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MULTILEVEL ADDITIVE PRECONDITIONER FOR ELLIPTIC PROBLEMS ON NON-NESTED MESHES

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Abstract. We will analyze multilevel additive Schwarz preconditioner based on non-nested multilevel meshes. Under some assumptions on the relations between neighbouring mesh levels, we prove that the condition number of the preconditioned system can be bounded by $O(L^2)$. Here L is the number of levels of non-nested meshes.

Key Words. Non-nested meshes, multilevel additive Schwarz preconditioner.

AMS(MOS) subject classification. 65N30, 65F10

1. Introduction. In this paper, we are mainly interested in applying multilevel additive Schwarz algorithms to solve linear systems of equations arising from the finite element discretizations of second-order elliptic problems based on non-nested meshes. For the nested meshes, there have existed a well developed theory for the multilevel algorithms. Algorithms of the kind was initially proposed by Bramble et al. [2], known as BPX algorithm, and a little later, was also independently discussed in Zhang [11]. It was proved there that the condition number of the BPX operator is bounded by $O(L^2)$ where L is the number of mesh levels. Later on, Oswald [9], Bramble and Pasciak [3] and Xu [10] improved the previous estimates of the condition number of BPX algorithm and showed that the condition number is $O(1)$. Dryja and Widlund [7] and Zhang [12] used the additive Schwarz framework to discuss a class of multilevel algorithms with BPX as its special case and proved that the condition number is $O(1)$ in these more general cases.

Unlike the nested meshes, for non-nested meshes the spaces of functions on the " coarser " meshes are no longer subspaces of that on the fine mesh. Therefore, both the theory and the algorithms developed for the nested meshes need to be modified to accomodate the present non-nested meshes. In this paper, we will construct the multilevel additive Schwarz preconditioner on the non-nested meshes and show that the condition number of the preconditioned system can be bounded by $O(L^2)$, under some assumptions on the relations between neighbouring mesh levels. Some of these restrictions on the meshes are: the fine and all the coarser meshes are assumed to be

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quasi-uniform, each coarser element contain sufficiently many finest elements, and all the boundaries of coarser domains are matching to the boundary of the fine domain.

2. The formulation of the problem. We consider the following self-adjoint Dirichlet elliptic problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v), \ \forall\ v \in H_0^1(\Omega)$$

where $\Omega \subset R^d (d=2,3)$ is a open convex polygonal (d=2) or polyhedral (d=3) domain and

$$a(u,v) = \int_\Omega (\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + b\, uv) dx, \quad f \in H^{-1}(\Omega).$$

We will solve the above variational problem by finite element methods. Suppose we are given a family of triangulations $\{\mathcal{T}^h\}$ on Ω, our finite element problem is stated as:

Find $u^h \in V^h$ such that

$$a(u^h, v^h) = f(v^h), \ \forall\ v^h \in V^h. \tag{1}$$

Let $A = (\, a(\phi_i, \phi_j)\,)$ with $\{\phi_i\}$ being the nodal basis functions of V^h. A is called the stiffness matrix. It is well-known that A is ill-conditioned, and our aim is to construct a good preconditioner M for A by the multilevel additive Schwarz method to be used with the preconditioned conjugate gradient method.

3. Multilevel additive Schwarz preconditioner. In this section, we will give the expression of the multilevel additive Schwarz precondioner. We do not suppose the triangulation $\mathcal{T}^h$ is obtained by the successive refinement of a coarse triangulation on Ω.

