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Almost Optimal Convergence of the Point Vortex Method for Vortex Sheets using Numerical Filtering

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Abstract

Standard numerical methods for the Birkhoff-Rott equation for a vortex sheet are ill-posed due to amplification of roundoff error by the Kelvin-Helmholtz instability. A nonlinear filtering method was used by Krasny to eliminate this spurious growth of round-off error and accurately compute the Birkhoff-Rott solution essentially up to the time it becomes singular. In this paper convergence is proved for the discretized Birkhoff-Rott equation with Krasny filtering and simulated roundoff error. The convergence is proved for a time almost up to the singularity time of the continuous solution. The proof is in an analytic function class and uses a discrete form of the abstract Cauchy-Kowalewski theorem. In order for the proof to work almost up to the singularity time, the linear and nonlinear parts of the equation, as well as the effects of Krasny filtering, are precisely estimated. The technique of proof applies directly to other ill-posed problems such as Rayleigh-Taylor unstable interfaces in incompressible, inviscid and irrotational fluids, as well as to Mullins-Sekerka unstable interfaces in Hele-Shaw cells.

Key words: vortex sheets, point vortices, numerical filtering, discrete Cauchy-Kowalewski theorem.

AMS subject classifications: primary 65M25; secondary 76C05.

1 Introduction

Standard numerical methods are generally not convergent for ill-posed problems. Typically, in an ill-posed problem, the linear growth rates increase unboundedly with increasing wavenumber.

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Such problems may have short time smooth solutions if the Fourier coefficients of the initial data have rapid enough decay (i.e. existence in analytic function spaces [5, 10, 21]). However, when standard numerical methods are used to compute them, the methods prove to be highly unstable. The reason is this: on the numerical level, the decay of the Fourier coefficients is limited by the numerical precision. For example, the Fourier coefficients of the initial data will decay only until the roundoff level is reached. Roughly speaking, all subsequent modes will be dominated by roundoff error. Since these highest modes are amplified the fastest, in time, the numerical solution becomes dominated by spurious error and the computation breaks down, even though the true solution may still be very smooth.

A prototypical ill-posed problem, and the one we will consider in this paper, is the evolution of a vortex sheet in an incompressible, inviscid and otherwise irrotational fluid. This is a classical problem in fluid dynamics and the sheet undergoes the Kelvin-Helmholtz instability. In this problem, the linear growth rate is proportional to the wavenumber. Moreover, singularity formation appears to be generic, even for vortex sheets initially near equilibrium [13, 6, 19]. One motivation for performing numerical simulations of the vortex sheet problem is to characterize the types of singularities that can form and to determine whether there is in fact a "generic" type.

To accurately compute the numerical evolution of a vortex sheet, one must overcome the spurious growth of roundoff error. This can be done using a numerical filter. However, standard *linear* filters, such as removing, or damping, a fixed band of modes, often "over-smooth" the details of the solution, making singularity characterization difficult. Moreover, through nonlinearity, the physically relevant spectrum typically expands in time into the region of artificially removed wavenumbers. If this region is fixed independently of the discretization parameters and of time, then this type of filtering scheme will clearly no longer converge at such times. On the other hand, a *nonlinear* filtering, introduced to this problem by Krasny [13], has proven very successful. The filtering scheme of Krasny simply says that Fourier modes should be *discarded* if they lie *below* a certain error tolerance and *kept* if they lie *above* the tolerance. It is nonlinear because the modes to which it applies depends on the function to which the filter is applied. Important consequences of this filter are that it allows nonlinearity to produce non-zero modes anywhere in the spectrum and that the linear growth rate is determined by the spatial *discretization* and not the filter. Using this nonlinear filter, Krasny [13] and subsequently Shelley [19] were able to accurately compute numerical solutions essentially up to the time they become singular.

In this paper, a convergence analysis is presented for the point vortex method (applied to the vortex sheet problem) with nonlinear filtering and in the presence of simulated round-off error. The proof is in an analytic function class and uses a discrete form of the Cauchy-Kowalewski theorem [7, 16, 17, 18]. The proof is presented for the case in which the sheet is initially near equilibrium and convergence is obtained nearly up to the singularity time. This result is nearly optimal and is referred to as a "long time" convergence theorem. This is a significant improvement over previous previous convergence theorems for this problem where the time of convergence was restricted to be much less than the singularity time [8, 12]. The near equilibrium case was studied on the continuous level in [5, 21]. If the near equilibrium condition is violated, convergence is obtained for a short time if the true solution remains smooth.

The improved result rests on two observations. First, the nonlinear filter must be included in the analysis to control the growth of the round-off error in time. We note that the previous convergence results did not include the nonlinear filter, as the analysis of it was incomplete at that time. Still, this is not enough to obtain a "long time" convergence theorem. Second, it also is necessary to separate the linear and nonlinear parts of the equation. Both parts of the equation must then be precisely estimated. This is analogous, in spirit, to the continuous analysis of [5] where the linear part of the equation is solved exactly (by integration along complex characteristics) and precise bounds were obtained for the nonlinear operator. The analysis in this paper applies directly to other numerical methods, such as the modified point vortex method [19], as well as to other ill-posed problems such as Rayleigh-Taylor unstable interfaces in incompressible, inviscid and irrotational fluids as well as to Mullins-Sekerka unstable interfaces in Hele-Shaw cells. Further, if surface tension is included so that the problems are well-posed [3, 2], then the analysis of Krasny filtering presented here, combined with the analysis presented in [4] can be used to prove convergence in that case also.

The outline of the paper is as follows. In section 2, the nonlinear filtering is introduced and a sequence of model equations is analyzed, providing an overall framework for our analysis. In section 3, the vortex sheet problem and point vortex discretization are introduced and the convergence theorem is given. In section 4, the discrete Cauchy-Kowalewski theorem is presented. In section 5, the main convergence theorem in proved. In section 6, the discrete Cauchy-Kowalewski theorem with filtering and roundoff error is proved. In Appendix 1, a continuous time version of the Cauchy-Kowaleski theorem is given. In Appendix 2, the consistency and stability of the nonlinear filtering are proved. In Appendix 3, the proof of the discrete Cauchy-Kowalewski theorem in the absence of numerical filtering and roundoff error is presented. Finally, in Appendix 4, an estimate concerning the time difference of certain nonlinear terms is proved.

2 Nonlinear Filtering and Model Problems

The nonlinear filter introduced by Krasny [13] can be considered as a projection operator in Fourier space. It is described as follows. Given an error tolerance τ, the projection operator P is given by

$$(\widehat{Pf})_k = \begin{cases} \hat{f}_k, & \text{if } |\hat{f}_k| \geq \tau \\ 0, & \text{if } |\hat{f}_k| < \tau. \end{cases} \tag{2.1}$$

Here $f(\alpha)$ is a periodic function, and $\hat{f}_k$ is the discrete Fourier transform of f. The filter P is nonlinear because the wavenumbers at which it is applied depend on the solution. Effective use of this filtering method depends on periodicity and analyticity of the function f, so that its transform $\hat{f}_k$ is rapidly decaying in k. It also requires high precision computations, since the filter level τ must be much larger than the round-off error size ε_r. Typical sizes for a double precision computation are $\varepsilon_r = 10^{-15}$, $\tau = 10^{-12}$.

The usefulness of this nonlinear filtering is that while it prevents the spurious growth of round-off error, it allows the linear growth rate to be determined by the numerical *discretization* rather than the filtering scheme, since the filtering is based on the amplitude of the solution and acts like the identity to modes that lie above the tolerance level. This is most effective for nonlinear problems, as the high wavenumbers grow due to nonlinear interactions as well as due to their own linear growth rate. Although it is difficult to explicitly write down a nonlinear example showing this, it is clearly seen in computations [13, 19].

In this section, we present a sequence of examples that show the essential effects of filtering, the necessity of using the abstract Cauchy- Kowalewski theorem and the overall strategy of our convergence proof. We begin with a linear example. Consider the simple model equation

$$\begin{cases} u_t = \frac{1}{2}\mathcal{H}(u_\alpha), \\ u(x,0) = u_0(\alpha) \end{cases} \tag{2.2}$$

in which $\mathcal{H}$ is the Hilbert transform; i.e. $(\widehat{\mathcal{H}u})_k = -isgn(k)\hat{u}_k$. Take the initial data to be $\hat{u}_0(k) = e^{-|k|\rho_0}$, so that the solution is

$$\hat{u}_k(t) = e^{-|k|(\rho_0 - t/2)}. \tag{2.3}$$

This solution develops a singularity at time $T_0 = 2\rho_0$, when the exponential decay of the Fourier components is lost. Of course, this singularity was "built" into the initial condition.

Now suppose that the initial data is perturbed by simulated roundoff error and solve equation (2.2) both with and without filtering. For simplicity, we will suppose there is no roundoff error in the equation. This will make the effect of filtering clearer. Moreover, because the equation is *linear*, the analysis of roundoff error in the equation essentially reduces to that given below for the case when initial data perturbed by roundoff error. This is because in the periodic case where k is an integer, multiplication by $|k|/2$ (for $|k| > 1$) ensures that if the initial data at mode k lies above the roundoff, then mode k lies above the roundoff at all subsequent times.

The roundoff error is simulated by a perturbation e_r with $\hat{e}_r(k) = \varepsilon_r$ in each Fourier mode (with $\varepsilon_r \approx 10^{-15}$). The perturbed problem without filtering is

$$\begin{cases} \hat{\partial}_t v_k = \frac{1}{2}|k|\hat{v}_k, \\ \hat{v}_k(0) = e^{-|k|\rho_0} + \varepsilon_r \end{cases} \tag{2.4}$$

which has solution

$$\hat{v}_k(t) = e^{-|k|(\rho_0 - t/2)} + \varepsilon_r e^{|k|t/2}. \tag{2.5}$$

Notice that the initial roundoff error is amplified exponentially in time with a rate proportional to $|k|$.

Now consider the perturbed problem with filtering. It is given by

$$\begin{aligned} \hat{\partial}_t w_k &= \frac{1}{2}|k|\hat{w}_k, \qquad (2.6) \\ \hat{w}_k(0) &= (\hat{Pu_0}) \\ &= \begin{cases} e^{-|k|\rho_0} & \text{if } e^{-|k|\rho_0} \geq \tau \\ 0 & \text{if } e^{-|k|\rho_0} < \tau \end{cases} \end{aligned}$$

which has solution

$$\hat{w}_k(t) = \begin{cases} e^{-|k|(\rho_0 - t/2)} & \text{if } e^{-|k|\rho_0} \geq \tau \\ 0 & \text{if } e^{-|k|\rho_0} < \tau. \end{cases} \tag{2.7}$$

The smallest wavenumber at which filtering is applied is

$$k_f = \rho_0^{-1}\log(\tau^{-1}). \tag{2.8}$$

Now compare the error $(v - u)$ and $(w - u)$ made in the two approximations. For the perturbed problem without filtering, the dominant contribution to $(v - u)$ is due to the growth of the largest wavenumber, $k_{max} = N/2$, so that

$$|v - u| \approx \varepsilon_r e^{Nt/4}. \tag{2.9}$$

The approximation fails when this error is of size $O(1)$, which occurs when $t = T_1 = 4N^{-1}$, a time that depends on the *discretization*, rather than on the singularity time of the continuous problem. If $N >> 1$ then $T_1 << T_0$; i.e., the solution with roundoff error but no filtering diverges from the unperturbed solution well before the singularity time.

On the other hand, for the problem with filtering the dominant contribution to $(w - u)$ comes from the smallest wavenumber k_f that is set to zero; i.e.,

$$|w - u| \approx e^{-|k_f|(\rho_0 - t/2)} = \tau^{1 - t/2\rho_0}. \tag{2.10}$$

This error becomes size $O(1)$ when $t = T_2 = 2\rho_0$ which is the same as the singularity time for the original problem.

These estimates for the errors show that the unfiltered problem with roundoff is close to the exact problem for only a short time; whereas the filtered problem is accurate almost up to the singularity time. This is precisely the behavior that has been observed in numerical simulation of the vortex sheet problem with and without filtering [13, 19].

