UCLA

COMPUTATIONAL AND APPLIED MATHEMATICS

First-Order System Least Squares Methods for Convection-Diffusion Equations on Unstructures Meshes

Tony F. Chan

Graeme Fairweather

Jian Ping Shao

October 1994

CAM Report 94-31

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

FIRST-ORDER SYSTEM LEAST SQUARES METHODS FOR CONVECTION-DIFFUSION EQUATIONS ON UNSTRUCTURED MESHES

TONY F. CHAN *, GRAEME FAIRWEATHER † AND JIAN PING SHAO ‡

Abstract. This paper studies first-order system least squares finite element methods for solving convection-diffusion equations on a mesh which is shape regular only. We do not assume that the mesh is quasi-uniform in our discussion. We give a general description of least squares finite element methods for first-order systems resulting from second-order elliptic equations by introducing the derivatives of the solution as new variables. We apply optimal iterative methods, such as multigrid methods and domain decomposition methods, to solve the associated discrete linear systems arising from least squares finite element methods. Several numerical results are presented to examine condition numbers, convergence behavior and error bounds. We also investigate the least squares functional with various weighted diffusion coefficients (the inverse of the Reynolds number). We compare the least squares methods with the standard Galerkin method and discuss how the small diffusion coefficient affects the error in each method. To overcome difficulties caused by a boundary layer, we apply an adaptive technique in the least squares finite element methods.

Key Words. convection-diffusion equation, first-order system, least squares finite element method, unstructured mesh, error estimates

AMS subject classifications: 65F10, 65N20.

1. Introduction. Recently, there has been considerable interest in the use of least squares finite element methods for solving first-order differential systems resulting from second-order differential equations. The application of such least squares methods to potential flows, Stokes and Navier-Stokes equations can be found in [4, 15, 21, 28]. Theoretically, there are two important approaches in the study of least squares finite element methods. One is based on the general first-order differential system theory of Agmon, Douglis and Nirenberg [1] to obtain *a priori* estimates for general partial differential equations [2] and Stokes equations [3]. The resulting stiffness matrices usually have condition numbers which are $0(h^{-4})$. The other approach is based on the Lax-Milgram theory by proving the coercivity and continuity of bilinear forms arising from least squares functionals [5, 12, 13, 22, 24].

The least squares finite element methods considered here for second-order partial differential equations basically consist of the following four main steps:

1. introducing the derivatives of the solution as new independent variables and transforming the second-order elliptic equation to a first-order system;
2. formulating a least squares functional as a sum of norms of preconditioned residuals from the first-order differential system;

* Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024-1555. Email:chan@math.ucla.edu. Supported in part by the National Science Foundation Grant ASC 92-01266, the Army Research Office under contract DAAL03-91-G-0150 and the Office of Naval Research under contract ONR-N00014-92-J-1890.

† Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887. Email: gfairwea@mines.colorado.edu. Supported in part by the National Science Foundation Grant EHR-9108764.

‡ Center for Computational Sciences, University of Kentucky, Lexington, KY 40506-0045. Email: shao@ms.uky.edu. Supported in part by the National Science Foundation Grant ASC-9310315.

3. minimizing the least squares functional and obtaining a variational problem;
4. discretizing the variational formulation by finite element methods and solving the resulting linear system directly or iteratively.

The main advantages of these least squares finite element methods are: (i) the solution and its derivatives, which usually have physical meanings such as fluxes, velocity, strains and stresses, etc., are obtained simultaneously from the minimization of the least squares functional; (ii) unlike mixed finite element methods [8, 26, 29], continuous piecewise polynomial spaces can be used as the approximation spaces for all unknowns without requiring the inf-sup conditions on these approximation spaces; (iii) the corresponding bilinear form is symmetric, coercive and continuous; (iv) unlike least squares methods directly applied to second-order elliptic equations [6], the least squares methods considered here require less smoothness of the solution, and the resulting stiffness matrices still have condition numbers which are $O(h^{-2})$.

The purpose of this paper is to study least squares finite element methods for solving first-order differential systems resulting from elliptic convection-diffusion equations with small diffusion. Cai et al. [12, 13] studied the coercivity of the associated bilinear form of the least squares method for general partial differential equations. However, their theory requires that the solution belongs to H^3. Bramble et al. [5] developed a least squares functional with minus one norm. This method requires only H^2 regularity on the solution. The above theories were developed primarily for a quasi-uniform mesh. However, it is very common in practice to use a non-quasi-uniform mesh for solving convection-diffusion equations efficiently, for convection dominated problems, in order to resolve boundary and internal layers. The main purpose of this paper is to extend the theory in [5] to non-quasi-uniform and unstructured meshes. Our main tools are borrowed from some recent works on the analysis of Schwarz domain decomposition methods for unstructured meshes; see, for example, Chan and Zou [17], Chan et al. [16] and Cai [9]. In addition, we have carried out extensive numerical experiments to examine the least squares methods, and show that multigrid and domain decomposition methods for the resulting discrete problems have optimal convergence rates without requiring the coarsest grid to be fine enough. We also analyze how the small diffusion coefficient and the choice of weights in the least squares methods affect the error behavior of the methods. Moreover, we apply an adaptive technique in the least squares methods for convection-diffusion equations to construct a nonuniform mesh automatically. Finally, we compare the least squares methods with the standard Galerkin methods.

This paper is organized as follows. In section 2, we introduce notation and the convection-diffusion equations, then we present a general form of the least squares finite element methods with various choices of weights and preconditioners as described in [5, 12, 13]. In section 3, we discuss and analyze some specific least squares methods and extend the methods on a shape regular only mesh instead of a quasi-uniform mesh. In section 4, we present the results of numerical experiments, and in the last section, we give some concluding remarks.

2. Convection-Diffusion Equations and General Least Squares Methods. Let Ω be a bounded open subset of R^d with a Lipschitz continuous boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where $\Gamma_D \cap \Gamma_N = \emptyset$. For m a nonnegative integer, $H^m(\Omega)$ denotes the classical Sobolev space with norm $||\cdot||_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$, and $H^0(\Omega) = L^2(\Omega)$. We assume that $\mathcal{A} = \mathcal{A}(x),\ x \in \Omega$, is a $d \times d$ matrix which is symmetric, uniformly positive definite and sufficiently smooth. We define spaces

$$H(\text{div};\Omega) = \{\mathbf{v}|\ \mathbf{v} \in (L^2(\Omega))^d \text{ and } \nabla\cdot\mathbf{v} \in L^2(\Omega)\},$$

and

$$H(\text{curl};\Omega) = \{\mathbf{v}|\ \mathbf{v} \in (L^2(\Omega))^d \text{ and } \nabla\times(\mathcal{A}^{-1}\mathbf{v}) \in (L^2(\Omega))^{2d-3},\ d \geq 2\},$$

with norms

$$||\mathbf{v}||_{H(\text{div};\Omega)} = (||\mathbf{v}||_{0,\Omega}^2 + ||\nabla\cdot\mathbf{v}||_{0,\Omega}^2)^{1/2},$$

and

$$||\mathbf{v}||_{H(\text{curl};\Omega)} = (||\mathbf{v}||_{0,\Omega}^2 + ||\nabla\times(\mathcal{A}^{-1}\mathbf{v})||_{0,\Omega}^2)^{1/2}.$$

We also introduce spaces

$$\mathbf{V} = \{\mathbf{v}|\ \mathbf{v} \in H(\text{div};\Omega) \text{ and } \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Gamma_N\},$$

$$\mathbf{W} = \{\mathbf{w}|\ \mathbf{w} \in H(\text{curl};\Omega) \text{ and } \mathbf{n}\times(\mathcal{A}^{-1}\mathbf{v}) = 0 \text{ on } \Gamma_N\},$$

and

$$\mathcal{Q} = \{q|\ q \in H^1(\Omega) \text{ and } q = 0 \text{ on } \Gamma_D\},$$

where $\mathbf{n}$ is the outer unit normal on the boundary Γ of Ω. Let $H^{-1}(\Omega)$ denote the dual space of $\mathcal{Q}$ with norm

$$||q||_{-1,\Omega} = \sup_{\phi\in\mathcal{Q}} \frac{(q,\phi)}{||\phi||_{1,\Omega}}.$$

We introduce finite element spaces to approximate $\mathbf{V}, \mathbf{W}$ and $\mathcal{Q}$. To this end, we first partition the domain Ω into simplices τ with diameter h_τ and obtain a triangulation $\Omega^h = \{\tau^h\}$ on Ω. Here $h = \max_{\tau\in\Omega^h} h_\tau$, and we denote the diameter of the largest ball in τ as ρ_τ. In this paper, we only assume that the triangulation Ω^h is **shape regular**, that is, there exists a positive constant c independent of τ such that $h_\tau/\rho_\tau \leq c$, $\forall\tau \in \Omega^h$. A triangulation is said to be **quasi-uniform** when the triangulation is shape regular and $h/h_\tau \leq c$, $\forall\tau \in \Omega^h$, for a positive constant c independent of τ.

