^i} \\ &= \int_\Omega P(d\omega)\, \widetilde{\mathbf{E}}^-_{\mathbf{e}^j} \cdot \left(I + \tilde{\Psi}\right) \widetilde{\mathbf{E}}^+_{\mathbf{e}^i} \\ &= \int_\Omega P(d\omega)\, \mathbf{e}^j \cdot \widetilde{\mathbf{D}}^+_{\mathbf{e}^i} \\ &= \sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j). \end{aligned}$$

In other words, σ_-^{eff} is the adjoint of σ_+^{eff}. Thus,

$$\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \frac{1}{2}\sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j) + \frac{1}{2}\sigma_-^{eff}(\mathbf{e}^j, \mathbf{e}^i) \tag{4.17}$$

which in turn equals

$$\begin{aligned} &\frac{1}{2}\int_\Omega P(d\omega)\widetilde{\mathbf{D}}^+_{\mathbf{e}^i} \cdot \widetilde{\mathbf{E}}^-_{\mathbf{e}^j} + \frac{1}{2}\int_\Omega P(d\omega)\widetilde{\mathbf{D}}^-_{\mathbf{e}^j} \cdot \widetilde{\mathbf{E}}^+_{\mathbf{e}^i} \\ &= \frac{1}{4}\int_\Omega P(d\omega)\left(\widetilde{\mathbf{D}}^+_{\mathbf{e}^i} + \widetilde{\mathbf{D}}^-_{\mathbf{e}^j}\right) \cdot \left(\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} + \widetilde{\mathbf{E}}^-_{\mathbf{e}^j}\right) \\ &\quad - \frac{1}{4}\int_\Omega P(d\omega)\left(\widetilde{\mathbf{D}}^+_{\mathbf{e}^i} - \widetilde{\mathbf{D}}^-_{\mathbf{e}^j}\right) \cdot \left(\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} - \widetilde{\mathbf{E}}^-_{\mathbf{e}^j}\right) \\ &= \int_\Omega P(d\omega)\widetilde{\mathbf{D}}_{ij} \cdot \widetilde{\mathbf{E}}_{ij} - \int_\Omega P(d\omega)\widetilde{\mathbf{D}}'_{ij} \cdot \widetilde{\mathbf{E}}'_{ij} \end{aligned} \tag{4.18}$$

and the mean field conditions (4.15) become

$$\begin{aligned}\int_\Omega P(d\omega)\widetilde{\mathbf{E}}'_{ij} &= \frac{\mathbf{e}^i - \mathbf{e}^j}{2}\,, \\ \int_\Omega P(d\omega)\widetilde{\mathbf{E}}_{ij} &= \frac{\mathbf{e}^i + \mathbf{e}^j}{2}\end{aligned} \tag{4.19}$$

In view of (4.9) and (4.10), (4.18) is equivalent to

$$\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \int_\Omega P(d\omega)\begin{pmatrix} -I & -\tilde{\Psi} \\ \tilde{\Psi} & I \end{pmatrix}\begin{pmatrix} \widetilde{\mathbf{E}}'_{ij} \\ \widetilde{\mathbf{E}}_{ij} \end{pmatrix}\cdot\begin{pmatrix} \widetilde{\mathbf{E}}'_{ij} \\ \widetilde{\mathbf{E}}_{ij} \end{pmatrix} \tag{4.20}$$

which admits a min-max variational characterization

$$\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \inf_{\substack{\nabla\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \sup_{\substack{\nabla\times\tilde{\mathbf{F}}'=0 \\ \langle\tilde{\mathbf{F}}'\rangle=(\mathbf{e}^i-\mathbf{e}^j)/2}} \int_\Omega P(d\omega)\begin{pmatrix} -I & -\tilde{\Psi} \\ \tilde{\Psi} & I \end{pmatrix}\begin{pmatrix} \tilde{\mathbf{F}}' \\ \tilde{\mathbf{F}} \end{pmatrix}\cdot\begin{pmatrix} \tilde{\mathbf{F}}' \\ \tilde{\mathbf{F}} \end{pmatrix} \tag{4.21}$$

since (4.12) are the Euler equations of (4.20). This is the min-max variational principle for the symmetrized cell problem that we will be using to extend the theory to unbounded coefficients.

The effective diffusivity σ^{eff} is not symmetric in general. But if, for example, the probability distribution $P(d\omega)$ is invariant under the transformation $\tilde{\Psi} \to -\tilde{\Psi}$, then σ^{eff} can be shown to be symmetric from (4.16) since $\sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \sigma_-^{eff}(\mathbf{e}^j, \mathbf{e}^i) = \sigma_+^{eff}(\mathbf{e}^j, \mathbf{e}^i)$. The last equality is due to the invariance of P with respect to change in sign for $\tilde{\Psi}$, [8].

Note that only the symmetric part of the effective tensor σ^{eff} appears in the final homogenized equation in Theorem 1. There is an identity for the symmetric part which will be useful later.

$$\frac{1}{2}\left\{\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) + \sigma^{eff}(\mathbf{e}^j, \mathbf{e}^i)\right\} = \langle \widetilde{\mathbf{E}}_{ii}\cdot\widetilde{\mathbf{E}}_{jj} - 2\tilde{\Psi}\widetilde{\mathbf{E}}_{ii}\widetilde{\mathbf{E}'}_{jj} - \widetilde{\mathbf{E}}'_{ii}\cdot\widetilde{\mathbf{E}}'_{jj}\rangle,\ \forall i,j. \tag{4.22}$$

Its derivation is straight forward. Using the definitions (4.5)-(4.6) and multiplying out the expressions the right hand side is equal to

$$\begin{aligned}&\frac{1}{2}\langle \widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^+_{\mathbf{e}^j} + \widetilde{\mathbf{E}}^+_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^-_{\mathbf{e}^j} - \tilde{\Psi}\left(\widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^+_{\mathbf{e}^j} - \widetilde{\mathbf{E}}^+_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^-_{\mathbf{e}^j} + \widetilde{\mathbf{E}}^+_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^+_{\mathbf{e}^j} - \widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^-_{\mathbf{e}^j}\right)\rangle \\ &= \langle\left(\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} + \tilde{\Psi}\widetilde{\mathbf{E}}^+_{\mathbf{e}^i}\right)\cdot\widetilde{\mathbf{E}}^-_{\mathbf{e}^j} + \left(\widetilde{\mathbf{E}}^-_{\mathbf{e}^i} - \tilde{\Psi}\widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\right)\cdot\widetilde{\mathbf{E}}^+_{\mathbf{e}^j} - \tilde{\Psi}\left(\widetilde{\mathbf{E}}^+_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^+_{\mathbf{e}^j} - \widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\right)\rangle\end{aligned} \tag{4.23}$$

after cancelling terms like $\widetilde{\mathbf{E}}^+_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^+_{\mathbf{e}^j}$ and $\widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\cdot\widetilde{\mathbf{E}}^-_{\mathbf{e}^i}$. This reduces further to

$$\frac{1}{2}\langle\left(\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} + \tilde{\Psi}\widetilde{\mathbf{E}}^+_{\mathbf{e}^i}\right)\cdot\mathbf{e}^j\rangle + \frac{1}{2}\langle\left(\widetilde{\mathbf{E}}^-_{\mathbf{e}^i} - \tilde{\Psi}\widetilde{\mathbf{E}}^-_{\mathbf{e}^i}\right)\cdot\mathbf{e}^j\rangle \tag{4.24}$$

because of (4.1)-(4.4) and the skew symmetry of $\widetilde{\Psi}$. Now observe that the first term of (4.24) is just $\sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j)$ and the second term $\sigma_-^{eff}(\mathbf{e}^i, \mathbf{e}^j)$. The identity (4.22) then follows immediately from (4.16).

4.1.2 Symmetrization of boundary value problem

Consider the inhomogeneous boundary value problem (3.3)-(3.4) and its adjoint, denoted with superscripts $+, -$, respectively:

$$\nabla \cdot (I + \Psi_n)\nabla \rho_n^+ = s + \nabla \cdot \mathbf{S}, \quad \text{in} \quad \mathcal{O}, \tag{4.25}$$

$$\rho_n^+ = 0, \quad \text{on} \quad \partial\mathcal{O}, \tag{4.26}$$

$$\nabla \cdot (I - \Psi_n)\nabla \rho_n^- = s + \nabla \cdot \mathbf{S}, \quad \text{in} \quad \mathcal{O}, \tag{4.27}$$

$$\rho_n^- = 0, \quad \text{on} \quad \partial\mathcal{O}. \tag{4.28}$$

in a bounded domain $\mathcal{O}$. Let ρ_n, ρ_n' be the sum and difference

$$\rho_n = \frac{1}{2}(\rho_n^+ + \rho_n^-) \tag{4.29}$$

$$\rho_n' = \frac{1}{2}(\rho_n^+ - \rho_n^-). \tag{4.30}$$

In terms of ρ_n, ρ_n' , we put (4.25)-(4.28) into symmetrized form by adding and subtracting (4.25) and (4.27)

$$\nabla \cdot \nabla \rho_n + \nabla \cdot \Psi_n \nabla \rho_n' = s + \nabla \cdot \mathbf{S}, \quad \text{in} \quad \mathcal{O}, \tag{4.31}$$

$$\nabla \cdot \nabla \rho_n' + \nabla \cdot \Psi_n \nabla \rho_n = 0, \quad \text{in} \quad \mathcal{O}, \tag{4.32}$$

$$\rho_n = \rho_n' = 0, \quad \text{on} \quad \partial\mathcal{O} \tag{4.33}$$

or, equivalently

$$(\nabla, \nabla) \cdot \begin{pmatrix} -I & -\Psi_n \\ \Psi_n & I \end{pmatrix} \begin{pmatrix} \nabla \rho_n' \\ \nabla \rho_n \end{pmatrix} = \begin{pmatrix} 0 \\ s + \nabla \cdot \mathbf{S} \end{pmatrix} \quad \text{in} \quad \mathcal{O}, \tag{4.34}$$

$$\rho_n = \rho_n' = 0, \quad \text{on} \quad \partial\mathcal{O}. \tag{4.35}$$

Equations (4.31)-(4.32) are formal and should be understood in the weak sense

$$\int_{\mathcal{O}} d\mathbf{x}\ \nabla \rho_n \cdot \nabla \phi + \int_{\mathcal{O}} d\mathbf{x}\ \Psi_n \nabla \rho_n' \cdot \nabla \phi = \int_{\mathcal{O}} d\mathbf{x}\ (\mathbf{S} \cdot \nabla \phi - s\phi) \tag{4.36}$$

$$\int_{\mathcal{O}} d\mathbf{x}\ \nabla \rho_n' \cdot \nabla \phi + \int_{\mathcal{O}} d\mathbf{x}\ \Psi_n \nabla \rho_n \cdot \nabla \phi = 0 \tag{4.37}$$

for all $\phi \in H_0^1(\mathcal{O})$ (recall that Ψ is assumed to be bounded in this section).

Clearly (4.34)-(4.35) are the Euler equations of the quadratic functional

$$J_n(s+\nabla\cdot\mathbf{S}) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \begin{pmatrix} -I & -\Psi_n \\ \Psi_n & I \end{pmatrix} \begin{pmatrix} \nabla\rho_n' \\ \nabla\rho_n \end{pmatrix} \cdot \begin{pmatrix} \nabla\rho_n' \\ \nabla\rho_n \end{pmatrix} + 2(s\rho_n - \mathbf{S}\cdot\nabla\rho_n), \quad (4.38)$$

that is, ρ_n and ρ_n' are a critical point of the min-max variational principle

$$J_n(s+\nabla\cdot\mathbf{S}) = \inf_{g|_{\partial\mathcal{O}}=0} \sup_{g'|_{\partial\mathcal{O}}=0} \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \begin{pmatrix} -I & -\Psi_n \\ \Psi_n & I \end{pmatrix} \begin{pmatrix} \nabla g' \\ \nabla g \end{pmatrix} \cdot \begin{pmatrix} \nabla g' \\ \nabla g \end{pmatrix} + 2(sg - \mathbf{S}\cdot\nabla g) \quad (4.39)$$

This is the variational characterization that we will use to extend the theory to unbounded coefficients.

4.2 Nonlocal (minimum and maximum) variational principles

We can get minimum (maximum) principles by eliminating the supremum (infimum) from the min-max variational principles (4.21) and (4.39). The resulting variational principles are nonlocal in nature because the solutions of the supremum (infimum) involve projection operators.

4.2.1 Abstract cell problem

The Euler equation for the supremum in (4.21) is

$$\tilde{\nabla}\cdot\tilde{\mathbf{F}}' + \tilde{\nabla}\cdot\tilde{\Psi}\tilde{\mathbf{F}} = 0\,. \quad (4.40)$$

Using the projection operator

$$\tilde{\Gamma} = \tilde{\nabla}\tilde{\Delta}^{-1}\tilde{\nabla}\cdot \quad (4.41)$$

that takes square integrable vector fields to curl free ones in $\mathcal{H}_g$, we write the solution of (4.40) in the form $\tilde{\mathbf{F}}' = -\tilde{\Gamma}\tilde{\Psi}\tilde{\mathbf{F}}$ and substituting it in (4.21)

$$\begin{aligned} \sigma^{eff}(\mathbf{e}^i,\mathbf{e}^j) &= \inf_{\substack{\tilde{\nabla}\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \int_\Omega P(d\omega)\left(\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}} - 2\tilde{\Psi}\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}}' - \tilde{\mathbf{F}}'\cdot\tilde{\mathbf{F}}'\right) \\ &= \inf_{\substack{\tilde{\nabla}\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \int_\Omega P(d\omega)(\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}} + \tilde{\Gamma}\tilde{\Psi}\tilde{\mathbf{F}}\cdot\tilde{\Gamma}\tilde{\Psi}\tilde{\mathbf{F}} \\ &\qquad -\tilde{\Psi}\tilde{\mathbf{F}}\cdot(\mathbf{e}^i-\mathbf{e}^j) - |\tfrac{\mathbf{e}^i-\mathbf{e}^j}{2}|^2) \end{aligned} \quad (4.42)$$

Note that (4.42) is nonlocal because of the projection operator $\tilde{\Gamma}$.

