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Abstract. We consider Arrow's model for an economy engaged in consuming a randomly distributed natural resource, and in exploring previously unexplored land to find more of the resource. After modifying the model so that each discovery reveals a random amount of the resource, we use dynamic programming techniques to derive the equations governing optimal rates of exploration, consumption, and pricing of the resource. We analyze these equations asymptotically when the typical amount discovered is small compared to the total amount of the resource, and approximately when the amount is medium or large. In both cases we obtain formulas for the optimal exploration, consumption, and pricing policies. We demonstrate the accuracy of these analytical results by comparing them to numerically-determined exact solutions, and discuss economic implications of these results.

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1. Introduction. Suppose that an economy has a reserve of an exhaustible resource which it consumes gradually [1]. Hotelling [1] showed how to determine the optimal rate of consumption, *i.e.* the rate that maximizes the present value of the resource. Later authors included deterministic land exploration to discover more of the resource, and determined the optimal exploration rate. Then uncertainty was introduced into the model. For example, Deshmukh and Pliska [2] let discovery of the resource be a random process with an infinite area of unexplored land. Arrow [3] formulated the corresponding problem for a finite area of unexplored land, and Arrow and Chang [4] partially analyzed it. Here we complete the analysis, obtaining explicit results from which we can deduce the economic consequences inherent in the model.

First we modify the model slightly to allow each discovery to reveal a random amount of the resource. This introduces a parameter ε, the ratio of the expected amount revealed by each discovery to the expected total reserve of the resource. Following Arrow and Chang [4], we use dynamic programming methods in §3 to obtain a set of equations characterizing the optimal policy. After briefly considering the deterministic case ($\varepsilon = 0$) in §4, in §5 we use singular perturbation techniques to analyze these equations when ε is small, obtaining explicit asymptotic expressions for the optimal policy. We then develop an approximate solution in §6 valid for any value of ε. In §7 we compare these results to the numerically-determined exact solution for several examples. There we find that the results of the asymptotic analysis agree well with the exact solution, and that the approximate solution agrees even better. Finally, we examine the economic implications of these results, and discover that even as the amount of uncertainty goes to zero, one does not completely recover the deterministic results.

2. Formulation of the model. Consider an economy engaged in consuming some natural resource and in exploring previously unexplored land to find more of the resource. With respect to these activities, the state of this economy at any time t is characterized by the rate of consumption $c(t)$, the rate of exploration $x(t)$, the amount $R(t)$ of known reserves of the resource, and the area $A(t)$ of unexplored land. We assume that the economy enjoys a utility of consumption $U(c)$ per unit time, and that it pays a price P to explore a unit area of land. The net increase in utility per unit time is $U(c(t)) - Px(t)$, so the *payoff*, the discounted present value of the resource, is

$$\int_0^\infty e^{-\rho t}\{U[c(t)] - Px(t)\}\,dt\,. \tag{2.1}$$

Here ρ is the discount rate for utility.

The amount of unexplored land $A(t)$ and the known reserves $R(t)$ must be non-negative, so consumption and exploration rates are limited by

$$A(t) = A(0) - \int_0^t x(t')\,dt' \geq 0\,, \tag{2.2a}$$

$$R(t) = R(0) - \int_0^t c(t')\,dt' + N(t) \geq 0\,. \tag{2.2b}$$

In 2.2b, $N(t)$ is the amount of resource discovered by exploration during the time interval 0 to t. To characterize $N(t)$, we assume that the resource is randomly distributed throughout the area A in discrete deposits of average amount ε. We assume that the probability of discovering a single deposit by exploring a small area dA is $\lambda/\varepsilon\, dA + O(dA)^2$, and that the probability of discovering two or more deposits is $O(dA)^2$. Let the amount of resource in a given deposit be εs, where the random variable s has probability density $g(s)$. In particular,

$$\int_0^\infty g(s)\,ds = 1\,, \qquad \int_0^\infty sg(s)\,ds = 1\,, \tag{2.3a}$$

since the deposits' average size is ε.

From these assumptions it follows that the expected amount of resource in any unexplored region of area A is λA, and that the variance in the amount is $\varepsilon M_2 \lambda A$, where M_2 is the second moment of g:

$$M_2 \equiv \int_0^\infty s^2 g(s)\,ds\,. \tag{2.3b}$$

The payoff is given by 2.1 with c(t) and x(t) subject to 2.2. Since the random variable N(t) occurs in 2.2, the payoff is also random. Therefore we seek a policy for choosing consumption rates c(t) and exploration rates x(t) which maximize the expected value of the payoff. These choices define our optimal policy.

The maximum expected payoff depends on the initial values A(0) = A and R(0) = R, so we denote it as V(A,R):

2.4 $$V(A,R) = \max_{x(\cdot),\, c(\cdot)} E\left\{ \int_0^\infty e^{-\rho t}\{ U[c(t)] - Px(t)\}\, dt \right\} .$$

In the next section we consider the problem of finding this maximum and the rates x(t) and c(t) at which it is achieved.

We assume that the utility function satisfies $U'(c) > 0$ with $U'(0) = +\infty$, and that it is concave, $U''(c) < 0$. To illustrate our results we will use

2.5a $$U(c) = -\frac{\alpha}{nc^n} , \qquad n > -1 ,$$

as examples, which we interpret as

2.5b $$U(c) = \alpha \log c$$

for n = 0.

For clarity, it is advantageous to nondimensionalize our variables. Let us measure area in units of a typical area A_0, such as the initial area, and measure reserves in units of λA_0, the expected quantity of the resource in A_0. We define the new variables

2.6a $$A' = A/A_0 , \qquad R' = R/\lambda A_0 , \qquad N' = N/\lambda A_0 ,$$

2.6b $$\varepsilon' = \varepsilon/\lambda A_0 \qquad x' = x/A_0 , \qquad c' = c/\lambda A_0 ,$$

2.6c $$P' = PA_0 , \qquad U'(c') = U(\lambda A_0 c') .$$

When we introduce these variables into the preceding equations, and omit the primes, the equations remain unchanged except that λ is replaced by unity. The new dimensionless ε is the ratio of the average amount of resource in a single deposit to the expected amount in the area A_0. It is also proportional to the variance of the amount of resource in the area, and thus measures the uncertainty in the amount of undiscovered resource.

3. Equations for the optimal policy. We determine the expected payoff V(A,R) by using dynamic programming [4]. Writing V at time t in terms of V a short time τ later, we find

$$3.1 \qquad V[A(t), R(t)] = \max_{x(\cdot),\, c(\cdot)} E\left\{ \int_t^{t+\tau} e^{-\rho(t'-t)}\{U[c(t')] - Px(t')\}\, dt' + e^{-\rho\tau} V[A(t+\tau), R(t+\tau)] \right\},$$

and expanding in powers of τ yields

$$3.2 \quad V[A(t), R(t)] = (1 - \rho\tau)V[A(t), R(t)] +$$

$$\max_{x(\cdot),\, c(\cdot)} \left\{ \tau[U(c(t)) - Px(t)] + E\{V[A(t+\tau), R(t+\tau)] - V[A(t), R(t)]\} \right\} + O(\tau^2).$$

In 3.2 we have used the fact that at time t, the rates x(t) and c(t) are determined by quantities which are known at time t, and thus are not random.

From 2.2,

$$3.3a \qquad A(t+\tau) = A(t) - \tau x(t) + O(\tau^2)$$

$$3.3b \qquad R(t+\tau) = R(t) - \tau c(t) + N(t+\tau) - N(t) + O(\tau^2).$$

Since an area $x(t)\tau + O(\tau^2)$ is explored in the time interval t to t + τ, the amount of resource discovered is

$$3.3c \qquad N(t+\tau) - N(t) = \begin{cases} 0 & \text{with probability} \quad 1 - x(t)\tau/\varepsilon + O(\tau^2) \\ \varepsilon s & \text{with probability density} \quad x(t)\tau/\varepsilon\, g(s)\, ds + O(\tau^2) \end{cases}.$$

Substituting 3.3 into 3.2, dividing by τ and letting τ tend to zero yields

$$3.4 \qquad \rho V(A,R) = \max_{x(\cdot),\, c(\cdot)} \left\{ U(c) - V_R(A,R)c + [\Delta V(A,R) - V_A(A,R) - P]x \right\}$$

where A, R, x, c are evaluated at time t. In the above

3.5 $$\Delta V(A,R) = \frac{1}{\varepsilon}\int_0^\infty g(s)\,[V(A,R+\varepsilon s) - V(A,R)]ds\ .$$

which is $1/\varepsilon$ times the expected return from making a discovery.

Note that $V_R(A, R)$ and $V_A(A, R)$ represent the (marginal) values of a unit amount of resource and unexplored land when the economy is in state A, R. Equivalently, $V_R(A, R)$ and $V_A(A, R)$ represent the cost of consuming a unit amount of resource and exploring a unit amount of land. Thus, $U(c) - V_R(A,R)c$ is the net rate at which the economy gains value by consuming the resource at rate c; likewise, $[\Delta V(A,R) - V_A(A,R) - P]\,dA$ is the expected gain from exploring an area dA. Clearly 3.4 articulates the balance between the benefits of exploring and consuming now, against the benefits of future exploration and consumption.

We must now select the consumption and exploration rates that maximize the right hand side of 3.4, 3.5, subject to the constraints that A and R remain positive. This yields differential and integro-differential equations for the payoff V(A,R). After selecting the optimal c and x, in this section we reduce the resulting equations to a single integro-differential equation for V. In §§4-6 we use asymptotic and approximate techniques to solve the integro-differential equation explicitly when $\varepsilon = 0$ (no uncertainty), $\varepsilon << 1$ (small uncertainty), and when ε is moderate or large. These formulas are then compared to exact numerical solutions in §7.

3.1. Optimal consumption and exploration rates. The maximum of 3.4 occurs when the marginal utility of consumption equals the cost of the resource:

3.6a $$U'(c) = V_R(A, R)\ .$$

The alternative, that the maximum occurs at $c = 0$, is excluded by the hypothesis that $U'(0) = +\infty$. Since U is concave, expression 3.6a can be inverted to give the optimal consumption rate as a function of the cost V_R:

3.6b $$c = C(V_R)\ .$$

The right side of 3.4 is linear in $x(t)$, so its maximum occurs at $x = 0$ if the coefficient of x is negative, and at $x = +\infty$ if the coefficient is positive. If the coefficient is zero, the expression is independent of x, so maximizing it leaves the value of x undetermined. Thus, if

$$\text{3.7} \qquad \Delta V(A,R) - V_A(A,R) - P \quad \begin{array}{ll} < 0 & \text{then } x = 0 \\ = 0 & \text{then } x \text{ is undetermined}\,. \\ > 0 & \text{then } x = +\infty \end{array}$$

3.2. Equations for V(A,R). Let us divide the quarter plane $A \geq 0$, $R \geq 0$ into two regions

$$\text{3.8a} \qquad D_0 = \{\text{all } (A,R) \text{ with } \Delta V(A,R) - V_A(A,R) - P \leq 0 \text{ and } x < \infty\}$$

$$\text{3.8b} \qquad D_\infty = \{\text{all } (A,R) \text{ with } \Delta V(A,R) - V_A(A,R) - P \geq 0 \text{ and } x = \infty\}\,,$$

and define the boundary between these regions to be $R_B(A)$. As shown in figure 1, we are assuming that D_0 contains the region $R > R_B(A)$ and D_∞ contains the region $R < R_B(A)$, since the value of making a discovery surely decreases as the reserves R increase. Note that $R_B(A)$ represents the minimum acceptable reserves in the economy.

