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Abstract

The performance of the recently introduced block-size reduction block-ILU factorization preconditioning is studied in the case of block-tridiagonal finite element matrices arising from the discretization of the 2D Navier equations of elasticity. Conforming triangle finite elements are used for discretization of the differential problem. For the model problem, an estimate of the relative condition number is derived. The efficiency of this incomplete factorization is based on the Sherman–Morrison–Woodbury formula, and of particular importance, this factorization exists for symmetric and positive definite block-tridiagonal matrices that are not necessarily M–matrices. The convergence rate of the preconditioner is controlled by the block–size reduction parameter. The presented numerical tests illustrate a strategy for coarse grid size selection which leads to efficient iteration algorithms for large scale problems, even near the incompressible limit.

1 Introduction

Consider a block-tridiagonal symmetric positive definite matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & & & 0 \\ A_{21} & A_{22} & A_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ 0 & & & A_{n,n-1} & A_{nn} \end{pmatrix},$$

where the blocks A_{ij} are assumed sparse of order $n_i \times n_j$. Several block-ILU techniques for such matrices have already been proposed in the literature (see e.g., Axelsson [1], Axelsson, Brinkkemper and Il'in [2], Axelsson, Polman [5], Concus, Golub and

Meurant [11], Meurant [15]). In general, these methods work very well for block-tridiagonal matrices arising in discretization of 2D, second order elliptic problems. It is also known, that the block-ILU methods are robust and allow for good vectorization of the algorithms [3], [15]. However, the above methods require the matrices to satisfy the *maximum principle* (to be M-matrices), which does not hold for the FEM discretization of elliptic problems in many cases. Such a requirement does not hold in general for matrices obtained after discretization of coupled elliptic systems and/or for higher degree piecewise polynomials for the finite element spaces.

We study in this paper the performance of the *block-size reduction block-ILU factorization* (recently introduced by Chan and Vassilevski [9]), applied to preconditioning linear algebraic systems of equations arising from the 2D elasticity finite element discretizations. The block-size reduction block-ILU factorization is well defined in a very general case, and in particular, it is well defined for positive definite block-tridiagonal matrices, which need not necessarily be M-matrices. We consider the coupled system of the Navier equations of elasticity, as a test problem, to demonstrate the more general abilities of this new block-ILU factorization technique. And one of the goals that we achieve here is to demonstrate that the method indeed works as fine as for the scalar elliptic case. And for the scalar case (see Chan and Vassilevski [9]) it has been demonstrated that the method is competitive with other types of block–factorization methods.

There are a lot of works dealing with preconditioning iterative solution methods for the Navier equations of elasticity. Here we will briefly comment on some of the used approaches. In an earlier paper, Axelsson and Gustafsson [4] have implemented modified point–ILU factorization for this problem. As the coupled system does not lead to an M-matrix, they construct their preconditioners based on the point–ILU factorization of the displacement decoupled block-diagonal part of the original matrix. This approach is based on Korn's inequality, and the convergence deteriorates when the Poisson ratio tends to $\frac{1}{2}$. The displacement decomposition remains up to now one of the most robust approaches (see also, e.g., [12], [7]). In contrast to these preconditioning methods, we stress that in the present paper we consider a block-ILU factorization based on the original matrix of the coupled elasticity system (and not just on its block–diagonal part). This we view as an advantage since this results in a more robust method in general.

There are also some recent papers, where the multigrid (see, e.g., [13], [16]) method is implemented for the elasticity problem. In particular, Brenner [8] has proved an optimal order convergence rate of the full multigrid method applied to the nonconforming mixed discretization of the problem. It is proved also in [14] an uniform (on the Poisson ratio) upper bound of the constant in the strengthened C.B.S. inequality for FEM 2D elasticity equations. It follows from this estimate, that the algebraic multilevel method as introduced by Axelsson and Vassilevski [6] has an optimal convergence rate. However the optimality of both these methods, their computational cost is significantly high for the almost incompressible case (when the Poisson ratio is near to the case $\nu = \frac{1}{2}$). The preconditioner studied in the present paper is not optimal. It has a convergence rate depending upon the Poisson ratio. However, by a proper choice of the coarse grid parameter we get an efficient algorithm not only for moderate values of the Poisson ratio, but even for values near the almost incompressible case.