Similarly, we can eliminate the infimum in (4.21) and derive a nonlocal maximum principle

$$\sigma^{eff}(\mathbf{e}^i,\mathbf{e}^j) = \sup_{\substack{\tilde{\nabla}\times\mathbf{F}'=0 \\ \langle\mathbf{F}'\rangle=(\mathbf{e}^i-\mathbf{e}^j)/2}} \int_\Omega P(d\omega)\Big(-\widetilde{\mathbf{F}'}\cdot\widetilde{\mathbf{F}'} - \tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}'}\cdot\tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}'} + \tilde{\Psi}\widetilde{\mathbf{F}'}\cdot(\mathbf{e}^i+\mathbf{e}^j) + \left|\tfrac{\mathbf{e}^i+\mathbf{e}^j}{2}\right|^2\Big) \tag{4.43}$$

4.2.2 Boundary value problem

The Euler equation for the supremum in (4.39) is

$$\Delta g' + \nabla\cdot\Psi_n\nabla g = 0, \quad \text{in} \quad \mathcal{O}, \tag{4.44}$$

$$g' = 0, \qquad \text{on} \quad \partial\mathcal{O} \tag{4.45}$$

Solving with the help of the projection operator Γ_0 defined by (3.9) we have $\nabla g' = -\Gamma_0\Psi_n\nabla g$ from (4.44), (4.45) and substituting this into the min-max principle (4.39), we obtain the minimum principle

$$J_n(s+\nabla\cdot\mathbf{S}) = \inf_{g|_{\partial\mathcal{O}}=0} \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ (\nabla g\cdot\nabla g + \Gamma_0\Psi_n\nabla g\cdot\Gamma_0\Psi_n\nabla g + 2sg - 2\mathbf{S}\cdot\nabla g) \tag{4.46}$$

which is nonlocal. Similarly, we can eliminate the infimum in (4.39) by solving

$$\Delta g + \nabla\cdot\Psi_n\nabla g' = s + \nabla\cdot\mathbf{S}, \quad \text{in} \quad \mathcal{O}, \tag{4.47}$$

$$g = 0, \qquad \text{on} \quad \partial\mathcal{O} \tag{4.48}$$

and obtain a nonlocal maximum principle. There is an extra difficulty in (4.47) due to the interaction of the oscillation in $\nabla\Psi_n\nabla g'$ and the macroscopic source term $s+\nabla\cdot\mathbf{S}$, which will be handled in Sections 8.4 and 8.5. Therefore, it is a bit clumsy to express this maximum principle in terms of g' as, for example, in

$$J_n(s+\nabla\cdot\mathbf{S}) = \sup_{g'|_{\partial\mathcal{O}}=0} \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ (-\nabla g'\cdot\nabla g' - \Gamma_0\Psi_n\nabla g'\cdot\Gamma_0\Psi_n\nabla g' - 2s\Delta^{-1}\nabla\cdot\Psi_n\nabla g' + 2\mathbf{S}\cdot\Gamma_0\Psi_n\nabla g' + s\Delta^{-1}s - \Gamma_0\mathbf{S}\cdot\Gamma_0\mathbf{S})\,. \tag{4.49}$$

The most economic form is

$$J_n(s+\nabla\cdot\mathbf{S}) = \sup_{g'|_{\partial\mathcal{O}}=0} \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ (\nabla g\cdot\nabla g - 2\Psi_n\nabla g\cdot\nabla g' - \nabla g'\cdot\nabla g' + 2sg - 2\mathbf{S}\cdot\nabla g) \tag{4.50}$$

where the supremum is subject to (4.47)-(4.48).

One can explore the duality between the the gradient fields($\mathcal{H}_g$) and the curl fields ($\mathcal{H}_c$) and deduce the dual min-max principles both for the cell problem (cf. [8]) and the boundary value problem. They are not used in this paper however.

5 Existence, uniqueness and a priori estimates

5.1 Existence

For a bounded stream matrix Ψ, the symmetrized Dirichlet problem leads to the system (4.36)-(4.37). In this section, we show that (4.36)-(4.37) are also solvable for unbounded but square integrable matrices $\tilde{\Psi}$.

As noted before, the natural function spaces are $H_0(\Psi_n, \mathcal{O})$ defined in (3.10), with the norm $\|\cdot\|_\Psi$ defined in (3.11). For the variational framework, it does not matter whether Ψ is bounded or unbounded as long as the Hilbert spaces $H_0(\Psi_n, \mathcal{O})$ contain all the smooth functions and for this square integrability is the minimum assumption. We state it as a lemma

Lemma 5.1 *If $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$, then*

$$C_0^\infty(\mathcal{O}) \subset H_0(\Psi_n, \mathcal{O}),\ \forall n \tag{5.1}$$

for almost all $\omega \in \Omega$.

Let us assume the validity of this lemma and prove it after stating the existence theorem. In terms of the norm $\|\cdot\|_\Psi$, the functional (4.46) is simply

$$J_n(s + \nabla\cdot \mathbf{S}) = \inf_{g\in H_0(\Psi_n,\mathcal{O})} \frac{1}{|\mathcal{O}|}(\|g\|^2_{\Psi_n} + 2sg - 2\mathbf{S}\cdot\nabla g). \tag{5.2}$$

The Euler equation of (5.2) is

$$\int_{\mathcal{O}} d\mathbf{x}\ \nabla\rho\cdot\nabla\phi + \int_{\mathcal{O}} d\mathbf{x}\ (-\Psi_n\Gamma_0\Psi_n\nabla\rho)\cdot\nabla\phi = \int_{\mathcal{O}} d\mathbf{x}\ (\mathbf{S}\cdot\nabla\phi - s\phi) \tag{5.3}$$

for $\forall\phi \in H_0(\Psi_n, \mathcal{O})$. Now the right hand side of (5.3) defines a bounded linear functional on $H_0(\Psi_n, \mathcal{O})$ for $\mathbf{S}, s \in L^2$ and the left hand side is the bilinear form associated with the norm $\|\cdot\|_{\Psi_n}$, so existence and uniqueness are guaranteed by the Riesz representation theorem.

We note that ρ' also belongs to $H_0(\Psi_n, \mathcal{O})$ since

$$\nabla\rho' = -\Gamma_0\Psi_n\nabla\rho \in L^2(\mathcal{O}) \tag{5.4}$$

$$\Gamma_0\Psi_n\nabla\rho' = -\nabla\rho + \nabla\Delta_0^{-1}(s + \nabla\cdot\mathbf{S}) \in L^2(\mathcal{O}). \tag{5.5}$$

Thus we have shown, granted the validity of Lemma 5.1,

Theorem 5.1 *If $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$, then there exist unique $\rho_n, \rho'_n \in H_0(\Psi_n, \mathcal{O})$ such that (4.36), (4.37) hold for all $\varphi \in H_0(\Psi_n, \mathcal{O})$, for almost all $\omega \in \Omega$.*

So once we have the natural spaces $H_0(\Psi_n, \mathcal{O})$, existence and uniqueness are standard.

Proof of Lemma 5.1: Since $|\tilde{\Psi}|^2$ is integrable with respect to $P(d\omega)$, we have

$$\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\; |\Psi_n|^2(\mathbf{x},\omega) = \frac{1}{n^d|\mathcal{O}|}\int_{n\mathcal{O}} dy|\Psi|^2(\mathbf{x},\omega) \overset{n\to\infty}{\longrightarrow} |\tilde{\Psi}|^2_{L^2(\Omega)} \tag{5.6}$$

for almost all ω, by ergodicity of P. That is, for almost all fixed $\omega \in \Omega$, given $\delta > 0$, there exists $n_0(\omega,\delta)$ such that, for $n > n_0(\omega,\delta)$

$$\int_{\mathcal{O}} d\mathbf{x}\; |\Psi_n|^2 < |\mathcal{O}||\tilde{\Psi}|^2_{L^2(\Omega)} + \delta \tag{5.7}$$

and this estimate for $|\Psi_n|^2_{L^2(\mathcal{O})}$ is uniform in n, for almost all fixed $\omega \in \Omega$. Clearly $\Psi_n\nabla\varphi \in (L^2(\mathcal{O}))^d$, for all $\varphi \in C_0^\infty(\mathcal{O})$, hence $\Gamma_0\Psi_n\nabla\varphi \in (L^2(\mathcal{O}))^d$. A simple L^2 estimate shows that

$$|\Gamma_0\Psi_n\nabla\varphi|_{L^2(\mathcal{O})} \le |\Psi_n\nabla\varphi|_{L^2(\mathcal{O})} \le \sup_{\mathbf{x}\in\mathcal{O}}|\nabla\varphi||\Psi_n|_{L^2(\mathcal{O})} \tag{5.8}$$

So $C_0^\infty(\mathcal{O}) \subset H_0(\Psi_n, \mathcal{O}) \subset H_0^1(\mathcal{O})$ and the Lemma is proved.

We now show that the space of test functions ϕ in Theorem 5.1 can be enlarged from $H_0(\Psi_n, \mathcal{O})$ to $H_0^1(\mathcal{O})$:

Theorem 5.2 *If $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$, then there exist unique $\rho_n, \rho'_n \in H_0(\Psi_n, \mathcal{O})$ such that (4.36), (4.37) hold for all $\varphi \in H_0^1(\mathcal{O})$, for almost all $\omega \in \Omega$.*

Proof: This theorem is an immediate consequence of Theorem 5.1, Lemma 5.1 and these facts:

(i) $C_0^\infty(\mathcal{O})$ is dense in $H_0^1(\mathcal{O})$

(ii)

$$\int_{\mathcal{O}} dx\ \Psi_n \nabla \rho_n' \cdot \nabla \phi = \int_{\mathcal{O}} dx\ \Psi_n \nabla \rho_n' \cdot \Gamma_0 \nabla \phi = \int_{\mathcal{O}} dx\ \Gamma_0 \Psi_n \nabla \rho_n' \cdot \nabla \phi \tag{5.9}$$

and similarly

$$\int_{\mathcal{O}} dx\ \Psi_n \nabla \rho_n \cdot \nabla \phi = \int_{\mathcal{O}} dx\ \Gamma_0 \Psi_n \nabla \rho_n \cdot \nabla \phi \tag{5.10}$$

(iii) $\Gamma_0 \Psi_n \nabla \rho_n', \Gamma_0 \Psi_n \nabla \rho_n \in L^2(\mathcal{O})$. So, if (4.36), (4.37) hold for $\phi \in C_0^\infty(\mathcal{O})$, then they also hold for $\phi \in H_0^1(\mathcal{O})$.

The existence result of the original (before symmetrization) Dirichlet boundary value problem follows from Theorem 5.2:

Corollary 5.1 *Assume $\langle |\tilde{\Psi}|^2 \rangle < \infty$. There exist unique $\rho_n^+, \rho_n^- \in H_0(\Psi_n, \mathcal{O})$ such that*

$$\int_{\mathcal{O}} dx\ \nabla \rho_n^+ \cdot \nabla \phi + \int_{\mathcal{O}} dx\ \Psi_n \nabla \rho_n^+ \cdot \nabla \phi = \int_{\mathcal{O}} dx\ (\mathbf{S} \cdot \nabla \phi - f \phi) \tag{5.11}$$

$$\int_{\mathcal{O}} dx\ \nabla \rho_n^- \cdot \nabla \phi - \int_{\mathcal{O}} dx\ \Psi_n \nabla \rho_n^- \cdot \nabla \phi = \int_{\mathcal{O}} dx\ (\mathbf{S} \cdot \nabla \phi - f \phi) \tag{5.12}$$

for all $\phi \in H_0^1(\mathcal{O})$, for almost all $\omega \in \Omega$.

Proof: By taking

$$\rho_n^+ = \rho_n + \rho_n' \tag{5.13}$$

$$\rho_n^- = \rho_n - \rho_n' \tag{5.14}$$

and adding and subtracting (4.36), (4.37) the Theorem follows.

5.2 Uniform estimates

Here we derive some n−uniform estimates for the solutions of (4.36)-(4.37) in Theorem 5.2. These uniform estimates come naturally as byproducts of the new Hilbert space $H_0(\Psi_n, \mathcal{O})$ formulation. We do not need them in the convergence proof and we present them here for completeness.

Theorem 5.3 *Assume $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$. Let ρ_n, ρ_n' be the solution of the system (4.36), (4.37) and ρ_n^+, ρ_n^- the the solution of (5.11), (5.12), respectively. We have*

$$\|\rho_n\|_{\Psi_n} \le C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.15}$$

$$\|\rho_n'\|_{\Psi_n} \le C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.16}$$

$$\|\rho_n^+\|_{\Psi_n} \le C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.17}$$

$$\|\rho_n^-\|_{\Psi_n} \le C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.18}$$

for some constant C depending only on the domain $\mathcal{O}$.