Consider the region $R > R_B(A)$. There $(P + V_A - \Delta V)x = 0$ because either the first factor is positive and $x = 0$, or the first factor is zero and x is finite. So eq. 3.4 yields

$$\text{3.9} \qquad \rho V(A,R) = U(c) - V_R(A,R)c \qquad \text{for } R > R_B(A)\,.$$

Substituting the optimal consumption rate $c = C(V_R)$ from 3.6, we obtain

$$\text{3.10} \qquad \rho V(A,R) = U(C(V_R)) - V_R(A,R)C(V_R) \qquad \text{for } R > R_B(A)\,.$$

This is an autonomous first order differential equation for V as a function of R. Therefore its solution can be written as

$$\text{3.11} \qquad V(A,R) = W(R + R_E(A)) \qquad \text{for } R > R_B(A)\,,$$

where the function W is determined by 3.10, and depends only on ρ and U(c). The 'integration constant' $R_E(A)$ is arbitrary as yet, and must be determined later. Note that it can be regarded as the resource equivalent of an area A of unexplored land, so we choose

3.12 $$R_E(0) = 0 .$$

Now consider the line A = 0. Since there is no unexplored land, clearly x = 0 and the economy remains on this line as the resource R is consumed. With x = 0, eq. 3.4 reduces to 3.9 which leads to 3.10 as before. Solving 3.10 then gives

3.13 $$V(0,R) = W(R) .$$

At A = 0, R = 0, there can be no consumption or exploration, so

3.14a $$W(0) = V(0,0) = \int_0^\infty e^{-\rho t} U(0)\, dt = U(0)/\rho \qquad \text{if } |U(0)| < \infty ,$$

3.14b $$W(0) = -\infty \qquad \text{if } U(0) = -\infty .$$

The differential equation 3.10, together with the initial condition 3.14, determine W(R) for any utility function U(c).

Finally, let us consider the region $R < R_B(A)$, in which the exploration rate x is infinite. From 3.4 we deduce that

3.15 $$V_A(A,R) = \Delta V(A,R) - P \qquad \text{for } R < R_B(A) ,$$

since otherwise V would be infinitely large. This is the desired equation for V when $R < R_B(A)$. (A careful derivation of 3.15 is given in Appendix A.) From the definition of $\Delta V(A,R)$ in 3.5, we note that the integral in ΔV may extend to values of V in D_0.

We view 3.15 as a first order differential equation for V as a function of A when $R < R_B(A)$. Then 3.13 provides the initial condition V(0,R) = W(R) at A = 0. We note that V and its normal derivative must be continuous along the boundary $R = R_B(A)$, since otherwise V could be increased by shifting the boundary appropriately. For example, if V were larger on the upper side of the boundary for some part of the curve $R = R_B(A)$, then V could be increased by shifting the boundary upward. A similar argument in Appendix B shows that the normal derivative of V must also be continuous across the boundary $R = R_B(A)$. The continuity of V and its normal derivative implies that both V_A and V_R are continuous at the boundary.

3.3. Summary of equations for V(A,R). For future reference, let us gather the equations and conditions determining V(A,R). We have

3.16a $$V(0,R) = W(R) \qquad \text{at } A = 0\,,$$

3.16b $$V_A = \Delta V(A,R) - P \qquad \text{for } 0 \le R \le R_B(A)\,,$$

3.16c $$V(A,R) = W(R + R_E(A)) \qquad \text{for } R > R_B(A)\,,$$

3.16d $$V,\ V_R,\ V_A \quad \text{continuous at } R = R_B(A)\,,$$

where

3.16e $$\Delta V(A,R) = \frac{1}{\varepsilon}\int_0^\infty g(s)\,[V(A,R+\varepsilon s) - V(A,R)]ds\ .$$

In addition, W(R) is the solution of the ordinary differential equation

3.17a $$\rho W = U(C(W')) - W'C(W')$$

with the initial condition $W(0) = U(0)/\rho$ if U(0) is finite and $W(0) = -\infty$ otherwise. Here the function $C(\chi)$ is defined implicitly by

3.17b $$U'(C) = \chi\,.$$

Note that differentiating 3.17a and using 3.17b yields the identity

3.17c $$C(W') = -\rho\,\frac{W'}{W''}$$

For example, if U(c) is given by 2.5, then 3.17b yields the optimal consumption rate

3.18a $$C(W') = \left(\frac{\alpha}{W'}\right)^{1/(n+1)},$$

so 3.17a becomes

$$\text{3.18b} \qquad \frac{dW}{dR} = \alpha\left(\frac{n\rho}{(n+1)\alpha}\right)^{(n+1)/n} |W|^{(n+1)/n} \qquad (n \neq 0)\,,$$

which results in

$$\text{3.19a} \qquad W(R) = -\,K/nR^{n} \qquad (n \neq 0)$$

with $K = \alpha\{(n+1)/\rho\}^{n+1}$. Similarly, when $n = 0$ we have

$$\text{3.19b} \qquad W(R) = (\alpha/\rho)\log \rho R/e \qquad (n = 0)$$

Equations 3.16a-3.16c were given by Arrow and Chang [4] with $\Delta V = V(A,R+1) - V(A,R)$. In addition, they gave two other equations, their (23) and (24), which are superfluous.

One can expect eqs. 3.16a-3.16e to have a unique solution. To see why, suppose that $R_B(A)$ and $R_E(A)$ are given smooth functions. Eq. 3.16c yields V for $R > R_B(A)$, and then eq. 3.16b has a unique solution for $0 \le R \le R_B(A)$ which satisfies the initial condition 3.16a. (This is easily proven using standard Picard iteration). In general, these solutions will not satisfy the continuity conditions 3.16d; therefore these conditions will provide two independent equations that determine $R_B(A)$ and $R_E(A)$. These equations are solved for $V(A,R)$, R_B, and R_E in subsequent sections. Once they have been solved, the consumption rate can be found from $c = C(V_R(A,R))$, and the optimal exploration rate is $x = 0$ when $R > R_B(A)$ and $x = \infty$ when $R < R_B(A)$.

3.4. Optimal pricing. The optimal resource price p_R and land price p_A are the marginal values $V_R(A,R)$ and $V_A(A,R)$, and are found easily once $V(A,R)$ is known. In particular,

$$\text{3.20} \qquad p_R = W'(R + R_E(A))\,, \qquad p_A = W'(R + R_E(A))R_E'(A)\,, \qquad \text{when } R \ge R_B(A)\,.$$

We can show that these prices rise exponentially in time at the discount rate ρ as long as the economy remains in the region $R \ge R_B(A)$. Since $dR/dt = -c = -C(W')$ and $dA/dt = -x = 0$ in this region, differentiating $W'(R + R_E(A))$ yields

$$3.21 \qquad \frac{1}{p_R}\frac{dp_R}{dt} = \frac{W''}{W'}\left(\frac{dR}{dt} + R_E'(A)\frac{dA}{dt}\right) = -\frac{W''}{W'}\,C(W') \qquad \text{for } R \geq R_B(A)\,.$$

Together with 3.17c, this shows that $\frac{dp_R}{dt} = \rho\, p_R$, so

$$3.22 \qquad p_R(t) = p_R(t_0)\, e^{\rho(t-t_0)} \qquad \text{for } R \geq R_B(A)\,.$$

Hence, if the economy is at $R(t_0) \geq R_B(A(t_0))$ at time t_0, the resource price rises exponentially as the economy evolves along the path $dR/dt = -c$, $A(t) = A(t_0)$ until it reaches the boundary $R = R_B(A)$. Since 3.20 shows that $p_A(t) = p_R(t)\, R_E'(A(t_0))$, clearly land prices also rise exponentially until the economy reaches the boundary.

Once the economy reaches the boundary, the exploration rate is infinite until enough discoveries are made to raise it above the curve $R = R_B(A)$. So the economy jumps to a new value of A and R. Surprisingly, we will find that the prices p_A and p_R after the jump equals, on average, the prices before the jump. So, in a certain sense (see §7), the exponential price rise continues through the jumps.

4. The deterministic case. Before considering the problem with uncertainty, we analyze the deterministic case ($\varepsilon = 0$). When $\varepsilon = 0$ there is no uncertainty in the discovery process. Exploring at rate x is certain to discover resource at rate x. The discount factor makes it disadvantageous to explore before necessary, so x should be zero when $R(t) > 0$. Thus, $\lim_{\varepsilon \to 0} R_B(A) \equiv 0$ and

$$4.1a \qquad V(A,R) = W(R + R_E(A)) \qquad \text{for all } R \geq 0,\ A \geq 0\,.$$

See 3.16c. When $R = 0$ it is disadvantageous to explore at a rate greater than is needed to make up for consumption, so the optimal exploration rate is

$$4.1b \qquad x = 0 \quad \text{when} \quad R > 0\,, \qquad x = c \quad \text{when} \quad R = 0\,.$$

The optimal consumption rate is (see 3.17)

$$4.1c \qquad c = C(W'[R + R_E(A)]) \qquad \text{for all } R \geq 0,\ A \geq 0\,,$$

while the optimal resource and land prices are

$$p_R = W'(R + R_E(A)), \quad p_A = W'(R + R_E(A))R_E'(A) \qquad \text{for all } R \geq 0, \; A \geq 0. \tag{4.1d}$$

Thus, if the economy starts at a point A_0, R_0 at time $t = 0$, the optimal policy is to not explore, so A remains at A_0 as $R(t)$ declines with consumption until reaching $R = 0$. Then the exploration rate is $x = c$, so the economy remains at $R = 0$ as A gradually declines towards zero.

The remaining problem is finding $R_E(A)$. Using continuity at $R = R_B = 0$ allows us to substitute $V(A,0) = W(R_E(A))$ into 3.16b, and from 3.16e we see that ΔV becomes V_R as $\varepsilon \to 0$. After dividing through by $W'(R_E)$, 3.16b yields

$$R_E'(A) = 1 - P/W'(R_E) \tag{4.3}$$

With the initial condition $R_E(0) = 0$, this differential equation determines $R_E(A)$ uniquely.