The remainder of the paper is organized as follows. Some background facts about the Navier equations of elasticity and their FEM approximation are presented in §2. The block-size reduction block-ILU algorithm is described in §3. In §4 we give a model analysis of the relative condition number of the studied preconditioner. A set of numerical tests illustrating the performance of the resulting preconditioned conjugate gradient algorithm are presented in the last section §5.

2 FEM approximation of the Navier equations of elasticity

We consider in this paper the plain strain problem of elasticity (the plane stress problem is a subclass from the mathematical point of view) in the weak formulation of the Navier system of equations. The unknown displacements $\underline{w}^t = (u, v)$ satisfy the following variational equations:

Find $(u, v) \in H_1^0 \times H_1^0$, such that

$$\begin{aligned} a(u,\tilde{u}) + e_{12}(v,\tilde{u}) &= f_1, \\ e_{21}(u,\tilde{v}) + b(v,\tilde{v}) &= f_2, \end{aligned} \qquad \forall (\tilde{u},\tilde{v}) \in H_1^0 \times H_1^0, \tag{1}$$

where $H_1^0 = \{w \in H_1(\Omega) : w|_{\Gamma_D} = 0\}$, and the related bilinear forms are defined by the formulas:

$$a(\phi,\psi) = \int_\Omega \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{(1-\tilde{\nu})}{2}\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} \right) d\Omega,$$

$$b(\phi,\psi) = \int_\Omega \left(\frac{(1-\tilde{\nu})}{2}\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} \right) d\Omega,$$

$$e_{12}(\psi,\phi) = e_{21}(\phi,\psi) = \int_\Omega \left(\tilde{\nu}\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} + \frac{(1-\tilde{\nu})}{2}\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x} \right) d\Omega.$$

Here $\tilde{\nu} = \nu/(1-\nu) \in (0,1)$ stands for the modified Poisson ratio ($\nu \in (0, 0.5)$ is the Poisson ratio). The notion *almost incompressible* is used for the case $\tilde{\nu} = 1-\delta$, where δ is a small positive number. Note that if $\tilde{\nu} = 1$ (the material is incompressible), the problem (1) is ill-posed.

Now, let ω be a square mesh, and let Ω be a polygonal domain, triangulated by right isosceles triangles $T \in \tau$ obtained by a diagonal bisection of the square cells of ω.

Let $W = W_1^0 \times W_1^0$, where $W_1^0 \in H_1^0$ is the finite element space of conforming piecewise linear functions with nodal Lagrangian basis $\{\phi_i\}_{i=1}^N$ corresponding to the triangulation τ. Then the finite element approximation (u^h, v^h) of the problem (1) is determined as follows:

Find $u^h = \sum_{i=1}^N u_i\phi_i$, $v^h = \sum_{i=1}^N v_i\phi_i$, such that

$$\begin{aligned} a(u^h,\phi_i) + e_{12}(v^h,\phi_i) &= f_{1,i}, \\ e_{21}(u^h,\phi_i) + b(v^h,\phi_i) &= f_{2,i}. \end{aligned} \qquad \forall i = 1,\dots,N\cdot \tag{2}$$