Let $\{\mathcal{T}^l\}_{l=0}^{L}$ be a series of quasi-uniform triangulations on Ω, they are not necessarily nested. h_l denotes the maximum diameter of all elements in $\mathcal{T}^l$. $\mathcal{T}^L$ is the finest triangulation on which the finite element space V^h, and our finite element problem (1), is defined. We consider only the matching boundary case, i.e. we assume

(A1) $\quad \bar{\Omega}^l = \cup_{\tau \subset \mathcal{T}^l} \bar{\tau} = \bar{\Omega}, \quad 0 \le l \le L.$

Let $\hat{\Omega}^l$ be the set of nodal points of the triangulation $\mathcal{T}^l$, $V^l \subset H_0^1(\Omega^l)$ the piecewise linear finite element space corresponding to the triangulation $\mathcal{T}^l$, $\{\phi_i^l, i \in \hat{\Omega}^l\}$ the set of usual nodal basis functions of V^l. For $l = L$, we use $\hat{\Omega} = \hat{\Omega}^L$.

Let $\{\Omega_k^l\}_{k=1}^{N_l}$ be an overlapping domain decomposition of Ω^l, which is obtained by extending a given nonoverlapping subdomain covering $\{\tilde{\Omega}_k^l\}_{k=1}^{N_l}$ of Ω^l such that

(A2) $\quad \mathrm{dist}(\partial\tilde{\Omega}_k^l, \partial\Omega_k^l) \ge \delta_l, \quad 1 \le k \le N_l.$

δ_l is called the l-th level overlapping size. Here we assume that the boundary of Ω_k^l aligns with the ones of the l-th level elements in $\mathcal{T}^l$, and

(A3) $\quad$ Any point in Ω is covered by a fixed number of subdomains in $\{\Omega_k^l\}_{k=1}^{N_l}$.

Let $\hat{\Omega}_k^l = \Omega_k^l \cap \hat{\Omega}^l$ be the set of nodal points in Ω_k^l.

By Π_h we denote the the standard nodal value interpolation operator from V^l to V^L, $0 \le l \le L-1$, and I_l the matrix representation of $\Pi_h : V^l \to V^L$. Further, by E_k^l we denote the matrix representation of the zero extension operator from $V_k^l \equiv V^l \cap H_0^1(\Omega_k^l)$ to V^l, $0 \le l \le L$, $1 \le k \le N_l$.

With the above preparations, we can give the expression of the multilevel additive Schwarz preconditioner M for A:

$$M = \sum_{l=0}^{L} \sum_{k=1}^{N_l} I_l \left(E_k^l (A_k^l)^{-1} (E_k^l)^T \right) I_l^T.$$

where A_k^l is the stiffness matrix corresponding to the subspace $V_k^l \subset V^l$, i.e.

$$A_k^l = a(\phi_i^l, \phi_j^l)_{i,j \in \hat{\Omega}_k^l}, 0 \le l \le L, 1 \le k \le N_l.$$

4. Estimation of the condition number $\kappa(MA)$. To evaluate $\kappa(MA)$, one needs to estimate the upper and lower bounds of the generalized Rayleigh quotient

$$\frac{(AMA\boldsymbol{u}, \boldsymbol{u})}{(A\boldsymbol{u}, \boldsymbol{u})}.$$

It is more convenient to express the above quotient in terms of the operators. To do so, for $0 \le l \le L, 1 \le k \le N_l$, we define projections $P_k^l : V^L \to V_k^l$ such that $P_k^l u \in V_k^l$ and

$$a(P_k^l u, \phi_i^l) = a(u, \Pi_h \phi_i^l), \ \forall\ \phi_i^l \in V_k^l,$$

we denote $\tilde{P}_k^l = \Pi_h P_k^l$, whose matrix representation is $I_l \left(E_k^l (A_k^l)^{-1} (E_k^l)^T \right) I_l^T A$. Let $\tilde{P} = \sum_{l=0}^{L} \sum_{k=1}^{N_l} \tilde{P}_k^l$. Then it is readily seen that

$$\frac{(AMA\boldsymbol{u}, \boldsymbol{u})}{(A\boldsymbol{u}, \boldsymbol{u})} = \frac{a(\tilde{P}u, u)}{a(u, u)}.$$

We need two more assumptions to estimate the above Rayleigh quotient:

(A4) Each element τ belonging to the coarser triangulation $\mathcal{T}^l, 0 \le l \le L-1$, contains some fine elements belonging to $\mathcal{T}^L$ such that the measure of those fine elements is proportional to the one of the coarse element τ.