Now, consider the following nonlinear modification of Eq. (2.2). Suppose that ε is a small parameter and take

$$\begin{cases} \eta_t = \frac{1}{2}\mathcal{H}[\eta_\alpha] + \frac{\varepsilon}{2}A[\eta](\alpha,t), \quad A[\eta] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\eta(\alpha)-\eta(\alpha'))^2}{(\alpha-\alpha')^3}d\alpha' \\ \hat{\eta}_k(0) = e^{-\rho_0|k|} \end{cases} \tag{2.11}$$

The choice of $A[\eta]$ is motivated by the expansion of the integral operator in the vortex sheet problem given in section 3.2 for the discrete case (see Eq. (5.7)). The parameter ε arises from rescaling the equation given small initial data. Although we cannot write the explicit solution to Eq. (2.11), we expect that it's solution remains smooth until $t \approx 2\rho_0$ since ε may be expected to keep the nonlinearity small. Roughly speaking, the operator $A[\eta]$ behaves like the product $\mathcal{H}[\eta_\alpha]\cdot\mathcal{H}[\eta_\alpha]$ (also see section 3.2). Therefore the nonlinearity contains derivatives of the same order as the linear term. This fact combined with the linear ill-posedness of the equation and the nonlocal nature of the nonlinear terms, requires the use of the abstract Cauchy-Kowalewski theorem to prove existence. The abstract Cauchy-Kowalewski theorem is a fundamental theorem on the existence of 'analytic' solutions of functional differential equations such as certain integro-differential equations. Actually, solutions are obtained in certain more general Banach spaces, but we always use analytic function spaces in this paper. A precise statement of this theorem is given in Appendix 1. To prove that solutions exist up to $t \approx 2\rho_0$, it is instructive to rewrite Eq. (2.11) by integrating in time and using an integrating factor. This gives

$$\eta(\alpha,t) = u(\alpha,t) + \frac{\varepsilon}{2}\int_0^t \tilde{A}[\eta](\alpha,t,t')dt' \text{ where} \tag{2.12}$$

$$\widehat{\tilde{A}}[\eta](k,t,t') = e^{\frac{1}{2}|k|(t-t')}\hat{A}[\eta](k,t') \tag{2.13}$$

and u is the solution to the *linear* Eq. (2.2). Thus, the linear part of the equation is integrated exactly. The abstract Cauchy-Kowalewski theorem in Appendix 1 can then be used to show existence of solutions to Eq. (2.12) for a time interval arbitrarily near $2\rho_0$ for ε close to 0. We will use a similar exact integration of the linear part of the numerical scheme when we prove the convergence of the point vortex method for long times in section 3.2.

Now, consider the case with both filtering and roundoff error. Since the equation (2.11) is *nonlinear*, the mode interaction makes the analysis of the effects of filtering and roundoff error in the equation (scheme) much more difficult than the case where filtering and roundoff error perturb only the initial condition. Therefore, we consider equation (2.11) with filtering and roundoff error in *both* the initial condition and the equation.

$$\begin{cases} \zeta_t = P\left\{\frac{1}{2}\mathcal{H}[\zeta_\alpha] + \frac{\varepsilon}{2}A[\zeta] + e_r\right\} \\ \hat{\zeta}_k(0) = e^{-\rho_0|k|} \text{ for } |k| \le \frac{1}{\rho_0}log\frac{1}{\tau} \text{ and } 0 \text{ otherwise} \end{cases} \tag{2.14}$$

Again, we expect that if $|e_r(k)| \le \varepsilon_r$ is small, that solutions to (2.14) exist for $t \approx 2\rho_0$ as well. The presence of the nonlinear filtering and roundoff error makes it difficult to directly apply the Cauchy-Kowalewski theorem of Appendix 1 to obtain this result. This is because additional assumptions are required to control the effects of the filtering and roundoff error. Using the appropriate assumptions, a careful mode by mode analysis shows that (2.14) does, in fact, have solutions existing for t

arbitrarily close to $2\rho_0$ when $\varepsilon, \varepsilon_r$ are close to 0. More specifically, it is shown in sections 4 and 6 how (on the discrete level) the assumptions of the abstract Cauchy-Kowalewski theorem, it's statement and it's proof, respectively, must be modified to accomodate numerical filtering and roundoff error. The continuous version follows analogously.

Finally, the difference $\eta - \zeta$ can also be analyzed using the Cauchy-Kowalewski theorem as follows. Let $\nu = \eta - \zeta$, then

$$\nu(\alpha,t) = u(\alpha,t) - w(\alpha,t) + F(\alpha,t) + \frac{\varepsilon}{2}\int_0^t \left(\tilde{A}[\eta](\alpha,t,t') - \tilde{A}[\eta-\nu](\alpha,t,t')\right) dt' \tag{2.15}$$

$$\widehat{F}(k,t) = \int_0^t e^{\frac{1}{2}|k|(t-t')}\hat{f}(k,t)dt' \quad \text{and} \tag{2.16}$$

$$f(\alpha,t) = (I-P)\left\{\frac{1}{2}\mathcal{H}[\zeta_\alpha] + \frac{\varepsilon}{2}A[\zeta]\right\} \tag{2.17}$$

where w is the solution to Eq. (2.6) and I is the identity operator. Therefore, treating F as a forcing function by using the fact that the solution, ζ, to (2.14) is smooth *and* using the consistency of the nonlinear filtering operator P, presented in lemmas 5.1-5.3 in section 5, *and* the fact that $|u-w| \approx \tau^{1-t/2\rho_0}$ is smooth, *then* the abstract Cauchy-Kowalewski theorem of Appendix 1 can be applied to show that smooth solutions to (2.15) exist in a slightly smaller time interval than for either η or ζ, but that this solution is, roughly speaking, of size $O(\tau^{1-\gamma} + \frac{\varepsilon_r}{\tau})$. This result is almost optimal because it holds nearly up to the singularity time of the smooth solution.

The above scenario provides an outline for the approach we take to prove the convergence of the point vortex method in the following sections of this paper.

3 Vortex Sheets and Main Result

The equation governing the motion of a periodic, planar vortex sheet, with single-signed vortex sheet strength, is called the Birkhoff-Rott equation and is given by

$$\frac{dz^*}{dt} = \frac{1}{4\pi i}PV\int_{-\pi}^{+\pi} \cot\left(\frac{z(\alpha,t)-z(\alpha',t)}{2}\right) d\alpha' \tag{3.1}$$

$$z(\alpha,0) = \alpha + s_0(\alpha) \tag{3.2}$$

in which $z(\alpha,t)$ is the complex position of the interface and α is the Lagrangian circulation variable. If the initial vortex sheet strength is not single-signed then the circulation variable cannot be used to parametrize the sheet and the vortex sheet strength must be explicitly introduced. Our analysis also applies to this case, however we omit it here for simplicity. The explicit inclusion of the vortex sheet strength only introduces minor modifications of the analysis presented here since the vortex sheet strength is time independent in the Lagrangian frame. See [14] for details.

In Eq. (3.1), the integral is a Cauchy principal value integral, due to the singularity at $\alpha' = \alpha$, and * denotes complex conjugate. The periodicity implies that

$$z(\alpha,t) = \alpha + s(\alpha,t) \tag{3.3}$$

in which $s(\alpha,t)$ is 2π periodic in α for each t. Since filtering can be applied only to functions that are periodic, the operator P will be applied to s, but cannot be directly applied to z. For simplicity of notation, however, we denote

$$Pz = \alpha + Ps. \tag{3.4}$$

Denote by $\tilde{z}_j$ the discrete approximation of $z(\alpha_j, t)$, in which $\alpha_j = jh = 2\pi j/N$. Without the numerical filtering, the usual point vortex approximation is

$$\begin{aligned} \frac{d\tilde{z}_j^*}{dt} &= \frac{h}{4\pi i} \sum_{\substack{l=-N/2+1 \\ l \neq j}}^{N/2} \cot\left(\frac{\tilde{z}_j - \tilde{z}_l}{2}\right) \\ &= \frac{h}{4\pi i} \sum_{\substack{l=-N/2+1 \\ l \neq j}}^{N/2} \cot\left(\frac{\alpha_j - \alpha_l + \tilde{s}_j - \tilde{s}_l}{2}\right). \end{aligned} \tag{3.5}$$

With the inclusion of simulated roundoff error e_r and the application of Krasny filtering P, the ODE's for $\tilde{z}$ become

$$\frac{d\tilde{z}_j^*}{dt} = P\{\frac{h}{4\pi i} \sum_{\substack{l=-N/2+1 \\ l \neq j}}^{N/2} \cot\left(\frac{\tilde{z}_j - \tilde{z}_l}{2}\right) + e_r\}_j. \tag{3.6}$$

A time discrete version can be obtained by applying any consistent time discretization. For simplicity, we consider the Euler time discretization and we also filter the full right hand side rather than just the $O(\Delta t)$ terms. The fully discrete method is then given by

$$\tilde{z}_j^{n+1*} = P\left\{\tilde{z}^{n*} + \Delta t \frac{1}{4\pi i} \sum_{\substack{l=-N/2+1 \\ l \neq j}}^{N/2} \cot\left(\frac{\tilde{z}_j - \tilde{z}_l}{2}\right) h + \Delta t \cdot e_r\right\}_j. \tag{3.7}$$

We refer the reader to [13, 19, 8, 12] for additional details.

We now introduce some notation. For $\rho > 0$, define a norm as follows

$$\|f\|_\rho = \sum_{k=-\infty}^{+\infty} |\hat{f}_k| e^{\rho|k|}. \tag{3.8}$$

Assuming that $\|f\|_\rho$ is finite is roughly equivalent to assuming that $f(\alpha)$ is analytic in the strip $|Im(\alpha)| < \rho$. Denote such analytic function spaces by B_ρ

$$B_\rho = \{f : \|f\|_\rho < \infty\}. \tag{3.9}$$

Moreover, if the function f is defined on the grid $\{\alpha_j = 2\pi j/N\}$ for $j = -N/2+1, \ldots, N/2$, then there is a corresponding discrete norm

$$\|f\|_\rho = \sum_{k=-N/2+1}^{N/2} |\hat{f}_k| e^{\rho|k|} \tag{3.10}$$

in which $\hat{f}_k$ is the k^{th} discrete Fourier coefficient for f. This is the norm in which convergence is proven.

In the continuous case, Caflisch & Orellana [5] showed the following near equilibrium result.

Theorem 3.1 *Long Time Existence, Caflisch & Orellana*
Let ϵ be sufficiently small and $z(\alpha, 0) = \alpha + s_0(\alpha)$ with

$$\|s_0\|_{\rho_0} \le \epsilon \quad \text{and} \quad \hat{s}_0(0) = 0. \tag{3.11}$$

Then, there exists $\kappa > 1$ such that for $0 \le t \le T_0 = 2\rho_0/\kappa$, the Birkhoff-Rott equation has an analytic solution $z(\alpha, t) = \alpha + s(\alpha, t)$ in which the perturbation s continues to have 0 mean and remains of size ϵ, ie.

$$\|s(t)\|_{\bar{\rho}(t)} \le \epsilon, \quad \text{and} \quad \hat{s}(0,t) = 0 \tag{3.12}$$

where $\bar{\rho}(t) = \rho_0 - \kappa t/2$ and moreover, κ is arbitrarily close to 1 when ϵ is close to 0.

For initial data in B_{ρ_0}, there may be a singularity at position α_* in the complex α plane with $\rho_* = |Im(\alpha_*)| > \rho_0$. For such data, linear theory predicts that a singularity will occur at time $t_* = 2\rho_*$. It was shown in [6] that for ε small and for a restricted set of initial data, that the nonlinear and linear solutions are nearly identical up to, and including, the singularity time. Therefore, the time of existence T_0 is nearly optimal.

The main result of this paper is to show that with roundoff error and filtering, the point vortex method converges to the types of solutions considered by Caflisch & Orellana for a time interval almost up to the singularity time.

Theorem 3.2 *Almost Optimal Convergence with Roundoff Error and Filtering*

Assume that $z(\alpha, t) = \alpha + s(\alpha, t)$ is a near equilibrium, periodic solution of the Birkhoff-Rott equation satisfying Eqs. (3.11) and (3.12). Suppose that $\tilde{z}_j^n$ solves the discretized Birkhoff-Rott equation (3.7) with simulated roundoff error and filtering. Then, for any $0 < \omega < 1$ there exist constants C, c independent of the numerical parameters but depending on ω and $z(\alpha, t)$ such that

$$\|z^n - \tilde{z}^n\|_{\rho_2(t_n)} \le C \left[\Delta t + h + \frac{\tau^{1-\omega}}{\Delta t} + \frac{\varepsilon_r}{\tau \Delta t} \right] \tag{3.13}$$

for a time interval $0 \le t_n \le T_2$, where $t_n = n\Delta t$ and in which

$$\begin{aligned} T_2 &= 2\omega\rho_0/(1 + c\sqrt{\varepsilon}) \\ \rho_2(t) &= \omega\rho_0 - (1 + c\sqrt{\varepsilon})t/2. \end{aligned} \tag{3.14}$$

for ε, Δt, h, $\tau^{1-\omega}/\Delta t$, and $\varepsilon_r/(\tau\Delta t)$ sufficiently small.

Remarks

1. The proof of theorem 3.2 relies on two versions of the discrete Cauchy-Kowalewski theorem, which will be presented in the next section. One version includes the effects of numerical filtering and simulated roundoff error. In addition, careful estimates must be obtained for the filter P and for the linear part of the discrete operator as well as the nonlinear part. The $\sqrt{\varepsilon}$ in the theorem arises naturally from the choice of constants in the application of the discrete Cauchy-Kowaleski theorem.

2. If the solution is far from equilibrium, then the careful estimate on the nonlinear part of the discrete operator breaks down. It still can be estimated, however, but only in a way that results in short time convergence (if the true solution is smooth).

3. The technique of proof can be used to prove similar convergence theorems for other discretizations, such as the modified point vortex method [19], as well as for many other ill-posed problems to which the abstract Cauchy-Kowaleski theorem can be used to prove existence of analytic solutions in the continuous (spatially and temporally) case. Such problems include Rayleigh-Taylor unstable interfaces in inviscid, incompressible and irrotational fluids as well as Mullins-Sekerka unstable interfaces in a Hele-Shaw cell. See [14, 15, 9, 22, 23, 1] for example. The appropriate convergence proofs are then obtained by carefully analyzing the particular numerical method in question, obtaining an error equation and then applying the discrete Cauchy-Kowalewski theorems to these cases. In order to be sure that the discrete Cauchy-Kowalewski theorem can be applied, two things are important. First, it must be possible to apply the continuous version to prove existence of analytic solutions. Second, it must be possible to write the spatial discretization so that it does not explicitly contain discrete derivatives of higher order than 1. This is because the C-K theorem applies only to $1st$ order operators. One consequence of this is that our proof cannot be directly applied to the case with surface tension as this contains high order derivatives. However, this case is in fact linearly well-posed [3] and our analysis of Krasny filtering, presented here, combined with the convergence analysis presented in [3, 2] can be used to prove convergence in that case as well.