Equations (2) are equivalent to the linear system

$$A\underline{w}_h = \underline{b},$$

where A is the stiffness matrix and $\underline{w}_h$ is the vector of the nodal unknowns $\{u_i\}_{i=1}^N$ and $\{v_i\}_{i=1}^N$. Following the usual FEM procedure we assemble the global stiffness matrix A by the element stiffness matrix A_T,

$$A_T = \begin{pmatrix} 3-\tilde{\nu} & -2 & -1+\tilde{\nu} & 1+\tilde{\nu} & -1+\tilde{\nu} & -2\tilde{\nu} \\ -2 & 2 & 0 & -2\tilde{\nu} & 0 & 2\tilde{\nu} \\ -1+\tilde{\nu} & 0 & 1-\tilde{\nu} & -1+\tilde{\nu} & 1-\tilde{\nu} & 0 \\ 1+\tilde{\nu} & -2\tilde{\nu} & -1+\tilde{\nu} & 3-\tilde{\nu} & -1+\tilde{\nu} & -2 \\ -1+\tilde{\nu} & 0 & 1-\tilde{\nu} & -1+\tilde{\nu} & 1-\tilde{\nu} & 0 \\ -2\tilde{\nu} & 2\tilde{\nu} & 0 & -2 & 0 & 2 \end{pmatrix}. \tag{3}$$

This element stiffness matrix corresponds to the triangle T with vertices P_1, P_2 and P_3 with respective coordinates $(0,0), (1,0)$ and $(0,1)$. The ordering of the unknowns is as follows, $[u(P_1), u(P_2), u(P_3), v(P_1), v(P_2), v(P_3)]$.

Finally, we consider the structure of the stiffness matrix A in the case of the model problem in $\Omega = (0,1)\times(0,1)$ with homogeneous Dirichlet boundary conditions. If a column by column (row by row) ordering of the nodes is used, then the matrix A admits a block–tridiagonal structure, explicitly presented by the formulas:

$$A \;=\; tridiag(A_{i,i-1}, A_{i,i}, A_{i,i+1}),$$

where

$$A_{i,i} = \begin{pmatrix} D & E & & & 0 \\ E & D & E & & \\ & \ddots & \ddots & \ddots & \\ & & E & D & E \\ 0 & & & E & D \end{pmatrix},$$

$$A_{i,i-1} = \begin{pmatrix} F & G & & & 0 \\ & F & G & & \\ & & \ddots & \ddots & \\ & & & F & G \\ 0 & & & & F \end{pmatrix}, \qquad A_{i,i+1} = \begin{pmatrix} F & & & & 0 \\ G & F & & & \\ & \ddots & \ddots & & \\ & & G & F & \\ 0 & & & G & F \end{pmatrix},$$

and where D, E, F and G are 2×2 scalar matrices depending on the problem parameter $\tilde{\nu}$, namely

$$D = \begin{pmatrix} 12 - 4\tilde{\nu} & 2 + 2\tilde{\nu} \\ 2 + 2\tilde{\nu} & 12 - 4\tilde{\nu} \end{pmatrix}, \qquad E = \begin{pmatrix} -2 + 2\tilde{\nu} & -1 - \tilde{\nu} \\ -1 - \tilde{\nu} & -4 \end{pmatrix},$$

$$F = \begin{pmatrix} -4 & -1 - \tilde{\nu} \\ -1 - \tilde{\nu} & -2 + 2\tilde{\nu} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 + \tilde{\nu} \\ 1 + \tilde{\nu} & 0 \end{pmatrix}.$$

3 Block-ILU method for positive definite block–tridiagonal matrices

The weak statement (2) corresponding to the finite element formulation leads to a symmetric positive definite block system and therefore is amenable to iterative solution by block schemes. In this section we present in some detail the block–ILU method proposed in Chan and Vassilevski [9], with the specific parameters that we choose for the present problem.

The method is defined for any given block–tridiagonal matrix A with positive definite symmetric part. Note that this is a much larger class than the class of M–matrices for which the more classical block–ILU methods (cf. Concus, Golub and Meurant [11], Axelsson and Polman [5]) have been proven to exist.

Consider the block tridiagonal matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & & & 0 \\ A_{21} & A_{22} & A_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ 0 & & & A_{n,n-1} & A_{nn} \end{pmatrix}.$$

The block–entries of A are assumed sparse.

The block–ILU factorization matrix C is defined as follows. Let $\{R_i\}$ denote a set of restriction matrices that transform vectors of the size of the block A_{ii} to a lower dimensional vector space of a small and fixed size m. Then we perform the following approximate factorization algorithm.