Proof: For arbitrary $\delta > 0$, $g \in H_0^1(\mathcal{O})$

$$\left|\int_{\mathcal{O}} (s + \nabla \cdot \mathbf{S})g\right| \le \delta |g|^2_{L^2(\mathcal{O})} + \frac{1}{\delta}|s|^2_{L^2(\mathcal{O})} + \delta|\nabla g|^2_{L^2(\mathcal{O})} + \frac{1}{\delta}|\mathbf{S}|^2_{L^2(\mathcal{O})} \tag{5.19}$$

$$\le (c+1)\delta|\nabla g|^2_{L^2(\mathcal{O})} + \frac{1}{\delta}(|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}) \tag{5.20}$$

where c is the constant associated with the Poincare inequality, depending only on the domain $\mathcal{O}$. Thus,

$$\|\rho_n\|^2_{\Psi_n} = \int_{\mathcal{O}} (\nabla\rho_n \cdot \nabla\rho_n + \Gamma_0\Psi_n\nabla\rho_n \cdot \Gamma_0\Psi_n\nabla\rho_n) \tag{5.21}$$

$$\le |\mathcal{O}|J_n(s + \nabla\cdot\mathbf{S}) + (c+1)\delta|\nabla\rho_n|^2_{L^2(\mathcal{O})} + \frac{1}{\delta}(|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}) \tag{5.22}$$

Consequently,

$$\begin{aligned}
&(1-(c+1)\delta)|\nabla\rho_n|^2_{L^2(\mathcal{O})} + |\Gamma_0\Psi_n\nabla\rho_n|^2_{L^2(\mathcal{O})} \\
&\le |\mathcal{O}|J_n(s+\nabla\cdot\mathbf{S}) + \frac{1}{\delta}((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}) \\
&\le \frac{1}{\delta}((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})})
\end{aligned} \tag{5.23}$$

since

$$J_n(s + \nabla\cdot\mathbf{S}) \le 0 \tag{5.24}$$

by taking the trial function $g \equiv 0$ in (5.2). Let $\delta = \frac{1}{2(c+1)}$, so that $1-(c+1)\delta = \frac{1}{2}$. We obtain

$$\|\rho_n\|^2_{\Psi_n} \le 4(c+1)(|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}). \tag{5.25}$$

From the identities

$$\nabla\rho_n' = -\Gamma_0\Psi_n\nabla\rho_n, \quad \Gamma_0\Psi_n\nabla\rho_n' = -\nabla\rho_n + \nabla\Delta_0^{-1}(s + \nabla\cdot\mathbf{S}) \tag{5.26}$$

we also have

$$\|\rho_n'\|^2_{\Psi_n} \le (8(c+1)+2)((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}), \tag{5.27}$$

after applying the energy estimate to $\nabla\Delta_0^{-1}(s+\nabla\cdot\mathbf{S})$. Since $\rho_n^+=\rho_n+\rho_n', \rho_n^-=\rho_n-\rho_n'$, it follows from (5.25), (5.27) that

$$\|\rho_n^+\|_{\Psi_n}^2 \leq C((|s|_{L^2(\mathcal{O})}^2+|\mathbf{S}|_{L^2(\mathcal{O})}^2) \tag{5.28}$$

$$\|\rho_n^-\|_{\Psi_n}^2 \leq C((|s|_{L^2(\mathcal{O})}^2+|\mathbf{S}|_{L^2(\mathcal{O})}^2), \tag{5.29}$$

for some constant C depending only on the domain. This completes the proof.

5.3 Cell problem and correctors

The effective diffusivity is defined by the cell problem. In this section we study the existence of the intensity and flux fields and give bounds for their norm. The method is completely analogous to that for the Dirichlet problems in Section 5.1, with some minor changes, such as replacing the projection operator Γ_0 (3.9) by $\widetilde{\Gamma}$ (4.41). We work in the variational framework on the the space of $\mathcal{H}_g(\widetilde{\Psi})$ defined in (3.28) and this makes the questions of existence and uniqueness standard.

We state the existence theorem and provide a brief explanation with details omitted.

Theorem 5.4 *Assume $\langle\tilde{\Psi}^2\rangle<\infty$. There exists unique $\tilde{\mathbf{E}}_{ij}-\langle\tilde{\mathbf{E}}_{ij}\rangle, \tilde{\mathbf{E}}_{ij}'-\langle\tilde{\mathbf{E}}_{ij}'\rangle \in \mathcal{H}_g(\tilde{\Psi})$ such that*

$$\int_\Omega P(d\omega)\tilde{\mathbf{E}}_{ij}\cdot\tilde{\mathbf{F}}+\int_\Omega P(d\omega)\tilde{\Psi}\tilde{\mathbf{E}}_{ij}'\cdot\tilde{\mathbf{F}}=0 \tag{5.30}$$

$$\int_\Omega P(d\omega)\tilde{\mathbf{E}}_{ij}'\cdot\tilde{\mathbf{F}}+\int_\Omega P(d\omega)\tilde{\Psi}\tilde{\mathbf{E}}_{ij}\cdot\tilde{\mathbf{F}}=0 \tag{5.31}$$

$$\langle\tilde{\mathbf{E}}_{ij}\rangle=\frac{\mathbf{e}^i+\mathbf{e}^j}{2} \tag{5.32}$$

$$\langle\tilde{\mathbf{E}}_{ij}'\rangle=\frac{\mathbf{e}^i-\mathbf{e}^j}{2} \tag{5.33}$$

for all $\tilde{\mathbf{F}}\in\mathcal{H}_g(\Omega), i,j=1,\ldots,d$.

Proof: The mean field conditions (5.32), (5.33) play the role of the inhomogeneous terms $s+\nabla\cdot\mathbf{S}$ in (5.30), (5.31), in the form

$$-\nabla\cdot\tilde{\Psi}\left(\frac{\mathbf{e}^i+\mathbf{e}^j}{2}\right) \quad \text{and} \quad -\nabla\cdot\tilde{\Psi}\left(\frac{\mathbf{e}^i-\mathbf{e}^j}{2}\right), \tag{5.34}$$

respectively. The L^2 integrability of $\tilde{\Psi}$ then implies existence and uniqueness as in Theorem 5.2.

The system (5.30)-(5.33) are Euler equations for the min-max principle

$$\sigma_{ij}^{eff} = \sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \inf_{\substack{\nabla\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=\frac{\mathbf{e}^i+\mathbf{e}^j}{2}}} \sup_{\substack{\nabla\times\tilde{\mathbf{F}}'=0 \\ \langle\tilde{\mathbf{F}}'\rangle=\frac{\mathbf{e}^i-\mathbf{e}^j}{2}}} \left\langle \begin{pmatrix} -I & -\tilde{\Psi} \\ \tilde{\Psi} & I \end{pmatrix} \begin{pmatrix} \mathbf{F}' \\ \mathbf{F} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{F}' \\ \mathbf{F} \end{pmatrix} \right\rangle, \quad i,j = 1\cdots d \tag{5.35}$$

which defines the effective diffusivity $\sigma^{eff} = (\sigma_{ij}^{eff})$. Note that $0 < \sigma_{ij}^{eff} < \infty, i,j = 1,\ldots,d$ because of the integrability condition, $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$ as can be seen by taking as trial fields $\tilde{\mathbf{F}} = \frac{\mathbf{e}^i+\mathbf{e}^j}{2}, \tilde{\mathbf{F}}' = \frac{\mathbf{e}^i-\mathbf{e}^j}{2}$.

The field $\tilde{\mathbf{E}}_{ij}$ can also be characterized as the minimizer of the minimum principle

$$\sigma_{ij}^{eff} = \sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \inf_{\substack{\nabla\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=\frac{\mathbf{e}^i+\mathbf{e}^j}{2}}} \{\langle\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}}\rangle + \langle\Gamma\tilde{\Psi}\tilde{\mathbf{F}}\cdot\Gamma\tilde{\Psi}\tilde{\mathbf{F}}\rangle\}. \tag{5.36}$$

The direct and adjoint intensity fields come from $\tilde{\mathbf{E}}_{ij}^{\pm} = \tilde{\mathbf{E}}_{ij} \pm \tilde{\mathbf{E}}'_{ij}, i,j = 1\cdots d$, and so we have existence and uniqueness for them.

Theorem 5.5 *The intensity fields $\tilde{\mathbf{E}}_{ij}^{\pm} \in \mathcal{H}_g(\Psi), i,j = 1\cdots d$ solve uniquely the cell problems*

$$\int_\Omega P(d\omega)(I+\tilde{\Psi})\tilde{\mathbf{E}}_{ij}^{+}\cdot\tilde{\mathbf{F}} = 0 \tag{5.37}$$

$$\int_\Omega P(d\omega)(I-\tilde{\Psi})\tilde{\mathbf{E}}_{ij}^{-}\cdot\tilde{\mathbf{F}} = 0 \tag{5.38}$$

for all $\tilde{\mathbf{F}} \in \mathcal{H}_g, i,j = 1\cdots d$.

We also have the a priori bounds

Theorem 5.6 *There is a constant C such that for $i,j = 1,\ldots,d$*

$$||\tilde{\mathbf{E}}_{ij}||_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.39}$$

$$||\tilde{\mathbf{E}}'_{ij}||_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.40}$$

$$||\tilde{\mathbf{E}}_{ij}^{+}||_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.41}$$

$$||\tilde{\mathbf{E}}_{ij}^{-}||_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.42}$$

In the theory of homogenization [4], [18], a prominent role is played by the *correctors* χ_j^+, χ_j^- which are defined, up to constant, by

$$\nabla \chi_j^+(\mathbf{x},\omega) = \mathbf{E}_j^+(\mathbf{x},\omega), \quad \nabla \chi_j^-(\mathbf{x},\omega) = \mathbf{E}_j^-(\mathbf{x},\omega). \tag{5.43}$$

Let us fix the constant by setting

$$\chi_j^+(0,\omega) = 0, \quad \chi_j^-(0,\omega) = 0.$$

The symmetrized correctors are

$$\chi_j = \frac{1}{2}(\chi_j^+ + \chi_j^-), \quad \chi_j' = \frac{1}{2}(\chi_j^+ - \chi_j^-) \tag{5.44}$$

and satisfy

$$\nabla \chi_j(\mathbf{x},\omega) = \mathbf{E}_j(\mathbf{x},\omega), \quad \nabla \chi_j'(\mathbf{x},\omega) = \mathbf{E}_j'(\mathbf{x},\omega). \tag{5.45}$$

The correctors are square integrable but not stationary in general. However, they satisfy certain sublinear growth condition for large $|\mathbf{x}|$ which play an essential role in the convergence proof and are analyzed in detail in Sections 8.1 and 8.2.

6 Convergence

We shall establish in this section the main result of this paper which is the strong convergence theorem of homogenization in the case of L^2 skew symmetric coefficients.

Theorem 6.1 *Assume that the stream matrix is square integrable* $\langle |\tilde{\Psi}|^2 \rangle < \infty$ *and let* $\chi_n^{+j} = \frac{1}{n}\chi_j^+(n\mathbf{x},\omega)$, $\chi_n^{-j} = \frac{1}{n}\chi_j^-(n\mathbf{x},\omega)$ *and similarly* $\chi_n^j, \chi_n'^j$ *be the scaled correctors with the unscaled ones defined by (5.43) and (5.44). Then*

$$\int_{\mathcal{O}} d\mathbf{x} \left(\nabla \left(\rho_n - \bar{\rho} - \sum_j \chi_n^j(\mathbf{x},\omega) \frac{\partial \bar{\rho}(\mathbf{x})}{\partial x_j} \right) \right)^2 \to 0 \tag{6.1}$$

$$\int_{\mathcal{O}} d\mathbf{x} \left(\nabla \left(\rho_n' - \sum_j \chi_n'^j(\mathbf{x},\omega) \frac{\partial \bar{\rho}(\mathbf{x})}{\partial x_j} \right) \right)^2 \to 0 \tag{6.2}$$

as $n \to \infty$, *for almost all* ω, *where* $\bar{\rho}$ *satisfies the homogenized problem*

$$\nabla \cdot \left(\frac{1}{2} \left(\sigma^{eff} + \sigma^{efft} \right) \nabla \bar{\rho} \right) = s + \nabla \cdot \mathbf{S}, \quad \textit{in} \quad \mathcal{O} \tag{6.3}$$

$$\bar{\rho} = 0, \qquad \textit{on} \quad \partial\mathcal{O} \tag{6.4}$$

We also have the following corollary of Theorem 6.1 :

Theorem 6.2 *Assume that* $\langle|\widetilde{\Psi}|^2\rangle < \infty$. *Then*

$$\int_{\mathcal{O}} d\mathbf{x} \; \left(\nabla\left(\rho_n^+ - \bar{\rho} - \sum_j \chi_n^{+j}(\mathbf{x},\omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \to 0 \tag{6.5}$$

$$\int_{\mathcal{O}} d\mathbf{x} \; \left(\nabla\left(\rho_n^- - \bar{\rho} - \sum_j \chi_n^{-j}(\mathbf{x},\omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \to 0 \tag{6.6}$$

as $n \to \infty$, *for almost all* ω, *where* $\bar{\rho}$ *is again the solution of (6.3),(6.4).*

Theorems 6.1, 6.2 are also valid when (6.1), (6.2), (6.5) and (6.6) are averaged over ω.

We will show below that $\int_{\mathcal{O}} d\mathbf{x}\; (\chi_n^{+j})^2 \to 0$, $\int_{\mathcal{O}} d\mathbf{x}\; (\chi_n^{-j})^2 \to 0, j = 1, ..., d$, with probability one. Therefore, we have the following corollary of Theorem 6.2

Corollary 6.1 *Assume that* $\langle|\widetilde{\Psi}|^2\rangle < \infty$. *Then*

$$\int_{\mathcal{O}} d\mathbf{x} \;\; (\rho_n^+ - \rho)^2 \to 0 \tag{6.7}$$

$$\int_{\mathcal{O}} d\mathbf{x} \;\; (\rho_n^- - \rho)^2 \to 0 \tag{6.8}$$

as $n \to \infty$, *for almost all* ω.

For the proof we use the minimum and maximum principles to obtain upper and lower bounds, respectively, for the functionals with suitably constructed trial functions which prove convergence of the functionals. The strong convergence results then follow from the convergence of functionals in view of the ellipticity of the problem.

To obtain minimum or maximum principles, the partial Euler equations (4.44,4.45) and (4.47,4.48) have to be solved for selected trial functions asymptotically as $n \to \infty$. This amounts to solving the Poisson equation with rapidly oscillatory right hand side. This is the most technical part of the paper, partly because of the singular behavior near the boundary of the domain which requires careful cut-off arguments. It is presented in Sections 8.4 and 8.5.

6.1 Convergence of functionals

As in the usual homogenization [4],[18] with the multiple scale expansion, we would like to show that the solutions of the inhomogeneous Dirichlet problems have the form

$$\rho_n^+(\mathbf{x}) \asymp \bar{\rho} + \sum_j \frac{1}{n}\chi^{+j}(n\mathbf{x})\frac{\partial \bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.9}$$

$$\rho_n^-(\mathbf{x}) \asymp \bar{\rho} + \sum_j \frac{1}{n}\chi^{-j}(n\mathbf{x})\frac{\partial \bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.10}$$

in the $H_0^1(\mathcal{O})$ sense or, in the symmetrized form,

$$\rho_n(\mathbf{x}) \asymp \bar{\rho} + \sum_j \frac{1}{n}\chi^{j}(n\mathbf{x})\frac{\partial \bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.11}$$

$$\rho_n'(\mathbf{x}) \asymp \sum_j \frac{1}{n}\chi'^{j}(n\mathbf{x})\frac{\partial \bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}). \tag{6.12}$$

Here $\chi^{+j}, \chi^{-j}, \chi^j, \chi'^j$ are the correctors defined by (5.43, 5.44), $\bar{\rho}$ is the exact solution of the homogenized problem (6.3,6.4) and $\alpha_n(\mathbf{x})$ is a suitable cut-off function that makes (6.9)-(6.12) satisfy the Dirichlet boundary conditions. The precise way of doing the cut-off is technically important and one of the essential elements of Section 8.4 and 8.5.