4 Discrete Cauchy-Kowalewski Theorem

The Cauchy-Kowalewski Theorem is a fundamental theorem on existence of analytic solutions of partial differential equations. In its abstract form [16, 17, 18] it is applicable to integro-differential equations such as the Birkhoff-Rott equation (3.1). The abstract form of the theorem is directly applicable to semi-discrete equations (with continuous time), and needs only superficial modification for equations with discrete time. A precise statement of the continuous time version is given in Appendix 1. Of course, for fully discrete equations, existence of solutions is trivial, and the real point of the theorem is to obtain uniform bounds on the solution. A discrete version of the theorem was proved in [8]. In this section, two versions of the discrete Cauchy-Kowalewski theorem are given. The first is a discrete version of the strengthened formulation and simplified proof of the Cauchy-Kowalewski theorem by Safonov [18]. It has been modified to serve as a result for estimating perturbations, as needed for the nearly optimal convergence result with filtering. The second version modifies the first by allowing the inclusion of simulated roundoff error and numerical filtering. Again, a nearly optimal bound results. This is necessary for the convergence proof (presented in the next section) by providing uniform bounds on the numerical solution of the point vortex method with filtering and roundoff error.

Consider first the discrete equation without roundoff error and filtering

$$\begin{aligned} u_{n+1} &= Lu_n + \Delta t A_n[u_n] \\ u_0 &= 0 \end{aligned} \tag{4.1}$$

in which $u_n = \{u_n^i\}$ is a discrete function in B_ρ. Suppose that the linear operator L satisfies

(i). L is a linear operator on B_ρ such that for any $\rho' > \rho > 0$ and any $u \in B_{\rho+\lambda_0\Delta t}$

$$||Lu||_\rho \leq ||u||_{\rho+\lambda_0\Delta t} \tag{4.2}$$

$$||(L-I)u||_\rho \leq \lambda_0\Delta t(\rho'-\rho)^{-1}||u||_{\rho'}. \tag{4.3}$$

Suppose further that the nonlinear operator A satisfies the following assumptions:

(ii). For any $0 < \rho < \rho' < \rho_0 - \lambda_0 n\Delta t$, A_n is a continuous mapping of $\{u \in B_{\rho'}, ||u||_{\rho'} \leq R\}$ into B_ρ.

(iii). For any $0 < \rho < \rho' < \rho_0 - \lambda_0 n\Delta t$, and for any $u, v \in B_{\rho'}$ with $||u||_{\rho'} \leq R, ||v||_{\rho'} \leq R$,

$$||A_n[u] - A_n[v]||_\rho \leq C_1(\rho'-\rho)^{-1}||u-v||_{\rho'} \tag{4.4}$$

where C_1 is a constant independent of u, v, ρ, ρ', n. It may depend on R.

(iv). For any $0 < \rho < \rho_0 - \lambda_0 n\Delta t$

$$||A_n[0]||_\rho \leq K. \tag{4.5}$$

where K is independent of ρ, n.

(v). For any $0 < \rho < \rho_0 - \lambda_0 n\Delta t$ and any $u \in B_{\rho'}$ with $||u||_{\rho'} \leq R$,

$$||A_{n+1}[u] - A_n[u]||_\rho \leq C_2(\rho' - \rho)^{-1}\Delta t \tag{4.6}$$

where C_2 is independent of ρ, ρ', u, n. It may depend on R and boundedly on Δt as $\Delta t \to 0$.

Theorem 4.1 *Discrete Cauchy-Kowalewski Theorem*

Suppose that L and A satisfy assumptions (i)-(v) for some positive constants $\rho_0, \lambda_0, K, C_1, C_2$ and R. Then, there is a constant λ (defined explicitly below), such that for $|n| \leq \rho_0/(\lambda\Delta t)$ the solution u_n of equation (4.1) satisfies $u_n \in B_{\rho_n}$ and

$$||u_n||_{\rho_n} \leq R \tag{4.7}$$

in which $\rho_n = \rho_0 - \lambda|n|\Delta t$ and λ is given by

$$\lambda = max \begin{cases} \lambda_0 + \frac{R_0\rho_0^{1-\gamma}}{R(1-\gamma)}, \\ \lambda_0\left(1 + \frac{R_0 2^{1+\gamma}}{R\gamma}\right), \\ \gamma^{-1}\left[C_1 2^{1+\gamma}(1+\frac{R}{R_0}) + 2C_2\frac{\rho_0^\gamma}{R_0}\right] \end{cases} \tag{4.8}$$

for any $0 < \gamma < 1$ and $R_0 \geq K\rho_0^\gamma$.

The bound (4.7) will be used to estimate the difference between the solutions of the Birkhoff-Rott equation and the discretized equation, in order to show convergence of the discretized solutions. Note that in assumption (v), which does not appear in the statement of the continuous version, the values of the operator A are compared at two different discrete time values n and $n+1$. In the application to the convergence theorem 3.2, the n dependence of A will be due to the time dependence of the exact solution. The proof of Theorem 4.1 will be given in Appendix 3.

Note that time interval of existence for the linear operator L alone would be ρ_0/λ_0. If the nonlinear operator A is small, as would be the case if the solution u were small, then the constants C_1, C_2, K and R could be taken to be small. By careful choice of these constants, the resulting value of λ will be only a small perturbation of λ_0; i.e. by separating the linear and nonlinear parts of the equation, we obtain a nearly optimal time of existence.

Now consider the discrete equation with filtering and roundoff error

$$\begin{aligned} v_{n+1} &= P\{Lv_n + \Delta t A_n[v_n] + \Delta t e_r\} \\ v_0 &= \text{given} \end{aligned} \tag{4.9}$$

where P is the nonlinear projection operator defined in Eq. (2.1) with filter level τ, and e_r is the simulated roundoff error which is assumed to satisfy the bound

(vi) $|\hat{e}_r(k)| < \varepsilon_r < \frac{\tau}{2\Delta t}$ for all wavenumbers k.

In this theorem, the filter level τ is allowed to depend on the wavenumber k. This is needed in the convergence proof for the Birkhoff-Rott equation, since the Cauchy-Kowalewski theorem will be applied to the derivative of the original equation.

The linear operator L, in addition to satisfying (i), is also assumed to be diagonalized by the Fourier transform, i.e.

(vii) $\widehat{Lu}(k) = l(k)\hat{u}(k)$.

The nonlinear operator A_n is assumed to satisfy assumptions (ii)-(v). Unlike the previous case, non-zero initial data, v_0, is allowed. This is because the nonlinearity of P makes it difficult to absorb the initial data into the equation. The projection is performed on the initial data, and it is further assumed to satisfy

(viii) $\|v_0\|_{\rho_0+\beta} \leq \delta R$ and $v_0 = Pv_0$, with $\delta < 1$.

for some $\beta > 0$. Further, let Π be an arbitrary *linear* Fourier projection operator such that $\widehat{\Pi u}(k)$ is either 0 or $\hat{u}(k)$ and set $0 < \gamma < 1$. Define the constants R_2 and R_1 such that

$$R_2 \geq \rho_0^\gamma \sup_{\Pi} \|A_0[\Pi v_0]\|_{\rho_0} \tag{4.10}$$

$$R_1 \geq \frac{\rho_0^\gamma}{\Delta t}\|(L-I)v_0\|_{\rho_0} \tag{4.11}$$

Note that assumptions (i), (iii), (iv) and (viii) imply that $R_2 \geq (\delta R C_1/\beta + K)\rho_0^\gamma$ and $R_1 \geq \delta \frac{\lambda_0 R}{\beta}\rho_0^\gamma$ satisfies (4.10) and (4.11).

Theorem 4.2 *Discrete Cauchy-Kowalewski Theorem with Roundoff Error and Filtering*

Suppose that P is defined by Eq. (2.1) and that L, A, v_0, e_r and τ satisfy assumptions (i)-(viii) for some positive constants $\rho_0, \lambda_0, K, C_1, C_2$, R, R_2, R_1 and ε_r. Then, there is a constant λ (defined explicitly below), such that for $n \leq \rho_0/(\lambda\Delta t)$ the solution v_n of equation (4.9) satisfies $v_n \in B_{\rho_n}$ and

$$\|v_n\|_{\rho_n} \leq R \tag{4.12}$$

in which $\rho_n = \rho_0 - \lambda|n|\Delta t$ and λ is given by

$$\lambda = max \begin{cases} \lambda_0 + \left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right)\left(1 - \delta - \frac{\varepsilon_r}{\tau}t\right)^{-1} \frac{R_2 \rho_0^{1-\gamma}}{R(1-\gamma)}, \\ \lambda_0\left[1 + \frac{R_2}{R_1\gamma}2^{1+\gamma}\left(1 + \frac{2\varepsilon_r\Delta t}{\tau}\right)\right] + \frac{\varepsilon_r}{\gamma\tau}\rho_0, \\ \frac{1}{\gamma}\left[C_1 2^{1+\gamma}\left(1 + \frac{R_1}{R_2} + \frac{2\varepsilon_r\Delta t}{\tau}\right) + 2C_2\frac{\rho_0^\gamma}{R_2} + \frac{2\varepsilon_r}{\tau}\frac{R}{R_2}\rho_0^\gamma\right]. \end{cases} \tag{4.13}$$

in which $t = \max(|n|)\Delta t$ and if the filter level τ depends on k, then $\tau = \min \tau(k)$.

The bound (4.12) will be used to estimate the solution of the point vortex method with filtering and roundoff error. This requires the additional assumptions (vi)-(viii). Furthermore, if $\delta \ll 1$ and if $\varepsilon_r \ll \gamma\tau$ then, λ given by Eq. (4.13) is close to that given by Eq. (4.8), which gives a nearly optimal result in the case of filtering and roundoff error.

Before giving the proof of Theorem 4.2, we first prove the convergence result stated in Theorem 3.2. The proof of Theorem 4.2 will be given in section 6.

5 Convergence Proof

In this section, the proof of the convergence Theorem 3.2 is presented. We begin by using the discrete Cauchy-Kowalewski Theorem 4.2 to prove uniform bounds for the numerical solution of the point vortex method with roundoff error and filtering. This bound plays an important role in the convergence proof by providing a control on the error introduced by the filtering.

In terms of the discrete periodic function $\tilde{s}_j$, i.e. $\tilde{s}_j = \tilde{z}_j - \alpha_j$ where $\alpha_j = jh$, the point vortex method is given by

$$\tilde{s}_j^{n+1*} = P\left\{\tilde{s}_j^{n*} + \Delta t \frac{h}{4\pi i}\sum_{\substack{l=-N/2+1 \\ l\neq j}}^{N/2} \cot\left(\frac{\tilde{z}_j^n - \tilde{z}_l^n}{2}\right) + \Delta t e_r\right\}_j . \tag{5.1}$$

It is convenient to expand the cotangent kernel as follows. Extend the discrete solution, $\tilde{s}_j$, periodically outside the interval $(-N/2+1 \leq j \leq N/2)$, i.e. $\tilde{s}_{j+N} = \tilde{s}_j$. Then, it is a straightforward computation to see that with $\tilde{z}_j = \alpha_j + \tilde{s}_j$ so extended, one gets

$$\frac{h}{4\pi i} \sum_{\substack{l=-N/2+1 \\ l \neq j}}^{N/2} \cot\left(\frac{\tilde{z}_j - \tilde{z}_l}{2}\right) = \lim_{M \to \infty} \frac{h}{2\pi i} \sum_{\substack{l=-N(M+1/2)+1 \\ l \neq j}}^{N(M+1/2)} \frac{1}{\tilde{z}_j - \tilde{z}_l}. \tag{5.2}$$

See [4] for details. Hereafter, we adopt the notation

$$F[\tilde{s}]_j \equiv \frac{h}{2\pi i} \sum_{l \neq j} \frac{1}{\tilde{z}_j - \tilde{z}_l} \equiv \lim_{M \to \infty} \frac{h}{2\pi i} \sum_{\substack{l=-N(M+1/2)+1 \\ l \neq j}}^{N(M+1/2)} \frac{1}{\tilde{z}_j - \tilde{z}_l}. \tag{5.3}$$

Some properties of F are given as follows. Define D to be the discrete spectral derivative, i.e. $\hat{D} = ik$ for $-N/2+1 \leq k \leq N/2$ and periodically extended to all k. Then, it was shown in [8] that if $\|Df\|_{\rho'} \leq 1/2$ and $\|Dg\|_{\rho'} \leq 1/2$, where $\|\cdot\|_\rho$ is the discrete norm in (3.10), then

$$\|DF[f] - DF[g]\|_\rho \leq \frac{c}{\rho' - \rho} \|Df - Dg\|_{\rho'}. \tag{5.4}$$

All subsequent norms in this paper are this discrete norm. Further, decompose F into a linear and nonlinear part

$$F[\tilde{s}]_j = F_L[\tilde{s}]_j + F_{NL}[\tilde{s}]_j \tag{5.5}$$

where

$$F_L = \frac{h}{2\pi i} \sum_{l \neq j} \frac{\tilde{s}_j - \tilde{s}_l}{(\alpha_j - \alpha_l)^2} \quad \text{and} \quad F_{NL} = F - F_L. \tag{5.6}$$

If $\|D\tilde{s}\|_\rho < 1$, then F_{NL} may be expanded in the series [8]

$$F_{NL} = \frac{h}{2\pi i} \sum_{l \neq j} \sum_{m=2}^{\infty} \frac{(\tilde{s}_j - \tilde{s}_l)^m}{(\alpha_j - \alpha_l)^{m+1}}. \tag{5.7}$$

Discrete Fourier analysis can be used to analyze F_L and F_{NL}. In fact, F_L is exactly 1/2 times the trapezoidal quadrature (omitting the singular point) of the continuous spatial derivative of the Hilbert transform

$$\mathcal{H}[f_\alpha](\alpha) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\alpha) - f(\alpha')}{(\alpha - \alpha')^2} d\alpha' \tag{5.8}$$

applied to periodic functions (see [11]). It is not difficult to see that

$$\hat{F}_L = \frac{1}{2}|k| \left(1 - \frac{|k|}{N}\right) \tag{5.9}$$

for $-N/2+1 \leq k \leq N/2$ and periodically extended to all k. See [4] for example. This should be contrasted with the continuous case, in which the symbol of the continuous version of F_L is given by $\frac{1}{2}|k|$ for all k, as well as with the symbols of other quadrature rules such as the alternating point quadrature rule [20] which has as its symbol $\frac{1}{2}|k|$ for $-N/2+1 \leq k \leq N/2$ and periodically extended to all k.