Definition 1 (Block–ILU factorization)

(i) Set

$$Z_1 = A_{11} \text{ and let } \tilde{Z}_1 = R_1^T Z_1 R_1;$$

(ii) For $i = 2, \ldots, n$

$$Z_i = A_{ii} - A_{i,i-1} R_{i-1}^T \tilde{Z}_{i-1}^{-1} R_{i-1} A_{i-1,i},$$

and let

$$\tilde{Z}_i = R_i^T Z_i R_i.$$

Then the block–ILU factorization matrix is defined as

$$
\begin{aligned}
C &= \begin{pmatrix} Z_1 & & & & 0 \\ A_{21} & Z_2 & & & \\ & \ddots & \ddots & & \\ & & A_{n-1,n-2} & Z_{n-1} & \\ 0 & & & A_{n,n-1} & Z_n \end{pmatrix} \\
&\times \begin{pmatrix} I & Z_1^{-1}A_{12} & & & 0 \\ & I & Z_2^{-1}A_{23} & & \\ & & \ddots & \ddots & \\ & & & I & Z_{n-1}^{-1}A_{n-1,n} \\ 0 & & & & I \end{pmatrix}.
\end{aligned} \tag{4}
$$

This algorithm requires the exact inverses of the reduced blocks $\tilde{Z}_i$ which are of relatively small size $m \times m$, and that size we can control. In [9] it is shown that the above algorithm is well–defined for block–tridiagonal matrices with positive definite symmetric part and for any choice of full rank restriction matrices. The restriction matrices in the present paper (used in section 5) are defined for any $i = 1, 2, \ldots, m$, as

$$
R_i = \begin{pmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \vdots \\ \mathbf{q}_m^T \end{pmatrix},
$$

where $\mathbf{q}_k$, $k = 1, \ldots, m$ are the first m eigenvectors of the $n \times n$ matrix

$$
T = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}.
$$

That is,

$$
\mathbf{q}_k = \sqrt{\frac{2}{n+1}} \left(\sin\left(\frac{kj\pi}{n+1} \right) \right)_{j=1}^{n}.
$$

Note that R_i can be viewed as a projection on the space spanned by the first m smooth (low oscillating) modes.

The block–ILU factorization matrix C is used in a preconditioned conjugate gradient (PCG) method for solving systems with the original matrix A. Since at every iteration step in the PCG method we have to solve a system of the form $C\mathbf{v} = \mathbf{w}$ for a residual vector $\mathbf{w}$, it is clear that those solutions are based on the standard forward and backward recurrences using the factored form in (4). These recurrences involve solution of systems with the block matrices Z_i and matrix vector products with the sparse matrices $A_{i,i-1}$ and $A_{i-1,i}$. For the solution of the systems with blocks Z_i we use the following Sherman–Morrison–Woodbury formula:

$$Z_i^{-1} = A_{ii}^{-1} + A_{ii}^{-1}A_{i,i-1}R_{i-1}^T\left(\tilde{Z}_{i-1} - R_{i-1}A_{i-1,i}A_{ii}^{-1}A_{i,i-1}R_{i-1}^T\right)^{-1}R_{i-1}A_{i-1,i}A_{ii}^{-1}.$$

Note that the $m \times m$ matrix

$$\tilde{Z}_{i-1} - R_{i-1}A_{i-1,i}A_{ii}^{-1}A_{i,i-1}R_{i-1}^T$$

can be formed explicitly based on m actions of A_{ii}^{-1} and then factored or inverted exactly. We assume that we can efficiently solve systems involving the blocks $\{A_{ii}^{-1}\}$. Note that for 2–D domains Ω these systems are banded. In 3–D one has to approximate them, but since they are well–conditioned this should not be as difficult, cf., e.g., Chan and Vassilevski [10].

4 Model analysis of the relative condition number

In this section we present a model analysis of the block–ILU factorization method presented in Section 3 for the elasticity problem of Section 2. The analysis fits in the framework of the convergence theory presented in Chan and Vassilevski [10].

