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Russel E. Caflisch
Peter Smereka

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

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Department of Mathematics
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ABSTRACT. Singularities are found in solutions of three systems that model the equations for axi-symmetric swirling flow. The first example is a simple first order system in "Jordan form." The second is a one dimensional analogue of the 2D Boussinesq system. In the third example, a complex solution of the axi-symmetric swirling flow equations is numerically constructed. This solution is a traveling wave with a complex wave speed that brings a singularity from the complex plane to a real position at a finite real time. The complex-valued flow can be understood as coming from a solution Moore's approximation for axi-symmetric swirling flow.

1. Introduction

A principal outstanding question in the mathematical theory of fluid dynamics is whether singularities can form in finite time for solutions of the 3D incompressible, inviscid Euler equations. The significance of this problem lies first in its relation to existence theory for these equations. Singularities, if they exist for even a small set of possible initial data, would probably also cause significant problems for numerical computations and could be a source of development of small scales in the onset of turbulence.

The purpose of this paper is to present several singular solutions of simple models equations for axi-symmetric flow with swirl. Some of the features of these examples may provide mathematical insight into the process of singularity formation: The first two examples show that, for a first order system in "Jordan form," the off-diagonal terms can cause singularity development. Since the swirling flow equations may be written as a first order system in Jordan form with the vortex stretching terms as off-diagonal elements, we conjecture that singularity formation for swirling flow (if it occurs) can be understood as caused by this Jordan structure.

The third example presents a numerically determined solution of the full equations for axi-symmetric swirling flow. The solution is complex-valued and can be understood as coming from Moore's approximation for swirling flow. A singularity in this solution is clearly seen through an analysis of the asymptotics of the Fourier coefficients. The simplicity of the resulting singularity structure suggests that it may be a generic singularity form. This example also shows that singularities may move around in the complex space plane until they hit the real line at some finite time.

The main analytic result on singularity formation for 3D inviscid, incompressible flow is that of Beale, Kato & Majda 1989 , who showed that if an initially smooth flow develops a singularity at time t, then

$$\int_0^t \sup_{\boldsymbol{x}} |\omega(\boldsymbol{x}\ ,t')|dt' = \infty. \tag{1.1}$$

A similar result, that loss of analyticity implies blow-up of vorticity, was proved by Bardos & Benachour 1977 . For inviscid, incompressible flow in two dimensions, an initially smooth velocity field $\boldsymbol{u}\ (\boldsymbol{x}\ ,0)$ will stay smooth for all time due to the conservation of vorticity.

The numerical search for singularities was started on the Taylor-Green flow by Brachet *et al.* 1983 , for which the results indicate a singularity for a complex value of t but not for real t. Experimental results showing vortex reconnection for high-Reynolds number flows inspired a study of singularities for a filament model of two interacting vortex tubes by Siggia 1985 . The singularities in this model equation are smoothed out however due to flattening of the vortex cores, as seen in computations of Anderson & Greengard 1989 , Pumir & Siggia 1990 and Shelley, Meiron & Orszag 1992 . More recently Kerr 1992 presented a new set of computations for this problem which exhibit intensification of vorticity that may indicate singularity formation. Several related studies of singularity formation are in Bhattacharjee & Wang 1992, Bell & Marcus 1992, Childress *et al.* 1989, Chorin 1981, Chorin 1982 and Stuart 1989 and in the contributions of Brachet *et al.*, Kerr and Pumir & Siggia in these Proceedings.

Swirl in axi-symmetric flows amplifies vorticity by stretching, and the axi-symmetry may prevent core flattening, so that such flows seem a likely candidate for singularities. Grauer & Sideris 1992 , Meiron & Shelley 1992 and Pumir & Siggia 1992 have performed computations of axi-symmetric flow with swirl that show significant vortex stretching but no singularities within the computational time. In Pumir & Siggia 1992 , however, an adaptive numerical method, with nonlinear scaling of $\boldsymbol{x}$ and t, was used on an asymptotically reduced equation to produce singularities for axi-symmetric flow with swirl. Some critical remarks on this results are presented in Bhattacharjee & Wang 1991 . A study of singularity formation for axi-symmetric, swirling vortex sheets was performed by Caflisch, Li & Shelley 1992 .