Apart from a few special cases, this equation cannot be solved explicitly for $R_E(A)$, even when $U(c)$ is given by 2.5, and hence W by 3.19. However, note that $W'(R_E(A)) = V_R(A,0)$. As A goes to zero, the marginal value of the resource surely goes to infinity, so the last term in 4.3 must become negligible. That is, the cost of exploration becomes negligible compared to the value of the resource. Dropping the last term shows that

$$R_E(A) \sim A \qquad \text{for A small}. \tag{4.4a}$$

On the other hand, as A becomes large, $R_E(A)$ will go exponentially to the root R_∞ of the right side of 4.3. So

$$R_E(A) \sim R_\infty \quad \text{where} \quad W'(R_\infty) = P \qquad \text{for A large}. \tag{4.4b}$$

Then 4.1d yields

$$p_R \sim P, \qquad p_A \sim 0 \qquad \text{for A large}. \tag{4.4c}$$

So the price of the resource becomes just the cost of exploration, and the price of land tends to zero as the amount of unexplored land becomes large.

We now consider how the optimal prices evolve with time. From §3.4 we know that both p_R and p_A rise exponentially at the discount rate when $R > 0$. So suppose that the economy reaches $R = 0$, $A = A_0$ at time $t = t_0$. Then it will remain at $R = 0$ with the amount of unexplored land A decreasing at the consumption rate $C(W'[R_E(A)])$. From 3.17c, then

4.5 $$\frac{dA}{dt} = \rho W'(R_E(A))/W''(R_E(A)) .$$

Using 4.3 we argue that

4.6 $$\frac{d}{dt}\{W'(R_E) - P\} = W''(R_E) R_E' \frac{dA}{dt} = \rho W'(R_E) R_E' = \rho\{W'(R_E) - P\} .$$

So once the economy reaches $R = 0$, then

4.7 $$W'(R_E(A)) = P + \{W'(R_E(A_0)) - P\} e^{\rho(t-t_0)} .$$

Now $p_R = W'(R_E)$, so this equation yields $p_R(t)$ explicitly. Since $p_A = W'(R_E) R_E' = W'(R_E) - P$, we have

4.8 $$p_A(t) = p_A(t_0) e^{\rho(t-t_0)}, \qquad p_R(t) = P + p_A(t_0) e^{\rho(t-t_0)} .$$

That is, the optimal resource price is the exploration cost plus the price of land, which grows exponentially at the discount rate. As we shall see, this is in contrast to prices in the presence of uncertainty, where the average prices do not converge to 4.8, even as the amount of uncertainty goes to zero. We will examine this seeming contradiction in §7.

5. Small uncertainty. Small uncertainty ($\varepsilon << 1$) is the most widely occurring case for most natuaral resources; it pertains whenever each discovery reveals a small fraction of the total amount of the undiscovered resource.

We need to solve 3.16b for $V(A,R)$ when $R < R_B(A)$, and then use continuity to determine $R_B(A)$ and $R_E(A)$. Expanding $V(A, R+\varepsilon s)$ in a Taylor series and substituting into 3.16b, we find

5.1 $$V_A = -P + V_R + \frac{\varepsilon}{2} M_2 V_{RR} + \frac{\varepsilon^2}{6} M_3 V_{RRR} + \ldots \qquad \text{for } R \le R_B(A)$$

Here M_2 is the second moment of g(s), defined in 2.3b, and M_3 is the third moment, defined similarly. This suggests finding V(A,R) via a power series expansion:

$$V(A,R) = V^{(0)}(A,R) + \varepsilon V^{(1)}(A,R) + \dots \qquad \text{for } R \le R_B(A)\,. \tag{5.2}$$

Substituting this expansion into 5.1 and the initial condition 3.16a, and equating like powers of ε yields

$$V_A^{(0)} - V_R^{(0)} = -P\,, \qquad V^{(0)}(0,R) = W(R)\,, \tag{5.3a}$$

$$V_A^{(1)} - V_R^{(1)} = \tfrac{1}{2} M_2 V_{RR}^{(0)}\,, \qquad V^{(1)}(0,R) = 0\,, \tag{5.3b}$$

through O(ε), and solving these equations yields

$$V(A,R) = W(R+A) - PA + \tfrac{\varepsilon}{2} M_2 A W''(R+A) + \dots \qquad \text{for } R \le R_B(A)\,. \tag{5.4}$$

Now, $V(A,R_B)$ and $V_R(A,R_B)$ must be continuous at $R = R_B(A)$. From expressions 5.4 and 3.16c, then

$$W(R_B + R_E) = W(R_B + A) - PA + \tfrac{\varepsilon}{2} M_2 A W''(R_B + A) + \dots, \tag{5.5a}$$

$$W'(R_B + R_E) = W'(R_B + A) + \tfrac{\varepsilon}{2} M_2 A W'''(R_B + A) + \dots. \tag{5.5b}$$

Solving these equations, we find that

$$R_B(A,\varepsilon) = R_0(\varepsilon) - A + \dots, \qquad R_E(A,\varepsilon) = A\{1 + \tfrac{\varepsilon}{2} M_2 W'''(R_0)/W''(R_0) + \dots\} \tag{5.6a}$$

where $R_0(\varepsilon)$ is defined by

$$W''(R_0) - W'(R_0)W'''(R_0)/W''(R_0) = 2P/\varepsilon M_2\,. \tag{5.6b}$$

The solution for V(A,R) in 5.4, with R_B and R_E given by 5.6, can be continued to higher powers of ε in the same manner.

5.1. Singular expansion when $A > R_0$. The preceding expansion fails when $A > R_0(\varepsilon)$, since it predicts that R_B is negative. In appendix C we use singular techniques to analyze 3.16b more carefully. There we discover that when $A/\varepsilon >> 1$,

5.7a $$R_E'(A) = 1 - \frac{P}{W'} + \frac{1}{2}\varepsilon M_2 \frac{W''}{W'} + \ldots ,$$

5.7b $$W'(R_B)e^{-a/\varepsilon} = -\varepsilon \frac{W''}{W'}\left\{P + \frac{1}{2}\varepsilon M_2\left(\frac{W'W'''}{W''} - W''\right) + \ldots\right\} .$$

Here

5.7c $$a = R_0(\varepsilon) \quad \text{if } A < R_0(\varepsilon) , \qquad a = A \quad \text{if } A > R_0(\varepsilon) ,$$

and the arguments of W are $R_E(A) + R_B(A)$ on the right hand sides of 5.7a, 5.7b.

Equations 5.7 sum up Arrow's model in the presence of uncertainty, at least when these uncertainties are small. The algebraic relation 5.7b implicitly determines R_B in terms of R_E and A, and this can be used to eliminate R_B in 5.7a, reducing it to an ordinary differential equation for R_E. After solving this equation, the payoff V can be found as $W(R + R_E(A))$ when $R \geq R_B(A)$, and the other economic quantities can be deduced from the payoff. See figure 2.

Note that the right hand side of 5.7b depends only on $R_E(A) + R_B(A)$, while the left hand side is transcendentally small unless $R_B(A)$ is transcendentally small. Let $R_0(\varepsilon)$ be the value of $R_B + R_E$ at which the right hand side is zero (see 5.6b). Then 5.7b yields $R_B(A) = R_0 - R_E(A)$ until R_B is transcendentally small. When $R_B(A)$ is transcendentally small, it can be neglected on the right hand side of 5.7b. Thus we obtain

5.8a $$R_B(A) = R_0(\varepsilon) - R_E(A) + \ldots \qquad \text{when } R_E(A) < R_0(\varepsilon) ,$$

5.8b $$W'(R_B) = \varepsilon e^{A/\varepsilon}\left\{-\frac{PW''}{W'} + \frac{1}{2}\varepsilon M_2\left(\frac{W''W''}{W'} - W'''\right) + \ldots\right\} \quad \text{when } R_E(A) > R_0(\varepsilon).$$

Here the argument of W on the right side of 5.8b is $R_E(A)$, so 5.8b determines $R_B(A)$ implicitly from R_E.

The arguments on the right side of 5.7a are $R_B + R_E$, which equals $R_0(\varepsilon)$ when $R_E(A) < R_0(\varepsilon)$. So $R_E'(A)$ is constant there, and using 5.6b we obtain

5.9a $$R_E(A) = A\{1 + \tfrac{\varepsilon}{2} M_2 W'''(R_0)/W''(R_0) + \dots\} \qquad \text{when } R_E(A) < R_0(\varepsilon)\,.$$

When $R_E(A) > R_0(\varepsilon)$, then $R_B(A)$ is transcendentally small and can be neglected in 5.7a. Hence, R_E is the solution of the differential equation

5.9b $$\frac{dR_E}{dA} = 1 - \frac{P}{W'(R_E)} + \tfrac{1}{2}\varepsilon M_2 \frac{W''(R_E)}{W'(R_E)} + \dots \qquad \text{when } R_E(A) > R_0(\varepsilon)\,,$$

with the initial condition

5.9c $$R_E(A) = R_0(\varepsilon) + \dots \qquad \text{at } A = R_0(\varepsilon)\{1 - \varepsilon M_2 W'''(R_0)/2W''(R_0) + \dots\}\,.$$

We see that 5.8 and 5.9 agree precisely with the regular expansion 5.6 when $R_E(A) < R_0(\varepsilon)$. The results differ when $R_E(A) > R_0(\varepsilon)$ because the transcendentally small term $e^{-A/\varepsilon}$ in 5.7 forces $R_B(A)$ to remain slightly positive. Now, $e^{-A/\varepsilon}$ is the probability that the unexplored area A contains no more resources, and recalling that $V(0,R) = W(R)$, we see that its co-factor $W'(R_B)$ is the marginal value (price) of the resource at $A = 0$, $R = R_B$. Thus, the only reason that the minimum acceptable reserve $R_B(A)$ remains non-zero when $R_E(A) > R_0(\varepsilon)$ is to guard against the unlikely possibility that no more discoveries will be made. More precisely, to limit the price shock that would occur in this unlikely event.

In addition, as A becomes large $R_E(A)$ will tend to a constant value R_∞, which satisfies

5.10 $$W'(R_\infty) + \tfrac{1}{2}\varepsilon M_2 W''(R_\infty) + \dots = P\,.$$

So when there is plenty of unexplored land, the average value of a discovery is just the cost εP of finding it.

5.2. Economic considerations. Equations 5.8 and 5.9 show that $R_B(A)$ declines linearly when A is small. This can be understood from an economic viewpoint as follows. Let Q denote the random amount of resource in the unexplored land, and suppose that the entire area is explored at once. Then the expected payoff $V_0(A,R)$ would be

$$5.11a \qquad V_0(A,R) = EV(0,R+Q) - PA = EW(R+Q) - PA$$

$$= W(R+A) + \tfrac{1}{2}\varepsilon M_2 A W''(R+A) + \dots - PA\,.$$

To obtain the last line we expanded W in $Q-A$, recalling that the mean of Q is A and its variance is $\varepsilon M_2 A$.