REMARK 1. *When the triangulation Ω^h is quasi-uniform, all of the following algorithms have been discussed in [5, 12, 13]. In this paper, we allow the elements τ to have quite different sizes and only assume that the triangulation Ω^h is shape regular.*

Let $\mathcal{Q}_h \subset \mathcal{Q}$ and $\mathbf{V}_h \subset \mathbf{V}$ be the continuous piecewise linear finite element spaces defined on the triangulation Ω^h. Let $\mathcal{R}_h : L^2 \to \mathcal{Q}_h$ be a local quasi-projection defined by

$$\mathcal{R}_h q = \sum_i (\mathcal{P}_i q)(x^{(i)})\phi_i^h,$$

where $\mathcal{P}_i q$ is a linear polynomial on the support O_i of the standard nodal basis function ϕ_i^h at a node $x^{(i)}$. The polynomial $\mathcal{P}_i q$ is determined by

$$\int_{O_i} (\mathcal{P}_i q)\rho dx = \int_{O_i} q\rho dx \qquad \text{for all linear polynomials } \rho\ .$$

When the triangulation is shape regular, we have the following approximation properties [18]. Given any $q \in H^s \cap \mathcal{Q}, 1 \le s \le 2$, the function $q_h = \mathcal{R}_h q \in \mathcal{Q}_h$ satisfies

$$(2.1) \qquad \|q - q_h\|_{0,\Omega}^2 \le C \sum_{\tau \in \Omega^h} h_\tau^{2s} \|q\|_{s,\tau}^2, \qquad \|q - q_h\|_{1,\Omega}^2 \le C \sum_{\tau \in \Omega^h} h_\tau^{2s-2} \|q\|_{s,\tau}^2,$$

and given any $\mathbf{v} \in (H^s)^d \cap \mathbf{V}, 1 \le s \le 2$, the vector $\mathbf{v}_h = \mathcal{R}_h \mathbf{v} \in \mathbf{V}_h$ satisfies

$$(2.2) \qquad \|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega}^2 \le C \sum_{\tau \in \Omega^h} h_\tau^{2s} \|\mathbf{v}\|_{s,\tau}^2, \qquad \|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega}^2 \le C \sum_{\tau \in \Omega^h} h_\tau^{2s-2} \|\mathbf{v}\|_{s,\tau}^2,$$

where C is a constant independent of h.

We consider the following elliptic convection-diffusion problem:

$$(2.3) \qquad \begin{cases} \mathcal{L}p = f & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma_N, \end{cases}$$

where $f \in L^2(\Omega)$ and $\dfrac{\partial p}{\partial \nu}$ is the co-normal derivative on Γ_N. Here the elliptic operator $\mathcal{L}$ has the form

$$\mathcal{L}p(x) = -\nabla \cdot \mathcal{A}(x) \nabla p(x) + \mathcal{X} p(x),$$

where the operator $\mathcal{X}$ is defined by

$$\mathcal{X} p(x) = \mathbf{b}(x) \cdot \nabla p(x) + c(x) p(x).$$

We assume that the vector $\mathbf{b}(x) \in R^d$ is sufficiently smooth.

Now we present the general least squares method for a first-order differential system. We first introduce several new variables from the derivatives of the solution:

$$\mathbf{u}(x) = -\mathcal{A}(x) \nabla p(x),$$

which is sometimes called the flux. Then the convection-diffusion problem (2.3) is transformed into a first-order system

$$(2.4) \qquad \begin{cases} \mathbf{u} + \mathcal{A} \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + \mathcal{X} p = f & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = 0 & \text{on } \Gamma_N. \end{cases}$$

REMARK 2. *Here we do not require that the finite element spaces $\mathcal{Q}_h \subset \mathcal{Q}$ and $\mathbf{V}_h \subset \mathbf{V}$ satisfy the inf-sup condition. This condition is, however, needed for mixed finite element methods [8, 26, 29].*

Let M and N be two preconditioning operators which are positive definite and symmetric. Then the solution $(p, \mathbf{u})$ of (2.4) is the solution of the minimization problem

$$(2.5) \qquad \begin{cases} J(p, \mathbf{u}) = \sup_{(q,\mathbf{v}) \in \mathcal{Q} \times \mathbf{V}} J(q, \mathbf{v}), \\ J(q, \mathbf{v}) = \|M^{1/2}(\nabla \cdot \mathbf{v} + \mathcal{X} q - f)\|_{0,\Omega}^2 + \|N^{1/2}(\mathbf{v} + \mathcal{A} \nabla q)\|_{0,\Omega}^2. \end{cases}$$

The associated bilinear form is

$$(2.6) \qquad a(p, \mathbf{u}; q, \mathbf{v}) = (M(\nabla \cdot \mathbf{u} + \mathcal{X} p),\ \nabla \cdot \mathbf{v} + \mathcal{X} q) + (N(\mathbf{u} + \mathcal{A} \nabla p),\ \mathbf{v} + \mathcal{A} \nabla q).$$

Thus the minimization problem (2.5) over the finite element space $\mathcal{Q}_h \times \mathbf{V}_h$ is equivalent to the variational problem: Find $(p_h, \mathbf{u}_h) \in \mathcal{Q}_h \times \mathbf{V}_h$ such that

$$(2.7) \qquad a(p_h, \mathbf{u}_h;\, q_h, \mathbf{v}_h) = (Mf,\, \nabla \cdot \mathbf{v}_h + \mathcal{X} q_h) \qquad \forall\, (q_h, \mathbf{v}_h) \in \mathcal{Q}_h \times \mathbf{V}_h.$$

The corresponding linear system is written as

$$(2.8) \qquad A_h U_h = b_h,$$

where A_h is the stiffness matrix, and U_h and b_h are vectors.

REMARK 3. *There are many choices of the operators M and N. The choice will affect the efficiency of the algorithm, the regularity requirements of the solution, and the accuracy of the least squares method.*

REMARK 4. *We can introduce boundary conditions or compatibility conditions into the quadratic functional (2.5). Then the functions in the finite element space are not required to satisfy the usual boundary conditions and/or the convergence rate of some iterative methods for solving the associated discrete problems may be improved.*

3. Several Least Squares Finite Element Methods. In this section, we give several choices of the preconditioned operators in the least squares method and discuss their important properties. We first present two least squares methods discussed in [12, 13, 14, 25]. Then we analyze the least squares method based on the minus one norm [5]. These three choices of least squares methods were proposed in [12, 13, 5]. We summarize them in this section. Again we emphasize that in our discussion we allow the elements of the triangulation to have quite different sizes and assume that the triangulation Ω^h is only shape regular. In the following discussion, c_0, c_1 and C denote generic positive constants which are independent of h.

Choice 1: (Cai et al. [12]) In this method, the authors choose the preconditioning operator M as the identity operator and N as $\mathcal{A}^{-1}$. Then the solution $(p, \mathbf{u}) \in \mathcal{Q} \times \mathbf{V}$ of problem (2.4) minimizes the quadratic functional (2.5) over the space $\mathcal{Q} \times \mathbf{V}$:

$$(3.1) \qquad \begin{cases} J(p, \mathbf{u}) = \inf_{(q,\mathbf{v}) \in \mathcal{Q} \times \mathbf{V}} J(q, \mathbf{v}), \\ J(q, \mathbf{v}) = ||\nabla \cdot \mathbf{v} + \mathcal{X} q - f||_{0,\Omega}^2 + ||\mathcal{A}^{-1/2}(\mathbf{v} + \mathcal{A} \nabla q)||_{0,\Omega}^2. \end{cases}$$

It is known [12] that the corresponding bilinear form

$$(3.2) \qquad \begin{aligned} a(p, \mathbf{u};\, q, \mathbf{v}) \;=\; & (\nabla \cdot \mathbf{v} + \mathcal{X} q,\, \nabla \cdot \mathbf{u} + \mathcal{X} p) \\ & + (\mathcal{A}^{-1/2}(\mathbf{v} + \mathcal{A} \nabla q),\, \mathcal{A}^{-1/2}(\mathbf{u} + \mathcal{A} \nabla p)) \end{aligned}$$

is (a) symmetric, that is, for any $(q, \mathbf{v}) \in \mathcal{Q} \times \mathbf{V}$ and $(p, \mathbf{u}) \in \mathcal{Q} \times \mathbf{V}$,

$$a(p, \mathbf{u};\, q, \mathbf{v}) = a(q, \mathbf{v};\, p, \mathbf{u}),$$

(b) coercive, that is, for any $(q, \mathbf{v}) \in \mathcal{Q} \times \mathbf{V}$,

$$(3.3) \qquad a(q, \mathbf{v};\, q, \mathbf{v}) \geq c_0(||q||_{1,\Omega}^2 + ||\mathbf{v}||_{H(\mathrm{div};\Omega)}^2)$$

and (c) continuous, that is, for any $(q, \mathbf{v}) \in \mathcal{Q} \times \mathbf{V}$ and $(p, \mathbf{u}) \in \mathcal{Q} \times \mathbf{V}$,

$$(3.4) \qquad a(p, \mathbf{u};\, q, \mathbf{v}) \leq c_1(||q||_{1,\Omega}^2 + ||\mathbf{v}||_{H(\mathrm{div};\Omega)}^2)^{1/2}(||p||_{1,\Omega}^2 + ||\mathbf{u}||_{H(\mathrm{div};\Omega)}^2)^{1/2}.$$