Since $|\hat{F}_L| \leq |k|/2$, it follows that for an appropriate grid function f

$$\begin{aligned} \|(I + \Delta t F_L)[f]\|_\rho &\leq \|f\|_{\rho+\Delta t/2} && (5.10) \\ \|F_l[f]\|_\rho &\leq \frac{1}{2}(\rho' - \rho)^{-1} \|f\|_{\rho'} && (5.11) \end{aligned}$$

with $\rho' > \rho > 0$.

The nonlinear term F_{NL} may be estimated exactly as in [8] where it was shown that

$$\|F_{NL}[f] - F_{NL}[g]\|_\rho \leq c\varepsilon \|Df - Dg\|_\rho \tag{5.12}$$

if $\|Df\|_\rho, \|Dg\|_\rho \leq \varepsilon$ and ε is sufficiently small and c is a constant independent of f, g.

Now, turn to Eq. (5.1). Applying the discrete derivative D to (5.1) and defining

$$\begin{aligned} v^{n+1} &= D\tilde{s}^{n+1} && (5.13) \\ Lv^n &= v^n + \Delta t F_L[v^n]^* && (5.14) \\ \mathcal{A}[v^n] &= DF_{NL}\left[D^{-1}v^n\right]^* = DF_{NL}\left[\tilde{s}^n\right]^* && (5.15) \\ \tilde{e}_r &= De_r^* && (5.16) \end{aligned}$$

one gets

$$v^{n+1} = \tilde{P}\left\{Lv^n + \Delta t \mathcal{A}\left[v^n\right] + \Delta t \tilde{e}_r\right\} \tag{5.17}$$

where $\tilde{P}$ denotes the projection operator with τ replaced by $\tau|k|$ for each Fourier mode $-N/2+1 \leq k \leq N/2$. Further, τ and ε_r are replaced similarly in the assumption (vi) which now applies to $\tilde{e}_r$. The reason for this is that the discrete derivative D has been passed through the original projection P.

Eq. (5.17) is now exactly in the form required to apply the discrete Cauchy-Kowalewski Theorem 4.2. It is straightforward to see that Eqs. (5.9)-(5.13) imply that requirements (i)-(iii), (vii), (4.10) and (4.11) are satisfied with

$$\begin{aligned} \lambda_0 &= \frac{1}{2},\ K = 0,\ C_2 = 0,\ R = \varepsilon,\ \delta = 1/2 && (5.18) \\ C_1 &= c\varepsilon,\ R_2 = \varepsilon^2 \frac{c2^\gamma \rho_0^\gamma}{\gamma\beta},\ R_1 = \lambda_0 \varepsilon \frac{\rho_0^\gamma}{2\beta}. && (5.19) \end{aligned}$$

Further, if $\varepsilon_r \ll \varepsilon\gamma\tau$ and $\varepsilon_r \ll \varepsilon^2\tau$, and taking initial data that satisfies (viii), Theorem 4.2 implies that

$$\|\tilde{s}^n\|_{\rho_n} \leq \varepsilon \tag{5.20}$$

where

$$\rho_n = \rho_0 - \lambda_f n \Delta t \tag{5.21}$$

and λ_f is a small perturbation of the linear result $\lambda_0 = 1/2$

$$\lambda_f = \frac{1}{2} + c\varepsilon \tag{5.22}$$

where c is a constant that can be bounded independently of $C_1, \beta, \gamma, \Delta t, \tau$ and ε_r provided that Δt, τ are small enough and ε_r is as above. The bound (5.20) will be used to control the effect of the filtering error in the full convergence proof.

Now, turn to the question of the convergence of the numerical scheme. In [8], it was shown that the continuous solution $s_j^n = z(\alpha_j, n\Delta t) - \alpha_j$ satisfies the discrete equation

$$s_j^{n+1} = s_j^n + \Delta t F[s^n]_j^* + \Delta t \left(\Delta t f_1^{n^*} + h f_2^{n^*}\right)_j \tag{5.23}$$

where F is as in (5.3) and f_1^n, f_2^n are the local temporal and spatial consistency errors respectively. They satisfy the bound

$$\|D f_i^n\|_\rho \leq c\varepsilon \left(\bar{\rho}(t) - \rho\right)^{-3} \quad \text{for } i = 1,2 \tag{5.24}$$

for $\bar{\rho}(t) = \rho_0 - \kappa t/2$ from Theorem 3.1. The -3 power is not of much significance and is probably not optimal. The bound can be controlled by keeping ρ sufficiently smaller than $\bar{\rho}$. For example, if $\rho < \omega\rho_0 - \kappa t/2$ with $0 < \omega < 1$, then

$$\|D f_i\|_\rho \leq c_\omega \varepsilon. \tag{5.25}$$

Define the error to be $d^n = s^n - \tilde{s}^n = z^n - \tilde{z}^n$ and letting $u^{n+1} = Dd^{n+1}$ gives the error equation

$$u^{n+1} = Lu^n + \Delta t A_n[u^n] \tag{5.26}$$

where L is defined as in Eq. (5.14) and the nonlinear operator, A_n, is given by

$$A_n[u^n] = DF_{NL}[s^n]^* - DF_{NL}\left[s^n - D^{-1}u^n\right]^* + \Delta t e_t^n + h e_s^n + e_f^n \tag{5.27}$$

and e_t^n, e_s^n, e_f^n denote the temporal, spatial and filtering errors respectively. They are given by

$$\begin{aligned} e_t^n &= D f_1^{n*} &(5.28)\\ e_s^n &= D f_2^{n*} &(5.29)\\ e_f^n &= \frac{1}{\Delta t} D\left[L\tilde{s}^n + \Delta t F_{NL}[\tilde{s}^n] - P\left\{L\tilde{s}^n + \Delta t F_{NL}[\tilde{s}^n] + \Delta t e_r\right\}\right]^*. &(5.30)\end{aligned}$$

Consequently, Eq. (5.26) is exactly in the form to which Theorem 4.1 may be applied once conditions (i)-(v) are verified. Eqs. (5.10) and (5.11) show that condition (i) is satisfied with $\delta = 1/2$. Eq. (5.12) shows that (ii) and (iii) are satisfied with $C_1 = cR$, where $\|u\|_{\rho'}, \|v\|_{\rho'} \leq R$ and $\rho' < \omega\rho_0 - \frac{1}{2}n\Delta t$ with $0 < \omega < 1$ (with $\omega\rho_0$ replacing ρ_0). It remains to verify conditions (iv) and (v).

Consider first condition (iv). Evaluating $A_n[0]$ gives

$$A_n[0] = \Delta t e_t^n + h e_s^n + e_f^n. \tag{5.31}$$

As we have seen from Eq. (5.25),

$$\|A_n[0]\|_\rho \leq c_\omega \varepsilon(\Delta t + h) + \|e_f\|_\rho \tag{5.32}$$

with $\rho < \omega\rho_0 - \kappa n\Delta t/2$. Thus, the filtering error, $\|e_f\|_\rho$, must be estimated. The following three lemmas, which show that the filtering operator P is stable and consistent, will be used for this purpose.

Lemma 5.1 *(Consistency property of P)*
Let $0 < \rho < \rho'$ and assume that $f \in B_{\rho'}$. Then

$$\|(I - P)f\|_\rho \leq \|f\|_{\rho'}^{\rho/\rho'}(2 + \rho^{-1} + (\rho' - \rho)^{-1})\tau^{1-\rho/\rho'}. \tag{5.33}$$

Lemma 5.2 *(Stability Property of P).*
Let $0 < \rho < \rho'$ and assume that $f \in B_{\rho'}$ and $g \in B_\rho$. Then

$$\|Pf - Pg\|_\rho \leq \|f - g\|_\rho + \|f\|_{\rho'}^{\rho/\rho'}(2 + \rho^{-1} + (\rho' - \rho)^{-1})\tau^{1-\rho/\rho'}. \quad (5.34)$$

Lemma 5.3 *(Filtering with Roundoff Error)*
Let $0 < \rho < \rho'$ and assume that $f \in B_{\rho'}$. Let e_r represents simulated roundoff error, with $|\hat{e}_r(k)| \leq \varepsilon_r < \tau/2$ for all k. Two estimates on the filtering in the presence of roundoff error are the following:

$$\begin{aligned} \|P(f + e_r) - Pf\|_\rho &\leq \frac{\varepsilon_r}{\rho'}\left(\frac{\|f\|_{\rho'}}{2\tau}\right)^{\rho/\rho'}\log\left(\frac{\|f\|_{\rho'}}{2\tau}\right) + 4\|f - P_{2\tau}f\|_{\rho'} && (5.35) \\ \|P(f + e_r) - Pf\|_\rho &\leq \frac{\varepsilon_r}{\tau}(\rho' - \rho)^{-1}\|f\|_{\rho'} + 4\|f - P_{2\tau}f\|_{\rho'} && (5.36) \end{aligned}$$

in which $P_{2\tau}$ is the filtering operator of (2.1) with τ replaced by 2τ.

The proofs of these lemmas are given in Appendix 2.

Now, estimate e_f^n by

$$\begin{aligned} \|e_f\|_\rho \leq\ & \frac{1}{\Delta t}(\rho' - \rho)^{-1}\|(I - P)\left(L\tilde{s}^n + \Delta t F_{NL}[\tilde{s}^n]\right)\|_{\rho'} \\ & + \frac{1}{\Delta t}(\rho' - \rho)^{-1}\|P\left(L\tilde{s}^n + \Delta t F_{NL}[\tilde{s}^n]\right) - P\left(L\tilde{s}^n + \Delta t F_{NL}[\tilde{s}^n] + \Delta t e_r\right)\|_{\rho'}. \end{aligned} \quad (5.37)$$

As we have seen by applying Theorem 4.2 to the discrete filtered equation, $\tilde{s}^n$ satisfies the uniform bound

$$\|\tilde{s}^n\|_{\tilde{\rho}} \leq \varepsilon \quad (5.38)$$

with $\tilde{\rho} \leq \rho_0 - \lambda_f n\Delta t$ with $\lambda_f = 1/2 + c\varepsilon$. Thus, by restricting ρ and ρ' in Eq. (5.37) by $0 < \rho < \rho' < \omega\rho_0 - \lambda_f n\Delta t$, Lemmas 5.1 and 5.3, together with the estimates (5.10)-(5.12) can be applied to show that

$$\|e_f\|_\rho \leq c_\omega \varepsilon\left(\frac{\tau^{1-\omega}}{\Delta t} + \frac{\varepsilon_r}{\tau\Delta t}\right). \quad (5.39)$$

Putting everything together shows that (iv) is satisfied with $K = c_\omega\varepsilon\left(\Delta t + h + \frac{\tau^{1-\omega}}{\Delta t} + \frac{\varepsilon_r}{\tau\Delta t}\right)$. Again, c_ω is a generic constant depending only on ω and $z(\alpha, t)$.