We compare this to the result obtained by first consuming at the rate c for a short time dt, and then exploring all the land. In this case the payoff is

$$5.11b \qquad V_1(A,R) = (1-\rho\,dt)V_0(A,R-c\,dt) + U(c)\,dt + O(dt)^2$$

$$= V_0(A,R) - [V_{0R}(A,R)c + \rho V_0(A,R) - U(c)]\,dt + O(dt)^2$$

$$= V_0(A,R) + [\tfrac{1}{2}\varepsilon M_2(\frac{W'W'''}{W''} - W'') + P]A\rho\,dt + O(dt)^2\,.$$

Here the argument of W is $R+A$, and to obtain the last expression we used $\rho W = U(c) - W'c$ and $c = -\rho W'/W''$. See 3.17a - 3.17c. Delaying for a time dt is a better policy if and only if the coefficient of dt in 5.11b is positive. The boundary between these cases occurs at $R+A = R_0(\varepsilon)$, that is, at $R = R_0(\varepsilon) - A$. Comparing with 5.6 shows that this boundary is $R_B(A)$ to leading order. In essence, the curve R_B is set by balancing the gain $PA\rho\,dt$ of delaying exploration against the expected losses $\varepsilon M_2 A\rho\,dt\,(W'' - W'W'''/W'')/2$ caused by choosing consumption rates based on the expected, and not actual reserves. Thus this latter term represents the inherent value of information in the economy.

The curve $R_B(A)$ for large A is determined by different considerations. At $R = R_B(A)$, the price of the resource $p_R = W'$ can be written as the sum of two terms. One is the value of the reserves at $A = 0$ multiplied by the probability $e^{-A/\varepsilon}$ that no more discoveries will be made. The other is the expected price after one discovery. This yields

$$5.12a \qquad W'(R_B + R_E) \approx W'(R_B)\,e^{-A/\varepsilon} + W'(R_B+R_E) + \varepsilon(1-R_E')W''(R_B+R_E) + \dots\,.$$

When A is large the price of reserves W' should be nearly the price of land $W'R_E'$ plus the unit cost of exploration P, so $W' = W'R_E' + P$, which yields $R_E' = 1 - P/W'$. From 5.12a, then

$$5.12b \qquad W'(R_B)\, e^{-A/\varepsilon} = -\varepsilon P W''(R_B + R_E)/W'(R_B + R_E) + \dots$$

which agrees with 5.8b for large A.

5.3. Explicit examples. Before considering larger uncertainties, let us illustrate the preceding results using the utility function U(c) in 2.5. Then W is given by 3.19, and solving 5.6b yields

$$5.13a \qquad R_0(\varepsilon) = (\varepsilon M_2 K/2P)^{1/(n+2)},$$

so R_0 goes to zero with ε. When $R_E(A) < R_0(\varepsilon)$ we have

$$5.13b \qquad R_B = R_0(\varepsilon) - R_E(A) + \dots, \qquad R_E = A\{1 - \varepsilon M_2(n+2)/2R_0(\varepsilon) + \dots\},$$

When $R_E(A) > R_0(\varepsilon)$, solving the implicit relation 5.8b yields

$$5.14a \qquad R_B = R_E e^{-A/\varepsilon(n+1)} \left\{ \frac{K R_E^2/\varepsilon(n+1)}{P R_E^{n+2} - \varepsilon M_2 K/2} \right\}^{1/(n+1)} + \dots \qquad \text{when } R_E(A) > R_0(\varepsilon),$$

explicitly showing that R_B is transcendentally small. Eq. 5.9b shows that R_E solves the equation

$$5.14b \qquad \frac{dR_E}{dA} = 1 - \frac{P}{K} R_E^{n+1} - \frac{1}{2}\varepsilon M_2 \frac{n+1}{R_E} + \dots$$

with

$$5.14c \qquad R_E = R_0(\varepsilon) \qquad \text{at } A = R_0(\varepsilon) + \varepsilon M_2(n+2)/2.$$

6. Moderate and large uncertainties. We now consider the case where ε is not small. When ε is small we found that $R_B(A)$ and $R_E(A)$ are initially nearly straight lines. So for the present case we expect that R_B and R_E can be represented by their Taylor series for moderately small values of A. Using 3.16a and 3.16b to expand V(A,R) in a power series in A for $R \le R_B(A)$, we find

$$6.1 \qquad V(A,R) = W(R) - (A/\varepsilon)\left\{\varepsilon P + W(R) - \int_0^\infty g(s) W(R+\varepsilon s)\, ds\right\}$$

$$+ (A^2/2\varepsilon^2)\left\{\varepsilon P + W(R) - \int_0^\infty g(s) W(R+\varepsilon s)\, ds + \varepsilon \int_0^\infty g(s) V_A(0, R+\varepsilon s)\, ds\right\} + O(A^3)$$

We evaluate V and V_R in 6.1 at $R = R_B(A)$, equate the results to $W(R_E + R_B)$ and $W'(R_E + R_B)$, and then expand in powers of A. This yields

$$6.2a \quad \varepsilon R_E'(0) = -\left\{ W[R_B(0)] + \varepsilon P - \int_0^\infty g(s) W[R_B(0) + \varepsilon s]\, ds \right\} / W'[R_B(0)],$$

$$6.2b \quad \varepsilon R_E'(0) = -\left\{ W'[R_B(0)] - \int_0^\infty g(s) W'[R_B(0) + \varepsilon s]\, ds \right\} / W''[R_B(0)],$$

$$6.2c \qquad R_B'(0) = -R_E'(0)/2, \qquad R_E''(0) = 0.$$

Therefore,

$$6.3 \qquad R_B(A) = R_B(0) - \tfrac{1}{2} R_E'(0) A + O(A^2), \qquad R_E(A) = R_E'(0) A + O(A^3)$$

for A small, where $R_B(0)$ can be obtained by equating 6.2a and 6.2b and solving the resulting algebraic equation.

For larger values of A, one equation can be obtained by using 3.16b to evaluate V_A at $R = R_B(A)$, and equating it with $W'(R_E + R_B) R_E'(A)$. This yields the exact expression

$$6.4a \quad R_E'(A) = \{\varepsilon W'[R_E + R_B]\}^{-1} \left\{ \int_0^\infty g(s) W(R_E + R_B + \varepsilon s)\, ds - W(R_E + R_B) - \varepsilon P \right\}.$$

Note that the right hand side depends only on $R_E(A) + R_B(A)$. Recalling that the price of land is $p_A = R_E'(A) W'(R_E + R_B)$ when $R \geq R_B(A)$, we see that eq. 6.4a simply states that p_A is equal to the expected increase in value due to making a discovery less the exploration cost at $R = R_B(A)$.

A second equation can be obtained from the integral equation C.2. Taking the derivative of C.2 with respect to R, and evaluating it at $R = R_B(A)$ yields

$$6.4b \quad W'[R_E + R_B] = \int_0^\infty g(s) \{ W'[R_E + R_B + \varepsilon s] - \varepsilon R_E'(A) W''[R_E + R_B + \varepsilon s] + \ldots \}\, ds$$

$$+ e^{-A/\varepsilon} \Big\{ W'(R_B) - \int_0^\infty g(s) W'[R_B + \varepsilon s]\, ds$$

$$- \int_0^\infty \int_0^\infty g(s) g(s') [W'(R_B + \varepsilon s) - W'(R_B + \varepsilon s + \varepsilon s')]\, ds'\, ds - \ldots \Big\}$$

This equation is only approximate since it neglects terms with V_{AA} and higher derivatives in C.2. We took the derivative with respect to R, rather than use C.2 directly, since prices are generally more sensitive to details — like precise values of $R_B(A)$ and $R_E(A)$ — than payoffs.

The pair of equations, 6.4a and 6.4b, determine $R_B(A)$ and $R_E(A)$. If we use 6.4a to eliminate $R_E'(A)$ in 6.4b, then 6.4b reduces to an algebraic equation. We have solved 6.4a, 6.4b numerically for $U(c) = \alpha \log c$, $g(s) = \delta(s-1)$, $P\rho/\alpha = 1$, with $\varepsilon = 0.1$ and 1.0. Figures 3 and 4 show the approximate $R_B(A)$ and $R_E(A)$ obtained, along with the exact values.

As A becomes large, $R_B(A)$ will become small enough that $R_E + R_B$ can be replaced by R_E. Then 6.4 simplifies to

6.5a $$R_E'(A) = \{\varepsilon W'[R_E]\}^{-1}\left\{\int_0^\infty g(s)W(R_E+\varepsilon s)\,ds - W(R_E) - \varepsilon P\right\},$$

6.5b $$e^{-A/\varepsilon}\,W'[R_B(A)] = W'[R_E] - \int_0^\infty g(s)\{W'[R_E+\varepsilon s] - \varepsilon R_E'(A)W''[R_E+\varepsilon s]\,ds\,.$$

In 6.5b we have also neglected the exponentially small terms, but not $e^{-A/\varepsilon}\,W'[R_B(A)]$ since this term is not small when R_B is small enough. The differential equation 6.5a is now uncoupled from 6.5b, and can be easity solved for $R_E(A)$; then 6.5b yields $R_B(A)$.

As A becomes infinite, 6.5a shows that $R_E'(A)$ tends to zero and $R_E(A)$ tends to the root R_∞ of

6.6 $$\int_0^\infty g(s)W(R_\infty+\varepsilon s)\,ds - W[R_\infty] = \varepsilon P + \ldots\,.$$

So as before, the average value of a discovery is the average cost εP of finding it when there is plenty of unexplored land.

7. Numerical solutions. Discussion. To test our analytical results we have calculated the exact curves $R_B(A)$ and $R_E(A)$ numerically for many cases. In figures 2 - 4 we portray the case $U(c) = \alpha \log c$, $g(s) = \delta(s-1)$, and $P\rho/\alpha$ for $\varepsilon = 0.01$, 0.1, and 1.0. For $\varepsilon = 0.01$, we compare the exact curves to the asymptotic results obtained by solving 5.7. In figures 2 - 4, we compare the exact solutions to 6.3 when A is moderately small, and to the approximate solutions obtained from 6.4 when A is larger. Note the surprising accuracy of the approximate solution.

Let us now consider our results. The optimal exploration policy is determined by the curve $R_B(A)$, which represents the minimum acceptable reserves for the economy. If at time t the proven reserves R(t) and unexplored land A(t) satisfy $R(t) > R_B(A(t)$, then the exploration rate is zero and the optimal consumption rate is $c(t) = -\rho W'[R(t) + R_E(A)]/W''[R(t) + R_E(A)]$. Thus R(t) decreases and A(t) remains unchanged until $R(t) = R_B[A(t)]$. Then exploration begins at an infinite rate until enough discoveries have been made to increase R above $R_B(A)$. This occurs instantly since the exploration rate is infinite, so A(t) decreases and R(t) increases discontinuously to new values $A(t^+)$ and $R(t^+)$, which are random since they depend on the vagaries of the exploration process. Then the cycle begins again.