(A5) $(h_l + h_{l-1})/\delta_l \le C$, where δ_l is the l-th level overlapping size defined in Section 3, and h_l and h_{l-1} are the maximum diameters of all elements in the l-th and $(l-1)$-th level trangulations $\mathcal{T}^l$ and $\mathcal{T}^{l-1}$, respectively.

In order to evaluate the condition number $\kappa(MA)$, we first give some lemmas. The first lemma states that the standard nodal value interpolation Π_h is H^1 stable and has the L^2 optimal approximation when it is restricted to the coarser finite element subspaces. The proof of the lemma was given in Chan et al. [4].

LEMMA 4.1. *For any coarse triangulation $\mathcal{T}^l$, $0 \le l \le L-1$, and any $u \in V^l$, we have*

$$\begin{aligned} |\Pi_h u|_1 &\le C |u|_1, \\ \|u - \Pi_h u\|_0 &\le Ch\,|u|_1. \end{aligned}$$

The second lemma states the properties of the Clément interpolation operator; see Clément [6].

LEMMA 4.2. *For any triangulation $\mathcal{T}^l$, $0 \le l \le L$, let $\mathcal{R}_l : L^2(\Omega) \to V^l$ be the Clément interpolation operator, then for any $u \in H_0^1(\Omega)$,*

$$
\begin{aligned}
(2) \qquad & |\mathcal{R}_l u|_1 &\le& \; C|u|_1, \\
(3) \qquad & \|u - \mathcal{R}_l u\|_0 &\le& \; Ch_l\,|u|_1.
\end{aligned}
$$

Moreover, let $\bar{V}^l \subset H^2(\Omega)$ be a higher order finite element defined on $\mathcal{T}^l$ and $\bar{\mathcal{R}}_l : L^2(\Omega) \to V^l$ be the corresponding Clément interpolation operator, then $\bar{\mathcal{R}}_l$ satisfies (2) and (3), and

$$|\bar{\mathcal{R}}_l u|_2 \le C|u|_2.$$

For each space V^l, $0 \le l \le L-1$, which is not a subspace of V^L, we define $\tilde{V}^l = I_h V^l$. Obviously, $\tilde{V}^l$ is a subspace of V^L.

Define $P^l : H_0^1(\Omega) \to \tilde{V}^l$ to be the orthogonal projection with respect to the scalar product $a(\cdot,\cdot)$.

LEMMA 4.1. *For any $v \in V^L$, we have*

$$\|v - P^l v\|_{0,\Omega} \le Ch_l\|v\|_{a,\Omega}.$$

Proof. We use Nitsche's trick to prove Lemma 4.1. First we show that for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a $\tilde{u}^l \in \tilde{V}^l$ such that

$$\|u - \tilde{u}^l\|_{a,\Omega} \le Ch_l\|u\|_{2,\Omega}. \tag{4}$$

Let $\tilde{u}^l = \Pi_h \mathcal{R}^l u \in \tilde{V}^l$, we have

$$\|u - \tilde{u}^l\|_{a,\Omega} = \|u - \mathcal{R}^l u\|_{a,\Omega} + \|\mathcal{R}^l u - \bar{\mathcal{R}}^l u\|_{a,\Omega} + \|\bar{\mathcal{R}}^l u - \Pi_h \mathcal{R}^l u\|_{a,\Omega}.$$