We assume that $\Omega = (0,1)^2$ the unit rectangle and that we have a square mesh of stepsize $h = \frac{1}{n+1}$ for a given integer n. Let $\Gamma_i = \{a_i\} \times (0,1)$, $a_i = ih$, $i = 0, 1, \ldots, n+1$ be the vertical grid–lines (including the boundary ones). Define also the subdomains $\Omega_i = (0, a_i) \times (0,1)$. For the analysis to follow we will need some facts concerning the following inhomogeneous Dirchlet problem:

Given a rectangular domain $D = (0,a) \times (0,b)$ and a function $g = g(y) \in H_0^1(0,b)$. Then consider

$$\begin{aligned} -\Delta u &= 0, && \text{in } D, \\ u &= g && \text{on } \Gamma = \{a\} \times (0,b), \\ u &= 0 && \text{on the rest of } \partial D. \end{aligned}$$

For a given small parameter $h \to 0$ split the boundary data $g = g^1 + g^2$, where $g^1 = \frac{2}{b}\sum\limits_{k \le h^{-1}} g_k \sin(\frac{k\pi y}{b})$ – the smooth part of g, and $g^2 = \frac{2}{b}\sum\limits_{k > h^{-1}} g_k \sin(\frac{k\pi y}{b})$ – the oscillatory part of g. Then the solution of the above inhomogeneous Dirichlet problem is split up in two corresponding components, $u = u^1 + u^2$ that are estimated as follows:

$$|u^1|_{2,D}^2 \le Ch^{-1}|g|_{1,(0,b)}^2, \tag{5}$$

where the constant C is h and domain independent. For the second component (corresponding to the oscillatory part of g) we have

$$|u^2|_{1,D}^2 \le Ch|g|_{1,(0,b)}^2, \tag{6}$$

again with constant C h–independent and domain independent. The estimates (5) and (6) are derived in Chan and Vassilevski [10]. We will use these estimates even for $a = O(h)$, i.e., for narrow domains D.

The main steps of the analysis are summarized as follows. We consider the solution of an inhomogeneous Dirichlet problem (see (7) below) for a given boundary data $\underline{g} \in \left(H_0^1(\Gamma_i)\right)^2$. Then we need two finite element approximations of this problem: a standard fine–grid approximation and a coarse–space approximation derived on the basis of a specifically constructed coarse finite element space that exploits coarse restriction of the coefficient vectors of the elements of the given fine finite element space. The specific construction of the restriction operators and of the coarse spaces is given first below. Then two error estimates (see (8) and (9) below) for the smooth and oscillatory components of the solution of (7) are proved. Finally, based on the a priori estimates (5) and (6), the difference between the fine and the coarse finite element approximations of the solution of (7) is estimated (this is estimate (10) below) from which the main desired estimate (11) is a corollary.

The final steps are the inequalities (12) and (13) which represent the lower and the upper bounds for the eigenvalues of $C^{-1}A$.

We proceed now with the already outlined steps of the analysis.

Consider the bilinear form corresponding to the variational problem (1). Using vector–function notation we denote its restriction to Ω_i by $A_{\Omega_i}(\underline{\chi}, \underline{\psi})$ for $\underline{\chi}, \underline{\psi} \in H^1(\Omega_i)$ that vanish on $\partial\Omega \cap \partial\Omega_i$. Consider then the inhomogeneous Dirichlet problem, similar to the scalar case above, for a given vector–functions $\underline{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \left(H_0^1(\Gamma_i)\right)^2$,

$$\begin{aligned} A_{\Omega_i}(\underline{u}, \underline{\psi}) &= 0 \quad \text{for all } \underline{\psi} \in (H_0^1(\Omega_i))^2, \\ \underline{u} &= \underline{g} \text{ on } \Gamma_i. \end{aligned} \tag{7}$$