The price of reserves $p_R(t)$ and land $p_A(t)$ both increase exponentially at the discount rate ρ during the consumption phase of each cycle, as shown by 3.22 *et seq.* They both change discontinuously at the instant when exploration occurs, jumping to new values $p_R(t^+) = V_R[A(t^+),R(t^+)]$ and $p_A(t^+) = V_A[A(t^+),R(t^+)]$. In Appendix D we show that, on average, the post-exploration prices are exactly equal to the prices immediately before exploration:

7.1 $$E\{p_R(t^+)\} = p_R(t)\,, \qquad E\{p_A(t^+)\} = p_A(t)\,.$$

This means that on average, both resource and land prices continue to increase exponentially as the economy goes through the exploration/consumption cycles, regardless of how little (or much) uncertainty there is in the exploration process. In the limit of small uncertainty ($\varepsilon \to 0$), we do *not* recover the deterministic result of §4:

7.2 $$p_R(t) = p_A(t) + P\,, \qquad p_A(t) = const. \times e^{\rho t}$$

To resolve this apparent contradiction, consider the small uncertainty case. In §5 we discovered that $R_B(A)$ is governed by 5.7b. Using 5.7a, we can re-write this equation as:

$$7.3\quad \{\varepsilon W''(R+R_E)+\tfrac{1}{2}\varepsilon^2 M_2 W'''(R+R_E)+\dots\} - \{\varepsilon W''(R+R_E)R_E'(A)+\dots\} + W'(R)e^{-A/\varepsilon} = 0$$

$$\text{at } R = R_B(A).$$

At $R = R_B(A)$ the optimal policy is to explore, and the first two terms represent the change in the resource price p_R after the first discovery is made. The first term is the price change due to the increase in reserves R, and the second is the change due to the decrease in unexplored land A. However, there is a small probability $e^{-A/\varepsilon}$ that there is no first discovery. The last factor is the expected change in resource price if no discovery is made. In essence 7.3 states that the expected price change during exploration is zero, but this zero average is made up of two terms: a price decrease that occurs after a discovery, and is virtually certain to happen; and a small possibility that an enormous price increase occurs because no more resource is discovered. As $\varepsilon \to 0$, the latter possibility becomes extremely remote, but the optimal policy reduces the minimum acceptable reserves $R_B(A)$ until the price increase that would occur is large enough to offset the remoteness of the possibility. So because the optimal policy must discount the price shock that would occur if no more discoveries are made, the economy enjoys a price decrease each time it passes through a discovery phase without this possibility occurring. Thus the model does not account for the slow growth of oil prices from 1950 to 1970, as proposed in [3].

Appendix A. The equation for $V(A,R)$ in the region D_∞ can be found by re-examining the exploration process. Suppose that a small area dA is explored. The probability (density) that a deposit of size εs is found while exploring is $g(s)\, dA/\varepsilon$, the probability that no deposit is found is $1 - dA/\varepsilon$, and the probability of finding two or more deposits is $O(dA)^2$. Since this area is explored infinitely fast, no consumption takes place during exploration. Therefore,

$$\text{A.1} \qquad V(A,R) \;=\; \frac{dA}{\varepsilon}\int_0^{\infty} g(s)V(A-dA,R+\varepsilon s)\,ds \;+\; (1-dA/\varepsilon)V(A-dA,R) \;-\; PdA \;+\; O(dA)^2$$

Expanding in dA now yields 3.15.

Appendix B. We now prove that the function $V(A,R)$ defined by 3.4, and its derivative V_R, are continuous across the curve $R = R_B(A)$. To do so, we assume that $W(R)$ is an increasing function of R (this is easily proven from 3.17), and that $U(c) < \alpha c + \beta$.

To prove that V is continuous, note that 2.4 shows that $V(A,R)$ is the maximum of the expected value of an integral over a class of admissible functions. These functions are restricted by 2.2a, 2.2b with $A(0) = A$ and $R(0) = R$. When A and R are changed slightly, this class of functions is changed slightly in the L^1 norm. Then, because $U(c)$ grows no faster than linearly, it follows from 2.4 that V changes only slightly. Thus V is a continuos function of A and R.

We prove that V_R is continuous by contradiction. Suppose that $V_R[A,R_B(A)^-]$ is larger than $V_R[A,R_B(A)^+]$ when $|A - A_0| \le \delta$ for some A_0 and $\delta \ge 0$. Then we can construct a new larger function $V^N(A,R)$ and corresponding functions $R_B^N(A)$ and $R_E^N(A)$ such that these functions satisfy 3.16a - 3.16c with V^N continuous; this will contradict the maximality of V.

To construct V^N, increase R_B by an amount $\Delta R(A) > 0$:

$$\text{B.1} \qquad R_B^N(A) \;=\; R_B(A) + \Delta R(A) \qquad \text{for } |A - A_0| < \delta\,,$$

requiring that $\Delta R(A) = 0$ for $|A - A_0| \ge \delta$. We define V^N by $V^N = V$ for $A \le A_0 - \delta$. When $A > A_0 - \delta$, we set $V^N = W[R + R_E^N(A)]$ for $R \ge R_B^N(A)$ and we require V^N to satisfy 3.16b for $0 \le R \le R_B^N(A)$, and to equal $V(A_0 - \delta, R)$ at $A = A_0 - \delta$. From this definition it follows that

B.2 $$V^N = V + O(\delta\Delta R) \qquad \text{for } |A - A_0| < \delta \text{ and } R < R_B .$$

The function $R_E^N(A)$ is equal to R_E for $A \le A_0 - \delta$, while for $A > A_0 - \delta$ it is determined by the continuity of V at R_B^N. By using B.2 in this condition, we obtain for $|A - A_0| < \delta$

B.3 $$W(R_B^N + R_E^N) = V^N(A,R_B^N) = V(A,R_B) + \Delta R V_R(A,R_B^-) + O(\Delta R)^2 + O(\delta\Delta R)$$

$$= W(R_B + R_E) + \Delta R V_R(A,R_B^-) + O(\Delta R)^2 + O(\delta\Delta R) .$$

From B.3 we find that

B.4 $$R_E^N(A) - R_E = \Delta R\,[V_R(A,R_B^-) - W'(R_B + R_E)]/W'(R_B + R_E) + O(\Delta R)^2 + O(\delta\Delta R)$$

$$\text{for } |A - A_0| < \delta .$$

Now, $W' > 0$ and we are assuming that $V_R(A,R_B^-) > V_R(A,R_B^+) = W'(R_B + R_E)$. So

B.5 $$R_E^N(A) - R_E > 0 \qquad \text{for } |A - A_0| < \delta$$

provided we select $\delta > 0$ and $\Delta R > 0$ sufficiently small.

Comparing V^N with V, we see they are equal when $A \le A_0 - \delta$. For $|A - A_0| < \delta$ and $R > R_B^N(A)$, the monotonicity of W ensures that

B.6 $$V^N(A,R) = W[R + R_E^N(A)] > W[R + R(A)] = V(A.R) ,$$

so $V^N > V$ in this region. We now use B.6 and 3.16b to conclude that $V_A^N \ge V_A$ in $|A - A_0| < \delta$, $R < R_B(A)$. It then follows that $V^N > V$ in this region. Similarly we find that $V^N \ge V$ for all $A > A_0 - \delta$.

Because $V^N \ge V$ everywhere, we conclude that $V_R[A_0,R_B(A_0)^-] > V_R[A_0,R_B(A_0)^+]$ for some A_0 cannot occur. If this inequality is reversed, we can construct a larger solution V^N simply by reversing the sign of ΔR. Thus we see that V_R cannot be discontinuous.

Appendix C. Singular analysis. The regular expansion fails near $A > R_0(\varepsilon)$ since it predicts that R_B is negative. To obtain a valid solution, let us return to 3.16b. We can re-write 3.16b and its initial condition 3.16a as

$$\text{C.1} \qquad V(A,R) = W(R)e^{-A/\varepsilon} - \varepsilon P(1-e^{-A/\varepsilon}) + \frac{1}{\varepsilon}\int_0^A e^{(a-A)/\varepsilon}\int_0^\infty g(s)V(a,R+\varepsilon s)\,ds\,da$$

for $R \le R_B(A)$. Repeatedly integrating the last term by parts yields

$$\text{C.2} \qquad V(A,R) = W(R)e^{-A/\varepsilon} - \varepsilon P(1-e^{-A/\varepsilon})$$

$$+ \int_0^\infty g(s)\{V(A,R+\varepsilon s) - \varepsilon V_A(A,R+\varepsilon s) + \varepsilon^2 V_{AA}(A,R+\varepsilon s) - \dots\}\,ds$$

$$- e^{-A/\varepsilon}\int_0^\infty g(s)\{V(0,R+\varepsilon s) - \varepsilon V_A(0,R+\varepsilon s) + \varepsilon^2 V_{AA}(0,R+\varepsilon s) - \dots\}\,ds\,.$$

We now evaluate C.2 at $R = R_B(A)$ and equate it with $W(R_E + R_B)$. In the first integral we can replace $V(A,R_B+\varepsilon s)$ by $W(R_E(A) + R_B(A) + \varepsilon s)$. In the last integral the initial condition 3.16a allows us to evaluate $V(0,R+\varepsilon s)$, the initial condition and 3.16b allows us to evaluate $V_A(0,R+\varepsilon s)$, etc. Hence, C.2 becomes

$$\text{C.3} \quad W(R_E+R_B) = W(R_B)e^{-A/\varepsilon} - \varepsilon P$$

$$+ \int_0^\infty g(s)\{W(R_E+R_B+\varepsilon s) - \varepsilon W_A(R_E+R_B+\varepsilon s) + \varepsilon^2 W_{AA}(R_E+R_B+\varepsilon s) - \dots\}\,ds$$

$$- e^{-A/\varepsilon}\int_0^\infty g(s)W(R_B+\varepsilon s)ds - e^{-A/\varepsilon}\int_0^\infty\int_0^\infty g(s)g(s')[W(R_B+\varepsilon s) - W(R_B+\varepsilon s+\varepsilon s')]\,ds'\,ds + \dots$$

The regular expansion in §5 is valid when $A < R_0(\varepsilon)$, so let us consider values of A large enough so that $A/\varepsilon >> 1$, and thus $e^{-A/\varepsilon}$ is transcendentally small. The integrals that are multiplied by $e^{-A/\varepsilon}$ can be relevant only if $R_B(A)$ is transcendentally small, so that $W(R_B+\varepsilon s)$ is large enough to balance out the factor $e^{-A/\varepsilon}$. However, since $W(R_B+\varepsilon s)$ occurs in the integrand, these integrals are smaller than the first term $W(R_B)e^{-A/\varepsilon}$, at least by a factor of R_B. So these integral terms are never relevant and can be neglected: they are either transcendentally small or transcendentally smaller than the first term $W(R_B)e^{-A/\varepsilon}$.