Finally, it remains to show that (v) is satisfied. We have

$$\begin{aligned} A_{n+1}[u] - A_n[u] =\ & DF_{NL}[s^{n+1}] - DF_{NL}[s^{n+1} - D^{-1}u] - DF_{NL}[s^n] + DF_{NL}[s^n - D^{-1}u] \\ & + \Delta t(e_t^{n+1} - e_t^n) + h(e_s^{n+1} - e_s^n) + e_f^{n+1} - e_f^n. \end{aligned} \quad (5.40)$$

It is not difficult to show that

$$\|\Delta t(e_t^{n+1} - e_t^n) + h(e_s^{n+1} - e_s^n) + e_f^{n+1} - e_f^n\|_\rho \leq \Delta t K \quad (5.41)$$

with $\rho \leq \omega\rho_0 - \tilde{\lambda}\Delta t$ with $\tilde{\lambda} = \max(\lambda_f, \kappa/2)$ and λ_f, K defined as above with a redefined constant c_ω. The estimate of the remaining terms in Eq. (5.40) is more subtle. A *straightforward* estimate of these terms yields an estimate of the form

$$\begin{aligned} \|DF_{NL}[s^{n+1}] &- DF_{NL}[s^{n+1} - D^{-1}u] - DF_{NL}[s^n] + DF_{NL}[s^n - D^{-1}u]\|_\rho \\ &\leq \|DF_{NL}[s^{n+1}] - DF_{NL}[s^n]\|_\rho + \|DF_{NL}[s^{n+1} - D^{-1}u] - DF_{NL}[s^n - D^{-1}u]\|\rho \\ & \hspace{30em} (5.42) \\ &\leq \frac{c}{\rho' - \rho}\|Ds^{n+1} - Ds^n\|_{\rho'} \\ &\leq \frac{c_\omega \varepsilon \Delta t}{\rho' - \rho} \hspace{25em} (5.43) \end{aligned}$$

for $\rho < \rho' \leq \omega\rho_0 - \tilde{\lambda}(n+1)\Delta t$ provided that $\|u\|_{\rho'} \leq 1/2$. See below and in Appendix 4 for further details. Estimate (5.43) implies that $C_2 = c_\omega \varepsilon$. However, it is not difficult to see that this estimate is not good enough to obtain convergence. This is because in the evaluation of λ from Eq. (4.8), the terms R_0/R, R/R_0 and C_2/R_0 must be evaluated. Since $R \to 0$ as the numerical parameters vanish (for convergence), we must similarly have $R_0 \to 0$ in order to obtain a finite estimate of λ. But, this implies that $c_\omega \varepsilon / R_0 \to \infty$ as c_ω, ε are independent of the numerical parameters. Clearly, it is important, to obtain a finite estimate of λ, that $C_2 \to 0$ as the numerical parameters vanish as well. This difficulty arises because the terms involving u and s^{n+1}, s^n have been handled in (5.42),(5.43) so that the dependence on u (the equivalent of the numerical error) is removed. It is exactly this dependence that is required for $C_2 \to 0$. It turns out that if the estimate is not broken up as in Eq. (5.42) that the highest order terms involving s^{n+1} and s^n cancel and only cross terms involving u and $s^{n+1} - s^n$ remain! This shows that C_2 in fact $\to 0$ as the numerical parameters vanish. The following lemma summarizes the result.

Lemma 5.4 *(Time Difference of Nonlinear Operator)*
Let $\|u\|_\rho \leq R$ and $\|Df\|_\rho, \|Dg\|_\rho \leq \tilde{R}$ with $R \leq \tilde{R} < 1/3$. Then

$$\|F_{NL}[f] - F_{NL}[f - D^{-1}u] - F_{NL}[g] + F_{NL}[g - D^{-1}u]\|_\rho \leq cR\|D(f-g)\|_\rho. \tag{5.44}$$

The constant c is independent of f, g and u.

The proof uses discrete Fourier analysis to obtain the explicit cancellation and will be presented in Appendix 4.

Applying Lemma 5.4, with $f = s^{n+1}$ and $g = s^n$, to Eq. (5.40) gives

$$\|A_{n+1}[u] - A_n[u]\|_\rho \leq \frac{c}{\rho' - \rho}\left[R\|D\left(s^{n+1} - s^n\right)\|_{\rho'} + \Delta t K\right] \tag{5.45}$$

where $\rho < \rho' < \omega\rho_0 - \tilde{\lambda}(n+1)\Delta t$, $\|u\|_{\rho'} \leq R$ and $\|Ds\|_{\bar{\rho}(t)} \leq \varepsilon$. Using Eq. (5.23), the estimates (5.10)-(5.12) and the analyticity of the smooth solution s, it is straightforward to show that

$$\|D\left(s^{n+1} - s^n\right)\|_{\rho'} \leq c_\omega \varepsilon \Delta t. \tag{5.46}$$

Using (5.46) in (5.45) gives

$$\|A_{n+1}[u] - A_n[u]\|_\rho \leq \frac{\Delta t}{\rho' - \rho}\left[c_\omega R\varepsilon + K\right] \tag{5.47}$$

This shows that (v) is satisfied with $C_2 = c_\omega R\varepsilon + K$ which $\to 0$ as the numerical parameters vanish. Consequently, the hypotheses of the theorem are satisfied by taking

$$K = c_\omega \varepsilon \left(\Delta t + h + \frac{\tau^{1-\omega}}{\Delta t} + \frac{\varepsilon_r}{\tau \Delta t} \right), \; C_2 = c_\omega R\varepsilon + K, \;\; C_1 = c_\omega \varepsilon, \;\; \gamma = \omega \tag{5.48}$$

$$\lambda_0 = \max(\lambda_f, \kappa/2), \; R = \frac{K}{\varepsilon\omega}, \;\; R_0 = \frac{K\rho_0^\omega}{\sqrt{\varepsilon}} \tag{5.49}$$

provided that $\Delta t, h, \tau$ and ε_r are small enough. This shows that

$$\|s^{n+1} - \tilde{s}^{n+1}\|_{\rho_{n+1}} \le \frac{c_\omega}{\omega} \left(\Delta t + h + \frac{\tau^{1-\omega}}{\Delta t} + \frac{\varepsilon_r}{\tau \Delta t} \right) \tag{5.50}$$

where

$$\rho_{n+1} = \omega \rho_0 - \lambda(n+1)\Delta t, \;\; \text{and } \lambda = 1/2 + c_\omega \sqrt{\varepsilon} \tag{5.51}$$

since $\lambda_0 = \max(\lambda_f, \kappa/2) = 1/2 + c\epsilon$ for some c. This completes the proof of Theorem 3.2.

6 Proof of Discrete C-K Theorem with Roundoff Error and Filtering

In this section, the proof of Theorem 4.2 is presented. This theorem provides uniform bounds for the numerical solution taking into account roundoff error and filtering. The proof of Theorem 4.2 is performed by carefully estimating each wavenumber separately and using induction. In spirit, it is very similar to that of Theorem 4.1 which is presented in Appendix 2.

Since the projection is applied to the right hand side of Eq. (4.9), the nth iterate v_n satisfies

$$|\hat{v}_n(k)| > \tau \;\; \text{or} \;\; \hat{v}_n(k) = 0. \tag{6.52}$$

This implies that there are 3 cases for the next iterate:

(a). $\hat{v}_{n+1}(k) = 0$

(b). $|\hat{v}_{n+1}(k)| > \tau$ and $\hat{v}_n(k) = 0$

(c). $|\hat{v}_{n+1}(k)| > \tau$ and $|\hat{v}_n(k)| > \tau$.

Cases (b) and (c) can be used to estimate the size of the Fourier coefficients of the roundoff error in terms of the nonlinear term, A_n, and the previous iterate, v_n, respectively. Consider (b) first. This implies that

$$|\widehat{A_n}[v_n](k) + \hat{e}_r(k)| > \tau/\Delta t. \tag{6.53}$$

Further, assumption (vi) implies that

$$|\widehat{A_n}[v_n](k)| > \frac{\tau}{2\Delta t}. \tag{6.54}$$

Again using (vi), this implies that

$$|\hat{e}_r(k)| < \frac{2\varepsilon_r \Delta t}{\tau} |\widehat{A_n}[v_n](k)|. \tag{6.55}$$

For case (c), it is straightforward to see that

$$|\hat{e}_r(k)| < \frac{\varepsilon_r}{\tau} |\hat{v}_n(k)|. \tag{6.56}$$

These bounds will be important in the proof.

Now, let $0 < \gamma < 1$, Π be a linear Fourier projection operator as described above, $m \leq n$, λ be as in Eq. (4.13) and R_2, R_1 be as in Eqs. (4.10), (4.11) respectively. Further, let ρ such that

$$0 < \rho < \rho_0 - \lambda m \Delta t \tag{6.57}$$

Suppose, by way of induction, that

$$\begin{aligned} \|v_m\|_\rho &\leq R & (6.58)\\ (\rho_0 - \rho - \lambda m\Delta t)^\gamma \|A_m[\Pi v_m]\|_\rho &\leq R_2 & (6.59)\\ (\rho_0 - \rho - \lambda m\Delta t)^\gamma \|(L-I)v_m\|_\rho &\leq \Delta t \cdot R_1 & (6.60) \end{aligned}$$

for any Fourier filter Π. It is straightforward to see that Eqs. (6.58)- (6.60) hold at $m = 0$ by assumption (viii) on the initial data and the definitions of R_2 and R_1. The proof will be complete when it is shown that (6.58)-(6.60) hold for $m = n+1$.

The bound (6.58) will be established first. For each wavenumber k, define m_k to be the largest integer such that

$$0 \leq m_k \leq n+1 \quad \text{and} \quad \hat{v}_{m_k}(k) = 0. \tag{6.61}$$

On the other hand, if

$$\hat{v}_m(k) \neq 0 \quad \text{for all} \quad 0 \leq m \leq n+1 \tag{6.62}$$

then set $m_k = 0$. Given m_k, the solution v_{n+1} can be written as follows

$$\hat{v}_{n+1}(k) = \sum_{m=m_k}^{n} l^{n-m}(\hat{v}_{m+1} - l\hat{v}_m) + l^{n+1-m_k}\hat{v}_{m_k} \tag{6.63}$$

where $\hat{L} = l$. Roughly speaking, m_k measures the amount of time a mode is inactive. Note that if $m_k \neq 0$, then the last term in Eq. (6.63) vanishes. Each Fourier mode of v_{n+1} will be estimated separately to establish (6.58). In this analysis, the effect of the filter P will be written out explicitly, so that the nonlinearity of P can be handled. There are now two cases to consider.

Case 1: $\hat{v}_{n+1}(k) = 0$

In this case, the bound is trivial.

Case 2: $|\hat{v}_{n+1}(k)| > \tau$

Estimate each term in (6.63). Since $|\hat{v}_{m+1}(k)| > \tau$, then the filter has no effect and the equation for $\hat{v}_{m+1}$ is

$$\hat{v}_{m+1} = l\hat{v}_m + \Delta t(\hat{A}_m[v_m] + \hat{e}_r) \tag{6.64}$$

Consider first the subcase $n \geq m > m_k$ or $n \geq m \geq m_k$ if $m_k = 0$. This implies that

$$|\hat{v}_m(k)| > \tau \tag{6.65}$$

since otherwise the solutions would be set to zero by the filter. Therefore, case (c) applies to give

$$\begin{aligned} |\hat{v}_{m+1} - l\hat{v}_m| &\leq \Delta t\left(|\widehat{A_m}[v_m]| + |\hat{e}_r|\right) \\ &\leq \Delta t\left(|\widehat{A_m}[v_m]| + \frac{\varepsilon_r}{\tau}|\hat{v}_m|\right). \end{aligned} \tag{6.66}$$

Now, consider the subcase $m = m_k$ with $m_k \neq 0$. This implies that

$$|\hat{v}_{m_{k+1}}| > \tau \quad \text{and} \quad \hat{v}_m = 0 \tag{6.67}$$

which is exactly case (b). This implies that

$$\begin{aligned}|\hat{v}_{m_{k+1}} - l\hat{v}_{m_k}| &\leq \Delta t \left(|\widehat{A_{m_k}}[v_{m_k}]| + |\hat{e}_r|\right) \\ &\leq \Delta t \left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right) |\widehat{A_{m_k}}[v_{m_k}]|. \end{aligned} \tag{6.68}$$

Finally, it remains to consider the term $\hat{v}_{m_k}$. If $m_k = 0$ then this term is equal to 0. Otherwise, it is $\hat{v}_0$. This completes the analysis of the $2nd$ case.