The discrete problem of minimizing the functional (3.1) can be written in the variational form: Find $(p_h, \mathbf{u}_h) \in \mathcal{Q}_h \times \mathbf{V}_h$ such that

$$(3.5) \qquad a(p_h, \mathbf{u}_h;\, q_h, \mathbf{v}_h) = (f,\, \nabla \cdot \mathbf{v}_h + \mathcal{X} q_h) \qquad \forall\, (q_h, \mathbf{v}_h) \in \mathcal{Q}_h \times \mathbf{V}_h.$$

According to the Lax-Milgram theorem, the coercivity and continuity of the bilinear form imply that the variational problem (3.5) has a unique solution. The coercivity (3.3) and the continuity (3.4) imply that the condition number of the associated stiffness matrix is $0(h^{-2})$. From (3.3), (3.4), (2.1) and (2.2), we can easily derive the error estimate

$$(3.6) \qquad ||p - p_h||^2_{1,\Omega} + ||\mathbf{u} - \mathbf{u}_h||^2_{H(\mathrm{div};\Omega)} \le Ch^2(||p||^2_{2,\Omega} + ||\mathbf{u}||^2_{2,\Omega}) \le Ch^2 ||p||^2_{3,\Omega}.$$

REMARK 5. *Note that the above discrete variational form is based on the use of the subspace $\mathcal{Q}_h \times V_h$ of $H^1 \times H(div;\Omega)$. No iterative method has been shown to have optimal convergence for the associated discrete linear system. To apply existing optimal iterative methods, such as multigrid and domain decomposition methods, we use another discrete variational form which is defined on a discrete subspace of $H^1(\Omega) \times (H^1(\Omega))^d$ instead of $H^1(\Omega) \times H(div;\Omega)$.*

Choice 2: (Cai et al. [13]). To introduce a variational form on $H^1 \times (H^1)^d$ instead of $H^1 \times H(\mathrm{div};\Omega)$, we add a compatibility constraint and corresponding boundary condition [13] to the first-order differential system (2.4):

$$(3.7) \qquad \begin{cases} \mathbf{u} + \mathcal{A}\nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + \mathcal{X} p = f & \text{in } \Omega, \\ \nabla \times (\mathcal{A}^{-1}\mathbf{u}) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = 0 & \text{on } \Gamma_N, \\ \mathbf{n} \times (\mathcal{A}^{-1}\mathbf{u}) = 0 & \text{on } \Gamma_D. \end{cases}$$

Then solving problem (3.7) over the finite element space $\mathcal{Q}_h \times (\mathbf{W}_h \cap \mathbf{V}_h)$ is equivalent to the minimization problem: Find $(p_h, \mathbf{u}_h) \in \mathcal{Q}_h \times (\mathbf{W}_h \cap \mathbf{V}_h)$ such that

$$(3.8) \qquad \begin{cases} J(p_h, \mathbf{u}_h) &= \inf_{(q_h, \mathbf{v}_h) \in \mathcal{Q}_h \times (\mathbf{W}_h \cap \mathbf{V}_h)} J(q_h, \mathbf{v}_h), \\ J(q_h, \mathbf{v}_h) &= ||\nabla \cdot \mathbf{v}_h + \mathcal{X} q_h - f||^2_{0,\Omega} + ||\nabla \times (\mathcal{A}^{-1}\mathbf{v}_h)||^2_{0,\Omega} \\ & \quad + ||\mathcal{A}^{-1/2}(\mathbf{v}_h + \mathcal{A}\nabla q_h)||^2_{0,\Omega}. \end{cases}$$

The corresponding bilinear form [13]

$$(3.9) \qquad \begin{aligned} \tilde{a}(p, \mathbf{u};\, q, \mathbf{v}) \;=\; & (\nabla \cdot \mathbf{u} + \mathcal{X} p,\, \nabla \cdot \mathbf{v} + \mathcal{X} q) + (\nabla \times (\mathcal{A}^{-1}\mathbf{u}),\, \nabla \times (\mathcal{A}^{-1}\mathbf{v})) \\ & + (\mathcal{A}^{-1/2}(\mathbf{u} + \mathcal{A}\nabla p),\, \mathcal{A}^{-1/2}(\mathbf{v} + \mathcal{A}\nabla q)), \end{aligned}$$

is symmetric, and coercive, that is, for any $(q, \mathbf{v}) \in \mathcal{Q} \times (\mathbf{W} \cap \mathbf{V})$,

$$\tilde{a}(q, \mathbf{v};\, q, \mathbf{v}) \ge c_0(||q||^2_{1,\Omega} + ||\mathbf{v}||^2_{1,\Omega}).$$

Moreover, it is continuous, that is, for any $(p, \mathbf{u}) \in \mathcal{Q} \times (\mathbf{W} \cap \mathbf{V})$ and $(q, \mathbf{v}) \in \mathcal{Q} \times (\mathbf{W} \cap \mathbf{V})$,

$$\tilde{a}(p, \mathbf{u};\, q, \mathbf{v}) \le c_1(||p||^2_{1,\Omega} + ||\mathbf{u}||^2_{1,\Omega})^{1/2}(||q||^2_{1,\Omega} + ||\mathbf{v}||^2_{1,\Omega})^{1/2}.$$

Then the minimization problem (3.8) is equivalent to: Find $(p_h, \mathbf{u}_h) \in \mathcal{Q}_h \times (\mathbf{W}_h \cap \mathbf{V}_h)$ such that

$$(3.10)\quad \tilde{a}(p_h, \mathbf{u}_h;\, q_h, \mathbf{v}_h) = (f,\, \nabla \cdot \mathbf{v}_h + \mathcal{X} q_h(x)) \qquad \forall\, (q_h, \mathbf{v}_h) \in \mathcal{Q}_h \times (\mathbf{W}_h \cap \mathbf{V}_h),$$

whose condition number is $0(h^{-2})$, and for which

$$(3.11)\qquad ||p - p_h||^2_{1,\Omega} + ||\mathbf{u} - \mathbf{u}_h||^2_{1,\Omega} \le Ch^2(||p||^2_{2,\Omega} + ||\mathbf{u}||^2_{2,\Omega}) \le Ch^2||p||^2_{3,\Omega}.$$

REMARK 6. *Note that the error estimates (3.6) and (3.11) of the finite element method for variational problems (3.5) and (3.10) are bounded by the H^3 norm of the exact solution, and thus require higher regularity on the solution than expected. Furthermore, the H^3 norm of the exact solution is much larger than its H^2 norm when the solution has a boundary layer. As we shall see, the next algorithm requires less smoothness of the solution.*

Choice 3: (Bramble et al. [5]). Now we present a choice of the preconditioned operators M and N in (2.5) which enables the error estimate to be bounded by the H^2 norm of the exact solution. We first introduce an operator $T : L^2(\Omega) \to \mathcal{Q}$ defined by $p = Tf \in \mathcal{Q}$ and

$$(p,\, \theta) + (\nabla p,\, \nabla \theta) = (f,\, \theta) \qquad \forall\, \theta \in \mathcal{Q}.$$

For this symmetric and positive definite operator T, we have

$$(Tq, q) = \sup_{\theta \in \mathcal{Q}} \frac{(q, \theta)^2}{||\theta||^2_{1,\Omega}} = ||q||^2_{-1,\Omega} \qquad \forall q \in H^{-1}(\Omega).$$

Then we take $M = T$ and $N = \mathcal{A}^{-1}$ in (2.5) to obtain

$$(3.12)\qquad \begin{cases} J(p, \mathbf{u}) &= \inf_{(q,\mathbf{v}) \in \mathcal{Q} \times \mathbf{V}} J(q, \mathbf{v}), \\ J(q, \mathbf{v}) &= (T(\nabla \cdot \mathbf{v} + \mathcal{X} q - f),\, \nabla \cdot \mathbf{v} + \mathcal{X} q - f) \\ & \quad + ||\mathcal{A}^{-1/2}(\mathbf{v} + \mathcal{A} \nabla q)||^2_{0,\Omega} \\ &= ||\nabla \cdot \mathbf{v} + \mathcal{X} q - f||^2_{-1,\Omega} + ||\mathcal{A}^{-1/2}(\mathbf{v} + \mathcal{A} \nabla q)||^2_{0,\Omega}. \end{cases}$$