Consider the first integral on the right hand side. Regardless of the value of A, the argument $R_E(A)+R_B(A)+\varepsilon s$ is always larger than about $R_0(\varepsilon)$. For if $A \le R_0(\varepsilon)$ then $R_E+R_B \sim R_0(\varepsilon)+\ldots$ (see 5.6), and for larger values of A we have $R_E(A)+R_B(A) \ge R_E(A) \ge R_0(\varepsilon)$ since $R_E(A)$ is an increasing function of A. Therefore we can expand the integrand in powers of ε. This yields

C.4a $$(R_E'-1)W' = W(R_B)\varepsilon^{-1}e^{-A/\varepsilon} - P + \varepsilon W''\{\tfrac{1}{2}M_2 - R_E' + (R_E')^2\} + \varepsilon W' R_E'' + \ldots$$

where unless specified otherwise, the argument of W is $R_E(A)+R_B(A)$.

In a similar manner we differentiate C.2 with respect to R and evaluate it at $R = R_B(A)$. Following the preceding arguments leads to

C.4b $$(R_E'-1)W'' = W'(R_B)\varepsilon^{-1}e^{-A/\varepsilon} + \varepsilon W'''\{\tfrac{1}{2}M_2 - R_E' + (R_E')^2\} + \varepsilon W'' R_E'' + \ldots$$

where all unspecified arguments of W are $R_E(A)+R_B(A)$ as above. Now consider the term $W(R_B)\varepsilon^{-1}e^{-A/\varepsilon}$ in C.4a. For this term to be relevant, R_B must be transcendentally small. But for small R_B, this term is a factor R_B smaller than the term $W'(R_B)\varepsilon^{-1}e^{-A/\varepsilon}$ in C.4b. Accordingly, we can neglect the term $W(R_B)\varepsilon^{-1}e^{-A/\varepsilon}$ in C.4a since it is either transcendentally small or transcendentally smaller than $W'(R_B)\varepsilon^{-1}e^{-A/\varepsilon}$ in C.4b. After neglecting this term, manipulation of eqs. C.4a and C.4b yields

C.5a $$R_E' = 1 - \frac{P}{W'} + \varepsilon\frac{W''}{W'}\{\tfrac{1}{2}M_2 - R_E' + (R_E')^2\} + \varepsilon R_E'' + \ldots,$$

C.5b $$W'(R_B)\varepsilon^{-1}e^{-A/\varepsilon} = -\frac{W''}{W'}\left\{P + \varepsilon\left(\frac{W'W'''}{W''} - W''\right)\left(\tfrac{1}{2}M_2 - R_E' + (R_E')^2\right) + \ldots\right\}.$$

Eq. C.5a shows that $R_E' = 1 - P/W'$ to leading order. Substituting this into the higher order terms in C.5a, C.5b yields

C.6a $$R_E' = 1 - \frac{P}{W'} + \tfrac{1}{2}\varepsilon M_2\frac{W''}{W'} + \ldots$$

and changes the last term in C.5b to

$$\varepsilon\left(\frac{W'W'''}{W''} - W''\right)\left(\tfrac{1}{2}M_2 - (1 - P/W')P/W'\right).$$

When $R_B + R_E$ is $O(1)$, this term is $O(\varepsilon)$ and can be neglected compared to P. When $R_B + R_E << 1$, then $(1 - P/W')P/W'$ can be neglected compared to $M_2/2$. In either case C.5b simplifies to

C.6b $$W'(R_B)\varepsilon^{-1}e^{-A/\varepsilon} = -\frac{W''}{W'}\left\{P + \tfrac{1}{2}\varepsilon M_2\left(\frac{W'W'''}{W''} - W''\right) + \ldots\right\}$$

to leading order.

Note that if the factor $e^{-A/\varepsilon}$ is set to zero in C.6b, then eq. C.6a and C.6b reduce to the regular perturbation solution of §5. Accordingly, we can devise a uniformly valid expansion by replacing this factor by one which is always transcendentally small:

C.7a $$W'(R_B)\varepsilon^{-1}e^{-a/\varepsilon} = -\frac{W''}{W'}\left\{P + \tfrac{1}{2}\varepsilon M_2\left(\frac{W'W'''}{W''} - W''\right) + \ldots\right\}$$

where

C.7b $$a = R_0(\varepsilon) \quad \text{if } A < R_0(\varepsilon)\,, \qquad a = A \quad \text{if } A > R_0(\varepsilon)\,.$$

Eqs. C.6a and C.7 are the equations needed in § 5.

Appendix D. Post-exploration prices. Suppose the economy is at some point A,R in the exploration region $R \le R_B(A)$. Let A^+ and R^+ be the point where the economy first ceases exploration, and let Z(A,R) be the expected land price p_A at this point:

D.1 $$Z(A,R) = E\{V_A(A^+, R^+)\}$$

There are three possibilities. With probability $e^{-A/\varepsilon}$, exploration ceases without making a discovery and the post-exploration price is $V_A(0,R)$. With probability $\varepsilon^{-1} e^{-(A-A')/\varepsilon} g(s)\, ds\, dA'$, the first discovery occurs at A' and increases the reserves to $R' = R + \varepsilon s$. If $R' < R_B(A')$ then exploration ceases and the post-exploration price is $V_A(A',R')$. If $R' < R_B(A')$ then exploration continues, and the expected price when the economy ceases exploration is Z(A',R'). Accounting for these possibilities, we have

D.2 $$Z(A,R) = e^{-A/\varepsilon}V_A(0,R) + \frac{1}{\varepsilon}\int_0^A e^{-(A-A')/\varepsilon}\int_0^\infty g(s)V_A(A',R+\varepsilon s)\,ds\,dA'$$

$$\frac{1}{\varepsilon^2}\int_0^A e^{-(A-A')/\varepsilon}\int_R^{R_B(A')} g([R'-R]/\varepsilon)[Z(A',R') - V_A(A',R')]\,dR'\,dA'\,.$$

Now, differentiating 3.16b with respect to A yields

D.3 $$\varepsilon V_{AA}(A,R) + V_A(A,R) = \int_0^\infty g(s)V_A(A,R+\varepsilon s)\,ds \qquad \text{for } R < R_B(A)\,,$$

and substituting into D.2 yields

D.4 $$Z(A,R) = e^{-A/\varepsilon}V_A(0,R) + \int_0^A \frac{d}{dA'}\left\{e^{-(A-A')/\varepsilon}V_A(A',R')\right\}dA'$$

$$\frac{1}{\varepsilon^2}\int_0^A\int_R^{R_B(A')} e^{-(A-A')/\varepsilon}g([R'-R]/\varepsilon)[Z(A',R') - V_A(A',R')]\,dR'\,dA'$$

Integrating D.4,

D.5 $$Z(A,R) - V_A(A,R) = \frac{1}{\varepsilon^2}\int_0^A\int_R^{R_B(A')} e^{-(A-A')/\varepsilon}g([R'-R]/\varepsilon)[Z(A',R') - V_A(A',R')]\,dR'\,dA'\,.$$

Equation D.5 is a homogeneous integral equation whose kernel is positive and satisfies

D.6 $$\frac{1}{\varepsilon^2}\int_0^A\int_R^{R_B(A')} e^{-(A-A')/\varepsilon}g([R'-R]/\varepsilon)\,dR'\,dA' < 1\,.$$

Consequently, the solution is $Z(A,R) - V_A(A,R) \equiv 0$, so

D.7 $$E\{V_A(A^+,R^+)\} = V_A(A,R) \qquad \text{for } R \le R_B(A)\,.$$

An indentical argument shows that

D.7b $$E\{V_R(A^+,R^+)\} = V_R(A,R) \qquad \text{for } R \le R_B(A)\,.$$

Consequently, once the economy enters the exploration phase, the expected price of land and resource immediately after exploration ceases is identical to the prices on entering.

Figure Captions.

Figure 1. In D_0, where the proven reserves R exceed minimumacceptable reserves $R_B(A)$, the optimal policy is no exploration (x = 0) and consuming at the rate $U'(c) = V_R(A,R)$. In D_∞, where $R < R_B(A)$, the optimal policy is to explore infinitely rapidly until new discoveries place the economy above the curve $R_B(A)$.

Figure 2. Comparison between the exact $R_B(A)$ and $R_E(A)$ (solid curves) and the asymptotic solutions given by 5.7 (dashed curves). The curves were computed using $U(c) = \alpha \log c$ with $g(s) = \delta(s-1)$ and $P\rho/\alpha = 1$, $\varepsilon = 0.01$. The asymptotic solution for $R_E(A)$ is indistinguishable from the exact solution.

Figure 3. Comparison between the exact $R_B(A)$ and $R_E(A)$ (solid curves) and the approximate results given by 6.4 (dashed curves). The curves were computed using $U(c) = \alpha \log c$ with $g(s) = \delta(s-1)$ and $P\rho/\alpha = 1$, $\varepsilon = 0.1$. The approximate solution for $R_E(A)$ is nearly indistinguishable from the exact solution.

Figure 4. Same as Figure 3, except that $\varepsilon = 1.0$.