Now combining the two cases and Eqs. (6.63), (6.66) and (6.68), an overestimate for (6.63) is obtained:

$$|\hat{v}_{n+1}(k)| \leq \Delta t \sum_{m=0}^{n} |l|^{n-m} \left[\left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right) |\widehat{A_m}[v_m](k)| + \frac{\varepsilon_r}{\tau} |\hat{v}_m(k)|\right] + |l|^{n+1} |\hat{v}_0|. \tag{6.69}$$

Recalling that the ρ-norm is given by

$$\|v\|_\rho = \sum_k e^{\rho|k|} |\hat{v}(k)|. \tag{6.70}$$

then Eq. (6.69) implies that

$$\begin{aligned}\|v_{n+1}\|_\rho &\leq \Delta t \sum_{m=0}^{n} \left[\left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right) \|L^{n-m} A_m[v_m]\|_\rho + \frac{\varepsilon_r}{\tau} \|L^{n-m} v_m\|_\rho\right] + \|L^{n+1} v_0\|_\rho \\ &\leq \Delta t \sum_{m=0}^{n} \left[\left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right) \|A_m[v_m]\|_{\rho+\lambda_0(n-m)\Delta t} + \frac{\varepsilon_r}{\tau} \|v_m\|_{\rho+\lambda_0(n-m)\Delta t}\right] \\ &\quad + \|v_0\|_{\rho+\lambda_0(n+1)\Delta t} \end{aligned} \tag{6.71}$$

using assumption (i). Let $\rho \leq \rho_n = \rho_0 - \lambda n \Delta t$. Since $\lambda > \lambda_0$, then $\rho + \lambda_0(n-m)\Delta t < \rho_0 - \lambda m \Delta t$ for $m \leq n$. So, the induction hypothesis (6.59) implies that

$$\|A_m[v_m]\|_{\rho+\lambda_0(n-m)\Delta t} \leq \frac{R_2}{[\rho_0 - (\rho + \lambda_0(n-m)\Delta t) - \lambda m \Delta t]^\gamma} \tag{6.72}$$

Writing $\lambda = \lambda_0 + \lambda'$ and using Eq. (6.72) and the induction hypothesis (6.58) in Eq. (6.71) gives

$$\|v_{n+1}\|_\rho \leq \left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right) R_2 \Delta t \sum_{m=0}^{n} \frac{1}{(\rho_0 - \rho - \lambda_0 n \Delta t - \lambda' m \Delta t)^\gamma} + \left(\delta + \frac{\varepsilon_r}{\tau} t\right) R \tag{6.73}$$

in which $t = n\Delta t$, where we have additionally used assumption (viii). Estimate the sum in Eq. (6.73) by the integral inequality

$$\begin{aligned}\Delta t \sum_{m=0}^{n} \frac{1}{(\rho_0 - \rho - \lambda_0 n \Delta t - \lambda' m \Delta t)^\gamma} &\leq \int_0^{n\Delta t} (\rho_0 - \rho - \lambda_0 n \Delta t - \lambda' t)^{-\gamma} \, dt \\ &\leq \frac{\rho_0^{1-\gamma}}{\lambda'(1-\gamma)}. \end{aligned} \tag{6.74}$$

Now, using Eq. (6.74) in Eq. (6.73) gives

$$\|v_{n+1}\|_\rho \leq \left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right) \frac{R_2 \rho_0^{1-\gamma}}{\lambda'(1-\gamma)} + \left(\delta + \frac{\varepsilon_r}{\tau} t\right) R. \tag{6.75}$$

Therefore, since

$$\lambda' = \lambda - \lambda_0 \geq \left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right)\left(1 - \delta - \frac{\varepsilon_r}{\tau}t\right)^{-1} \frac{R_2 \rho_0^{1-\gamma}}{R(1-\gamma)} \tag{6.76}$$

by Eq. (4.13), the estimate

$$\|v_{n+1}\|_{\rho_n} \leq R \tag{6.77}$$

is proved. Since $\rho_{n+1} < \rho_n$, this poves the induction hypothesis (6.58) for $m = n+1$, and in fact proves the theorem once the $n+1$ induction step is proved for Eqs. (6.59) and (6.60).

Next, turn to the proof of (6.59) for $m = n+1$. Begin by defining ρ' to be

$$\rho' = \frac{1}{2}(\rho_0 - \lambda n \Delta t + \rho) \tag{6.78}$$

for any ρ. Thus, if

$$0 < \rho < \rho_0 - \lambda(n+1)\Delta t \ \text{ then } \ \rho < \rho' < \rho_0 - \lambda n \Delta t. \tag{6.79}$$

Further, for any Fourier projection Π define the Fourier projections Π_{n+1} and Π' by

$$\widehat{\Pi}_{n+1} = \begin{cases} 0 & \text{if } \hat{v}_{n+1} = 0 \\ 1 & \text{otherwise} \end{cases} \tag{6.80}$$

$$\Pi' = \Pi_{n+1}\Pi. \tag{6.81}$$

Since $\rho' < \rho_n$, (6.77) implies that $\|v_{n+1}\|_{\rho'} < R$. Thus we may apply (v), as well as (iii) and the induction hypothesis (6.59) for Π' on v_n to obtain

$$\begin{aligned} \|A_{n+1}[\Pi v_{n+1}]\|_\rho &= \|A_{n+1}[\Pi' v_{n+1}]\|_\rho \\ &\leq \|A_n[\Pi' v_n]\|_\rho + \|A_{n+1}[\Pi' v_{n+1}] - A_n[\Pi' v_{n+1}]\|_\rho + \|A_n[\Pi' v_{n+1}] - A_n[\Pi' v_n]\|_\rho \\ &\leq \|A_n[\Pi' v_n]\|_\rho + \frac{C_1}{\rho' - \rho}\|\Pi' v_{n+1} - \Pi' v_n\|_{\rho'} + \frac{C_2 \Delta t}{\rho' - \rho}. \end{aligned} \tag{6.82}$$

The middle term is handled by estimating each Fourier mode separately as follows. There are two cases

Case 1: $\hat{v}_{n+1}(k) = 0$

By the definition of Π', this implies that

$$\widehat{\Pi' v}_{n+1}(k) - \widehat{\Pi' v}_n(k) = 0 \tag{6.83}$$

so the bound is trivial.

Case 2: $\hat{v}_{n+1}(k) \neq 0$

This implies that $|\hat{v}_{n+1}(k)| > \tau$ so that

$$\hat{v}_{n+1}(k) = l\hat{v}_n(k) + \Delta t \widehat{A_n}[v_n](k) + \Delta t \hat{e}_r(k). \tag{6.84}$$

For such k,

$$\hat{v}_{n+1}(k) - \hat{v}_n(k) = (l-1)\hat{v}_n(k) + \Delta t \widehat{A_n}[v_n](k) + \Delta t \hat{e}_r(k). \tag{6.85}$$

The roundoff error $\hat{e}_r(k)$ is estimated by combining cases (b) and (c) to give

$$|\hat{e}_r(k)| \leq \frac{\varepsilon_r}{\tau}\left(|\hat{v}_n(k)| + 2\Delta t |\widehat{A_n}[v_n](k)|\right). \tag{6.86}$$

Now, combine Eqs. (6.85) and (6.86) and use the linearity of Π' to estimate the middle term in Eq. (6.82) by

$$\begin{aligned} \|\Pi' v_{n+1} - \Pi' v_n\|_{\rho'} &\leq \|v_{n+1} - v_n\||\rho' \\ &\leq \|(L-I)v_n\|_{\rho'} + \Delta t\left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right)\|A_n[v_n]\|_{\rho'} + \frac{\varepsilon_r \Delta t}{\tau}\|v_n\|_{\rho'}. \end{aligned} \tag{6.87}$$

This completes the analysis of case 2.

Apply the induction hypotheses (6.58)-(6.60) and the two cases, with the definition of ρ' from (6.78), to Eq. (6.82) to yield

$$\begin{aligned} \|A_{n+1}[\Pi v_{n+1}]\|_\rho \leq\; & R_2\left(\rho_0 - \rho - \lambda n\Delta t\right)^{-\gamma} + \left[C_1 \frac{2\varepsilon_r \Delta t}{\tau} R + 2C_2\Delta t\right]\left(\rho_0 - \rho - \lambda n \Delta t\right)^{-1} \\ & + \Delta t C_1 2^{1+\gamma}\left[R_1 + \left(1 + \frac{2\varepsilon_r\Delta t}{\tau}\right)R_2\right]\left(\rho_0 - \rho - \lambda n\Delta t\right)^{-1-\gamma}. \end{aligned} \tag{6.88}$$

Now, another integral inequality implies that

$$\left(\rho_0 - \rho - \lambda(n+1)\Delta t\right)^{-\gamma} \geq \left(\rho_0 - \rho - \lambda n\Delta t\right)^{-\gamma} + \gamma\lambda\Delta t\left(\rho_0 - \rho - \lambda n\Delta t\right)^{-1-\gamma}. \tag{6.89}$$

Using Eq. (6.89) in (6.88) and

$$\gamma\lambda \geq C_1 2^{1+\gamma}\left[\frac{R_1}{R_2} + \left(1 + \frac{2\varepsilon_r\Delta t}{\tau}\right)\right] + 2C_2\frac{\rho_0^\gamma}{R_2} + C_1\frac{2\varepsilon_r}{\tau}\frac{R\rho_0^\gamma}{R_2} \tag{6.90}$$

as guaranteed by Eq. (4.13), yields the bound

$$\left(\rho_0 - \rho - \lambda(n+1)\Delta t\right)^\gamma \|A_{n+1}[\Pi v_{n+1}]\|_\rho \leq R_2 \tag{6.91}$$

with $0 < \rho < \rho_0 - \lambda(n+1)\Delta t$. This proves the $n+1$ induction step for (6.59).

It now remains to prove Eq. (6.60) for $m = n+1$. Again, estimate each wavenumber individually and as before, there are two cases.

Case 1: $\hat{v}_{n+1}(k) = 0$

This implies that $(l-1)\hat{v}_{n+1}(k) = 0$ and the bound is trivial.

Case 2: $\hat{v}_{n+1}(k) \neq 0$

As before, this implies that

$$\hat{v}_{n+1}(k) = l\hat{v}_n(k) + \Delta t\widehat{A_n}[v_n](k) + \Delta t\hat{e}_r(k). \tag{6.92}$$

This implies that

$$(l-1)\hat{v}_{n+1}(k) = (l-1)l\hat{v}_n(k) + \Delta t(l-1)\widehat{A_n}[v_n](k) + \Delta t(l-1)\hat{e}_r(k). \tag{6.93}$$

Now, using (b),(c) gives the bound

$$|(l-1)\hat{v}_{n+1}(k)| \leq |(l-1)l\hat{v}_n(k)| + \Delta t\left(1 + \frac{2\varepsilon_r\Delta t}{\tau}\right)|(l-1)\widehat{A_n}[v_n](k)| + \frac{\varepsilon_r\Delta t}{\tau}|(l-1)\hat{v}_n(k)|. \tag{6.94}$$

This completes the analysis of case 2.

Combining cases 1 and 2, the induction hypotheses, assumption (i) and the definition of ρ' gives the bound

$$\begin{aligned}\|(L-I)v_{n+1}\|_\rho \quad \le \quad & \Delta t R_1 \left(\rho_0 - \rho - (\lambda n + \lambda_0)\Delta t\right)^{-\gamma} + \frac{\varepsilon_r \Delta t^2}{\tau} R_1 \left(\rho_0 - \rho - \lambda n \Delta t\right)^{-\gamma} \\ & + \Delta t^2 \lambda_0 R_2 2^{1+\gamma}\left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right)\left(\rho_0 - \rho - \lambda n \Delta t\right)^{-1-\gamma}. \end{aligned} \tag{6.95}$$

Another integral inequality gives

$$\begin{aligned}\left(\rho_0 - \rho - \lambda(n+1)\Delta t\right)^{-\gamma} \quad \ge \quad & \left(\rho_0 - \rho - (\lambda n + \lambda_0)\Delta t\right)^{-\gamma} \\ & + \Delta t \gamma (\lambda - \lambda_0)\left(\rho_0 - \rho - (\lambda n + \lambda_0)\Delta t\right)^{-1-\gamma}. \end{aligned} \tag{6.96}$$

Thus, using (6.96) in (6.95) as well as the fact that

$$\lambda \ge \lambda_0 + \frac{\lambda_0 R_2}{\gamma R_1}\left(1 + \frac{2\varepsilon_r \Delta t}{\tau}\right) 2^{\gamma+1} + \frac{\varepsilon_r}{\gamma \tau}\rho_0 \tag{6.97}$$

as guaranteed by Eq. (4.13), yields the bound

$$\left(\rho_0 - \rho - \lambda(n+1)\Delta t\right)^{\gamma} \|(L-I)v_{n+1}\|_\rho \le \Delta t \cdot R_1 \tag{6.98}$$

with $0 < \rho < \rho_0 - \lambda(n+1)\Delta t$. This proves the $n+1$ induction step for (6.60) and completes the proof of the theorem.

7 Appendix 1: Abstract Cauchy-Kowalewski Theorem, Continuous Version

In this appendix, a time continuous version of the abstract Cauchy-Kowalewski theorem is presented. This version is a modification of the strengthened formulation of Safonov [18] and is specialized for perturbative problems. See also [16, 17, 7]. Consider the equation

$$u(\alpha, t) = \varepsilon \int_0^t \tilde{N}[u](\alpha, t, t')dt' \tag{7.1}$$

Let B_ρ be a family of Banach spaces for $0 < \rho < \rho_0$ with norm $\|\cdot\|_\rho$ such that $B_{\rho'} \subset B_\rho$ and $\|u\|_\rho \le \|u\|_{\rho'}$ for $0 < \rho < \rho'$. Suppose that $\tilde{N}$ satisfies the following assumptions:

(i). For any $0 < \rho < \rho' < \rho_0 - \lambda_0 t$ and $t \ge t'$ suppose that

$$\|\tilde{N}[u](\cdot, t, t')\|_\rho \le \|N[u](\cdot, t)\|_{\rho + \lambda_0(t-t')} \tag{7.2}$$

where N is a continuous mapping from $\{u \in B_{\rho'}, \|u\|_{\rho'} \le R\}$ into B_ρ.

Moreover, N satisfies

(ii). For any $0 < \rho < \rho' < \rho_0 - \lambda_0 t$ and for any u, v with $u, v \in B_{\rho'}$ and $\|u\|_{\rho'}, \|v\|_{\rho'} \le R$, then there exists a constant C such that

$$\|N[u](\cdot, t) - N[v](\cdot, t)\|_\rho \le \frac{C}{\rho' - \rho}\|u - v\|_{\rho'} \tag{7.3}$$

where C is independent of u, v, ρ, ρ', t.