The difficulty with (6.11)-(6.12) is that we do not know if the expansions are admissible. Is the right hand side of (6.11), (6.12) in $H_0(\Psi_n, \mathcal{O})$? The square integrability of

$$\Gamma_0\left\{\Psi_n \sum_j \nabla\chi^j(n\mathbf{x})\frac{\partial \bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x})\right\} + \Gamma_0\left\{\Psi_n \sum_j \frac{1}{n}\chi^j(n\mathbf{x})\nabla\left[\frac{\partial \bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x})\right]\right\} \tag{6.13}$$

is questionable because we do not know that if $\Psi_n \sum_j \nabla\chi^j$ or $\Psi_n\chi^j$ is square integrable. The estimates we obtained in Section 5 are not enough to ensure that we stay in the right spaces. This is also the difficulty encountered in Tartar's proof [19] when applied to this case.

One of the advantages of the variational framework is that we do not have to work with the exact solutions for which we have insufficient knowledge because we can always resort to nice trial functions which approximate the exact solutions.

To make the right hand side of (6.11)-(6.12) admissible for the maximum and minimum principles, let

$$h_n(\mathbf{x}) = \rho + \sum_j \frac{1}{n} f^j(n\mathbf{x})\frac{\partial \rho}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.14}$$

$$l'_n(\mathbf{x}) = \sum_j \frac{1}{n} g'^j(n\mathbf{x}) \frac{\partial \rho}{\partial x_j} \alpha_n(\mathbf{x}) \tag{6.15}$$

where $\rho \in C_0^\infty(\mathcal{O})$ and $f^j(\mathbf{x})$ and $g'^j(\mathbf{x})$ satisfy

$$f^j(\mathbf{0}) = 0, \quad g'^j(\mathbf{0}) = 0 \tag{6.16}$$

and have essentially bounded derivatives

$$\begin{aligned} \mathbf{F}^j &= \nabla f^j \in L^\infty \\ \mathbf{G}^{j'} &= \nabla g'^j \in L^\infty. \end{aligned} \tag{6.17}$$

To verify that

$$h_n, l'_n \in H_0(\Psi_m, \mathcal{O}) \tag{6.18}$$

we need the following lemma

Lemma 6.1 *If $\nabla f = \mathbf{F}$ is essentially bounded and $\langle \mathbf{F} \rangle = 0$ then*

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathcal{O}} f_n^2(\mathbf{x},\omega) = 0, \quad \textit{for almost all } \omega. \tag{6.19}$$

Here $f_n(\mathbf{x},\omega) = \frac{1}{n} f(n\mathbf{x},\omega)$, with $f(0,\omega) = 0$.

We note that the normalization $f(0,\omega) = 0$ is essential for the result to hold. The proof is given in Section 8.2.

This Lemma eliminates the difficulties that we have with the integrability of terms of the form (6.13) when Ψ in only L^2-stationarity and not in L^∞. It then follows that (6.14), (6.15) are admissible. To approximate (6.11) and (6.12) we need also a density lemma for the space of essentially bounded, curl-free fields

$$\mathcal{B} = \{\widetilde{\mathbf{F}} \in \mathcal{H}_g | \widetilde{\mathbf{F}} \text{ is essentially bounded}\}. \tag{6.20}$$

Lemma 6.2 *The space $\mathcal{B}$ is dense in $\mathcal{H}_g(\widetilde{\Psi})$, so that the class of essentially bounded fields is the appropriate trial field space for the variational principle (4.21) of the cell problem.*

It is easy to see by the argument of the proof of Lemma 5.1 that

$$\mathcal{B} \subseteq \mathcal{H}_g(\widetilde{\Psi}). \tag{6.21}$$

It is also clear that

$$\widetilde{\Psi}\widetilde{\mathbf{F}} \in \mathcal{H}^d = (L^2(\Omega,\mathcal{F},\mathcal{P}))^d \tag{6.22}$$

for all bounded $\widetilde{\mathbf{F}} \in \mathcal{B}$. The proof of Lemma 6.2 is given in Section 8.3.

We now show that the functional $J_n(s + \nabla \cdot \mathbf{S})$ can be bounded with arbitrarily small error from above and below with trial functions of the form (6.14), (6.15), respectively.

Theorem 6.3 *Assume that $\langle|\widetilde{\Psi}|^2\rangle < \infty$. Given $\epsilon > 0$ there exists $\delta > 0$ such that for all essentially bounded $\widetilde{\mathbf{F}}^j, \widetilde{\mathbf{G}}'^j$ satisfying*

$$||\widetilde{\mathbf{F}}^j + \mathbf{e}^j - \widetilde{\mathbf{E}}_{jj}||_{\widetilde{\Psi}} \leq \delta \tag{6.23}$$

$$||\widetilde{\mathbf{G}}'^j - \widetilde{\mathbf{E}}'_{jj}||_{\widetilde{\Psi}} \leq \delta \tag{6.24}$$

we have

$$\begin{aligned} &\overline{\lim}_{n\to\infty} J_n(s + \nabla \cdot \mathbf{S}) \\ &\leq \inf_{\rho \in C_0^\infty(\mathcal{O})} \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \quad \sum_{i,j} \left(\left(\mathbf{e}^i + \widetilde{\mathbf{F}}^i\right) \cdot \left(\mathbf{e}^j + \widetilde{\mathbf{F}}^j\right) - 2\widetilde{\Psi}\left(\mathbf{e}^i + \widetilde{\mathbf{F}}^i\right) \cdot \widetilde{\mathbf{F}}'^j - \widetilde{\mathbf{F}}'^i \cdot \widetilde{\mathbf{F}}'^j \right) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} \\ &\qquad +2\rho(s + \nabla \cdot \mathbf{S}) \\ &\leq \underline{\lim}_{n\to\infty} J_n(s + \nabla \cdot \mathbf{S}) \quad +\epsilon \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} &\underline{\lim}_{n\to\infty} J_n(s + \nabla \cdot \mathbf{S}) \\ &\geq \inf_{\rho \in C_0^\infty(\mathcal{O})} \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \quad \sum_{i,j} \left(\left(\mathbf{e}^i + \widetilde{\mathbf{G}}^i\right) \cdot \left(\mathbf{e}^j + \widetilde{\mathbf{G}}^j\right) - 2\widetilde{\Psi}\left(\mathbf{e}^i + \widetilde{\mathbf{G}}^i\right) \cdot \widetilde{\mathbf{G}}'^j - \widetilde{\mathbf{G}}'^i \cdot \widetilde{\mathbf{G}}'^j \right) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} \\ &\qquad +2\rho(s + \nabla \cdot \mathbf{S}) \\ &\geq \overline{\lim}_{n\to\infty} J_n(s + \nabla \cdot \mathbf{S}) \quad -\epsilon \end{aligned} \tag{6.26}$$

for almost all ω, where $\widetilde{\mathbf{E}}_{jj}, \widetilde{\mathbf{E}}'_{jj}$ are defined in (4.5)-(4.6) and $\widetilde{\mathbf{F}}^j, \widetilde{\mathbf{F}}'^j$ are related through the Poisson equation (4.40) (in the form (8.38)-(8.39)), and $\widetilde{\mathbf{G}}^j, \widetilde{\mathbf{G}}'^j$ are similarly related through a Poisson equation (in the form(8.74)-(8.75)). In particular, from the density lemma 6.2 we have

$$\lim_{n\to\infty} J_n(s+\nabla\cdot\mathbf{S}) = \inf_{\rho \in C_0^\infty(\mathcal{O})} \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \left(\sum_{i,j} 1/2 \left(\sigma_{ij}^{eff} + \sigma_{ij}^{eff\dagger}\right) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} + 2\rho(s + \nabla \cdot \mathbf{S}) \right), \tag{6.27}$$

for almost all ω.

We remark that the theorem is valid also when all the expressions are averaged over ω with respect to P.

Before getting into the proof, let us explain the notation we will use. We denote by $f^j(\mathbf{x},\omega), f'^j(\mathbf{x},\omega), g^j(\mathbf{x},\omega), g'^j(\mathbf{x},\omega)$, $j = 1,...,d$, non-stationary random functions whose gradients

$$\nabla f^j(\mathbf{x},\omega) = \mathbf{F}^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{F}}^j(\omega) \tag{6.28}$$

$$\nabla f'^j(\mathbf{x},\omega) = \mathbf{F}'^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{F}}'^j(\omega) \tag{6.29}$$

$$\nabla g^j(\mathbf{x},\omega) = \mathbf{G}^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{G}}^j(\omega) \tag{6.30}$$

$$\nabla g'^j(\mathbf{x},\omega) = \mathbf{G}'^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{G}}'^j(\omega) \tag{6.31}$$

are in $\mathcal{H}_g(\Omega)$, the space of L^2, stationary, gradient fields with zero mean.

The scaled functions $f_n^j(\mathbf{x},\omega), f_n'^j(\mathbf{x},\omega), g_n^j(\mathbf{x},\omega), g_n'^j(\mathbf{x},\omega)$, are defined by

$$f_n^j(\mathbf{x},\omega) = \frac{1}{n} f^j(n\mathbf{x},\omega), \quad f_n'^j(\mathbf{x},\omega) = \frac{1}{n} f'^j(n\mathbf{x},\omega) \tag{6.32}$$

$$g_n^j(\mathbf{x},\omega) = \frac{1}{n} g^j(n\mathbf{x},\omega), \quad g_n'^j(\mathbf{x},\omega) = \frac{1}{n} g'^j(n\mathbf{x},\omega). \tag{6.33}$$

and are uniquely determined up to constant. The normalization constant is essential in determining the right trial functions.

6.1.1 Upper bound

For the minimum principle of section 4.2.2, consider the trial function

$$h_n(\mathbf{x},\omega) = \rho(\mathbf{x}) + \sum_j f_n^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.34}$$

where f^j satisfies (6.17) and $\alpha_n(\mathbf{x})$ is the cut-off function defined in the Section 8.4, 8.5. We show in Lemma 8.3 that

$$h_n'(\mathbf{x},\omega) = \sum_j f_n'^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}). \tag{6.35}$$

solves the Poisson problem for the minimum principle asymptotically in the norm of $H_0^1(\mathcal{O})$, under the assumptions of Lemma 8.3 and if $\widetilde{\mathbf{F}}^j(\omega), \widetilde{\mathbf{F}}'^j(\omega), j = 1,...,d$, satisfy (8.38)-(8.39) with $f_n'^j(\mathbf{x},\omega)$ satisfying the normalization (8.41).

Thus $h_n(\mathbf{x},\omega), h'_n(\mathbf{x},\omega)$ are a legitimate pair of trial functions for the minimum principle of section 4.2.2 in the asymptotic sense as $n \to \infty$. Substituting (6.34), (6.35) into the minimum principle and collecting similar terms we get

$$J_n(s+\nabla\cdot\mathbf{S}) \le \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ \left(\nabla h_n\cdot\nabla h_n - 2\Psi_n\nabla h_n\cdot\nabla h'_n - \nabla h'_n\cdot\nabla h'_n + 2h_n(s+\nabla\cdot\mathbf{S})\right) + o(1)$$
$$\le \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ \Bigg(\sum_{i,j}\Big((\mathbf{e}^i+\nabla f_n^i)\cdot(\mathbf{e}^j+\nabla f_n^j) - 2\Psi_n(\mathbf{e}^i+\nabla f_n^i)\cdot\nabla f_n'^j$$
$$-\nabla f_n'^i\cdot\nabla f_n'^j\Big)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s+\nabla\cdot\mathbf{S})\Bigg) + o(1) \qquad (6.36)$$

in view of (8.43), (8.42) and (8.66). Passing to the limit in (6.36) gives

$$\limsup_{n\to\infty} J_n(s+\nabla\cdot\mathbf{S}) \le \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ \sum_{i,j}\Big(\left(\mathbf{e}^i+\widetilde{\mathbf{F}}^i\right)\cdot\left(\mathbf{e}^j+\widetilde{\mathbf{F}}^j\right) - 2\widetilde{\Psi}\left(\mathbf{e}^i+\widetilde{\mathbf{F}}^i\right)\cdot\widetilde{\mathbf{F}}'^j$$
$$-\widetilde{\mathbf{F}}'^i\cdot\widetilde{\mathbf{F}}'^j\Big)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s+\nabla\cdot\mathbf{S}) \qquad (6.37)$$

Minimizing the right side of (6.37) over $\widetilde{\mathbf{F}}^i, i = 1,...,d,$, bearing in mind the density lemma (6.2) and the identity (4.22), we obtain

$$\limsup_{n\to\infty} J_n(s+\nabla\cdot\mathbf{S}) \le \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ \left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff}+\sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s+\nabla\cdot\mathbf{S})\right) \qquad (6.38)$$

where we use the variational definition of the effective diffusivity (Section 4.2.1). Note that only its symmetric part appears on the right side.

6.1.2 Lower bound

To get lower bounds, we use the maximum principle of section 4.2.2. Consider the trial function l'_n defined by (6.15). As stated in Lemma 8.5, l_n defined by

$$l_n(\mathbf{x}) = \rho + \sum_j g_n^j(\mathbf{x})\frac{\partial\rho}{\partial x_j}\alpha_n, \qquad (6.39)$$

with $\nabla g^j = \widetilde{\mathbf{G}}^j$ and $\nabla g'^j = \widetilde{\mathbf{G}}'^j$, satisfies (8.74)-(8.75) and g_n^j satisfies the normalization (8.78).