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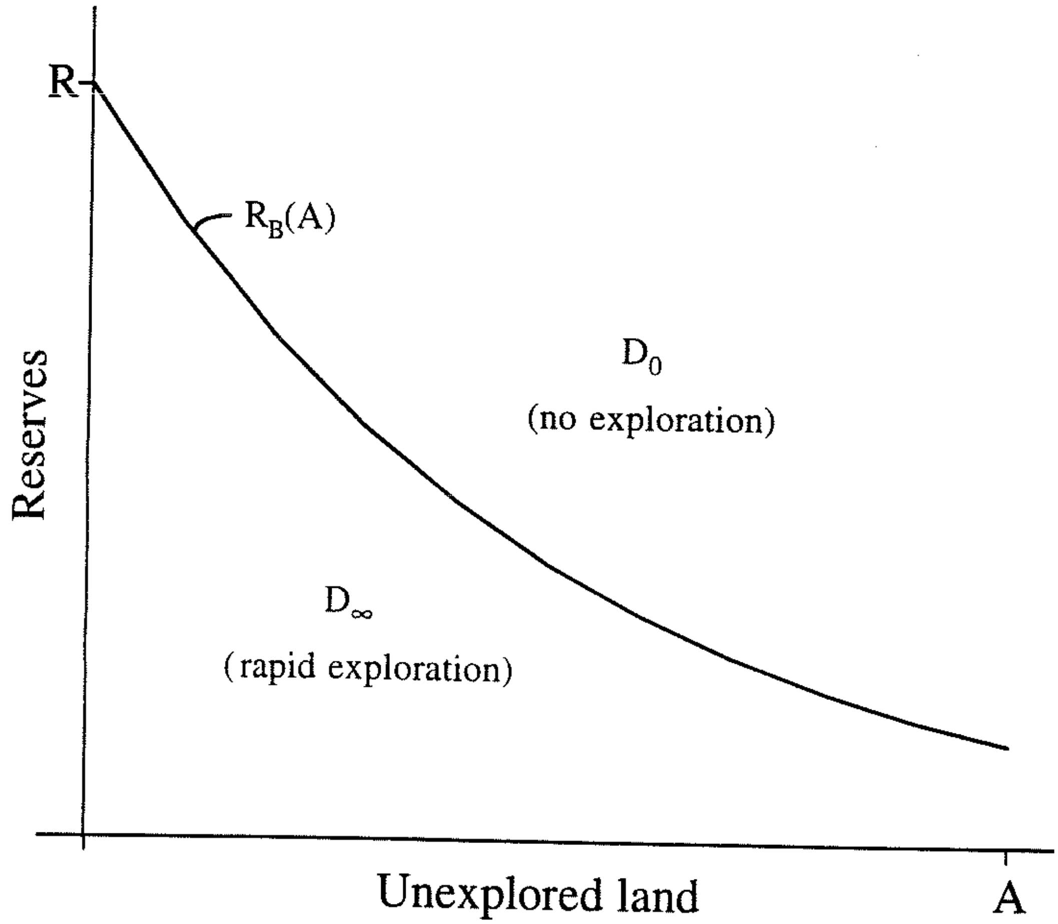
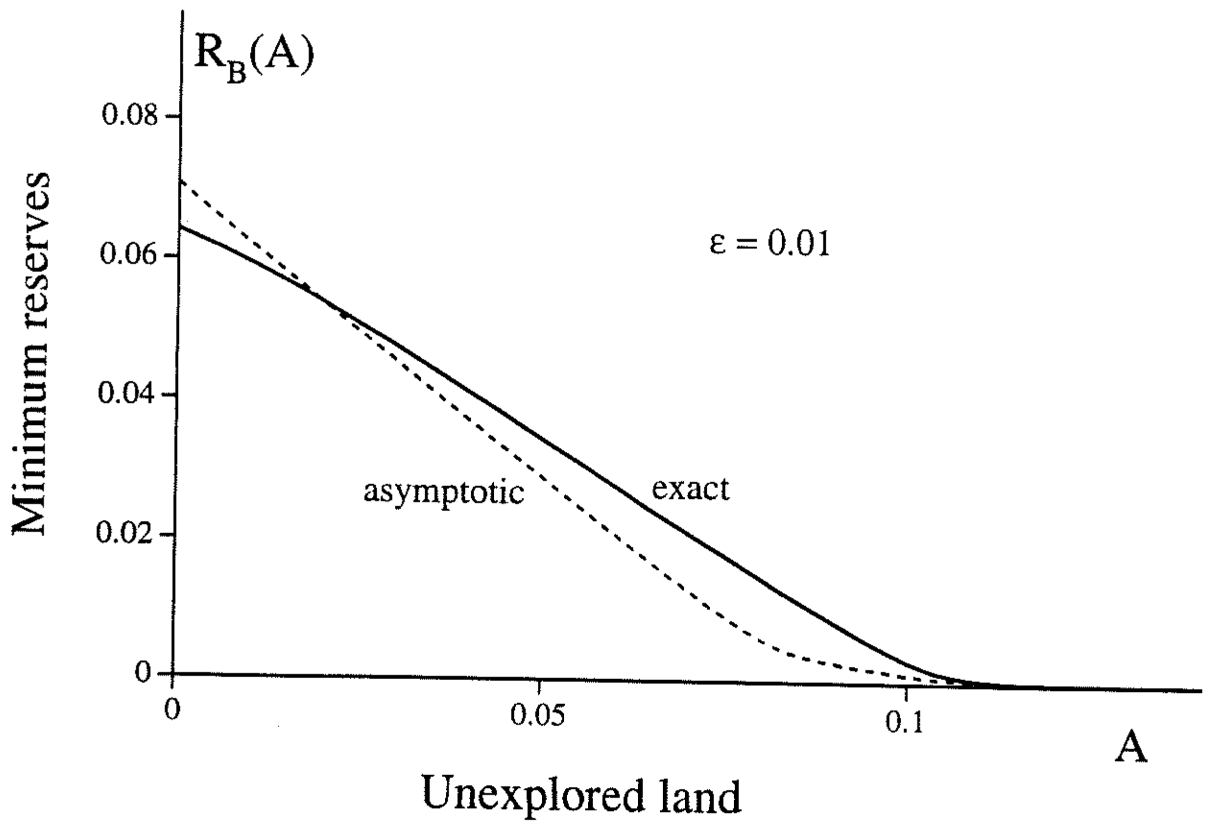


R
$R_B(A)$
D_0
(no exploration)
D_∞
(rapid exploration)
Reserves
Unexplored land
A

Figure 1



$R_B(A)$
0.08
0.06
0.04
0.02
0
Minimum reserves
ε = 0.01
asymptotic
exact
0
0.05
0.1
A
Unexplored land

Figure 2a

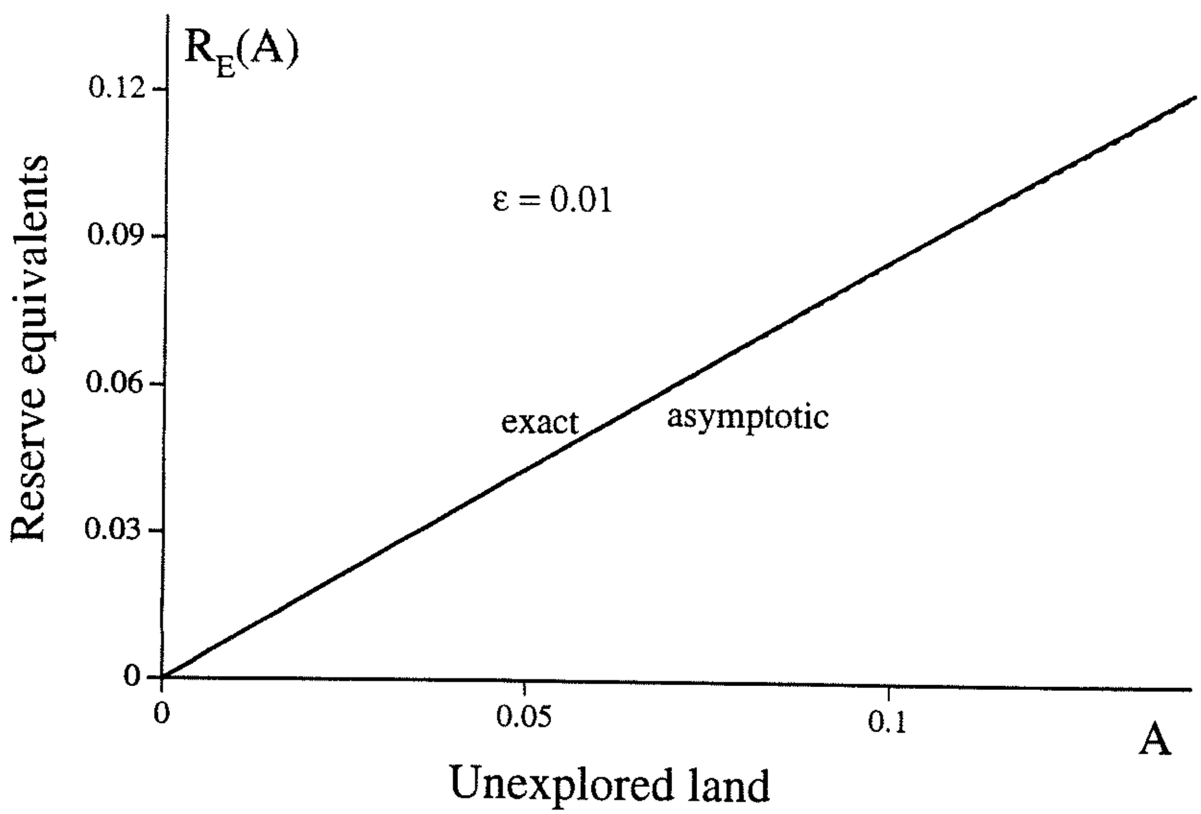
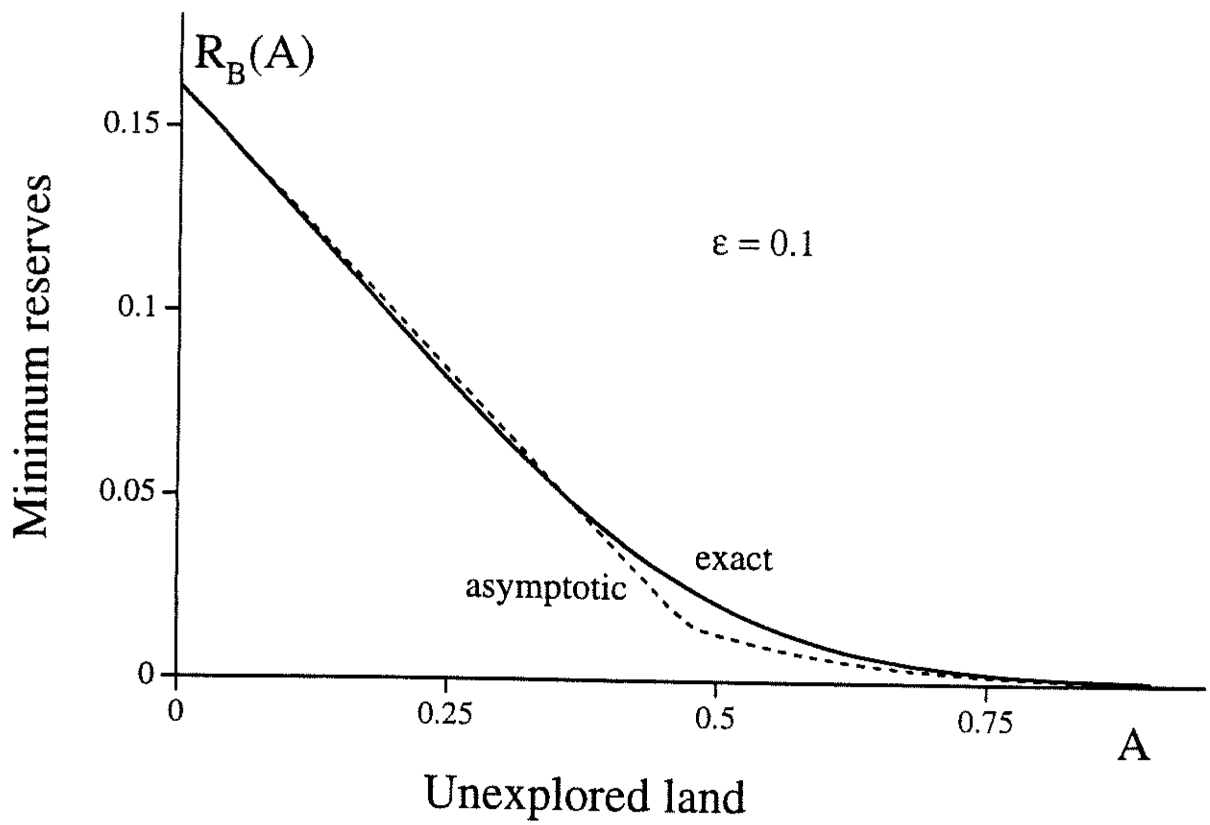