(iii). Finally, suppose that $N[0](\alpha, t)$ is a continuous function of t for $0 \le t < \rho_0/\lambda_0$ with values in B_ρ for $\rho < \rho_0$ such that

$$\|N[0](\cdot, t)\|_\rho \le K \tag{7.4}$$

for $\rho < \rho_0 - \lambda_0 t$ and some K independent of t, ρ.

Theorem 7.1 *Abstract Cauchy-Kowalewski Theorem*

For any R, K, C, ρ_0, λ_0 and $0 < \beta < 1$, such that (i)-(iii) are satisfied, there exists λ and a unique solution u to Eq. (7.1) such that

$$\|u\|_\rho \leq R \tag{7.5}$$

for $0 < \rho < \rho_0 - \lambda t$ and

$$\lambda = max \begin{cases} \lambda_0 + \frac{\epsilon R_0 \rho_0^{1-\beta}}{R(1-\beta)}, \\ \lambda_0 + 2^{2+\beta} \frac{C\epsilon}{\beta^2} \end{cases} \tag{7.6}$$

where $R_0 \geq K\rho_0^\gamma$ and C is the constant in assumption (ii).

The proof of this theorem follows closely that presented in [18] and we do not give it here.

Finally, this theorem can be applied to the nonlinear model problems (2.12) and (2.15) to show existence of solutions as follows. Take the analytic norm

$$\|f\|_\rho = \sum_k e^{\rho|k|} |\hat{f}_k| \tag{7.7}$$

and differentiate (2.12) and (2.15) to obtain the equations in quasi-linear form. Further, shift the solutions by u and $u - w + F$ respectively. That is, take as the new variable χ where $\chi = \eta_\alpha - u_\alpha$ for Eq. (2.12) and $\chi = \nu_\alpha - u_\alpha + w_\alpha - F_\alpha$ for Eq. (2.15). In addition, suppose that the mean of the initial data $\eta(0,t)$ and $\nu(0,t)$ is exactly 0. Then,

$$\tilde{N}[\chi] = \partial_\alpha \tilde{A}[\partial_\alpha^{-1}\chi + u], \quad \text{where} \quad \partial_\alpha^{-1}\chi = \eta - u \tag{7.8}$$

for Eq. (2.12) and

$$\tilde{N}[\chi] = \partial_\alpha \tilde{A}[\partial_\alpha^{-1}\chi + u - w + F], \quad \text{where} \quad \partial_\alpha^{-1}\chi = \nu - u + w - F \tag{7.9}$$

for Eq. (2.15). Then, it can be shown that assumptions (i)-(iii) are satisfied for some R, C, K and with ρ_0 replaced by $\gamma_1\rho_0$ for Eq. (2.12) and $\gamma_2\rho_0$ for Eq. (2.15) where $0 < \gamma_1 < \gamma_2 < 1$. The reason for this is that the initial conditions given for these equations in section 2 are in B_ρ for $\rho < \rho_0$, i.e. $\eta_k(0) = e^{-\rho_0|k|}$ is not in B_{ρ_0}.

8 Appendix 2: Proof of Filtering Lemmas

In this section, the proofs of lemmas 5.1-5.3 are given.

Proof of Lemma 5.1

By the definition of P, we have

$$\widehat{(I - P)f}_k = \begin{cases} \hat{f}_k, & \text{if } |\hat{f}_k| < \tau \\ 0, & \text{if } |\hat{f}_k| \geq \tau. \end{cases} \tag{8.1}$$

To bound $|\hat{f}_k|$ when $|\hat{f}_k| < \tau$, use $|\hat{f}_k| < \|f\|_{\rho'} e^{-\rho'|k|}$ if $e^{-\rho'|k|} < \sigma$ (i.e. if $|\log\sigma|/\rho' < |k|$), and use $|\hat{f}_k| < \tau$ if $\sigma < e^{-\rho'|k|}$ (i.e. if $|\log\sigma|/\rho' > |k|$). It follows that

$$\begin{aligned} \|\widehat{(I - P)}f\|_\rho &\leq \sum_{|\hat{f}_k|<\tau} |\hat{f}_k| e^{\rho|k|} \\ &\leq \sum_{|\log\sigma|/\rho'<|k|} \|f\|_{\rho'} e^{(\rho-\rho')|k|} + \sum_{|k|<|\log\sigma|/\rho'} \tau e^{\rho|k|} \\ &\leq \|f\|_{\rho'} \frac{a^n}{1-a} + \tau \frac{b^{n+1}-1}{b-1} \\ &\leq \|f\|_{\rho'} \sigma^{1-\rho/\rho'} (1 + (\rho' - \rho)^{-1}) + \tau\sigma^{-\rho/\rho'}(1 + \rho^{-1}) \end{aligned} \tag{8.2}$$

in which

$$\begin{aligned} n &= |\log \sigma|/\rho' \\ a &= e^{(\rho-\rho')} \\ b &= e^{\rho}. \end{aligned} \tag{8.3}$$

Choose $\sigma = \tau/||f||_{\rho'}$ to obtain

$$||(\widehat{I-P})f||_{\rho} \leq (2+\rho^{-1}+(\rho'-\rho)^{-1})||f||_{\rho'}^{\rho/\rho'}\tau^{1-\rho/\rho'}. \tag{8.4}$$

This proves Lemma 5.1.

Proof of Lemma 5.2

Decompose the norm $||Pf - Pg||$ into four parts, as

$$\begin{aligned} ||Pf - Pg||_{\rho} & \\ &= ||Pf - Pg||_{\rho}^{(1)} + ||Pf - Pg||_{\rho}^{(2)} \\ &\quad + ||Pf - Pg||_{\rho}^{(3)} + ||Pf - Pg||_{\rho}^{(4)} \end{aligned} \tag{8.5}$$

in which

$$||Pf - Pg||_{\rho}^{(i)} = \sum_{k \in K_i} |\hat{P}f(k) - \hat{P}g(k)| e^{\rho|k|} \tag{8.6}$$

with

$$\begin{aligned} K_1 &= \{k : |\hat{f}(k)| > \tau,\ |\hat{g}(k)| > \tau|\} \\ K_2 &= \{k : |\hat{f}(k)| < \tau,\ |\hat{g}(k)| < \tau|\} \\ K_3 &= \{k : |\hat{f}(k)| < \tau,\ |\hat{g}(k)| > \tau|\} \\ K_4 &= \{k : |\hat{f}(k)| > \tau,\ |\hat{g}(k)| < \tau|\}. \end{aligned} \tag{8.7}$$

For $k \in K_1$, $Pf = f$ and $Pg = g$, so that

$$||Pf - Pg||_{\rho}^{(1)} = ||f - g||_{\rho}^{(1)}. \tag{8.8}$$

For $k \in K_2$, $Pf = 0$ and $Pg = 0$, so that

$$||Pf - Pg||_{\rho}^{(2)} = 0. \tag{8.9}$$

For $k \in K_3$, $Pf = 0$ and $Pg = g$, so that

$$\begin{aligned} ||Pf - Pg||_{\rho}^{(3)} &= ||g||_{\rho}^{(3)} \\ &= ||f - g||_{\rho}^{(3)} + ||f - Pf||_{\rho}^{(3)} \\ &\leq ||f - g||_{\rho}^{(3)} \\ &\quad + (||f||_{\rho'}^{(3)})^{\rho/\rho'}(2+\rho^{-1}+(\rho'-\rho)^{-1})\tau^{1-\rho/\rho'}. \end{aligned} \tag{8.10}$$

For $k \in K_4$,

$$\begin{aligned} ||Pf - Pg||_{\rho}^{(4)} &= ||f||_{\rho}^{(4)} \\ &= S_1 + S_2 \end{aligned} \tag{8.11}$$

in which S_1 is the sum over $k \in K_4$ satisfying $e^{-\rho'|k|} < \sigma$ and S_2 is the sum over $k \in K_4$ satisfying $e^{-\rho'|k|} > \sigma$. If $k \in K_4$ and $|k| > |\log \sigma|/\rho'$, then $|\hat{f}(k)| < ||f||_{\rho'}^{(4)} e^{-\rho'|k|}$ and

$$\begin{aligned} S_1 &\leq ||f||_{\rho'}^{(4)} \sum_{|k|>|\log\sigma|/\rho'} e^{-(\rho'-\rho)|k|} \\ &\leq ||f||_{\rho'}^{(4)} \left(1 + (\rho' - \rho)^{-1}\right) \sigma^{1-\rho/\rho'}. \end{aligned} \tag{8.12}$$

Similarly for $k \in K_4$, $e^{-\rho'|k|} > \sigma$ (and $\tau > |\hat{g}_k|$), so that

$$\begin{aligned} S_2 &\leq \sum_{e^{-\rho'|k|}>\sigma,\ \tau>|\hat{g}_k|} \left(|\hat{f}_k - \hat{g}_k| + \tau\right) e^{\rho|k|} \\ &\leq ||f-g||_{\rho}^{(4)} + \tau \sum_{|k|<|\log\sigma|/\rho'} e^{\rho|k|} \\ &\leq ||f-g||_{\rho}^{(4)} + \tau\sigma^{-\rho/\rho'}(1+\rho^{-1}). \end{aligned} \tag{8.13}$$

Choose $\sigma = \tau/||f||_{\rho'}$ as above, to obtain

$$||Pf - Pg||_{\rho}^{(4)} \leq ||f-g||_{\rho}^{(4)} + (2 + \rho^{-1} + (\rho'-\rho)^{-1})||f||_{\rho'}^{\rho/\rho'} \tau^{1-\rho/\rho'}. \tag{8.14}$$

Combine these four sums together to obtain the result of Lemma 2. This proves Lemma 5.2.

Proof of Lemma 5.3

Decompose the sum $||P(f+e_r) - Pf||_\rho$ into three parts, as

$$||P(f+e_r) - Pf||_\rho = \left(\sum_{|\hat{f}_k|>2\tau} + \sum_{|\hat{f}_k|<\tau/2} + \sum_{\tau/2<|\hat{f}_k|<2\tau} \right) e^{\rho|k|} |(\widehat{P(f+e_r)} - \widehat{Pf})(k)|. \tag{8.15}$$

In the first sum, $(\widehat{P(f+e_r)} - \widehat{Pf})(k) = \hat{e}_r(k)$, so that $|(\widehat{P(f+e_r)} - \widehat{Pf})(k)| \leq \varepsilon_r$. Moreover, $2\tau < |\hat{f}_k| \leq ||f||_{\rho'} e^{-\rho'|k|}$, so that $|k| \leq k_1 = \frac{1}{\rho'}\log(||f||_{\rho'}/2\tau)$. Thus

$$\begin{aligned} \sum_{|\hat{f}_k|>2\tau} e^{\rho|k|} |(\widehat{P(f+e_r)} - \widehat{Pf})(k)| &\leq \varepsilon_r \sum_{k\leq k_1} (||f||_{\rho'}/2\tau)^{\rho/\rho'} \\ &\leq \frac{\varepsilon_r}{\rho'} (||f||_{\rho'}/2\tau)^{\rho/\rho'} \log(||f||_{\rho'}/2\tau). \end{aligned} \tag{8.16}$$

A cruder bound on this sum is

$$\begin{aligned} \sum_{|\hat{f}_k|>2\tau} e^{\rho|k|}\varepsilon_r &\leq \frac{\varepsilon_r}{2\tau} \sum e^{\rho|k|} |\hat{f}_k| \\ &\leq \frac{\varepsilon_r}{2\tau} ||f||_{\rho'} \sum e^{-(\rho'-\rho)|k|} \\ &\leq \frac{\varepsilon_r}{2\tau} (1 + (\rho'-\rho)^{-1}) ||f||_{\rho'}. \end{aligned} \tag{8.17}$$

In the second sum, $\widehat{P(f+e_r)}(k) = \widehat{Pf}(k) = 0$. Thus it contributes nothing.

In the third sum, a straightforward argument shows that $|(\widehat{P(f+e_r)} - \widehat{Pf})(k)| \leq 2\tau \leq 4|\hat{f}_k| = 4|(\hat{f} - \widehat{P_{2\tau}f})(k)|$. Thus

$$\sum_{\tau/2<|\hat{f}_k|<2\tau} \leq 4||f - P_{2\tau}f||_\rho. \tag{8.18}$$

This proves Lemma 5.3.

9 Appendix 3: Discrete Cauchy-Kowalewski Theorem

In this section, the proof of the discrete Cauchy-Kowalewski Theorem is presented in the absence of simulated roundoff error and numerical filtering.