2. Axi-Symmetric Flow with Swirl

The Euler equations for incompressible, inviscid flow are

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p &= 0 \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned} \tag{2.1}$$

The simplest flow in which there is non-trivial vortex stretching is *axi-symmetric flow with swirl* for which the velocity $\boldsymbol{u} = (u_r, u_\theta, u_z)$ in cylindrical coordinates satisfies $\partial_\theta \boldsymbol{u} = 0$. The swirl is the azimuthal velocity u_θ. The Euler equations for axi-symmetric swirling flow are

$$\partial_z u_z + r^{-1}\partial_r(r u_r) = 0 \tag{2.2}$$

$$\partial_t u_z + \boldsymbol{u} \cdot \nabla u_z = -\partial_z p \tag{2.3}$$

$$\partial_t u_r + \boldsymbol{u} \cdot \nabla u_r - r^{-1}u_\theta^2 = -\partial_r p \tag{2.4}$$

$$\partial_t u_\theta + \boldsymbol{u} \cdot \nabla u_\theta + r^{-1}u_r u_\theta = 0 \tag{2.5}$$

in which $\boldsymbol{u} \cdot \nabla = u_z\partial_z + u_r\partial_r$. The vorticity is

$$\omega = (\omega_r, \omega_\theta, \omega_z) = (-\partial_z u_\theta,\ \partial_z u_r - \partial_r u_z,\ r^{-1}\partial_r(r u_\theta)). \tag{2.6}$$

A convenient reformulation of these equations can be made in terms of stream function and vorticity. Define circulation $\Omega = r u_\theta$, azimuthal vorticity $\zeta = -r\omega_\theta$ and stream function ψ with

$$(u_r, u_z) = r^{-1}(-\partial_z\psi, \partial_r\psi). \tag{2.7}$$

Denote

$$D^2 = r\partial_r(r^{-1}\partial_r) + \partial_z^2. \tag{2.8}$$

Then the Euler equations for axi-symmetric swirling flow are equivalent to

$$\begin{aligned} (\partial_t + \boldsymbol{u} \cdot \nabla)\Omega &= 0 \\ (\partial_t + \boldsymbol{u} \cdot \nabla)(r^{-2}\zeta) &= -r^{-4}\partial_z(\Omega^2) \\ D^2\psi &= \zeta. \end{aligned} \tag{2.9}$$

The characteristic structure of these equations is most easily seen by transformation to first order form. Set

$$\begin{aligned} \rho &= \Omega^2 \\ \gamma &= r^{-2}\zeta \\ u &= u_r + iu_z \\ v &= u_r - iu_z \\ \alpha &= r + iz \\ \beta &= r - iz \end{aligned} \tag{2.10}$$

Then the axi-symmetric swirling flow equations (2.9) are equivalent to

$$
\begin{aligned}
(\partial_t + u\partial_\alpha + v\partial_\beta)\rho &= 0 \\
(\partial_t + u\partial_\alpha + v\partial_\beta)\gamma &= -ir^{-4}(\partial_\alpha - \partial_\beta)\rho \\
\partial_\alpha u &= a \\
\partial_\beta v &= b
\end{aligned}
\tag{2.11}
$$

in which

$$
\begin{aligned}
a &= -r^{-1}(u+v) + 2ir\gamma \\
b &= -r^{-1}(u+v) - 2ir\gamma.
\end{aligned}
\tag{2.12}
$$

In this complex formulation of the Euler equations, the system has been diagonalized, with characteristic velocity vectors $(1, u, v)$, $(0, 1, 0)$ and $(0, 0, 1)$ in (t, α, β). The first of these velocities, representing the particle speed, is a double characteristic, and there is an irremovable "off-diagonal" term $ir^{-4}(\partial_\alpha - \partial_\beta)\rho$ in the second equation. This term is analogous to the off-diagonal element in a Jordan block, since it consists of a derivative applied to the function ρ, a characteristic variable for the same speed We will refer to this form of the Euler equations as their *Jordan form.* Note that the off-diagonal term is exactly the vortex stretching term.