$R_E(A)$
0.12
0.09
0.06
0.03
0
Reserve equivalents
$\varepsilon = 0.01$
exact
asymptotic
0
0.05
0.1
A
Unexplored land

Figure 2b



$R_B(A)$
0.15
0.1
0.05
0
Minimum reserves
$\varepsilon = 0.1$
exact
asymptotic
0
0.25
0.5
0.75
A
Unexplored land

Figure 3a

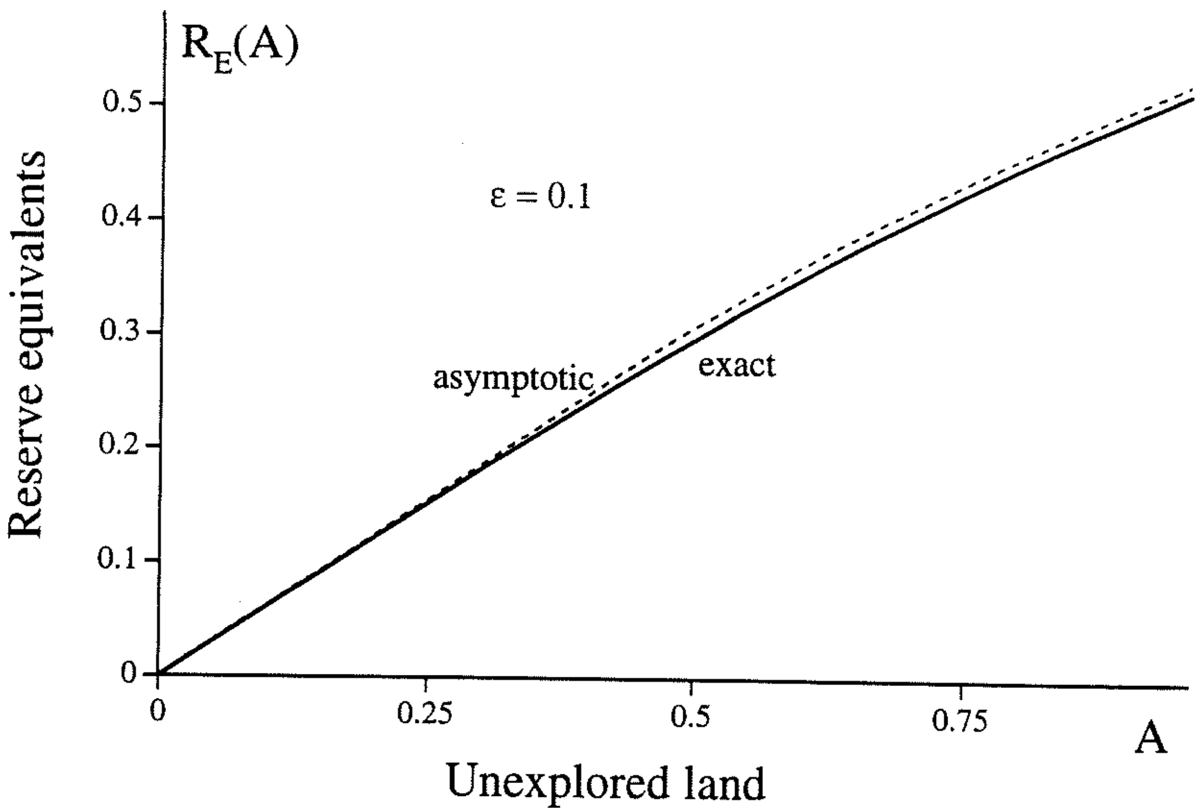
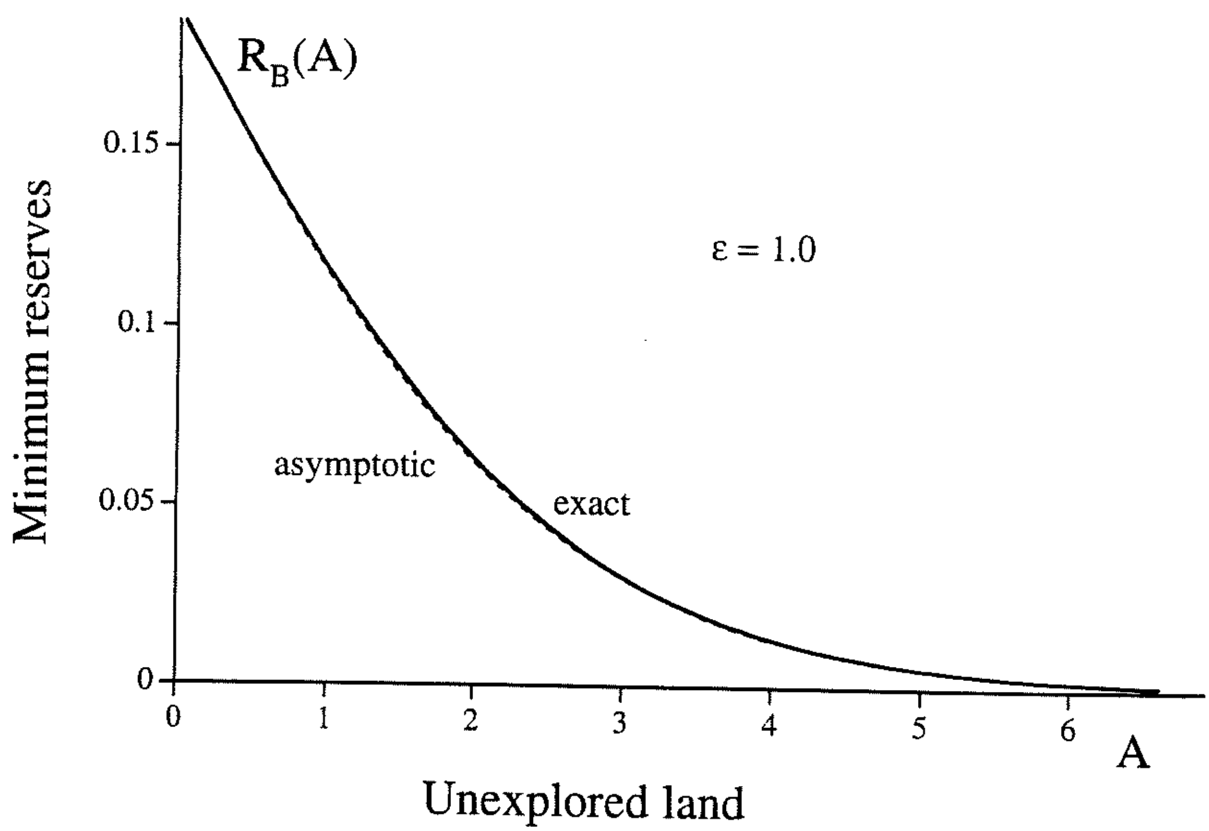
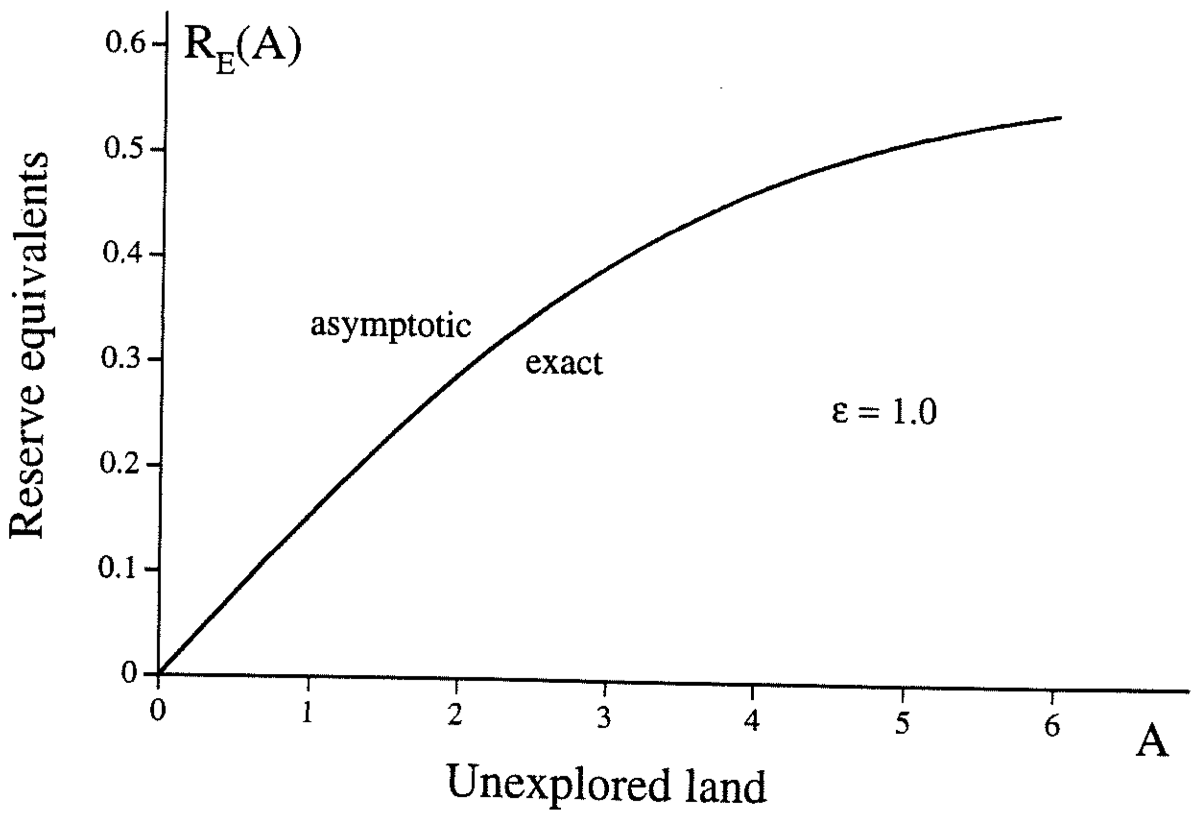

$R_E(A)$
0.5
0.4
0.3
0.2
0.1
0
Reserve equivalents
$\varepsilon = 0.1$
asymptotic
exact
0
0.25
0.5
0.75
A
Unexplored land

Figure 3b



$R_B(A)$
0.15
0.1
0.05
0
Minimum reserves
$\varepsilon = 1.0$
asymptotic
exact
0
1
2
3
4
5
6
A
Unexplored land

Figure 4a



0.6
0.5
0.4
0.3
0.2
0.1
0
$R_E(A)$
Reserve equivalents
asymptotic
exact
$\varepsilon = 1.0$
0
1
2
3
4
5
6
A
Unexplored land

Figure 4b