Thus l'_n, l_n are a legitimate pair of trial functions for the maximum principle of section 4.2.2 in the asymptotic sense as $n \to \infty$. Substituting l'_n, l_n and passing to the limit, using the ergodicity, gives

$$\liminf_{n\to\infty} J_n(s+\nabla\cdot\mathbf{S}) \geq \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \sum_{i,j}\left(\left(\mathbf{e}^i+\widetilde{\mathbf{G}}^i\right)\cdot\left(\mathbf{e}^j+\widetilde{\mathbf{G}}^j\right) - 2\widetilde{\Psi}\left(\mathbf{e}^i+\widetilde{\mathbf{G}}^i\right)\cdot\widetilde{\mathbf{G}}'^j \right.$$
$$\left. -\widetilde{\mathbf{G}}'^i\cdot\widetilde{\mathbf{G}}'^j\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s+\nabla\cdot\mathbf{S}) \tag{6.40}$$

where $\rho(\mathbf{x})$ satisfies

$$\sum_j(\mathbf{e}^j+\widetilde{\mathbf{G}}^j)\cdot\frac{\partial}{\partial x_j}\nabla\rho = s+\nabla\cdot\mathbf{S}, \quad \text{in } \mathcal{O} \tag{6.41}$$

$$\rho = 0, \quad \text{on } \partial\mathcal{O}. \tag{6.42}$$

Maximizing the right side of (6.40) over $\widetilde{\mathbf{G}}'^j$, $j=1,...,d$, and using the density lemma and the identity (4.22), we obtain

$$\liminf_{n\to\infty} J_n(s+\nabla\cdot\mathbf{S}) \geq \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff}+\sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s+\nabla\cdot\mathbf{S})\right) \tag{6.43}$$

with the help, of the density lemma 6.2. Note that for the corresponding $\widetilde{\mathbf{G}}^j, j=1,...,d$, we have

$$\left(\mathbf{e}^j+\widetilde{\mathbf{G}}^j\right)\cdot\mathbf{e}^i = 1/2\left(\sigma_{ij}^{eff}+\sigma_{ij}^{eff\dagger}\right). \tag{6.44}$$

Therefore, (6.43) is equivalent to

$$\liminf_{n\to\infty} J_n((s+\nabla\cdot\mathbf{S})) \geq \min_\rho \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff}+\sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s+\nabla\cdot\mathbf{S})\right). \tag{6.45}$$

because of (6.41). In view of (6.38), we then conclude that

$$\lim_{n\to\infty} J_n((s+\nabla\cdot\mathbf{S})) = \min_\rho \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff}+\sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s+\nabla\cdot\mathbf{S})\right). \tag{6.46}$$

This completes the proof of Theorem 6.3.

6.2 Strong convergence

In this section we complete the proof of Theorem 6.1- 6.2, stated in the beginning of Section 6, using Theorem 6.3. Because of the variational structure, the convergence of functionals established in the preceding section is very close to the proof of Theorem 6.1.

Define the differences

$$\mathbf{r}_n = \nabla\rho_n - \nabla h_n \tag{6.47}$$

$$\mathbf{r}'_n = \nabla\rho'_n - \nabla h'_n. \tag{6.48}$$

where ρ_n, ρ'_n are the solutions of the system of symmetrized inhomogeneous boundary value problems and h_n, h'_n are given by (6.34), (6.35). Then

$$\begin{aligned}
&\int_{\mathcal{O}} d\mathbf{x}\ (\mathbf{r}_n \cdot \mathbf{r}_n + \mathbf{r}'_n \cdot \mathbf{r}'_n) \\
&= \int_{\mathcal{O}} d\mathbf{x}\ (\nabla\rho_n \cdot \nabla\rho_n + \nabla\rho'_n \cdot \nabla\rho'_n) \\
&+ \int_{\mathcal{O}} d\mathbf{x}\ (\nabla h_n \cdot \nabla h_n + \nabla h'_n \cdot \nabla h'_n) \\
&-2 \int_{\mathcal{O}} d\mathbf{x}\ (\nabla\rho_n \cdot \nabla h_n + \nabla\rho'_n \cdot \nabla h'_n)
\end{aligned} \tag{6.49}$$

But, since ρ_n is the solution of the symmetrized boundary value problem

$$\nabla \cdot (I - \Psi_n\Gamma_0\Psi_n)\nabla\rho_n = s + \nabla\cdot\mathbf{S}, \quad \text{in} \quad \mathcal{O} \tag{6.50}$$

$$\rho_n = 0, \qquad \text{on} \quad \mathcal{O} \tag{6.51}$$

we have

$$\int_{\mathcal{O}} d\mathbf{x}\ (\nabla g \cdot \nabla\rho_n + \Gamma_0\Psi_n\nabla g \cdot \Gamma_0\Psi_n\nabla\rho_n) = -\int_{\mathcal{O}} d\mathbf{x}\ g(s + \nabla\cdot\mathbf{S}) \tag{6.52}$$

for all $g \in H_0^1(\mathcal{O})$, which implies, according to Lemma 8.3,

$$\int_{\mathcal{O}} d\mathbf{x}\ (\nabla h_n \cdot \nabla\rho_n + \nabla h'_n \cdot \nabla\rho'_n) \sim -\int_{\mathcal{O}} d\mathbf{x}\ h_n(s + \nabla\cdot\mathbf{S}). \tag{6.53}$$

Thus

$$\begin{aligned}
&\int_{\mathcal{O}} d\mathbf{x}\ (\mathbf{r}_n \cdot \mathbf{r}_n + \mathbf{r}'_n \cdot \mathbf{r}'_n) \\
&\sim \int_{\mathcal{O}} d\mathbf{x}\ (\nabla\rho_n \cdot \nabla\rho_n + \nabla\rho'_n \cdot \nabla\rho'_n) \\
&+ \int_{\mathcal{O}} d\mathbf{x}\ (\nabla h_n \cdot \nabla h_n + \nabla h'_n \cdot \nabla h'_n) \\
&+2 \int_{\mathcal{O}} d\mathbf{x}\ h_n(s + \nabla\cdot\mathbf{S})
\end{aligned} \tag{6.54}$$

which in the limit can be made as small as one please according to Theorem 6.3.

We recall that the way we prove Theorem 6.3 is to first let $n \to \infty$, and then minimize over essentially bounded $\mathbf{F}^i$ and maximize over essentially bounded $\mathbf{F}'^i$ for the upper and lower bounds respectively. With this observation in mind, we summarize what we have shown up to this point. We have shown that given $\epsilon > 0$, there exists $\delta > 0$ such that for all essentially bounded $\widetilde{\mathbf{F}}^j_n(\omega), j = 1, ..., d$, satisfying

$$||\widetilde{\mathbf{F}}^j - \widetilde{\mathbf{E}}_{jj}||_{\widetilde{\Psi}} \leq \delta \tag{6.55}$$

we have

$$\limsup_{n\to\infty} \int_{\mathcal{O}} d\mathbf{x} \ \ (\mathbf{r}_n \cdot \mathbf{r}_n + \mathbf{r}'_n \cdot \mathbf{r}'_n) \leq \epsilon. \tag{6.56}$$

Here $\widetilde{\mathbf{E}}_{jj}, j = 1, ..., d$, are the solutions of the symmetrized cell problems.

From this it follows easily that

$$\int_{\mathcal{O}} d\mathbf{x} \ \left(\nabla\left(h_n - \rho - \sum_j \chi^j_n(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\right)\right)^2 \leq c\delta \tag{6.57}$$

$$\int_{\mathcal{O}} d\mathbf{x} \ \left(\nabla\left(h'_n - \sum_j \chi'^j_n(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\right)\right)^2 \leq c\delta \tag{6.58}$$

for some constant c, provided (6.55) holds. This proves the strong convergence theorem.

7 Probabilistic convergence theorem: compactness of the processes

In this section we prove that the processes $\mathbf{x}_n(\cdot)$ defined by (1.2) satisfy the tightness condition

$$\lim_{h\downarrow 0} \overline{\lim_{n\downarrow\infty}} \ \mathrm{Prob}\left\{\sup_{\substack{0<s<t<T\\ |s-t|\leq h}} |\mathbf{x}_n(s) - \mathbf{x}_n(t)| > \delta\right\} = 0 \tag{7.1}$$

for each $\delta > 0$ and $T < \infty$. The unboundedness of Ψ makes the use of the Nash estimates, or generalizations [17], impossible. Thus, the compactness of the processes $\mathbf{x}_n(t)$ is no longer straightforward. We reduce the estimate of the probability in (7.1) to a resolvent estimate which we can study using variational methods except that in this case we need an L^∞, rather than L^2, estimate. This comes when we average over the ensemble of flows $\mathbf{u}$. So we no longer have convergence with probability one, as in Theorem 1, but convergence in measure with respect to the flows (cf. Theorem 7.1).

It is enough for (7.1) to obtain the estimate

$$\lim_{h\downarrow 0}\frac{1}{h}\overline{\lim_{n\downarrow\infty}}\ \text{Prob}\left\{\sup_{0<s<h}|\mathbf{x}_n(s)-\mathbf{x}_n(0)|>\delta\right\}=0 \tag{7.2}$$

for each $\delta>0$. To prove (7.2) it suffices to obtain for each component $x_i(s)$ an estimate of the form

$$\lim_{h\downarrow 0}\frac{1}{h}\overline{\lim_{n\downarrow\infty}}\ \text{Prob}\left\{\sup_{0<s<hn^2}|x_i(s)-x_i(0)|\geq\delta n\right\}=0 \tag{7.3}$$

for each $\delta>0$. Let τ_L be the time it takes for $x_i(t)$ to reach level L assuming $x_i(0)=0$. Then (7.3) reduces to

$$\lim_{h\downarrow 0}\frac{1}{h}\overline{\lim_{n\downarrow\infty}}\ \text{Prob}\left\{\tau_{\delta n}\leq hn^2\right\}=0 \tag{7.4}$$

for each $\delta>0$. From the Tchebyshev inequality

$$\text{Prob}\left\{\tau_{\delta n}\leq hn^2\right\}\leq e^{\alpha h}E\left\{e^{-n^{-2}\alpha\tau_{\delta n}}\right\}. \tag{7.5}$$

Therefore, (7.4) can be deduced from

$$\overline{\lim_{n\downarrow\infty}}E\left\{e^{-n^{-2}\alpha\tau\delta n}\right\}\leq M(\alpha,\delta), \tag{7.6}$$

for each $\alpha>0$, and

$$\inf_{\alpha>0}e^{\alpha h}M(\alpha,\delta)=o(h) \tag{7.7}$$

as $h\to 0$ for each $\delta>0$.

Let $\mathcal{L}$ be the generator of the processes

$$\mathcal{L}=\nabla\cdot[(\sigma I+\Psi_n(\mathbf{x}))\nabla\ \cdot] \tag{7.8}$$

and consider the solution of

$$\alpha\rho_\alpha-\mathcal{L}\rho_\alpha=0 \tag{7.9}$$

for $x_1<L$ with

$$\rho_\alpha=0,\text{for}\quad x_1=-\infty,\quad \rho_\alpha=1\ \text{ for }\ x_1=L \tag{7.10}$$

Write $\mathbf{x}=(x_1,\mathbf{x}^1)$ with $\mathbf{x}^1=(x_2,\ldots,x_d)$. and let

$$\int P(d\omega)\rho_\alpha((0,\mathbf{x}^1),L,\omega)=\theta_\alpha(L), \tag{7.11}$$

which does not depend on $\mathbf{x}^1$, by stationarity. To get (7.7) we need an estimate for $\theta_{n^{-2}\alpha}(\delta n)$ as $n\to\infty$.

Since averaging with respect to ω allows us to use stationarity, starting from zero and going to L is equivalent to starting from $-L$ and going to level 0. Therefore, we may consider

$$\alpha\rho_\alpha - \mathcal{L}\rho_\alpha = 0, \qquad x_1 < 0 \tag{7.12}$$

$$\rho_\alpha = 0, \quad \text{for} \quad x_1 = -\infty, \quad \rho_\alpha = 1, \quad \text{for} \quad x_1 = 0 \tag{7.13}$$

and let

$$\int P(d\omega)\rho_\alpha(\mathbf{x}, L, \omega) = \theta_\alpha(x_1) \tag{7.14}$$

We will study the asymptotic behavior of $\theta_{n^{-2}\alpha}(-\delta n)$ as $n \to \infty$.

The idea is to show that the averaged moment-generating functions $\theta_{n^{-2}\alpha}(-\delta n)$ of the exit time for the processes $\mathbf{x}_n$ is very close to that of Brownian motion for which we have the estimate (7.7). With slight modifications, it is routine to check that strong convergence holds. In particular,

$$\lim_{Q\to R^{d-1}} \frac{1}{Q}\int_Q dx^1 \int_{-\infty}^0 dx_1 \int_\Omega P(d\omega)\left(\rho_{n^{-2}\alpha}(n\mathbf{x},\omega) - \bar{\rho}_\alpha(\mathbf{x})\right)^2 \to 0 \tag{7.15}$$

as $n \to \infty$. Here $\bar{\rho}_\alpha(\mathbf{x})$ is the moment generating function of the exit time for Brownian motion with variance coefficient $\frac{1}{2}(\sigma^{eff} + \sigma^{eff\dagger})$. The techniques developed in the previous sections apply equally well here, with some modifications needed to account for the α dependence and the semi-infinite domain $\{\mathbf{x}|x_1 \le 0\}$. Therefore,

$$\int_{-\infty}^0 dx_1 \left(\theta_{n^{-2}\alpha}(nx_1) - M(\alpha, x_1)\right)^2 \to 0, \tag{7.16}$$

as $n \to \infty$, where $M(\alpha, x_1)$ is the moment-generating function of the exit time for the one-dimensional Brownian motion starting at x_1. Here we use the ω-average version of homogenization theorems as noted in the remark after the statements of Theorem 6.1and 6.2. However, from (7.6) and (7.7) we see that what is needed here is not the $L^2(dx_1)$ convergence but convergence pointwise in x_1. But both $\theta_{n^{-2}\alpha}(nx_1)$ and $M(\alpha, x_1)$ are monotone so $L^2(dx_1)$ convergence actually implies uniform convergence

$$\sup_{x_1 \le 0} \left(\theta_{n^{-2}\alpha}(nx_1) - M(\alpha, x_1)\right)^2 \to 0 \tag{7.17}$$

as $n \to \infty$. From the Laplace transform of the heat equation on the semi-infinite line, we know that as $\alpha \to \infty$

$$M(\alpha, \delta) = O(\sqrt{\alpha}e^{-c\delta\sqrt{\alpha}}). \tag{7.18}$$

for some positive constant c. The tightness condition (7.1) then follows from

$$\inf_{\alpha\geq 0} e^{\alpha h} M(\alpha,\delta) = O(\frac{1}{h}e^{-c_1 1/h}), \tag{7.19}$$

as $h \to 0$, for some positive constant c_1, by taking $\alpha = \frac{c_2^2}{h^2}$, for some positive but sufficiently small constant $c_2 = c_2(c,\delta)$. We have thus proven Theorem 2 of the Introduction, which we restate here.