It is easy to show that the corresponding bilinear form

$$(3.13)\qquad \begin{aligned} \hat{a}(p, \mathbf{u};\, q, \mathbf{v}) &= (T(\nabla \cdot \mathbf{u} + \mathcal{X} p),\, \nabla \cdot \mathbf{v} + \mathcal{X} q) \\ & \quad + (\mathcal{A}^{-1/2}(\mathbf{u} + \mathcal{A} \nabla p),\, \mathcal{A}^{-1/2}(\mathbf{v} + \mathcal{A} \nabla q)) \end{aligned}$$

is symmetric, coercive, that is, for any $(q, \mathbf{v}) \in \mathcal{Q} \times \mathbf{V}$,

$$(3.14)\qquad \hat{a}(q, \mathbf{v};\, q, \mathbf{v}) \ge c_0(||q||^2_{1,\Omega} + ||\mathbf{v}||^2_{0,\Omega}),$$

and continuous, that is, for any $(p, \mathbf{u}) \in \mathcal{Q} \times \mathbf{V}$ and $(q, \mathbf{v}) \in \mathcal{Q} \times \mathbf{V}$,

$$(3.15)\qquad \hat{a}(p, \mathbf{u};\, q, \mathbf{v}) \le c_1(||p||^2_{1,\Omega} + ||\mathbf{u}||^2_{0,\Omega})^{1/2}(||q||^2_{1,\Omega} + ||\mathbf{v}||^2_{0,\Omega})^{1/2}.$$

We replace the operator T by $\mathcal{T}_h$ (to be chosen later) which is spectrally equivalent to T, that is,

$$(3.16)\qquad c_0(Tq, q) \le (\mathcal{T}_h q, q) \le c_1(Tq, q) \qquad \forall q \in L^2(\Omega).$$

Then the minimization problem (3.12) can be replaced by

$$(3.17)\qquad \begin{cases} J(p,\mathbf{u}) &= \inf_{(q,\mathbf{v})\in\mathcal{Q}\times\mathbf{V}} J(q,\mathbf{v}) \\ J(q,\mathbf{v}) &= (\mathcal{T}_h(\nabla\cdot\mathbf{v}+\mathcal{X}q-f),\ \nabla\cdot\mathbf{v}+\mathcal{X}q-f) \\ & \quad +\|\mathcal{A}^{-1/2}(\mathbf{v}+\mathcal{A}\nabla q)\|_{0,\Omega}^2. \end{cases}$$

From this quadratic functional, we introduce the bilinear form

$$(3.18)\qquad \begin{aligned} \hat{a}_h(p,\mathbf{u};q,\mathbf{v}) &= (\mathcal{T}_h(\nabla\cdot\mathbf{u}+\mathcal{X}p),\ \nabla\cdot\mathbf{v}+\mathcal{X}q) \\ &\quad +(\mathcal{A}^{-1/2}(\mathbf{u}+\mathcal{A}\nabla p),\ \mathcal{A}^{-1/2}(\mathbf{v}+\mathcal{A}\nabla q)). \end{aligned}$$

When inequalities (3.16) are satisfied, we can show that the bilinear $\hat{a}_h$ is also coercive and continuous as in (3.14) and (3.15).

To construct an operator $\mathcal{T}_h$ which satisfies (3.16), let $T_h : L^2 \to \mathcal{Q}$ be the discrete form of T defined by $p_h = T_h f \in \mathcal{Q}_h$ and

$$(p_h,\ \theta_h) + (\nabla p_h,\ \nabla\theta_h) = (f,\ \theta_h) \qquad \forall\, \theta_h \in \mathcal{Q}_h.$$

It is easy to prove that

$$(T_h q, q) = \sup_{\theta_h\in\mathcal{Q}_h} \frac{(q,\theta_h)^2}{\|\theta_h\|_{1,\Omega}^2}.$$

Then we define $\mathcal{T}_h = h^2 I + B_h$, where I denotes the identity operator, and $B_h : L^2 \to \mathcal{Q}_h$ is a symmetric operator satisfying

$$c_0(T_h q, q) \le (B_h q, q) \le c_1(T_h q, q) \qquad \forall\, q \in L^2(\Omega).$$

When Ω^h is qusi-uniform, the following lemma has been proved in [5]. We now extend this lemma to unstructured grids.

LEMMA 3.1. *Assume that the triangulation Ω^h is only shape regular. Then*

$$(3.19)\qquad \|q\|_{-1,\Omega}^2 \le c_0(\mathcal{T}_h q, q) \le c_0(h^2\|q\|_{0,\Omega}^2 + \|q\|_{-1,\Omega}^2) \qquad \forall\, q \in L^2(\Omega).$$

Proof. The approximation properties (2.1) imply that, given any $q \in L^2(\Omega)$, we have

$$\sup_{\theta\in\mathcal{Q}} \frac{(q,\theta-\mathcal{R}_h\theta)}{\|\theta\|_{1,\Omega}} \le ch\|q\|_{0,\Omega}, \qquad \frac{\|\mathcal{R}_h\theta\|_{1,\Omega}}{\|\theta\|_{1,\Omega}} \le C,$$

and

$$\sup_{\theta\in\mathcal{Q}} \frac{(q,\mathcal{R}_h\theta)}{\|\theta\|_{1,\Omega}} = \sup_{\theta\in\mathcal{Q}} \frac{(q,\mathcal{R}_h\theta)}{\|\mathcal{R}_h\theta\|_{1,\Omega}} \frac{\|\mathcal{R}_h\theta\|_{1,\Omega}}{\|\theta\|_{1,\Omega}} \le C \sup_{\theta_h\in\mathcal{Q}_h} \frac{(q,\theta_h)}{\|\theta_h\|_{1,\Omega}} = C\sqrt{(T_h q, q)}.$$

Therefore, it follows that

$$\|q\|_{-1,\Omega}^2 = \sup_{\theta\in\mathcal{Q}} \frac{(q,\theta)^2}{\|\theta\|_{1,\Omega}^2} = \sup_{\theta\in\mathcal{Q}} \left(\frac{(q,\theta-\mathcal{R}_h\theta)}{\|\theta\|_{1,\Omega}} + \frac{(q,\mathcal{R}_h\theta)}{\|\theta\|_{1,\Omega}} \right)^2 \le C\{h^2(q,q) + (T_h q, q)\},$$

which gives the desired lower bound.

To obtain the upper bound, we note that from the definition of $\mathcal{T}_h$, we have $\forall\, q \in L^2(\Omega)$

$$(\mathcal{T}_h q, q) = h^2(q,q) + (B_h q, q) \le h^2(q,q) + c_1 \sup_{\theta_h \in \mathcal{Q}_h} \frac{(q,\theta_h)^2}{||\theta_h||^2_{1,\Omega}} \le h^2||q||^2_{0,\Omega} + c_1||q||^2_{-1,\Omega},$$

which completes the proof. □

LEMMA 3.2. *Assume that the triangulation Ω^h is only shape regular. Then*

$$c_0(||q||^2_{1,\Omega} + ||v||^2_{0,\Omega}) \le \hat{a}_h(q, v;\, q, v) \le c_1(h^2||\nabla\cdot v||^2_{0,\Omega} + ||v||^2_{0,\Omega} + ||q||^2_{1,\Omega}) \qquad (3.20)$$

for any $(q, v) \in \mathcal{Q} \times V$.

Proof. Inequalities (3.19) imply that

$$||\nabla\cdot \mathbf{v} + \mathcal{X}q||^2_{-1,\Omega} \le c_0(\mathcal{T}_h(\nabla\cdot\mathbf{v} + \mathcal{X}q),\ \nabla\cdot\mathbf{v} + \mathcal{X}q), \quad \forall (q,\mathbf{v}) \in \mathcal{Q}\times\mathbf{V}.$$

Then we obtain

$$\hat{a}(q,\mathbf{v};\, q,\mathbf{v}) \le c_0 \hat{a}_h(q,\mathbf{v};\, q,\mathbf{v}).$$

Combining this inequality with the coercive property of the bilinear form $\hat{a}$ gives the desired lower bound on $\hat{a}_h(q,\mathbf{v};\, q,\mathbf{v})$.