Here $\underline{u} \in (H^1(\Omega_i))^2$ and vanishes on $\partial\Omega \cap \partial\Omega_i$. Consider finally two finite element approximations of the last problem. The first one corresponds to the test functions from $W_i \subset W$, i.e., to the vector–functions from W that vanish outside Ω_i and the second one corresponds to the test functions from the coarse space $\tilde{W}_i \subset W_i$. This coarse space is defined grid–line by grid–line using semi–coarsering. Given Γ_s, a vertical grid line, consider the set of nodal basis functions $\{\phi_{s,r}\}_{r=1}^n$ that correspond to the nodes $x_{s,r}$ on Γ_s. Then given a function $v \in W_1^0$ (this is a scalar function) we expand it in terms of $\phi_{s,r}$ and get a coefficient vector $\mathbf{c} = (c_{s,r})$, where $c_{s,r} = v(x_{s,r})$. For a fixed s the restriction of the coefficient vector to Γ_s is denoted by $\mathbf{c}_s$. The coarse–grid restriction $R_s\mathbf{c}_s$ is then defined then by $\tilde{\mathbf{c}}_s = \left(\mathbf{q}_k^T \mathbf{c}_s\right)_{k=1}^m$, where $\mathbf{q}_k = \sqrt{\frac{2}{n+1}}(\sin(\frac{k\pi j}{n+1}))_{j=1}^n$, $k = 1, 2, \ldots, m$ and $m + 1 = \frac{1}{H}$. Here H can be viewed as a coarse–grid size. The coarse–finite element space $\tilde{V}$ is then defined by

$$\text{span}\{\sum_{r=1}^n (R_s^T \tilde{\mathbf{c}}_s)_r \phi_{s,r}\}$$

where $\tilde{\mathbf{c}}_s$ runs over $\mathbf{R}^m$ for $s = 1, 2, \ldots, n$.

The final estimates that we are actually needing are as follows. Consider $\underline{u}_h$ and $\underline{u}_H$ the finite element solutions of the above inhomogeneous Dirichlet problem using respectively W_i and $\tilde{W}_i$ as discretization spaces, i.e., we first construct $\underline{u}^0$ that satisfies the boundary condition and then for the difference $\underline{u} - \underline{u}^0$ we get a problem with a certain right hand side but with homogeneous boundary conditions. This problem we approximate in the corresponding finite element spaces. The particular solution $\underline{u}^0$ is constructed by solving Laplace problem in (the possibly) narrow domain Ω_i near Γ_i of width $a_i = ih$. This we do for each component of $\underline{u}^0$. Then we have the following standard error estimate for both (smooth and oscillatory) components of $\underline{u} - \underline{u}_H$,

$$A(\underline{u}^1 - \underline{u}_H^1, \underline{u}^1 - \underline{u}_H^1) \leq CH^2 |\underline{u}^1|_{2,\Omega_i}^2, \tag{8}$$

and for the oscillatory component of the solutions we get

$$A(\underline{u}^2 - \underline{u}_H^2, \underline{u}^2 - \underline{u}_H^2) \leq C|\underline{u}^2|_{1,\Omega_i}^2. \tag{9}$$

Similar estimates hold for both components of the fine–grid approximation $\underline{u}_h$.

Since $|\underline{u}^1|_{2,\Omega_i}^2 \leq C|(\underline{u}^0)^1|_{2,\Omega_i}^2 \leq Ch^{-1}|\underline{g}|_{1,\Gamma_i}^2$ and also $|\underline{u}^2|_{1,\Omega_i}^2 \leq C|(\underline{u}^0)^2|_{1,\Omega_i}^2 \leq Ch|\underline{g}|_{1,\Gamma_i}^2$ we finally obtain the desired basic estimate:

$$A_{\Omega_i}(\underline{u}_h - \underline{u}_H, \underline{u}_h - \underline{u}_H) \leq C\frac{H^2}{h}|\underline{g}|_{1,\Gamma_i}^2. \tag{10}$$

Note that all constants are mesh independent and domain independent.