Theorem 7.1 *The family $\mathbf{x}_n(t)$ of stochastic processes defined by (1.2) is uniformly tight in measure with respect to the ensemble of media $P(d\omega)$ and therefore we have weak convergence to Brownian motion in the space of continuous functions in R^d, in measure with respect to $P(d\omega)$.*

8 The proofs of some technical lemmas

8.1 L^2-sublinear growth of random functions with L^2-derivatives

The main result of this section is the proof of the almost sure L^2-sublinear growth estimate of Lemma 8.2 which is the strengthened version of the standard L^2-sublinear growth estimate stated in the following lemma 8.1. Lemma 8.2 is needed for the proof of Lemma 6.1, 8.3 and 8.5.

Lemma 8.1 *Let $\tilde{\mathbf{F}} \in \mathcal{H}_g$. There exists a uniquely defined process $f(\mathbf{x},\omega) \in H^1_{\text{loc}}(R^d; L^2(\Omega))$, it is not stationary, $f(0,\omega) = 0$ and*

$$\nabla f(\mathbf{x},\omega) = \mathbf{F}(\mathbf{x},\omega) = \tilde{\mathbf{F}}(\tau_{-\mathbf{x}}\omega). \tag{8.1}$$

For any compact subset $K \subset R^d$, we have

$$\lim_{n\to\infty} \sup_K \langle \left[\frac{1}{n} f(n\mathbf{x},\omega)\right]^2 \rangle = 0. \tag{8.2}$$

Proof: This proof follows Papanicolaou and Varadhan[18]. Define $f(\mathbf{x},\omega)$ by

$$f(\mathbf{x},\omega) = \int_{R^d} \frac{e^{i\mathbf{x}\cdot\mathbf{k}} - 1}{|\mathbf{k}|^2}(-i\mathbf{k}) \cdot U(d\mathbf{k})\tilde{\mathbf{F}}(\omega) \tag{8.3}$$

where $U(d\mathbf{k})$ is the spectral resolution of the unitary group $\{T_\mathbf{x}\}$, i.e.,

$$T_\mathbf{x} = \int_{\mathbf{k}\in R^d} e^{i\mathbf{k}\cdot\mathbf{x}} U(d\mathbf{k}). \tag{8.4}$$

The process $f(\mathbf{x},\omega)$ is not stationary because it is not of the form $f(\mathbf{x},\omega) = T_\mathbf{x}\tilde{f}(\omega)$. It is easy to see that $f(0,\omega) = 0$ and $\nabla f(\mathbf{x},\omega) = \mathbf{F}(\mathbf{x},\omega)$ and, as a consequence, it is in $H^1_{\text{loc}}(R^d; L^2(\Omega))$. It remains to show (8.2). We have the identity

$$\int_\Omega P(d\omega)\left(\frac{1}{n}f(n\mathbf{x},\omega)\right)^2 \tag{8.5}$$

$$= \int_{R^d} \left|\frac{e^{in\mathbf{x}\cdot\mathbf{k}}-1}{n\mathbf{k}}\right|^2 \sum_{i,j=1}^{d} \frac{k_i k_j}{|\mathbf{k}|^2}\hat{R}_{ij}(d\mathbf{k}) \tag{8.6}$$

where

$$\hat{R}_{ij}(d\mathbf{k}) = \int_\Omega P(d\omega)U(d\mathbf{k})\tilde{\mathbf{F}}_i(\omega)\tilde{\mathbf{F}}_j(\omega) \tag{8.7}$$

is the power spectral measure of $\mathbf{F}_i(\mathbf{x},\omega) = \tilde{\mathbf{F}}_j(\tau_\mathbf{x}\omega)$. From the estimate

$$\frac{1}{|\mathbf{k}|^2}\sum_{i,j=1}^{d} k_i k_j \hat{R}_{ij}(d\mathbf{k}) \leq \sum_{i=1}^{d}\hat{R}_{ii}(d\mathbf{k}), \tag{8.8}$$

we obtain

$$\int_\Omega P(d\omega)\left(\frac{1}{n}f(n\mathbf{x},\omega)\right)^2 \leq \int_{R^d}\left|\frac{e^{in\mathbf{x}\cdot\mathbf{k}}-1}{n\mathbf{k}}\right|^2 \sum_{i=1}^{d}\hat{R}_{ii}(d\mathbf{k}). \tag{8.9}$$

By ergodicity and $\langle\tilde{\mathbf{F}}\rangle = 0$, it follows that $\hat{R}_{ii}(\{0\}) = 0$. The Lebesque convergence theorem then yields the result.

Let

$$f_n(\mathbf{x},\omega) = \frac{1}{n}f(n\mathbf{x},\omega). \tag{8.10}$$

Then (8.2) implies that

$$\langle\int_K d\mathbf{x}\ f_n^2\rangle \to 0 \tag{8.11}$$

as $n\to\infty$. Consider also

$$f_n'(\mathbf{x},\omega) \equiv f_n(\mathbf{x},\omega) - a_n(\omega) \tag{8.12}$$

where $a_n(\omega) = \frac{1}{|K|}\int_K d\mathbf{x}\ f_n(\mathbf{x},\omega)$. It is easy to see that (8.11) implies that

$$\langle\int_K d\mathbf{x}\ (f_n')^2\rangle \to 0 \tag{8.13}$$

as $n\to\infty$, since

$$\langle a_n^2\rangle \leq \langle\frac{1}{|K|}\int_K d\mathbf{x}\ f_n^2\rangle \to 0 \tag{8.14}$$

as $n \to \infty$.

The constant $a_n(\omega)$ is essential for the proof of the following strengthened version of (8.13) that the convergence holds without the average $\langle \cdot \rangle$.

Lemma 8.2 *For P almost all $\omega \in \Omega$*

$$\int_K d\mathbf{x}\ (f_n')^2 \to 0, \quad \text{as} \quad \text{n} \to \infty. \tag{8.15}$$

Proof: Without loss of generality, we may assume $K = \{|\mathbf{x}| < a\}$, for some $a > 0$. From the definition (8.12), we have

$$\nabla f_n'(\mathbf{x}, \omega) = \mathbf{F}(n\mathbf{x}, \omega) \in L^2_{\text{loc}}(R^d) \tag{8.16}$$

for almost all $\omega \in \Omega$. Furthermore, given $\delta > 0$, there exists $n_0(\omega, \delta)$ such that, for $n > n_0(\omega, \delta)$

$$\frac{1}{|K|} \int_K d\mathbf{x}\ (\nabla f_n')^2 \leq |\tilde{\mathbf{F}}|_{L^2(\Omega)} + \delta \tag{8.17}$$

for almost all $\omega \in \Omega$, by ergodicity. The uniform estimate (8.17) and the mean zero property $\int_K d\mathbf{x}\ f_n' = 0$ imply that $\{f_n'\}$ is precompact in the strong L^2 sense. Consider any convergent subsequence, still denoted by $\{f_n'\}$. There exists a function $g(\mathbf{x}, \omega) \in L^2(K)$ such that

$$\int_K d\mathbf{x}\ (f_n' - g)^2 \to 0 \tag{8.18}$$

as $n \to \infty$, for almost all ω.

On the other hand, (8.13) implies that the sequence of positive random variables $\{\int_K d\mathbf{x}\ (f_n')^2\}$ converges to zero in probability with respect to P and in particular there exists a subsequence $\{\int_K d\mathbf{x}\ (f_{nj}')^2\}$ converging to zero for P almost all $\omega \in \Omega$. Thus $g(\mathbf{x}, \omega) = 0$, for almost all ω. This proves the lemma.

8.2 L^∞-sublinear growth of random functions with L^∞-derivatives

In this section we prove Lemma 6.1 which is essential in our estimates for the Poisson problems in Section 8.4 and 8.5. Note that, contrary to the constant $a_n(\omega)$, a different normalization has been taken in Lemma 6.1.

Proof of Lemma 6.1: In view of boundedness of the domain $\mathcal{O}$ and $\nabla f_n(\mathbf{x},\omega) = \mathbf{F}(n\mathbf{x},\omega)$, the pointwise convergence to zero

$$\lim_{n\to\infty} f_n^2(\mathbf{x},\omega) = 0, \quad \forall \mathbf{x} \in \mathcal{O} \tag{8.19}$$

implies the uniform convergence (6.19). It remains to prove (8.19) and this is done by contradiction.

Suppose (8.19) fails at a point $\mathbf{x}_o \in \mathcal{O}$. We select a convergent subsequence, still denoted by $f_n(\mathbf{x}_o,\omega)$ such that

$$f_n(\mathbf{x}_o,\omega) \to \alpha_\omega \neq 0. \tag{8.20}$$

By the boundedness of $\widetilde{\mathbf{F}}$ and the normalization $f_n(0,\omega) = 0$ it follows that there exists a $\delta > 0$ and $n_0 > 0$ such that for all $n > n_0$, we have

$$|f_n(\mathbf{x},\omega) - \alpha_\omega| \leq \alpha_\omega/3, \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_o| < \delta \tag{8.21}$$

$$|f_n(\mathbf{x},\omega)| \leq \alpha_\omega/3, \qquad \text{for} \quad |\mathbf{x}| < \delta \tag{8.22}$$

Now consider the cylinder set $\mathcal{O}'$ of radius δ, with $\overline{\mathbf{ox}_o}$ as its axis. By ergodicity and the zero mean property $\langle \widetilde{\mathbf{F}} \rangle = 0$, we have

$$\frac{1}{|\mathcal{O}'|}\int_{\mathcal{O}'} d\mathbf{x}\ \mathbf{F}(n\mathbf{x},\omega) \to 0, \quad \text{for almost all } \omega \tag{8.23}$$

But, from (8.21)-(8.22), it follows that $\frac{1}{|\mathcal{O}'|}\int_{\mathcal{O}'} d\mathbf{x}\ \nabla f_n(\mathbf{x},\omega)$ has a nonzero component in the direction of $\overline{\mathbf{ox}_o}$, which is larger than

$$\frac{\alpha_\omega |B_\delta^{d-1}|}{3|\mathcal{O}'|} > 0. \tag{8.24}$$

Here B_δ^{d-1} is the $d-1$ dimensional ball of radius δ. Thus, $\alpha_\omega = 0$ and the proof is complete.

8.3 Density lemma

Proof of Lemma 6.2: We decompose the space $\mathcal{H}_g(\widetilde{\Psi})$ into the closure of $\mathcal{B}$ and its orthogonal complement $\mathcal{A}$

$$\mathcal{H}_g(\widetilde{\Psi}) = \bar{\mathcal{B}} \oplus \mathcal{A} \tag{8.25}$$

with respect to the norm $||\cdot||_{\widetilde{\Psi}}$ of $\mathcal{H}_g(\widetilde{\Psi})$. For every $\widetilde{\mathbf{A}} \in \mathcal{A}$, we have $\widetilde{\mathbf{A}} \perp \bar{\mathcal{B}}$, that is,

$$\langle \widetilde{\mathbf{A}} \cdot \widetilde{\mathbf{F}} \rangle + \langle \Gamma\widetilde{\Psi}\widetilde{\mathbf{A}} \cdot \Gamma\widetilde{\Psi}\widetilde{\mathbf{F}} \rangle = 0, \quad \forall \widetilde{\mathbf{F}} \in \mathcal{B} \tag{8.26}$$

or, equivalently,

$$\langle (I - \widetilde{\Psi}\Gamma\widetilde{\Psi})\widetilde{\mathbf{A}} \cdot \widetilde{\mathbf{F}} \rangle = 0, \quad \forall \widetilde{\mathbf{F}} \in \mathcal{B}. \tag{8.27}$$

This implies that

$$(I - \widetilde{\Psi}\Gamma\widetilde{\Psi})\widetilde{\mathbf{A}} = 0, \quad \text{for almost all } \omega, \tag{8.28}$$

which in turn implies that

$$\widetilde{\mathbf{A}} = 0 \tag{8.29}$$

in view of the positive-definiteness of the operator $I - \widetilde{\Psi}\Gamma\widetilde{\Psi}$. This completes the proof.

8.4 Poisson problem for upper bound

Without loss of generality, let the domain $\mathcal{O}$ be the square $|x_i| \leq 1, \ i = 1, ..., d$. Consider the inhomogeneous boundary value problem

$$\Delta g_n(\mathbf{x}, \omega) + \nabla \cdot \Psi_n \mathbf{F}_n = 0, \quad \text{in } \mathcal{O} \tag{8.30}$$

$$g_n = 0, \qquad \text{on } \partial\mathcal{O}. \tag{8.31}$$

where the inhomogeneous term $\mathbf{F}_n(\mathbf{x}, \omega)$ has the form

$$\mathbf{F}_n(\mathbf{x}, \omega) = \sum_j \nabla(\alpha_n(\mathbf{x}) f_n^j(\mathbf{x}, \omega) \frac{\partial \rho}{\partial x_j}(\mathbf{x})) + \nabla\rho(\mathbf{x}), \quad \text{in } \mathcal{O}. \tag{8.32}$$

Here $\rho(\mathbf{x}) \in C_0^\infty(\mathcal{O})$ and $f_n^j(\mathbf{x}, \omega), j = 1, ..., d$ are non-stationary random functions whose gradients are

$$(\nabla f_n^j)(\mathbf{x}, \omega) = \mathbf{F}^j(n\mathbf{x}, \omega) = T_{n\mathbf{x}} \widetilde{\mathbf{F}}^j(\omega) \in \mathcal{H}_g(\Omega). \tag{8.33}$$

We take the gradients $\widetilde{\mathbf{F}}^i, i = 1, ..., d$, to be *essentially bounded.* The cut-off function is

$$\alpha_n(\mathbf{x}) = \prod_{i=1}^d \gamma(\frac{1 + x_i}{\tau_n})\gamma(\frac{1 - x_i}{\tau_n}) \tag{8.34}$$

with $\gamma(s) \in C^\infty(R)$ such that

$$0 \leq \gamma(s) \leq 1 \tag{8.35}$$

$$\gamma(s) = \begin{cases} 1, & |s| \geq 2, \\ 0, & |s| \leq 1. \end{cases} \tag{8.36}$$

and τ_n is a decreasing sequence of positive numbers with a rate that will be determined later and depends on $f^j(\mathbf{x},\omega), j = 1, ..., d$. We denote the set $\{\mathbf{x} \mid \alpha_n(\mathbf{x}) = 1\}$ by $\mathcal{O}'$.