Now we obtain the upper bound. From the definition of $\hat{a}$ and Lemma 3.1, we can show that

$$\hat{a}_h(q,\mathbf{v};\, q,\mathbf{v}) \le h^2||\nabla\cdot\mathbf{v} + \mathcal{X}q||^2_{0,\Omega} + \hat{a}(q,\mathbf{v};\, q,\mathbf{v}).$$

The continuity of $\hat{a}$ and the triangle inequality imply that

$$\hat{a}_h(q,\mathbf{v};\, q,\mathbf{v}) \le c_1(h^2||\nabla\cdot\mathbf{v}||^2_{0,\Omega} + ||v||^2_0 + ||q||^2_{1,\Omega}),$$

as desired. □

On the finite element space $\mathcal{Q}_h \times \mathbf{V}_h$, we define a finite element method for problem (3.12) by: Find $(p_h, \mathbf{u}_h) \in \mathcal{Q}_h \times \mathbf{V}_h$ such that

$$\hat{a}_h(p_h,\mathbf{u}_h;\, q_h,\mathbf{v}_h) = (\mathcal{T}_h f,\ \nabla\cdot\mathbf{v}_h + \mathcal{X}q_h) \qquad \forall\,(q_h,\mathbf{v}_h) \in \mathcal{Q}_h\times\mathbf{V}_h. \qquad (3.21)$$

REMARK 7. *From Lemma 3.2 and the domain decomposition framework [7, 16, 17, 19, 20], we can easily show that both the multigrid method and the domain decomposition method have optimal convergence rates for problem (3.21) without requiring the coarsest mesh to be fine enough.*

THEOREM 3.3. *Assume that the triangulation Ω^h is only shape regular and that approximation properties (2.1) and (2.2) are satisfied. Then*

$$||p - p_h||^2_{1,\Omega} + ||u - u_h||^2_{0,\Omega} \le Ch^2||p||^2_{2,\Omega}. \qquad (3.22)$$

Proof. We have

$$\hat{a}_h((p-p_h),(\mathbf{u}-\mathbf{u}_h);\, q_h,\mathbf{v}_h) = 0 \qquad \forall (q_h,\mathbf{v}_h) \in \mathcal{Q}_h\times\mathbf{V}_h.$$

Then Lemma 3.2 implies that

$$||p-p_h||^2_{1,\Omega} + ||\mathbf{u}-\mathbf{u}_h||^2_{0,\Omega} \le C \inf_{(q_h,\mathbf{v}_h)\in\mathcal{Q}_h\times\mathbf{V}_h} (h^2|\mathbf{u}-\mathbf{v}_h|^2_{1,\Omega} + ||\mathbf{u}-\mathbf{v}_h||^2_{0,\Omega} + ||p-q_h||^2_{1,\Omega}).$$

Applying approximation properties (2.1) and (2.2) gives

$$\|p - p_h\|_{1,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 \le Ch^2(\|\mathbf{u}\|_{1,\Omega}^2 + \|p\|_{2,\Omega}^2) \le Ch^2\|p\|_{2,\Omega}^2.$$

□

In the remainder of this section, we discuss the condition number of the linear system arising from (3.5), (3.10) or (3.21) when the mesh Ω^h is quasi-uniform. Assume that the inverse inequality is satisfied for the functions in the finite element spaces:

$$|q_h|_{1,\Omega} \le Ch^{-1}\|q_h\|_{0,\Omega} \qquad \forall\, q_h \in \mathcal{Q}_h,$$

and

$$|\mathbf{v}_h|_{1,\Omega} \le Ch^{-1}\|\mathbf{v}_h\|_{0,\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h.$$

Let $\{\phi_i\}$ and $\{\psi_j\}$ be bases for the finite element spaces $\mathcal{Q}_h$ and $\mathbf{V}_h$, respectively. Then for any $q_h = \sum_i \eta_i\phi_i \in \mathcal{Q}_h$ and $\mathbf{v}_h = \sum_j \zeta_j\psi_j \in \mathbf{V}_h$, we have

$$c_0h^d|\eta|_{l_2} \le \|q_h\|_{0,\Omega} \le c_1h^d|\eta|_{l_2}, \qquad\qquad c_0h^d|\zeta|_{l_2} \le \|\mathbf{v}_h\|_{0,\Omega} \le c_1h^d|\zeta|_{l_2},$$

where $|\eta|_{l_2}$ and $|\zeta|_{l_2}$ are the discrete l_2-norms of η and ζ. The condition numbers of Choice 1 and Choice 2 have been estimated in [12, 13]. Here we give the condition number estimates for all three choices.

THEOREM 3.4. *If the triangulation Ω^h is quasi-uniform, the condition number of the linear system (2.8) resulting from (3.5), (3.10) or (3.21) is* $0(h^{-2})$.

Proof. From the coercivity and continuity of the bilinear forms, we have, by using the inverse inequality,

$$c(\|\mathbf{v}_h\|_{0,\Omega}^2 + \|q_h\|_{0,\Omega}^2) \le a(q_h, \mathbf{v}_h;\, q_h, \mathbf{v}_h) \le Ch^{-2}(\|\mathbf{v}_h\|_{0,\Omega}^2 + \|q_h\|_{0,\Omega}^2),$$

where the bilinear form a represents any one of the three bilinear forms. Since the l_2-norm $|\cdot|$ is equivalent to $\|\cdot\|_{0,\Omega}$ on the finite element space, it follows that

$$ch^d(|\eta|^2 + |\zeta|^2) \le a(q_h, \mathbf{v}_h;\, q_h, \mathbf{v}_h) \le Ch^{d-2}(|\eta|^2 + |\zeta|^2).$$

These inequalities imply that the condition number of the associated matrix A_h is $0(h^{-2})$. □

4. Numerical Experiments. In this section, we describe numerical experiments involving the least squares finite element methods and the Galerkin finite element method for a model convection-diffusion equation, and compare their results. We study the condition number of the stiffness matrix and examine the convergence rates of various preconditioned methods for this linear system. We also discuss the accuracy of the least squares methods and investigate the use of the least squares functional with various weights. To overcome difficulties caused by a boundary layer, we use a nonuniform mesh Ω^h when the position of the boundary layer is known, or an adaptive technique when this location is not known a priori. We consider the problem in one dimension, which is sufficient for our purpose of testing the different treatments of the convection term. We are mainly interested in the limit as $\epsilon \to 0$ as well as $h \to 0$.

Consider the model problem

$$\begin{cases} -\epsilon p'' + p' = f & \text{in } \Omega = [0,1], \\ p(0) = 0, \quad p(1) = 0. \end{cases} \tag{4.1}$$

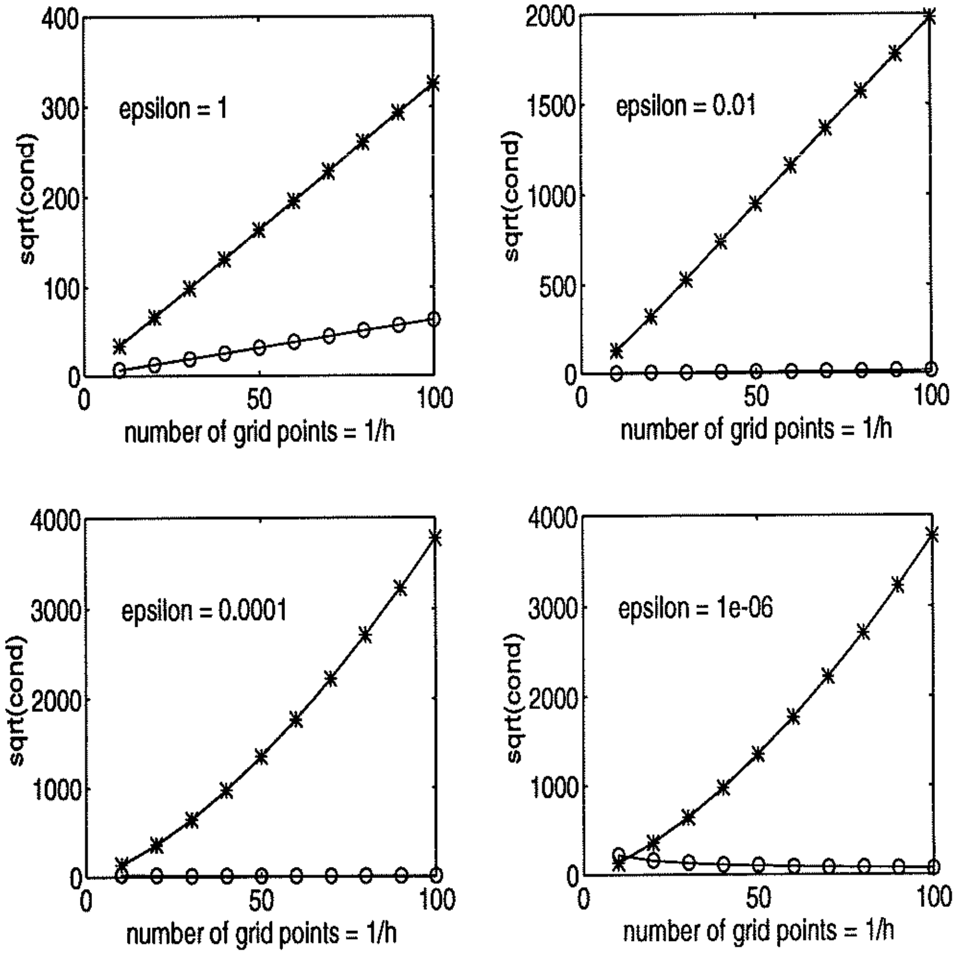

FIG. 4.1. *Comparison of Condition Numbers*
$*$ refers to A_h and o refers to G_h.

Each of the least squares finite element methods described above leads to a linear system (2.8), that is, $A_h U_h = b_h$. Let $G_h p_h = b_h$ be the linear system arising from applying the standard Galerkin method directly to (4.1). Since the least squares method (3.1) is equivalent to the least squares method (3.8) when $\Omega \subset R$, we only consider the least squares methods (3.1) and (3.12) in our numerical tests.