We note now that Z_i defined in the block–ILU algorithm is the Schur complement of precisely the coarse–finite element discretization matrix corresponding to the above problem (7). Let also S_i be the same Schur complement of the fine finite element discretization matrix corresponding to the same problem. Then the following estimate is a reformulation of the last estimate (10):

$$((Z_i - S_i)\underline{\mathbf{g}}_i, \underline{\mathbf{g}}_i) \leq C\frac{H^2}{h}|\underline{g}_i|_{1,\Gamma_i}^2. \tag{11}$$

for any given $\underline{g}_i$. Note also that S_i is a Schur complement computed with respect to a larger space (i.e., W_i) than the expression $A_{ii} - A_{i,i-1}Z_{i-1}^{-1}A_{i-1,i}$ which is a Schur complement computed from the same bilinear form as S_i (i.e., $A_{\Omega_i}(.,.)$) but in a smaller (coarser) space (namely, the space obtained from the space W_i using line restrictions on all but the last vertical grid lines in Ω_i (i.e., without Γ_i)). That is, having in mind the symmetry and positive definiteness of the bilinear form $A_{\Omega_i}(.,.)$ and the minimization property of the corresponding Schur complements, the following inequality is immediate,

$$(S_i\underline{\mathbf{v}}_i, \underline{\mathbf{v}}_i) \leq ((A_{ii} - A_{i,i-1}Z_{i-1}^{-1}A_{i-1,i})\underline{\mathbf{v}}_i, \underline{\mathbf{v}}_i)$$

for all $\underline{\mathbf{v}}_i$. Then using this inequality and estimate (11) we arrive at the first desired estimate:

$$\begin{aligned} ((C - A)\underline{\mathbf{v}}, \underline{\mathbf{v}}) &= \textstyle\sum_{i=2}^n ((Z_i - (A_{ii} - A_{i,i-1}Z_{i-1}^{-1}A_{i-1,i}))\underline{\mathbf{v}}_i, \underline{\mathbf{v}}_i) \\ &\leq \sum_{i=2}^n ((Z_i - S_i)\underline{\mathbf{v}}_i, \underline{\mathbf{v}}_i) \\ &\leq C\tfrac{H^2}{h}\textstyle\sum_{i=2}^n |\underline{v}_i|_{1,\Gamma_i}^2 \\ &\leq C\left(\tfrac{H}{h}\right)^2 |\underline{v}|_{1,\Omega}^2 \\ &\leq C\tfrac{1}{1-\bar{\nu}}\left(\tfrac{H}{h}\right)^2 A_\Omega(\underline{v}, \underline{v}). \end{aligned} \tag{12}$$

Here we have also used the inequality:

$$h\sum_{i=1}^{n} |\underline{v}_i|^2_{1,\Gamma_i} \leq C|\underline{v}|^2_{1,\Omega},$$

valid for finite element functions and also Korn's inequality (see, e.g., [4]).

Together with the estimate (see Chan and Vassilevski [9])

$$((C-A)\underline{v},\underline{v}) \geq 0, \tag{13}$$

we have proved the following main result.

Theorem 1 *The relative condition number of the block–ILU factorization preconditioner C with respect to A is bounded by $\frac{1}{1-\tilde{\nu}}(\frac{H}{h})^2$. Here $H = O(m^{-1})$ is the coarse–grid size along each vertical grid line of the original fine discretization with characteristic size h.*

5 Numerical tests

In this section, we analyze the performance rate of our preconditioned iterative method, varying the size parameters and the Poisson ratio ν. The computations are done with double precision on a Silicon Graphics SGIstation.

The test problem we consider is the system of Navier equations of elasticity in the unique square $\Omega = (0,1) \times (0,1)$, where the Poisson ratio is a problem parameter. Following the notations from the previous sections, we actually vary the modified Poisson ratio $\tilde{\nu} = \frac{\nu}{1-\nu}$, $\tilde{\nu} \in (0,1)$. The case $\tilde{\nu} = -1$ does not have a physical meaning, but it is interesting from a computational point of view, because then the Navier system of equations is decoupled (namely, it is split into two independent Laplace equations). We recall, that the almost incompressible case corresponds to $\tilde{\nu} = 1-\delta$, where δ is a small positive number.