We shall show how to solve (8.30), (8.31) in terms of $f_n'^i(\mathbf{x},\omega), i = 1, ..., d$, whose gradients

$$(\nabla f_n'^i)(\mathbf{x},\omega) = \mathbf{F}'^i(n\mathbf{x},\omega) = T_{n\mathbf{x}}\widetilde{\mathbf{F}}'^i(\omega) \in \mathcal{H}_g(\Omega) \tag{8.37}$$

satisfy

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{F}}'^i + \widetilde{\nabla} \cdot \widetilde{\Psi}(\widetilde{\mathbf{F}}^i + \mathbf{e}^i) = 0, \tag{8.38}$$

$$\langle \widetilde{\mathbf{F}}'^i \rangle = 0 \tag{8.39}$$

We impose the normalization conditions

$$f_n^i(0,\omega) = 0 \tag{8.40}$$

$$\int_{\mathcal{O}} d\mathbf{x}\ f_n'^i(\mathbf{x},\omega) = 0, \tag{8.41}$$

so that, by Lemmas 8.2 and 6.1,

$$\int_{\mathcal{O}} d\mathbf{x}\ (f_n'^i)^2(\mathbf{x},\omega) \to 0, \tag{8.42}$$

$$\sup_{\mathbf{x}\in\mathcal{O}} (f_n^i)^2(\mathbf{x},\omega) \to 0, \tag{8.43}$$

in the limit $n \to \infty$, with probability one.

We prove

Lemma 8.3 *Let z_n be defined by*

$$z_n(\mathbf{x},\omega) = g_n(\mathbf{x},\omega) - \sum_j f_n'^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}), \tag{8.44}$$

assume that $\langle|\widetilde{\Psi}|^2\rangle < \infty$, $\widetilde{\mathbf{F}}^j \in \mathcal{H}_g(\Omega), j = 1, ..., d$, is essentially bounded and the normalization conditions (8.40), (8.41) hold. Then

$$\int_{\mathcal{O}} d\mathbf{x}\ (\nabla z_n)^2 \to 0, \quad as \quad n \to \infty, \tag{8.45}$$

for almost all ω.

Proof of Lemma 8.3:

Under the same assumptions of Lemma 8.3, we first prove a Lemma which implies that z_n tends to zero in the L^2-norm in the limit $n \to \infty$.

Lemma 8.4 *For almost all* ω

$$\int_{\mathcal{O}} d\mathbf{x}\ g_n^2 \to 0 \tag{8.46}$$

as $n \to \infty$.

Proof of Lemma 8.4: The energy estimate for (8.30,8.31) gives

$$\begin{aligned}\int_{\mathcal{O}} d\mathbf{x}\ (\nabla g_n)^2 &= -\int_{\mathcal{O}} d\mathbf{x}\ \Psi_n \mathbf{F}_n \cdot \nabla g_n \\ &\leq |\Psi_n \mathbf{F}_n|_{L^2(\mathcal{O})} |\nabla g_n|_{L^2(\mathcal{O})} \\ &\leq (|\widetilde{\Psi}\widetilde{\mathbf{F}}|_{L^2(\Omega)} + \delta) |\nabla g_n|_{L^2(\mathcal{O})}\end{aligned} \tag{8.47}$$

for any given $\delta > 0$ and $n > n_o(\delta, \omega)$, by ergodicity. Hence

$$|\nabla g_n|_{L^2(\mathcal{O})} \leq c \tag{8.48}$$

where c is a constant independent of n. This implies that $\{g_n\}$ is precompact in the strong L^2 sense. Multiplying (8.30) by any test function $\phi \in C_0^\infty(\mathcal{O})$ and integrating by parts gives

$$\int_{\mathcal{O}} d\mathbf{x}\ \nabla\phi \cdot \nabla g_n + \int_{\mathcal{O}} d\mathbf{x}\ \Psi_n \mathbf{F}_n \cdot \nabla\phi = 0 \tag{8.49}$$

The second integral vanishes in the limit. It follows that $\{\nabla g_n\}$ converges weakly to zero. Thus, by the strong compactness, $\{g_n\}$ converges strongly to zero.

We return now to the proof of Lemma 8.3 and note that Lemma 8.4 and (8.42) imply that

$$\int_{\mathcal{O}} d\mathbf{x}\ z_n^2 \to 0 \tag{8.50}$$

as $n \to \infty$. From equations (8.30), (8.38) it follows that

$$\begin{aligned}\Delta z_n &= -\nabla \cdot \Psi_n \left(\nabla \left(\sum_j f_n^j \frac{\partial \rho}{\partial x_j} \alpha_n \right) + \nabla \rho \right) - \sum_j \Delta f_n^{\prime j} \frac{\partial \rho}{\partial x_j} \alpha_n - \sum_j \nabla f_n^{\prime j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \\ &\quad - \sum_j \nabla \cdot \left(f_n^{\prime j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \right) \\ &= -\nabla \cdot \Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j} \alpha_n - \nabla \cdot \Psi_n \sum_j f_n^j \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) - \nabla \cdot \Psi_n \nabla \rho \\ &\quad + \nabla \cdot \Psi_n \sum_j (\nabla f_n^j + \mathbf{e}^j) \frac{\partial \rho}{\partial x_j} \alpha_n - \sum_j \nabla f_n^{\prime j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) - \sum_j \nabla \cdot \left(f_n^{\prime j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \right) \\ &= -\nabla \cdot \Psi_n \nabla \rho (1 - \alpha_n) - \nabla \cdot \Psi_n \sum_j f_n^j \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) - \sum_j \nabla f_n^{\prime j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \\ &\quad - \sum_j \nabla \cdot \left(f_n^{\prime j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \right)\end{aligned} \tag{8.51}$$

The major terms $\nabla \cdot \Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j} \alpha_n$, $\nabla \cdot \Psi_n \nabla \rho$ and $\sum_j \Delta f_n^{'j} \frac{\partial \rho}{\partial x_j} \alpha_n$ nearly cancel because of (8.38) and the residual is $\nabla \cdot \Psi_n \nabla \rho (1-\alpha_n)$. Multiplying (8.51) by z_n, integrating by parts and using the Schwartz inequality gives

$$
\begin{aligned}
&\int_{\mathcal{O}} d\mathrm{x}\ (\nabla z_n)^2 \\
&= -\int_{\mathcal{O}} d\mathrm{x}\ \Psi_n \nabla \rho (1-\alpha_n) \cdot \nabla z_n - \int_{\mathcal{O}} d\mathrm{x}\ \Psi_n \nabla \rho \cdot \nabla (1-\alpha_n) z_n \\
&\quad - \int_{\mathcal{O}} d\mathrm{x}\ \Psi_n \sum_j \nabla f_n^j \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) z_n - \int_{\mathcal{O}} d\mathrm{x}\ \Psi_n \sum_j f_n^j \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \cdot \nabla z_n \\
&\quad + \int_{\mathcal{O}} d\mathrm{x}\ \sum_j \nabla f_n^{'j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) z_n - \int_{\mathcal{O}} d\mathrm{x}\ \sum_j f_n^{'j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \cdot \nabla z_n \\
&\leq \int_{\mathcal{O}} d\mathrm{x} \left| \Psi_n \nabla \rho (1-\alpha_n) \cdot \nabla z_n \right| + \int_{\mathcal{O}} d\mathrm{x} \left| \Psi_n \nabla \rho \cdot \nabla (1-\alpha_n) z_n \right| \\
&\quad + \int_{\mathcal{O}} d\mathrm{x} \left| \Psi_n \sum_j \nabla f_n^j \cdot \nabla \frac{\partial \rho}{\partial x_j} \alpha_n z_n \right| + \int_{\mathcal{O}} d\mathrm{x} \left| \Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j} \cdot \nabla \alpha_n z_n \right| \\
&\quad + \int_{\mathcal{O}} d\mathrm{x} \left| \Psi_n \sum_j f_n^j \nabla \frac{\partial \rho}{\partial x_j} \alpha_n \cdot \nabla z_n \right| + \int_{\mathcal{O}} d\mathrm{x} \left| \Psi_n \sum_j f_n^j \frac{\partial \rho}{\partial x_j} \nabla \alpha_n \cdot \nabla z_n \right| \\
&\quad + \int_{\mathcal{O}} d\mathrm{x} \left| \sum_j \nabla f_n^{'j} \nabla \frac{\partial \rho}{\partial x_j} \alpha_n z_n \right| + \int_{\mathcal{O}} d\mathrm{x} \left| \sum_j \nabla f_n^{'j} \frac{\partial \rho}{\partial x_j} \cdot \nabla \alpha_n z_n \right| \\
&\quad + \int_{\mathcal{O}} d\mathrm{x} \left| \sum_j f_n^{'j} \nabla \frac{\partial \rho}{\partial x_j} \alpha_n \cdot \nabla z_n \right| + \int_{\mathcal{O}} d\mathrm{x} \left| \sum_j f_n^{'j} \frac{\partial \rho}{\partial x_j} \nabla \alpha_n \cdot \nabla z_n \right| \\
&\leq \left(\int_{\mathcal{O}} d\mathrm{x}\ (\Psi_n \nabla \rho (1-\alpha_n))^2 \right)^{1/2} \left(\int_{\mathcal{O} \setminus \mathcal{O}'} d\mathrm{x}\ (\nabla z_n)^2 \right)^{1/2} \\
&\quad + \left(\int_{\mathcal{O}} d\mathrm{x}\ (\Psi_n \nabla \rho \cdot \nabla (1-\alpha_n))^2 \right)^{1/2} \left(\int_{\mathcal{O} \setminus \mathcal{O}'} d\mathrm{x}\ z_n^2 \right)^{1/2} \\
&\quad + \left(\int_{\mathcal{O}} d\mathrm{x} \left(\Psi_n \sum_j \nabla f_n^j \cdot \nabla \frac{\partial \rho}{\partial x_j} \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O}} d\mathrm{x}\ z_n^2 \right)^{1/2} \\
&\quad + \left(\int_{\mathcal{O}} d\mathrm{x} \left(\Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j} \cdot \nabla \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O} \setminus \mathcal{O}'} d\mathrm{x}\ z_n^2 \right)^{1/2} \\
&\quad + \left(\int_{\mathcal{O}} d\mathrm{x} \left(\Psi_n \sum_j f_n^j \nabla \frac{\partial \rho}{\partial x_j} \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O}} d\mathrm{x}\ (\nabla z_n)^2 \right)^{1/2} \\
&\quad + \left(\int_{\mathcal{O}} d\mathrm{x} \left(\Psi_n \sum_j f_n^j \frac{\partial \rho}{\partial x_j} \nabla \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O} \setminus \mathcal{O}'} d\mathrm{x}\ (\nabla z_n)^2 \right)^{1/2}
\end{aligned}
$$

$$+\left(\int_{\mathcal{O}} dx \left(\sum_j \nabla f_n'^j \nabla \frac{\partial \rho}{\partial x_j} \alpha_n\right)^2\right)^{1/2} \left(\int_{\mathcal{O}} dx\ z_n^2\right)^{1/2}$$

$$+\left(\int_{\mathcal{O}} dx \left(\sum_j \nabla f_n'^j \frac{\partial \rho}{\partial x_j} \cdot \nabla \alpha_n\right)^2\right)^{1/2} \left(\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ z_n^2\right)^{1/2}$$

$$+\left(\int_{\mathcal{O}} dx \left(\sum_j f_n'^j \nabla \frac{\partial \rho}{\partial x_j} \alpha_n\right)^2\right)^{1/2} \left(\int_{\mathcal{O}} dx\ (\nabla z_n)^2\right)^{1/2}$$

$$+\left(\int_{\mathcal{O}} dx \left(\sum_j f_n'^j \frac{\partial \rho}{\partial x_j} \nabla \alpha_n\right)^2\right)^{1/2} \left(\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ (\nabla z_n)^2\right)^{1/2} \tag{8.52}$$

There are no boundary contributions because of the boundary condition for z_n and the cut-off function α_n.