4.1. Condition numbers. It has been shown that the condition numbers $\kappa(A_h)$ and $\kappa(G_h)$ are $0(h^{-2})$ when the mesh Ω^h is quasi-uniform [12, 23]. Therefore, the square root of each condition number should be a linear function of $n \equiv 1/h$. In Figure 4.1, we plot the square root of the condition numbers as functions of $1/h$ for various values of ϵ. In Figure 4.2, we show how the condition numbers depend on the Reynolds numbers $1/\epsilon$ when the grid size h is fixed.

OBSERVATION 1. *Figure 4.1 shows that both slopes are linear, but the slope of* $\sqrt{\kappa(A_h)}$ *is much steeper than the slope of* $\sqrt{\kappa(G_h)}$, *and that* $\kappa(A_h)$ *is much larger than* $\kappa(G_h)$. *Furthermore,* $\kappa(A_h)$ *is increasing as* ϵ *tends to zero.*

OBSERVATION 2. *In Figure 4.1, even as* ϵ *goes to 0,* $\kappa(A_h)$ *behaves linearly in* h, *which shows that* A_h *behaves like an elliptic operator in the limit. This shows that the*

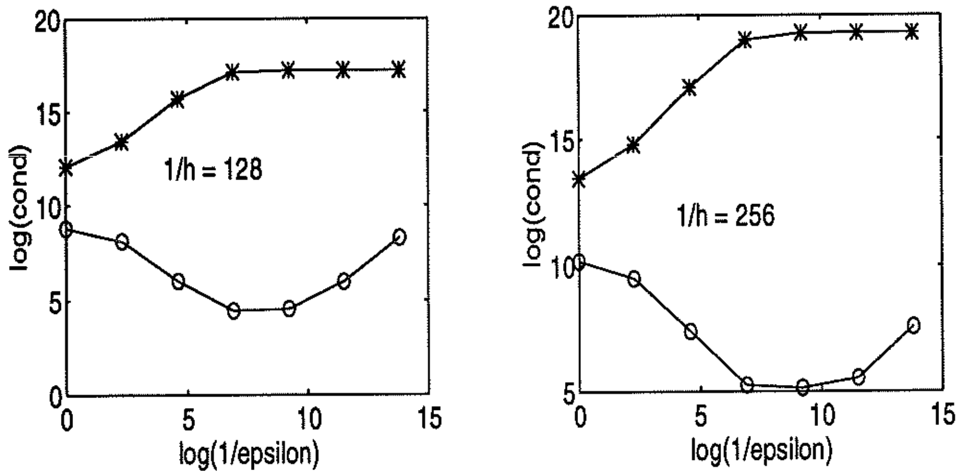

FIG. 4.2. *Comparison of Condition Numbers*
$*$ refers to A_h and o refers to G_h.

least squares formulation adds a bit of dissipation to A_h even when ϵ is small. This can be viewed as a regularization, or some sort of artificial dissipation if you will.

OBSERVATION 3. *From Figure 4.2, when the grid size is fixed and ϵ tends to zero, the effect of the diffusion term on the condition number diminishes while the effect of the convection term increases. Hence, the condition numbers have asymptotic upper bounds as $\epsilon \to 0$ and h is fixed.*

4.2. Convergence rates of multigrid and domain decomposition methods. It is well known that a large condition number of a linear system significantly slows down the convergence of most iterative methods for solving this linear system. Therefore, it is crucial to find a good preconditioner so that the preconditioned iterative method has an optimal convergence rate for solving the least squares finite element systems. In this subsection, we directly apply multigrid and domain decomposition methods to the least squares linear system (2.8) arising from the discrete problem (3.5). We use the same domain decomposition and multigrid methods as preconditioners in the preconditioned conjugate gradient method for the discrete problem (3.21). It is easy to prove that multigrid methods and domain decomposition methods have optimal convergence rates for the linear system (2.8) from the coercivity and continuity of the associated bilinear form, and domain decomposition framework [7, 16, 17, 19, 20]. We also note that these optimal convergence rates do not require the coarse grids to be fine enough unlike the results in [10, 11, 27].

We examine how the convergence of preconditioned iterative methods depends on small diffusion coefficients and mesh parameters. Let U_h be a randomly generated solution of (2.8) and set $b_h = A_h U_h$. In the following tables, we present the numbers of iterations required to reduce the L^2-norm of the residual $r^{(n)} = b_h - A_h U_h^{(n)}$ by a factor of 10^{-8}. We list these iteration numbers for discrete problems with various diffusion coefficients, coarse mesh size H and fine mesh size h.

In Table 4.1, we directly apply the multigrid algorithm and the two grid algorithm to the least squares finite element (LSFE) problem (3.5) with block Gauss-Seidel iteration as smoother. In multigrid methods, we use linear interpolation and its transform as interpolation and restriction operators, respectively. We assume that the coarsest

TABLE 4.1
Iteration Numbers of Multigrid Method for Problem (3.5)

Methods	Two Grid Method				Multigrid Method			
mesh	ϵ				ϵ			
h^{-1}	1	10^{-2}	10^{-4}	10^{-6}	1	10^{-2}	10^{-4}	10^{-6}
128	4	4	5	4	10	11	10	10
256	4	4	5	4	10	10	10	10
512	4	4	5	4	10	10	10	10
1024	4	4	4	4	10	10	10	10
2048	4	4	4	4	10	10	10	10
4096	4	4	4	4	10	10	10	10

TABLE 4.2
Iteration Numbers of PCG with Multigrid Preconditioner

	LSFE (3.5)				LSFE (3.21)			
mesh	ϵ				ϵ			
h^{-1}	1	10^{-2}	10^{-4}	10^{-6}	1	10^{-2}	10^{-4}	10^{-6}
64	6	9	10	10	6	9	10	10
128	6	8	8	8	6	8	8	8
256	6	8	9	9	6	8	9	9
512	6	8	9	9	6	8	9	9
1024	6	8	9	9	6	8	9	9
2048	6	8	6	6	6	8	6	6
4096	6	6	6	6	6	8	6	5

grid size is 1/4 in the multigrid method. In Table 4.2, we use the same multigrid method as the preconditioner in the preconditioned conjugate gradient (PCG) algorithm for solving problems (3.5) and (3.21). Table 4.3 shows the iteration numbers of preconditioned conjugate gradient (PCG) for problems (3.5) and (3.21) with the same additive Schwarz preconditioner.

OBSERVATION 4. *Both multigrid and domain decomposition methods have optimal convergence rates. The iteration numbers are not sensitive to the diffusion coefficient ϵ nor to H and h. This convergence does not require the coarse grid to be fine enough.*

4.3. Accuracy and Adaptive Method. Consider the convection-diffusion equation

$$\begin{cases} -\epsilon p'' + p' = 0 & \text{in } \Omega = [0,1], \\ p(0) = 1, & p(1) = 0, \end{cases} \tag{4.2}$$

which has the analytical solution

$$p(x) = a\left(1 - \exp(-\frac{1-x}{\epsilon})\right) \qquad a = \left(1 - \exp(-\frac{1}{\epsilon})\right)^{-1}.$$

The solution $p(x)$ has a boundary layer near $x = 1$. Furthermore, we have $||p||_{2,\Omega} = 0(\epsilon^{-3/2})$ and $||p||_{3,\Omega} = 0(\epsilon^{-2})$. We have already noticed that the constants C in the error bounds (3.6), (3.11), and (3.22) depend on the diffusion coefficient ϵ and $||p||_{3,\Omega}$

TABLE 4.3

Iteration Numbers of Domain Decomposition Method

Problem	LSFE (3.5)				LSFE (3.21)			
mesh	ϵ				ϵ			
H^{-1}/h^{-1}	1	10^{-2}	10^{-4}	10^{-6}	1	10^{-2}	10^{-4}	10^{-6}
2/64	10	11	11	11	10	11	11	11
4/64	11	13	14	13	10	13	14	14
8/64	11	14	15	14	11	14	14	15
16/64	13	15	15	16	12	14	16	16
32/64	14	15	16	16	14	16	16	16
2/128	9	9	10	10	10	11	10	10
4/128	10	13	12	13	10	13	13	12
8/128	10	14	13	14	10	12	14	13
16/128	11	13	14	14	11	13	15	15
32/128	12	14	15	15	12	14	15	15
64/128	14	15	16	17	14	15	16	16
2/256	8	10	10	10	8	10	9	10
4/256	9	13	12	12	9	13	12	12
8/256	9	14	13	13	9	13	13	13
16/256	10	13	13	13	10	13	13	14
32/256	11	13	14	14	11	13	14	14
64/256	12	12	15	15	12	13	15	15
128/256	14	14	16	16	14	14	16	16
2/512	9	17	9	9	9	17	10	10
4/512	8	12	11	11	8	14	11	11
8/512	8	15	12	12	8	14	12	12
16/512	9	14	13	12	9	14	13	13
32/512	10	13	13	13	9	13	13	14
64/512	10	12	14	13	10	12	14	14
128/512	12	12	15	14	12	12	15	15
256/512	14	14	17	17	14	14	17	17