Proof of Theorem 4.1

Set $0 < \gamma < 1$ and define

$$\begin{aligned} R_0 &= K\rho_0^\gamma \\ \lambda &= \max\left(\lambda_0 + \frac{R_0\rho_0^{1-\gamma}}{R(1-\gamma)}, \lambda_0\left(1 + \frac{R_0 2^{1+\gamma}}{R_1\gamma}\right), \gamma^{-1}\left(C_1 2^{1+\gamma}(1+\frac{R_1}{R_0}) + C_2\frac{\rho_0^\gamma}{R_0}\right)\right). \end{aligned} \tag{9.1}$$

Since $u_0 = 0$, the value of u_{n+1} is given by

$$u_{n+1} = \Delta t\sum_{k=0}^{n} L^{n-k}A_k[u_k]. \tag{9.2}$$

Suppose by way of induction that for $k \leq n$ and for $0 < \rho \leq \rho_0 - \lambda k\Delta t$

$$||u_k||_\rho \leq R \tag{9.3}$$

$$(\rho_0 - \rho - \lambda k\Delta t)^\gamma ||A_k[u_k]||_\rho \leq R_0 \tag{9.4}$$

$$(\rho_0 - \rho - \lambda k\Delta t)^\gamma ||(L-I)u_k||_\rho \leq \Delta t \cdot R_1. \tag{9.5}$$

At $k = 0$, the first and third are clearly satisfied since $u_0 = 0$; the second is satisfied by using assumption (iii) and the fact that $K\rho_0^\gamma = R_0$. We shall show that the bounds (9.3), (9.4) and (9.5) are also true for $k = n + 1$.

First, denote $\lambda = \lambda_0 + \lambda'$. Assuming that $\rho \leq \rho_n = \rho_0 - \lambda n\Delta t$, estimate

$$\begin{aligned} ||u_{n+1}||_\rho &\leq \Delta t\sum_{k=1}^{n} ||L^{n-k}A_k[u_k]||_\rho \\ &\leq \Delta t\sum_{k=0}^{n} ||A_k[u_k]||_{\rho+(n-k)\lambda_0\Delta t} \\ &\leq \Delta t\sum_{k=0}^{n} R_0(\rho_0 - \rho - \lambda_0 n\Delta t - \lambda' k\Delta t)^{-\gamma} \\ &\leq R_0\int_0^{n\Delta t}(\rho_0 - \rho - \lambda_0 n\Delta t - \lambda' t)^{-\gamma}dt \\ &\leq \frac{R_0}{(1-\gamma)\lambda'}(\rho_0 - \rho - \lambda n\Delta t)^{1-\gamma} \\ &\leq \frac{R_0\rho^{1-\gamma}}{(1-\gamma)\lambda'} \\ &\leq R \end{aligned} \tag{9.6}$$

since $\lambda' \geq \frac{K\rho_0}{R(1-\gamma)} = \frac{R_0\rho_0^{1-\gamma}}{R(1-\gamma)}$. This shows that

$$||u_{n+1}||_{\rho_n} \leq R. \tag{9.7}$$

Since $\rho_{n+1} < \rho_n$, this implies (9.3) for $m = n + 1$.

It remains now to prove (9.4) and (9.5). Consider (9.4). Let $0 < \rho < \rho_0 - \lambda(n+1)\Delta t$ and set

$$\rho' = (\rho_0 - \lambda n\Delta t + \rho)/2 \tag{9.8}$$

so that $\rho < \rho' < \rho_0 - \lambda n \Delta t$ and

$$\rho' - \rho = \rho_0 - \rho' - \lambda n\Delta t = (\rho_0 - \rho - \lambda n\Delta t)/2. \tag{9.9}$$

Since $\rho' < \rho_n$, equation (9.7) implies that $\|u_{n+1}\|_{\rho'} < R$. Thus we may apply assumption (v), as well as assumptions (iii) and the induction hypothese, to obtain

$$\begin{aligned} ||A_{n+1}[u_{n+1}]||_\rho &\leq ||A_n[u_n]||_\rho + ||A_{n+1}[u_{n+1}] - A_n[u_{n+1}]||_\rho + ||A_n[u_{n+1}] - A_n[u_n]||_\rho \\ &\leq ||A_n[u_n]||_\rho + C_2\Delta t(\rho'-\rho)^{-1} + C_1(\rho'-\rho)^{-1}||u_{n+1} - u_n||_{\rho'} \\ &\leq ||A_n[u_n]||_\rho + C_2\Delta t(\rho'-\rho)^{-1} + C_1(\rho'-\rho)^{-1}(||(L-I)u_n||_{\rho'} + \Delta t||A_n[u_n]||_{\rho'}) \\ &\leq R_0(\rho_0-\rho-\lambda n\Delta t)^{-\gamma} + C_2\Delta t(\rho'-\rho)^{-1} \\ &\quad +\Delta t(R_0+R_1)C_1(\rho'-\rho)^{-1}(\rho_0-\rho'-\lambda n\Delta t)^{-\gamma} \end{aligned} \tag{9.10}$$

Now, choosing ρ' as in Eq. (9.8), we get

$$\begin{aligned} ||A_{n+1}[u_{n+1}]||_\rho &\leq R_0(\rho_0-\rho-\lambda n\Delta t)^{-\gamma} + 2C_2\Delta t(\rho_0-\rho-\lambda n\Delta t)^{-1} \\ &\quad + 2^{1+\gamma}\Delta t C_1(R_0+R_1)(\rho_0-\rho-\lambda n\Delta t)^{-1-\gamma}. \end{aligned} \tag{9.11}$$

Again by using an integral inequality, one gets

$$\begin{aligned} (\rho_0-\rho-\lambda(n+1)\Delta t)^{-\gamma} &- (\rho_0-\rho-\lambda n\Delta t)^{-\gamma} \\ &\geq \gamma\lambda\Delta t(\rho_0-\rho-\lambda n\Delta t)^{-1-\gamma} \end{aligned} \tag{9.12}$$

this shows that

$$\begin{aligned} R_0\,(\rho_0-\rho-\lambda(n+1)\Delta t)^{-\gamma} &\geq R_0\,(\rho_0-\rho-\lambda n\Delta t)^{-\gamma} + R_0\gamma\lambda\Delta t\,(\rho_0-\rho-\lambda n\Delta t)^{-1-\gamma} \\ &\geq R_0\,(\rho_0-\rho-\lambda n\Delta t)^{-\gamma} + 2C_2\Delta t(\rho_0-\rho-\lambda n\Delta t)^{-1} \\ &\quad (\rho_0-\rho-\lambda n\Delta t)^{-1-\gamma} \end{aligned} \tag{9.13}$$

since $\gamma\lambda \geq C_1 2^{1+\gamma}(1+\frac{R_1}{R_0}) + 2C_2\frac{\rho_0^\gamma}{R_0}$. Therefore, using Eq. (9.13) in Eq. (9.11) proves the bound (9.4).

It remains to show (9.5). Consider

$$\begin{aligned} ||(L-I)u_{n+1}||_\rho &\leq ||L(L-I)u_n||_\rho + \Delta t||(L-I)A_n[u_n]||_\rho \\ &\leq ||(L-I)u_n||_{\rho+\lambda_0\Delta t} + (\Delta t)^2\lambda_0(\rho'-\rho)^{-1}||A_n[u_n]||_{\rho'} \\ &\leq R_1\Delta t\{(\rho_0-\rho-(\lambda n+\lambda_0)\Delta t)^{-\gamma} \\ &\quad +\frac{R_0}{R_1}\Delta t\lambda_0(\rho'-\rho)^{-1}(\rho_0-\rho'-\lambda n\Delta t)^{-\gamma}\} \\ &\leq R_1\Delta t(\rho_0-\rho-\lambda(n+1)\Delta t)^{-\gamma} \end{aligned} \tag{9.14}$$

by using assumption (i), the induction hypothesis, an integral inequality and $\lambda \geq \lambda_0(1+2^{1+\gamma}\frac{R_0}{R_1\gamma})$. This establishes the induction hypothesis for $k = n+1$, so that the bounds (9.3)-(9.5) are true for all k, hence (9.7) holds and the discrete Cauchy-Kowalewski theorem 4.1 is proven.

10 Appendix 4: Proof of Time Difference of Nonlinear Operator

In this section, Lemma 5.4 is proved by explicitly analyzing the discrete Fourier transform of the nonlinear operator. Begin by writing

$$F_{NL}[f] = \sum_{m=2}^{\infty}(-1)^m F_{NL}^m[f] \tag{10.1}$$

where

$$F_{NL}^m[f]_j = \frac{h}{2\pi i}\sum_{l\neq j}\frac{(f_j - f_l)^m}{(\alpha_j - \alpha_l)^{m+1}}. \tag{10.2}$$

A straightforward computation shows that

$$\widehat{F}_{NL}[f]_k = \sum_{k_1+\cdots+k_m=k}\left(\Pi_{r=1}^m \hat{f}_{k_r}\right)\tilde{I}(k_1,\ldots,k_m,k) \tag{10.3}$$

where

$$\tilde{I}(k_1,\ldots,k_m,k) = \frac{h}{2\pi i}\sum_{l\neq 0}\frac{1}{\alpha_l}\Pi_{r=1}^m\left(\frac{1-e^{ik_r\alpha_l}}{\alpha_l}\right) \tag{10.4}$$

Using the analysis presented in [8], it is straightforward to show that

$$|\tilde{I}(k_1,\ldots,k_m,k)| \leq c\Pi_{r=1}^m|k_r| \tag{10.5}$$

where c is independent of all k_r and m. Consequently, one gets

$$\begin{aligned}\widehat{F}_{NL}[f]_k - \widehat{F}_{NL}[f-v] - \widehat{F}_{NL}[g] + \widehat{F}_{NL}[g-v] = &\sum_{k_1+\cdots+k_m=k}\tilde{I}(k_1,\ldots,k_m,k)\cdot \\ &\left[\Pi_{r=1}^m\hat{f}_{k_r} - \Pi_{r=1}^m(\hat{f}_{k_r}-\hat{v}_{k_r}) - \Pi_{r=1}^m\hat{g}_{k_r} + \Pi_{r=1}^m(\hat{g}_{k_r}-\hat{v}_{k_r})\right].\end{aligned} \tag{10.6}$$

It is clear now that the terms $\Pi_{r=1}^m\hat{f}_{k_r}$, $\Pi_{r=1}^m\hat{g}_{k_r}$ $\Pi_{r=1}^m\hat{v}_{k_r}$ cancel in the bracketed term on the RHS of Eq. (10.6) leaving only the cross terms remaining. For example, if $m=2$, this yields

$$\hat{v}_{k_2}(\hat{f}_{k_1} - \hat{g}_{k_1}) + \hat{v}_{k_1}(\hat{f}_{k_2} - \hat{g}_{k_2}) \tag{10.7}$$

and for $m=3$ one gets

$$\begin{aligned}\hat{v}_{k_3}(\hat{f}_{k_1} - \hat{g}_{k_1}) &+ \hat{v}_{k_3}\hat{g}_{k_1}(\hat{f}_{k_2} - \hat{g}_{k_2}) - \hat{v}_{k_1}\hat{v}_{k_2}(\hat{f}_{k_3} - \hat{g}_{k_3}) + \hat{v}_{k_1}(\hat{f}_{k_2} - \hat{g}_{k_2})(\hat{f}_{k_3} - \hat{v}_{k_3}) \\ &+ \hat{v}_{k_1}\hat{g}_{k_2}(\hat{f}_{k_3} - \hat{g}_{k_3}) + \hat{v}_{k_2}\left[(\hat{f}_{k_1} - \hat{g}_{k_1})(\hat{f}_{k_3} - \hat{v}_{k_3}) + \hat{g}_{k_1}(\hat{f}_{k_3} - \hat{g}_{k_3})\right].\end{aligned} \tag{10.8}$$

The higher order terms are handled analogously and after tedious computation, exact formulae can be obtained.

Now, using Eqs. (10.5) and (10.7) one gets

$$\begin{aligned}\|F_{NL}[f] &- F_{NL}[f-v] - F_{NL}[g] + F_{NL}[g-v]\|_\rho \\ &\leq c\sum_{|k|}e^{\rho|k|}\sum_{k_1+k_2=k}\left[|k_1\hat{v}_{k_1}|k_2(\hat{f}_{k_2} - \hat{g}_{k_2})| + |k_2\hat{v}_{k_2}|k_1(\hat{f}_{k_1} - \hat{g}_{k_1})|\right] + \sum_{|k|}e^{\rho|k|}|\text{H.O.T.}| \\ &\leq 2c\|Dv\|_\rho\|D(f-g)\|_\rho + \sum_{|k|}e^{\rho|k|}|\text{H.O.T.}|\end{aligned} \tag{10.9}$$

where H.O.T. stands for the higher order terms. A straightforward, but tedious computation shows that if $\|Dv\|_\rho \leq R$ and $\|Df\|_\rho, \|Df\| \leq \tilde{R}$ with $R \leq \tilde{R} \leq \frac{1}{3}$, then the H.O.T. are estimated by

$$\begin{aligned} \sum_{|k|} e^{\rho|k|}|\text{H.O.T.}| &\leq c\|Dv\|_\rho \|D(f-g)\|_\rho \sum_{m=3}^{\infty} \tilde{R}^{m-2}\left[2(m-1)+(m+3)2^{m-2}\right] \\ &\leq c\|Dv\|_\rho \|D(f-g)\|_\rho \end{aligned} \tag{10.10}$$

where c is independent of v, f, g since

$$\sum_{m=3}^{\infty} \tilde{R}^{m-2}\left[2(m-1)+(m+3)2^{m-2}\right] \leq \tilde{R}\left[\frac{\tilde{R}+6}{(1-\tilde{R})^2}+\frac{28}{(1-2\tilde{R})^2}\right] \leq c. \tag{10.11}$$

Finally, putting Eq. (10.9) together with Eqs. (10.10) and (10.11) proves Lemma 5.4.

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