We note that the Poincare inequality for z_n in $\mathcal{O} \setminus \mathcal{O}'$ gives

$$\begin{aligned}\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ z_n^2 &\le c_1 \tau_n^2 \int_{\mathcal{O}\setminus\mathcal{O}'} dx\ (\nabla z_n)^2 \\ &\le c_1 \tau_n^2 \int_{\mathcal{O}} dx\ (\nabla z_n)^2,\end{aligned} \tag{8.53}$$

since $\mathcal{O} \setminus \mathcal{O}'$ is a strip of width τ_n near $\partial\mathcal{O}$ and z_n vanishes on $\partial\mathcal{O}$. The estimate (8.53) holds for all $H^1(\mathcal{O})$-functions with zero Dirichlet data and the constant c_1 depends on the domain $\mathcal{O}$. For the cutoff function we have the estimates

$$\left|\nabla \alpha_n\right|^2 \le \frac{c_2}{\tau_n^2}, \tag{8.54}$$

$$\int_{\mathcal{O}} dx\ (\nabla \alpha_n)^2 \le \frac{c_3}{\tau_n}, \tag{8.55}$$

We now use (8.53), (8.54) and (8.55) whenever the integral $\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ z_n^2$ occurs. Lemma 8.3 then follows from (8.50) and

$$\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ \Psi_n^2 \to 0, \tag{8.56}$$

$$\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ (\nabla f_n^j)^2 \to 0, \quad j = 1, ..., d \tag{8.57}$$

$$\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ \left(\nabla f_n'^j\right)^2 \to 0, \quad j = 1, ..., d \tag{8.58}$$

$$\int_{\mathcal{O}\setminus\mathcal{O}'} dx\ (\Psi_n \nabla f_n^j)^2 \to 0 \tag{8.59}$$

$$\frac{1}{\tau_n^2}\int_{\mathcal{O}} d\mathbf{x}\ (f_n'^j)^2 \to 0, \quad j = 1, \dots, d, \tag{8.60}$$

$$\frac{1}{\tau_n^2}\left(\sup_{\mathbf{x}\in\mathcal{O}}(f_n^j)^2\right)\left(\int_{\mathcal{O}} d\mathbf{x}\ \Psi_n^2\right) \to 0, \quad j = 1, \dots, d. \tag{8.61}$$

for almost all ω. The estimate (8.61) is used for the integrals whose integrands involve $\Psi_n f_n^j$.

The estimates (8.56)-(8.59) are immediate consequences of the ergodicity, since $\tau_n \to 0$. To get (8.60) and (8.61), we first note that

$$\int_{\mathcal{O}} d\mathbf{x}\ \Psi_n^2 \le \langle|\widetilde{\Psi}|^2\rangle + \delta \tag{8.62}$$

for any $\delta > 0$, for n sufficiently large. So we must now choose a proper cut-off rate τ_n. Let

$$\eta_n = \max\left\{\int_{\mathcal{O}} d\mathbf{x}\ (f_n'^i)^2,\ \sup_{\mathbf{x}\in\mathcal{O}}(f_n^i)^2,\ i = 1, \dots, d\right\}. \tag{8.63}$$

We know from (8.43), (8.42) that

$$\eta_n \to 0, \quad \text{as} \quad n \to \infty, \quad \text{with probability one.} \tag{8.64}$$

The desired result (8.45) and Lemma 8.3 follow from

$$\tau_n \to 0, \quad \frac{\eta_n}{\tau_n^2} \to 0, \quad \text{as } n \to \infty. \tag{8.65}$$

when we let $\tau_n = \eta_n^{1/4}$.

Once Lemma 8.3 is proved, the cut-off function α_n can be omitted in the limit as $n \to \infty$. We have

$$\int_{\mathcal{O}} d\mathbf{x}\ (\nabla z_n')^2 \to 0, \tag{8.66}$$

where

$$z_n'(\mathbf{x},\omega) = g_n(\mathbf{x},\omega) - \sum_j f_n'^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}. \tag{8.67}$$

because

$$\int_{\mathcal{O}} dx\ \left(\sum_j \nabla f_n'^j \frac{\partial\rho}{\partial x_j}(1-\alpha_n)\right)^2 \to 0, \tag{8.68}$$

$$\int_{\mathcal{O}} dx\ \left(\sum_j f_n'^j \nabla\frac{\partial\rho}{\partial x_j}(1-\alpha_n)\right)^2 \to 0, \tag{8.69}$$

$$\int_{\mathcal{O}} dx\ \left(\sum_j f_n'^j \frac{\partial\rho}{\partial x_j}\nabla\alpha_n\right)^2 \to 0, \tag{8.70}$$

as $n \to \infty$, for almost all ω.

We note that Lemma 8.3 is also valid if the left side of (8.45) is averaged over ω. This can be seen by applying the Lebesque dominated convergence theorem.

8.5 Poisson problem for lower bound

We again assume, without loss of generality, that the domain $\mathcal{O}$ is the square $|x_i| \leq 1,\ i = 1, ..., d$.

The Poisson problem in this case is

$$\Delta g_n'(\mathbf{x}, \omega) + \nabla \cdot \Psi_n \mathbf{F}_n' = s + \nabla \cdot \mathbf{S}, \quad \text{in } \mathcal{O} \tag{8.71}$$

$$g_n' = 0, \quad \text{on } \partial\mathcal{O}. \tag{8.72}$$

where the inhomogeneous term $\mathbf{F}_n'(\mathbf{x}, \omega)$ has the form

$$\mathbf{F}_n'(\mathbf{x}, \omega) = \nabla \rho + \sum_j \nabla(\alpha_n(\mathbf{x}) f_n'^j(\mathbf{x}, \omega) \frac{\partial \rho}{\partial x_j}(\mathbf{x})) + \nabla \rho(\mathbf{x}), \quad \text{in } \mathcal{O}. \tag{8.73}$$

Here $\rho(\mathbf{x}) \in C_0^\infty(\mathcal{O})$, α_n is a cutoff function to be determined later and $f_n'^j(\mathbf{x}, \omega)$ satisfy (8.37). The abstract problems whose solutions we use to solve (8.71) is to determine $\mathbf{F}^i$ such that

$$\tilde{\nabla} \cdot \tilde{\mathbf{F}}^i + \tilde{\nabla} \cdot \tilde{\Psi} \tilde{\mathbf{F}}'^i = 0, \quad i = 1, ..., d \tag{8.74}$$

$$\langle \tilde{\mathbf{F}}^i \rangle = 0, \quad i = 1, ..., d \tag{8.75}$$

and the $\mathbf{F}^i(\mathbf{x}, \omega)$ can be written as gradients of nonstationary random functions $f^i(\mathbf{x}, \omega)$. Let the rescaled random functions be, as before,

$$f_n^i(\mathbf{x}, \omega) = \frac{1}{n} f^i(n\mathbf{x}, \omega). \tag{8.76}$$

and similarly for $f_n'^j(\mathbf{x}, \omega)$. They are determined up to constant which we fix so that

$$f_n'^i(0, \omega) = 0 \tag{8.77}$$

$$\int_{\mathcal{O}} d\mathbf{x}\ f_n^i(\mathbf{x}, \omega) = 0. \tag{8.78}$$

By Lemma 8.2 and 6.1,

$$\int_{\mathcal{O}} d\mathbf{x}\ (f_n^i)^2(\mathbf{x},\omega) \to 0, \tag{8.79}$$

$$\sup_{\mathbf{x}\in\mathcal{O}} (f_n^{\prime i})^2(\mathbf{x},\omega) \to 0, \tag{8.80}$$

in the limit $n \to \infty$.

We prove

Lemma 8.5 *Assume that* $\langle|\widetilde{\Psi}|^2\rangle < \infty$. *Let* $\widetilde{\mathbf{F}}^{\prime j} \in \mathcal{H}_g(\Omega), j = 1, ..., d$, *be essentially bounded and let the normalization conditions (8.77), (8.78) hold. Then*

$$\int_{\mathcal{O}} d\mathbf{x}\ (\nabla z_n)^2 \to 0, \quad as \quad n \to \infty, \tag{8.81}$$

for almost all ω, *where*

$$z_n(\mathbf{x},\omega) = g_n'(\mathbf{x},\omega) - \rho - \sum_j f_n^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}). \tag{8.82}$$

Here ρ *satisfies*

$$\sum_j \langle \mathbf{e}^j + \widetilde{\mathbf{F}}^j\rangle \cdot \frac{\partial}{\partial x_j}\nabla\rho = s + \nabla\cdot\mathbf{S}, \quad in\ \mathcal{O} \tag{8.83}$$

$$\rho = 0, \quad on\ \partial\mathcal{O}. \tag{8.84}$$

Proof:

First we observe that, as in the energy estimate (8.47) of Lemma 8.4,

$$|\nabla g_n'|_{L^2(\mathcal{O})} \le c \tag{8.85}$$

where c is a constant independent of n. Thus $g_n' \in H_0^1(\mathcal{O})$. From the equations (8.71), (8.74) it follows that

$$\begin{aligned}
\Delta z_n = s + \nabla\cdot\mathbf{S} &- \left(\nabla\cdot\Psi_n\sum_j \nabla f_n^{\prime j}\frac{\partial\rho}{\partial x_j}\alpha_n + \nabla\cdot\Psi_n\sum_j f_n^{\prime j}\nabla\left(\frac{\partial\rho}{\partial x_j}\alpha_n\right)\right) - \left(\sum_j \Delta f_n^j\frac{\partial\rho}{\partial x_j}\alpha_n\right. \\
&\left. + \Delta\rho + \sum_j \nabla f_n^j\cdot\nabla\left(\frac{\partial\rho}{\partial x_j}\alpha_n\right) + \sum_j \nabla\cdot\left(f_n^j\nabla\left(\frac{\partial\rho}{\partial x_j}\alpha_n\right)\right)\right) \\
&= \left(s + \nabla\cdot\mathbf{S} - \Delta\rho - \sum_j \nabla f_n^j\cdot\frac{\partial\nabla\rho}{\partial x_j}\alpha_n\right) - \nabla\cdot\Psi_n\sum_j f_n^{\prime j}\nabla\left(\frac{\partial\rho}{\partial x_j}\alpha_n\right) - \sum_j \nabla f_n^j\cdot\frac{\partial\rho}{\partial x_j}\nabla\alpha_n \\
&- \sum_j \nabla\cdot\left(f_n^j\nabla\left(\frac{\partial\rho}{\partial x_j}\alpha_n\right)\right)
\end{aligned} \tag{8.86}$$

The leading order $O(n)$-terms $\nabla \cdot \Psi_n \sum_j \nabla f_n'^j \frac{\partial \rho}{\partial x_j} \alpha_n$ and $\sum_j \Delta f_n^j \frac{\partial \rho}{\partial x_j} \alpha_n$ cancel because of (8.74). Multiplying this identity by z_n and integrating makes the $O(1)$-terms nearly zero in view of (8.83) and (8.85), since

$$\begin{aligned}
&\int_{\mathcal{O}} d\mathbf{x} \left(s + \nabla \cdot \mathbf{S} - \Delta \rho - \sum_j \nabla f_n^j \cdot \frac{\partial \nabla \rho}{\partial x_j} \alpha_n \right) z_n \\
&= \int_{\mathcal{O}} d\mathbf{x} \sum_j \left(\langle \mathbf{e}^j + \widetilde{\mathbf{F}}^j \rangle - \mathbf{e}^j - \nabla f_n^j \right) \cdot \frac{\partial \nabla \rho}{\partial x_j} z_n \\
&+ \int_{\mathcal{O}} d\mathbf{x} \sum_j \nabla f_n^j \cdot \frac{\partial \nabla \rho}{\partial x_j} (1 - \alpha_n) z_n
\end{aligned} \tag{8.87}$$

The first integral vanishes in the limit $n \to \infty$ by ergodicity and the L^2 integrability of $\mathbf{F}^j(\mathbf{x}, \omega)$. With (8.87) in mind, integrating by parts and using Schwartz inequality yield the estimate

$$\begin{aligned}
&\int_{\mathcal{O}} d\mathbf{x} \, (\nabla z_n)^2 \\
&\leq \left(\int_{\mathcal{O}} d\mathbf{x} \sum_j \left(\langle \mathbf{e}^j + \widetilde{\mathbf{F}}^j \rangle - \mathbf{e}^j - \nabla f_n^j \right)^2 \left(\frac{\partial \nabla \rho}{\partial x_j} \right)^2 \right)^{1/2} \left(\int_{\mathcal{O}} d\mathbf{x} \, z_n^2 \right)^{1/2} \\
&+ \left(\int_{\mathcal{O} \setminus \mathcal{O}'} \left(\sum_j \nabla f_n^j \cdot \frac{\partial \nabla \rho}{\partial x_j} \right)^2 \right)^{1/2} \left(\int_{\mathcal{O} \setminus \mathcal{O}'} z_n^2 \right)^{1/2} \\
&+ \left(\int_{\mathcal{O}} d\mathbf{x} \left(\Psi_n \sum_j f_n'^j \nabla \frac{\partial \rho}{\partial x_j} \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O}} d\mathbf{x} \, (\nabla z_n)^2 \right)^{1/2} \\
&+ \left(\int_{\mathcal{O}} d\mathbf{x} \left(\Psi_n \sum_j f_n'^j \frac{\partial \rho}{\partial x_j} \nabla \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O} \setminus \mathcal{O}'} d\mathbf{x} \, (\nabla z_n)^2 \right)^{1/2} \\
&+ \left(\int_{\mathcal{O}} d\mathbf{x} \left(\sum_j f_n^j \nabla \frac{\partial \rho}{\partial x_j} \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O}} d\mathbf{x} \, (\nabla z_n)^2 \right)^{1/2} \\
&+ \left(\int_{\mathcal{O}} d\mathbf{x} \left(\sum_j f_n^j \frac{\partial \rho}{\partial x_j} \nabla \alpha_n \right)^2 \right)^{1/2} \left(\int_{\mathcal{O} \setminus \mathcal{O}'} d\mathbf{x} \, (\nabla z_n)^2 \right)^{1/2}
\end{aligned} \tag{8.88}$$

Lemma 8.5 then follows from the ergodicity of $\mathbf{F}^j(\mathbf{x}, \omega)$,

$$\frac{1}{\tau_n^2} \int_{\mathcal{O}} d\mathbf{x} \, (f_n^j)^2 \to 0, \tag{8.89}$$

and

$$\frac{1}{\tau_n^2}\left(\sup_{\mathbf{x}\in\mathcal{O}}(f_n'^j)^2(\mathbf{x},\cdot)\right)\left(\int_{\mathcal{O}} d\mathbf{x}\ \Psi_n^2\right) \to 0, \quad j=1,...,d. \tag{8.90}$$

for almost all ω. As in the proof of Lemma 8.3, we have to choose the cutoff rate τ_n to satisfy (8.89) and (8.90). This completes the proof.

As with (8.66), we have

$$\int_{\mathcal{O}} d\mathbf{x}\ (\nabla z_n')^2 \to 0, \tag{8.91}$$

where

$$z_n'(\mathbf{x},\omega) = g_n(\mathbf{x},\omega) - \rho - \sum_j f_n^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}. \tag{8.92}$$

This says that the effect of the cut-off functions is negligible.

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