The next two sections present simple examples of singularity formation for first order systems in Jordan form. On this basis, we conjecture that singularity formation for the Euler equations (if it occurs) can be understood as caused by the Jordan form of the equations.

3. A Simple System in Jordan Form

The simplest first order system in Jordan form is just Burgers equation with a second equation such as

$$
\begin{aligned}
f_t + ff_x &= 0 \\
g_t + (fg)_x &= 0.
\end{aligned}
\tag{3.1}
$$

This system was studied earlier by Forestier & LeFloch 1992 who noted that the Riemann problem for this system does not have a general solution. In fact the solution of this system can be written in a simple implicit form as

$$
f(x,t) = f_0(x_0) \tag{3.2}
$$

$$
g(x,t) = \frac{g_0(x_0)}{1 + tf_0'(x_0)} \tag{3.3}
$$

$$
x = x_0 + tf_0(x_0). \tag{3.4}
$$

Since

$$\frac{\partial}{\partial x} f = \frac{\partial f_0}{\partial x_0}\left(\frac{\partial x}{\partial x_0}\right)^{-1}$$
$$= \frac{f_0'(x_0)}{1+tf_0'(x_0)} \quad (3.5)$$

then singularities in the derivative of f occur where $1+tf_0'(x_0)=0$, and also cause g to blowup. In particular $g \to \infty$ when a shock forms in f.

A simple variation of this system is also exactly solvable but has singularities of much worse form. Consider the system

$$f_t + ff_x = 0$$
$$g_t + fg_x + g^n f_x = 0. \quad (3.6)$$

The solution of this equation for $n \neq 1$ is

$$f(x,t) = f_0(x_0) \quad (3.7)$$
$$g(x,t) = \left((1-n)\log(1+tf_0'(x_0)) + g_0(x_0)^{1-n}\right)^{\frac{1}{1-n}} \quad (3.8)$$
$$x = x_0 + tf_0(x_0). \quad (3.9)$$

Note that the form of the singularity is now a power of a logarithm.

4. A 1D Model of the Boussinesq System

As pointed out by Pumir & Siggia 1992 , a singularity for axi-symmetric swirling flow should be approximately described by the slightly simpler Boussinesq system

$$(\partial_t + u\partial_r + v\partial_z)\Omega = 0$$
$$(\partial_t + u\partial_r + v\partial_z)\zeta = \partial_z(\Omega^2) \quad (4.1)$$
$$(\partial_r^2 + \partial_z^2)\psi = \zeta.$$
$$(u,v) = (-\partial_r\psi, \partial_z\psi)$$

A simple one dimensional model for this system is the system

$$f_t + uf_x = 0$$
$$g_t + ug_x = f_x$$
$$\psi_{xx} = g \quad (4.2)$$
$$u = \psi_x$$

The modified system (4.2) has a Jordan form similar to the of (4.1); since it is only one-dimensional, however, it does not include incompressibility.

The "stream function" ψ can be eliminated by replacing the last two equation of (4.2) with

$$u_x = g \tag{4.3}$$

This system can be solved using a Lagrangian coordinate ξ, defined so that

$$\frac{\partial}{\partial t}x(\xi,t) = u. \tag{4.4}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t}f(\xi,t) &= 0 \\ \frac{\partial}{\partial t}g(\xi,t) &= f_x = f_\xi x_\xi^{-1} \\ u_\xi &= g x_\xi \end{aligned} \tag{4.5}$$

with initial data

$$(f,g,x)(\xi,0) = (f_0,g_0,x_0)(\xi). \tag{4.6}$$

Elimination of f and x yields the equation

$$g_{tt} = -g g_t \tag{4.7}$$

so that

$$g_t = -\frac{1}{2}g^2 + \frac{1}{2}g_0^2 + f_0'/x_0'. \tag{4.8}$$

The solution is

$$g = \sqrt{2d}\cot[(t-t_0)\sqrt{d/2}] \tag{4.9}$$

in which

$$t_0 = \sqrt{2/d}\left(\frac{\pi}{2} + \tan^{-1}(\frac{g_0}{\sqrt{2d}})\right) \tag{4.10}$$