TABLE 4.4

Errors $||p - p_h||_{L^2(\Omega)}$ of Least Squares and Galerkin Methods

ϵ	Method	n = 16	n = 32	n = 64	n = 128
1	LSFE(3.5)	3.16×10^{-5}	7.91×10^{-6}	1.97×10^{-6}	4.91×10^{-7}
	LSFE(3.21)	2.88×10^{-5}	7.23×10^{-6}	1.81×10^{-6}	4.53×10^{-7}
	Galerkin	2.84×10^{-5}	7.10×10^{-6}	1.78×10^{-6}	4.44×10^{-7}
10^{-1}	LSFE(3.5)	6.11×10^{-3}	1.56×10^{-3}	3.92×10^{-4}	9.81×10^{-5}
	LSFE(3.21)	6.07×10^{-3}	1.55×10^{-3}	3.89×10^{-4}	9.74×10^{-5}
	Galerkin	5.30×10^{-3}	1.30×10^{-3}	3.22×10^{-4}	8.04×10^{-5}
10^{-2}	LSFE(3.5)	2.49×10^{-1}	1.26×10^{-1}	4.35×10^{-2}	1.21×10^{-2}
	LSFE(3.21)	2.49×10^{-1}	1.26×10^{-1}	4.35×10^{-2}	1.20×10^{-2}
	Galerkin	1.51×10^{-1}	4.73×10^{-2}	1.15×10^{-2}	2.66×10^{-3}

TABLE 4.5

Comparison of Errors $||p - p_h||_{L^2(\Omega)}$

$\epsilon = h$	Galerkin	Least Squares Method (4.3)				
h=1/16	0.011143	0.01625	0.01630	0.01705	0.03035	0.17832
h=1/32	0.007880	0.01801	0.01803	0.01849	0.03426	0.28409
h=1/64	0.005572	0.01883	0.01884	0.01909	0.03625	0.38543
h=1/128	0.003940	0.01923	0.01923	0.01936	0.03725	0.46336
h=1/256	0.002786	0.01943	0.01942	0.01949	0.03775	0.51381
h=1/512	0.001970	0.01945	0.01952	0.01955	0.03800	0.54294
weight factor (ω_1, ω_2)		$(\epsilon^{-2}, \epsilon)$	$(\epsilon^{-1}, \epsilon)$	$(1, \epsilon)$	$(1, 1)$	$(1, \epsilon^{-1})$

or $||p||_{2,\Omega}$. It is well known that the direct application of the Galerkin method to problem (4.2) has the error bound [23]

$$||p - p_h||_{1,\Omega}^2 \leq c\frac{h^2}{\epsilon}(\epsilon + h)|p|_{2,\Omega}^2 \approx 0(\frac{h^2}{\epsilon^3} + \frac{h^3}{\epsilon^4}),$$

where c is a positive constant independent of both h and ϵ.

In Table 4.4, the L_2-norm errors $||p - p_h||_{0,\Omega}$ are presented for both least squares methods (3.5) and (3.21) as well as the Galerkin method. The results show that the Galerkin method has better accuracy than the least squares finite element methods.

Now we introduce a weight factor pair (ω_1, ω_2) in the least squares method (3.17), to obtain a weighted least squares method

$$(4.3) \qquad \left\{ \begin{array}{rcl} J(p, \mathbf{u}) &=& \inf_{(q,\mathbf{v}) \in \mathcal{Q} \times \mathbf{V}} J(q, \mathbf{v}), \\ J(q, \mathbf{v}) &=& \omega_1(\mathcal{T}_h(\nabla \cdot \mathbf{v} + \mathcal{X} q - f),\ \nabla \cdot \mathbf{v} + \mathcal{X} q - f) \\ && + \omega_2 ||\mathcal{A}^{-1/2}(\mathbf{v} + \mathcal{A} \nabla q)||_{0,\Omega}^2. \end{array} \right.$$

Table 4.5 shows how the weight factor pairs (ω_1, ω_2) affect the accuracy of the least squares method when $h = \epsilon$. We can also introduce a weight factor pairs in the least squares finite element methods (3.1) and (3.8). Since the numerical results are similar, we only present the results related to (4.3) in Table 4.5.

OBSERVATION 5. *The numerical results in Table 4.4 and 4.5 show that the standard Galerkin method has better accuracy than the least squares finite element methods (3.5)*

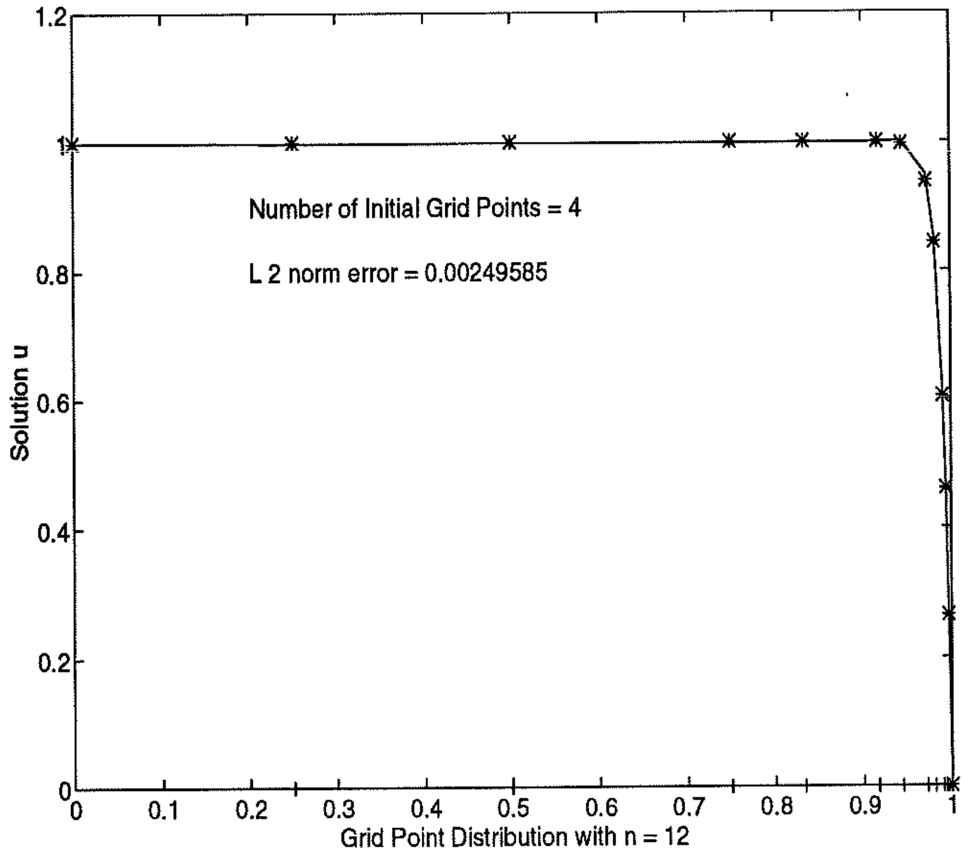

FIG. 4.3. *Adaptive Least Squares Method when* $\epsilon = 10^{-2}$

and (3.21). The error $||p - p_h||_{0,\Omega}$ *of the least squares methods increases as* $h = \epsilon$ *decreases. However, in the same situation, the Galerkin method has an error* $||p-p_h||_{0,\Omega}$ *which decreases slowly.*

OBSERVATION 6. *The numerical results in Table 4.5 show that the use of the weight factor pairs* $(\epsilon^{-2}, \epsilon)$, $(\epsilon^{-1}, \epsilon)$ *and* $(1, \epsilon)$ *improves the accuracy of the least squares finite element methods. The errors differ slightly for the least squares finite element methods with these three weight factor pairs. However, according to our numerical tests, the linear system associated with* $(1, \epsilon)$ *has better condition number than those associated with the other two weight factor pairs.*

To improve the accuracy, we can use higher order piecewise-polynomials or a non-uniform mesh in the finite element methods. Here we only consider the latter. In this case, by refining only a small region we do not increase the total number of unknowns significantly. This kind of refinement significantly improves the precision. When we know the boundary layer region, we usually impose a fine enough mesh on this small region. When the location of the boundary layer region is not clear, we use an adaptive technique to determine the approximate boundary layer region and refine the mesh in that region automatically. In our numerical experiments, we refine the mesh (by dividing τ into two elements of equal size) in the region τ where $|u|_{1,\tau} > H$, H being the initial grid size. Figures 4.3 and 4.4 present the exact solution and the adaptive least squares finite element solution for various large Reynolds numbers $1/\epsilon$.