Tables 1–4 show the number of iterations as a measure of the convergence rate of the preconditioners. The iteration stopping criterion is $\|r^{N_{it}}\|/\|r^0\| < 10^{-9}$, where r^j stands for the residual at the jth iteration step of the preconditioned conjugate gradient method. It can be seen from the numerical results that the block-size reduction block–ILU (BSR BILU) preconditioners are characterized by the estimate $\kappa(C^{-1}A) = O(\frac{H^2}{h^2(1-\tilde{\nu})})$, where h stands for the grid size, and respectively, H is the coarse grid size corresponding to the used incomplete factorization. The size of the discrete problem is respectively equal to $2N = n^2$, where $n = h^{-1}$.

In general, the presented data are in a good agreement with the theoretical estimate. They clearly illustrate the flexibility of the iterative algorithm. The proper choice of the factorization parameter $m \equiv H^{-1}$, or actually of the coarse grid size H, provides the opportunity to control effectively the convergence rate of the iterative solver.

The theoretical estimates, confirmed by the numerical tests, show that the block-size reduction block-ILU factorization can be used as a robust technique for preconditioning of FEM 2D elasticity systems for moderate values of the Poisson ratio, and by a proper choice of the coarse grid size, even for values near the incompressible case.

At the end of this section, we will briefly comment on the present results in comparison to the results from two recent papers, where optimal order multigrid/multilevel methods for elasticity problems are reported.

S. Brenner proved (cf. [8]) an optimal order convergence rate for the full multigrid algorithm, in the case of nonconforming mixed FEM approximation of the pure displacements elasticity problem. The presented numerical tests (see [8]) demonstrate that the convergence rate does not deteriorate in the almost incompressible case. However, there remains the requirement that the number of the inner iterations must be chosen large enough.

It has also been proved (cf. [14]), that the constant γ in the strengthened C.B.S. inequality is bounded uniformly on the Poisson ratio ($\gamma^2 < \frac{3}{4}$) in the case of conforming linear triangle FEM discretization of the Navier system. As a result, the algebraic multilevel iteration (AMLI) (see [6] for more details) method is of optimal convergence rate, uniformly in the Poisson ratio. However, the condition numbers of the first pivot blocks in the AMLI factorization (which are uniformly bounded on the mesh parameter) increase, when the problem tends to the incompressible case. This automatically implies the need for a special treatment of this first block which is similar to the requirement for large enough number of the inner iterations in the full multigrid algorithm.

Analyzing the presented in Tables 1–4 test data, we can make the conclusion that for some values near the incompressible case, the proper choice of the coarse grid size in the BSR BILU algorithm plays a similar role as the large enough number of inner iterations in the above mentioned multigrid/multilevel algorithms.

Table 1: Number of iterations for BSR BILU preconditioner, $\tilde{\nu} = -1.0$.

H^{-1} \ h^{-1}	128	64	32	16	8
4	60	31	16	9	5
8	37	19	10	5	
16	16	10	6		
32	11	6			
64	6				

Table 2: Number of iterations for BSR BILU preconditioner, $\tilde{\nu} = 0.5$.

H^{-1} \ h^{-1}	128	64	32	16	8
4	110	57	30	16	9
8	70	36	19	11	
16	40	21	11		
32	22	12			
64	12				

Table 3: Number of iterations for BSR BILU preconditioner, $\tilde{\nu} = 0.7$.

H^{-1} \ h^{-1}	128	64	32	16	8
4	140	73	38	20	11
8	89	47	25	13	
16	52	27	14		
32	28	15			
64	15				

Table 4: Number of iterations for BSR BILU preconditioner, $\tilde{\nu} = 0.9$.

H^{-1} \ h^{-1}	128	64	32	16	8
4	237	124	63	32	15
8	128	82	42	21	
16	90	46	24		
32	48	25			
64	26				

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