and

$$d = -\frac{1}{2}g_0^2 - f_0'/x_0'. \tag{4.11}$$

We also find

$$u_\xi = -\frac{f_\xi}{\sqrt{2d}}\sin[(t-t_0)\sqrt{2d}]. \tag{4.12}$$

Therefore we see that the solution of (4.8) blows up with the asymptotic form

$$\begin{aligned} g &\approx \frac{2}{t-t_0} \\ x_\xi &\approx -\frac{1}{2}f_\xi(t-t_0)^2 \\ u_\xi &\approx -f_\xi(t-t_0) \end{aligned} \tag{4.13}$$

near the singularity. In particular this shows that the "vorticity" g blows up, while the "velocity" u does not blow up at the singularity.

5. Moore's Approximation for Axi-Symmetric Flow with Swirl

Moore's approximation keeps nonlinear terms that transfer energy to higher wavenumbers, which is expected to be the dominant process in singularity formation, but neglects terms that transfer energy back to smaller wavenumbers. First formulated by Moore 1979 and 1985 for singularity formation on vortex sheets during Kelvin-Helmholtz instability, this method was generalized by Caflisch, Orellana & Siegel 1989 and applied to singularity formation in the Rayleigh-Taylor problem by Baker, Caflisch & Siegel 1992 . We will describe here the application of Moore's approximation to axi-symmetric flow with swirl, as carried out in Caflisch 1992 .

First decompose the velocity $\boldsymbol{u}$ into three pieces, as

$$\boldsymbol{u} = \boldsymbol{u}_+ + \boldsymbol{u}_- + \boldsymbol{u}_0 \tag{5.1}$$

in which $\boldsymbol{u}_+$ contains the positive z-wavenumbers, $\boldsymbol{u}_-$ contains the negative wavenumbers and $\boldsymbol{u}_0$ is constant in z; i.e.,

$$\begin{aligned} \boldsymbol{u}_+ &= \sum_{k=1}^{\infty} \hat{\boldsymbol{u}}_k(r,t)e^{ikz} \\ \boldsymbol{u}_- &= \sum_{k=-1}^{-\infty} \hat{\boldsymbol{u}}_k(r,t)e^{ikz} \\ \boldsymbol{u}_0 &= \boldsymbol{u}_0(r). \end{aligned} \tag{5.2}$$

Moore's approximation splits the Euler equations (2.2-2.5) into two sets of Euler equations, each with the same form as (2.2-2.5). The first system is for $\boldsymbol{u}_+ + \boldsymbol{u}_0$, and the second, which is the conjugate of the first, is for $\boldsymbol{u}_- + \boldsymbol{u}_0$.

We now look for solutions $\boldsymbol{u}$ for (2.2-2.5) of the form

$$\boldsymbol{u} = \sum_{k=0}^{\infty} \hat{\boldsymbol{u}}_k(r,t)e^{ikz}. \tag{5.3}$$

As discovered by Siegel 1989 (and further described in Baker, Caflisch & Siegel 1992) for the Rayleigh-Taylor problem, solutions of this form may be conveniently sought as traveling waves with an imaginary wave speed $i\sigma$ in the z direction. Thus we look for $\boldsymbol{u} = \boldsymbol{u}(r, z - i\sigma t)$; i.e.,

$$\boldsymbol{u} = \sum_{k=0}^{\infty} \hat{\boldsymbol{u}}_k(r)e^{ikz+k\sigma t}. \tag{5.4}$$

This solution can also be interpreted as consisting only of purely growing modes.