OBSERVATION 7. *Figures 4.3 and 4.4 show that the adaptive least squares finite*

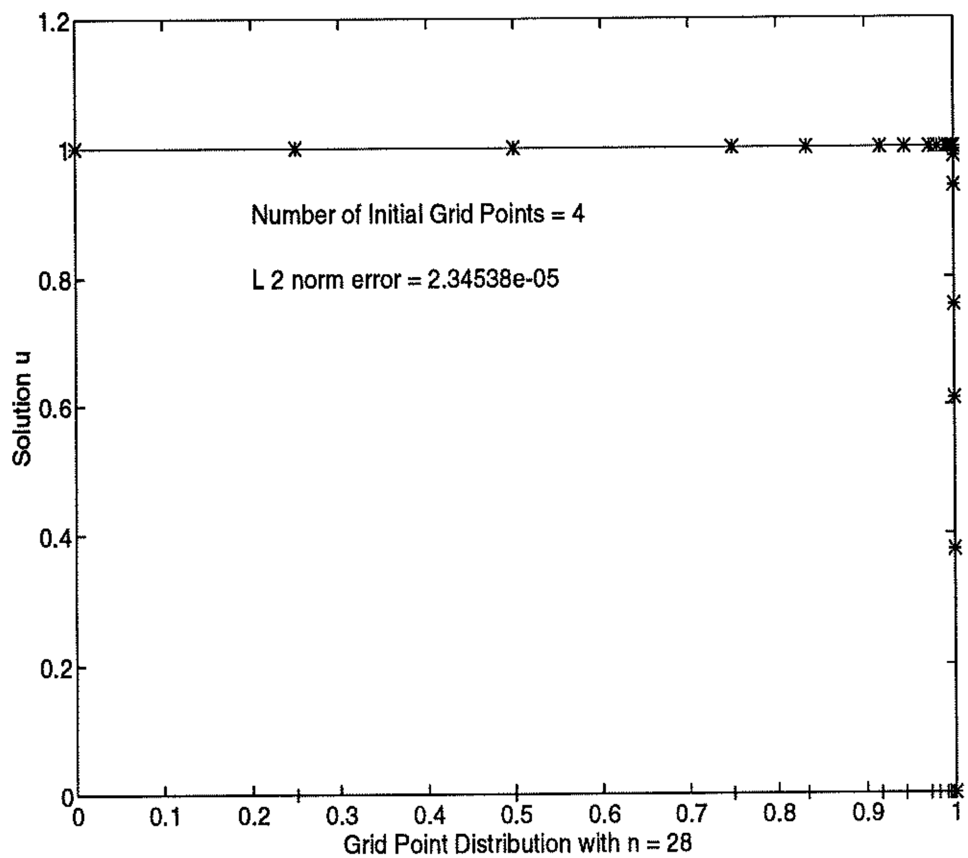

FIG. 4.4. *Adaptive Least Squares Method when* $\epsilon = 10^{-6}$

element methods are very efficient for the convection-diffusion equations with large Reynolds number $1/\epsilon$. Only 12 grid points are needed to obtain an error of 2.4×10^{-3} *when* $\epsilon = 10^{-2}$; *only 28 grid points yield an error of* 2.345×10^{-5} *even when* $\epsilon = 10^{-6}$.

5. Concluding Remarks. In this paper, we have extended the least squares finite element method based on the minus one norm to unstructured meshes. We have given a general description of least squares finite element methods for a first-order differential system resulting from a second-order partial differential equation, and have successfully applied multigrid and domain decomposition methods to solve the associated linear systems. We have conducted numerical experiments to examine the condition number, and convergence behavior of multigrid methods and domain decomposition methods. Our numerical results show that the convergence rate of multigrid or domain decomposition methods is insensitive to a small diffusion coefficient ϵ and mesh parameters. We have also examined the error of the weighted least squares methods with various weight pairs for boundary layer problems. Numerical results indicate that the standard Galerkin method has a smaller error than the errors of the least squares finite element methods. However, it is shown that the adaptive technique significantly improves the precision and efficiency of the least squares finite element methods for convection-diffusion equations with very small diffusion coefficients ϵ.

REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math., 17 (1964), pp. 35–92.

[2] A. K. Aziz, R. B. Kellogg, and A. B. Stephens, *Least squares methods for elliptic systems*, Math. Comp., 44 (1985), pp. 53–70.

[3] P. Bochev and M. D. Gunzburger, *Analysis of least-squares methods for the Stokes equations*, Math. Comp., to appear.

[4] ———, *Accuracy of least-squares methods for the Navier-Stokes equations*, Comput. Fluids, 22 (1993), pp. 549–563.

[5] J. H. Bramble, R. D. Lazarov, and J. P. Pasciak, *A least-squares approach based on a discrete minus one inner product for first order systems*, preprint.

[6] J. H. Bramble and J. A. Nitsche, *A generalized Ritz-least-squares method for Dirichlet problems*, SIAM J. Numer. Anal., 10 (1973), pp. 81–93.

[7] J. H. Bramble, J. E. Pasciak, J. Wang, and J. Xu, *Convergence estimates for product iterative methods with applications to domain decomposition*, Math. Comp., 57 (1991), pp. 1–21.

[8] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[9] X.-C. Cai, *A non-nested coarse space for Schwarz type domain decomposition methods*, SIAM J. Sci. Statist. Comput., 16(1) (1995, to appear).

[10] X.-C. Cai and O. B. Widlund, *Domain decomposition algorithms for indefinite elliptic problems*, SIAM J. Sci. Statist. Comput., 13 (1992), pp. 243–258.

[11] ———, *Multiplicative Schwarz algorithm for some nonsymmetric and indefinite elliptic problems*, SIAM J. Numer. Anal., 30 (1993), pp. 936–952.

[12] Z. Cai, R. D. Lazarov, T. A. Manteuffel, and S. F. McCormick, *First-order system least squares for second-order partial differential equations: Part I*, SIAM J. Numer. Anal., to appear.

[13] Z. Cai, T. A. Manteuffel, and S. F. McCormick, *First-order system least squares for second-order partial differential equations: Part II, to appear.*

[14] G. F. Carey, A. I. Pehlivanov, and P. S. Vassilevski, *Least-squares mixed finite element methods for non-selfadjoint elliptic problems, II: Performance of block-ILU factorization methods*, Tech. Rep. CAM 93-28, Department of Mathematics, UCLA, July 1993.

[15] G. F. CAREY AND Y. SHEN, *Convergence studies of least-squares finite elements for first-order systems*, Comm. Appl. Numer. Meth., 5 (1989), pp. 427–434.

[16] T. F. CHAN, B. SMITH, AND J. ZOU, *Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids*, Tech. Rep. CAM 94-8, Department of Mathematics, UCLA, 1994.

[17] T. F. CHAN AND J. ZOU, *Additive Schwarz domain decomposition for elliptic problems on unstructured meshes*, Tech. Rep. CAM 93-40, Department of Mathematics, UCLA, Dec. 1993, to appear in Numer. Alg.

[18] P. CLÈMENT, *Approximation by finite element functions using local regularization*, R.A.I.R.O. Numer. Anal., R-2 (1975), pp. 77–84.

[19] M. DRYJA AND O. B. WIDLUND, *Towards a unified theory of domain decomposition algorithms for elliptic problems*, Tech. Rep. 486, also Ultracomputer Note 167, Department of Computer Science, Courant Institute, 1989.

[20] ———, *Domain decomposition algorithms with small overlap*, Tech. Rep. TR-606, Courant Institute, New York Univ., 1992. SIAM J. Sci. Comp., to appear.

[21] B. N. JIANG AND C. L. CHANG, *Least-squares finite elements for the Stokes problem*, Comp. Methods Appl. Mech. Eng., 78 (1990), pp. 297–311.

[22] ———, *Optimal least-squares finite element method for elliptic problems*, Comp. Methods Appl. Mech. Eng., 102 (1993), pp. 199–212.

[23] C. JOHNSON, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, Cambridge, 1990.

[24] A. I. PEHLIVANOV, G. F. CAREY, AND R. D. LAZAROV, *Least-squares mixed finite element methods for second-order elliptic problems*, SIAM J. Numer. Anal., 31(5) (1994), pp. 1368–1377.

[25] A. I. PEHLIVANOV, G. F. CAREY, AND P. S. VASSILEVSKI, *Least-squares mixed finite element methods for non-selfadjoint elliptic problems, I: Error estimates*, preprint, 1993.

[26] P. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for 2-nd order elliptic problems*, in Mathematical Aspects of Finite Element Methods, A. Dold and B. Eckmann, eds., Springer, New York, 1975. Lecture Notes in Mathematics, Volume 606.

[27] J. P. SHAO, *Some domain decomposition methods for indefinite elliptic problems on unstructured meshes*, preprint, 1994.

[28] L. Q. TANG AND T. T. H. TSANG, *A least-squares finite element method for doubly-diffusive convection*, Int. J. Comput. Fluid Dyn., 3 (1994), pp. 1–17.

[29] M. F. WHEELER AND R. GONZALEZ, *Mixed finite element methods for petroleum reservoir engineering problems*, in Computing Methods in Applied Sciences and Engineering, VI, R. Glowinski and J. L. Lions, eds., North-Holland, New York, 1984, pp. 639–658.