After substitution into the system (2.2-2.5) and some manipulation, the equations for $f_k(r) = \hat{u}_r(k,r)$ are

$$\partial_r(r^{-1}\partial_r(rf_k)) - (k^2+\kappa)f_k = D_k \tag{5.5}$$

in which $D_k = D_k(f_0, ..., f_k - 1)$ depends only on f_l for $l < k$ and

$$\kappa(r) = \frac{2\bar{u}_\theta\bar{\omega}_z}{\sigma^2 r} \tag{5.6}$$

with $\bar{u}_\theta, \bar{\omega}_z$ the azimuthal velocity and vorticity for the zeroth mode $\boldsymbol{u}_0$. The zeroth component $\boldsymbol{u}_0 = \bar{u}_\theta(r)\boldsymbol{\theta}$ is a simple steady swirling flow. The first term f_1 represents a linearly unstable mode with growth rate σ for the steady flow $\boldsymbol{u}_0$, satisfying the homogeneous equation

$$\partial_r(r^{-1}\partial_r(rf_1)) - (1+\kappa)f_1 = 0. \tag{5.7}$$

The remarkable property of this solution, as found by Siegel 1989 , is that the traveling wave speed is determined by the linear eigenvalue problem (5.7), and that the equations (5.7) for f_k are "lower triangular." This makes their numerical solution fairly straightforward.

This problem was numerically solved for flow in an annulus $1 < r < 3$ with boundary conditions $u_r(r=1) = u_r(r=3) = 0$; i.e.,

$$f_k(1) = f_k(3) = 0 \tag{5.8}$$

for all k. First a basic state $\bar{u}_\theta(r)$ and a linearly unstable mode $f_1(r)$ were determined as a smoothed out version of the explicit steady flow and unstable mode for a cylindrical vortex sheet in an annulus, as determined in (Caflisch 1992). This smoothing was performed in a way that did not require solving a new eigenvalue problem, so that numerical error was minimized.

Once $\bar{u}_\theta(r)$ and $f_1(r)$ were determined, the equations (5.5) for f_k with $k > 1$ were solved without any ambiguity. Solution of the iteration equations (5.5) was delicate, however, because of an instability with respect to increase of the wavenumber k. This numerical instability amplified round-off error, as in the problem of singularity formation for vortex sheets. Krasny was able to remove the spurious growth of roundoff error in the Kelvin-Helmholtz problem through use of either a nonlinear filter (Krasny 1986a) or a smoothing parameter (Krasny 1986b) . In the present problem, the filtering method did not work since the problem is not periodic in r and the smoothing method was found to be too crude to yield information on the singularity formation. Instead, for this simple problem the effect of roundoff error was adequately minimized by the use of ultra-high precision arithmetic. Precision levels of 10^{-64} and 10^{-128} were implemented using the package MPFUN developed by David Bailey at NASA Ames (Bailey 1991a and 1991b) . This package comes with an automated translator that translates a standard FORTRAN program into a program that calls special high precision arithmetic subroutines for every arithmetic statement.

The forcing terms D_k in equations (5.5) were evaluated through a hybrid spectral-pseudospectral method with no aliasing or closure error. The computational complexity was $O(N_r N_z^{3/2})$ for N_r points in r and N_z wavenumbers. Since there was no closure error in the computation of the first N_z modes, we chose a fixed number of modes in z, and then refined the discretization of r until the results converged. The most refined computation for $N_z = 64$ used $N_r = 2048$ and agreed well with the results for $N_r = 1024$.

Singularities were analyzed through the asymptotics of the Fourier components f_k. Since the solution computed here has a symmetry with respect to reflection about $z = 0$, two singularities were expected to occur at positions

$$z_\pm(r) = -i\rho_1 \pm \rho_2. \tag{5.9}$$

Near each of these singularities the structure of the radial velocity u_r was sought in the form

$$u_r(z,r) \simeq c_1 e^{ic_2}(z - z_\pm)^{-1-(\alpha_1 - i\alpha_2)} \tag{5.10}$$

for z near $z_\pm$. As shown in Sulem *et al.* 1983 , this implies that the Fourier coefficients f_k will have the asymptotic form

$$f_k \cong c_1 k^{-\alpha_1} e^{-\rho_1 k} \sin(c_2 + \alpha_2 \log k + \rho_2 k) \tag{5.11}$$

for $k >> 1$. In (5.9- 5.11) the parameters $(c_1, c_2, \alpha_1, \alpha_2, \rho_1, \rho_2)$ depend on r, i.e.,

$$(c_1, c_2, \alpha_1, \alpha_2, \rho_1, \rho_2) = (c_1, c_2, \alpha_1, \alpha_2, \rho_1, \rho_2)(r). \tag{5.12}$$

Following Baker, Caflisch & Siegel 1992, Pugh 1989 and Pugh & Cowley 1992 , $(c_1, c_2, \alpha_1, \alpha_2, \rho_1, \rho_2)$ were determined through a sliding fit; i.e., for each k, the 6 parameters were chosen to exactly fit the 6 values $f_k, f_{k+1}, \cdots, f_{k+5}$. The asymptotic fit was successful since the values $(c_1, c_2, \alpha_1, \alpha_2, \rho_1, \rho_2)$ were found to be (nearly) independent of the starting index k, as well as independent of the discretization size $dr = 2\pi/N_r$.

The parameters α_1 and α_2 were found to be $\alpha_1 = -\frac{2}{3}\pm.01$ and $\alpha_2 = 0.\pm.01$ independent of r. The amplitude term c_1 is also nearly constant; while the phase c_2 and the real position ρ_2 are approximately linear in r. Since the singularity moves toward the real axis at speed σ in the imaginary z direction, the first singularity in time corresponds to the minimum value of the complex position $\rho_1(r)$ for $r = r_{\min}$. This was seen to occur at approximately $r = 2$, near which ρ_1 was found to be approximately quadratic in $r - r_{\min}$.

Now the time dependence is included by replacing z with $z - i\sigma t$. Using the above results, for r near $r_{\min}$ and z near $z_+ + i\sigma t$ we approximate

$$\begin{aligned} c_1 &= \bar{c}_1, \quad c_2 = \bar{c}_2 \\ \rho_1 &= \bar{\rho}_1 + \bar{\rho}_{11}(r - r_{\min})^2 \\ \rho_2 &= \bar{\rho}_2 + \bar{\rho}_{21}(r - r_{\min}) \\ \alpha_1 &- i\alpha_2 = -2/3. \end{aligned} \tag{5.13}$$

Define new orthogonal variables centered at the real singularity position and time by

$$\begin{aligned}\tau &= \sigma t - \bar{\rho}_1 \\ \zeta &= (z - \bar{\rho}_2) - \bar{\rho}_{21}(r - r_{\min}) \\ \eta &= \sqrt{\bar{\rho}_{11}}((r - r_{\min}) + \bar{\rho}_{21}(z - \bar{\rho}_2))\end{aligned} \tag{5.14}$$

The form of the singularity in (5.10) is then

$$u_r(z, r, t) \cong c\xi^{-1/3} \tag{5.15}$$

in which the singularity variable ξ is

$$\xi = \eta^2 - \tau - i\zeta. \tag{5.16}$$

The corresponding stream function is

$$\psi(z, r, t) \cong c\xi^{2/3}. \tag{5.17}$$

This shows that the singularity occurs at the centers of oval-shaped rolls which are flattening as the singularity forms. Further details of this solution, as well as for the numerical method, can be found in Caflisch 1992 .

6. Conclusions

The examples above provide some relatively simple contexts in which singularity formation can be seen more or less explicitly. Two principles seem to be at work in these examples: First the off diagonal elements in the Jordan form for a first order system can lead to singularity formation. Second, singularities can be understood as moving in the complex plane, until the real singularity time when they hit the real axis. This same point of view is espoused by the contributions of Baker & Tanveer and Cowley in these proceedings.

The ultimate goal of this investigation is to describe singularity formation for the 3D Euler equations (or to show that singularities do not form). A second goal is to identify the generic forms of singularities for 3D Euler, as performed for first order systems in diagonal form with two speeds in Caflisch, Ercolani, Hou & Landis 1992 . The simplicity of the singularity found in Section 5 suggests that such a form may be generic.

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