By using Lemmas 4.1 and 4.2, we obtain $\|u - \mathcal{R}^l u\|_{a,\Omega} \le Ch_l|u|_{H^2(\Omega)}$, and

$$
\begin{aligned}
\|\mathcal{R}^l u - \bar{\mathcal{R}}^l u\|_{a,\Omega} &\le \|u - \mathcal{R}^l u\|_{a,\Omega} + \|u - \bar{\mathcal{R}}^l u\|_{a,\Omega} \le Ch_l|u|_{2,\Omega}, \\
\|\bar{\mathcal{R}}^l u - I_l \mathcal{R}^l u\|_{a,\Omega} &\le \|\bar{\mathcal{R}}^l u - \Pi_h \bar{\mathcal{R}}^l u\|_{a,\Omega} + \|\Pi_h \bar{\mathcal{R}}^l u - \Pi_h \mathcal{R}^l u\|_{a,\Omega} \\
&\le Ch_l|\bar{\mathcal{R}}^l u|_{H^2(\Omega)} + \|\bar{\mathcal{R}}^l u - \mathcal{R}^l u\|_{a,\Omega} \\
&\le Ch_l|\bar{\mathcal{R}}^l u|_{2,\Omega} \le Ch_l|u|_{2,\Omega},
\end{aligned}
$$

which proves (4).

To prove Lemma 4.1, for any $v \in V^L$, let $w \in H_0^1(\Omega)$ be the solution of the following Dirichlet problem

$$a(w,u) = (v - P^l v, u), \ \forall\, u \in H_0^1(\Omega).$$

and w_h be its finite element solution in $\tilde{V}^l$ such that

$$a(w_h,u) = (v - P^l v, u), \ \forall\, u \in \tilde{V}^l.$$

Since Ω is convex, we know $w \in H^2(\Omega) \cap H_0^1(\Omega)$. Hence from the previous result, there exists $\tilde{w}^l \in \tilde{V}^l$ such that

$$\|w - \tilde{w}^l\|_{a,\Omega} \le Ch_l|w|_{2,\Omega},$$

and by the standard a priori estimate, cf. Grisvard [8],

$$||w||_{2,\Omega} \leq C||v - P^l v||_{0,\Omega}.$$

Thus from the last two inequalities and the definitions of w and w_h, we obtain

$$\begin{aligned} ||v - P^l v||_{0,\Omega}^2 &= a(w, v - P^l v) = a(w - w_h, v - P^l v) \\ &\leq ||w - w_h||_{a,\Omega}||v - P^l v||_{a,\Omega} \leq ||w - \tilde{w}^l||_{a,\Omega}||v||_{a,\Omega} \\ &\leq Ch_l|w|_{2,\Omega}||v||_{a,\Omega} \leq Ch_l||v - P^l v||_{0,\Omega}||v||_{a,\Omega}. \end{aligned}$$

Then Lemma 4.1 follows immediately. □

LEMMA 5. *For $v^l \in V^l$ and $s = 0, 1$,*

$$|v^l|_{s,\Omega} \leq C|\Pi_h v^l|_{s,\Omega}. \tag{5}$$

Proof. Obviously, we can write,

$$|v^l|_{1,\Omega}^2 = \sum_{\tau \in T^l} |v^l|_{1,\tau}^2.$$

We know from **(A4)** that the meausre of those elements which are contained in $\tau \in T^l$ is proportional to $|\tau|$. Note that on these elements, $\Pi_h v^l$ is equal to v^l, so we derive by noting that v^l is linear on each τ,

$$|v^l|_{1,\tau}^2 \equiv c_\tau|\tau| \leq C \sum_{\tau^h \subset \tau} c_\tau|\tau^h| \leq C|\Pi_h v^l|_{1,\tau}^2,$$

which gives (5) with $s = 1$.

Next we prove (5) with $s = 0$. Using Lemma 4.1, the proved (5) with $s = 1$ and the inverse inequality of the finite element functions,

$$\begin{aligned} ||v^l||_{0,\Omega} &\leq ||v^l - \Pi_h v^l||_{0,\Omega} + ||\Pi_h v^l||_{0,\Omega} \leq Ch\,|v^l|_{1,\Omega} + ||\Pi_h v^l||_{0,\Omega} \\ &\leq Ch\,|\Pi_h v^l|_{1,\Omega} + ||\Pi_h v^l||_{0,\Omega} \leq C||\Pi_h v^l||_{0,\Omega}, \end{aligned}$$

that proves (5) with $s = 0$. □

With the previous preparations, we are now in a position to prove our main theorem:

THEOREM 5.1. *Under the assumptions **(A1) - (A5)**, we have*

$$\frac{1}{C(1+L)} \leq \frac{(AMA\boldsymbol{u}, \boldsymbol{u})}{(A\boldsymbol{u}, \boldsymbol{u})} = \frac{a(\tilde{P}u, u)}{a(u, u)} \leq C(1+L).$$

Proof. We first estimate the upper bound. Note that

$$a(P_k^l u, P_k^l u) = a(u, \Pi_h P_k^l u) = a(u, \tilde{P}_k^l u).$$

Therefore we have

$$(AMA\boldsymbol{u}, \boldsymbol{u}) = \sum_{l=0}^{L} \sum_{k=1}^{N_l} a(P_k^l u, P_k^l u).$$

Let $\bar{O}_k^l = \cup_{\tau \in \mathcal{T}^L, \tau \cap \Omega_k^l \neq \emptyset} \bar{\tau}$, we obtain by Cauchy-Schwarz inequality that

$$\begin{aligned} ||P_k^l u||_{a,\Omega}^2 &= a(u, \tilde{P}_k^l u) = a_{O_k^l}(u, \tilde{P}_k^l u) \\ &\leq C||u||_{a,O_k^l} ||\tilde{P}_k^l u||_{a,\Omega} \leq C||u||_{a,O_k^l} ||P_k^l u||_{a,\Omega}, \end{aligned}$$

that implies $||P_k^l u||_{a,\Omega} \leq C||u||_{a,O_k^l}$.

From above we get

$$(AMA\boldsymbol{u}, \boldsymbol{u}) = \sum_{l=0}^{L} \sum_{k=1}^{N_l} a(P_k^l u, P_k^l u) \leq C(1+L)||u||_{a,\Omega}^2,$$

where we need assume that any point in Ω can be covered by only a finite number of subdomains $\{O_k^l\}_{k=1}^{N_l}$.

Next we estimate the lower bound of the condition number. To do so, we need a proper decomposition for any finite element function u in V^L which is defined as follows:

$$\begin{aligned} &u^0 = P^0 u, \\ &u^1 = P^1(u - u^0), \\ &u^2 = P^2(u - u^0 - u^1), \\ &\cdots \\ &u^l = P^l(u - u^0 - u^1 - \cdots - u^{l-1}), \\ &\cdots \\ &u^L = P^L(u - u^0 - u^1 - \cdots - u^{L-1}) = u - u^0 - u^1 - \cdots - u^{L-1}. \end{aligned}$$

It is readily seen that $u^l \in \tilde{V}^l$ and

$$u = u^0 + u^1 + \cdots + u^L.$$

Let $w^l = u - \sum_{i=0}^{l-1} u^i$, then $u^l = P^l w^l$ and

$$w^l = w^{l-1} - P^{l-1} w^{l-1} = w^{l-1} - u^{l-1}.$$

By the definition of P^l, we derive for $0 \leq l \leq L$ that

$$||u^l||_{a,\Omega} \leq ||w^l||_{a,\Omega} \leq ||w^{l-1}||_{a,\Omega} \leq \cdots \leq ||w^1||_{a,\Omega} \leq ||u||_{a,\Omega},$$

and by Lemma 4.1,

$$\begin{aligned} ||u^l||_{0,\Omega} &= ||P^l w^l||_0 \leq ||w^l||_0 + ||w^l - P^l w^l||_0 \\ &= ||w^{l-1} - P^{l-1} w^{l-1}||_0 + ||w^l - P^l w^l||_0 \\ &\leq C(h_{l-1} + h_l)||u||_{a,\Omega}. \end{aligned}$$

Since $u^l \in \tilde{V}^l$, we can write $u^l = \Pi_h v^l$, $v^l \in V^l$. We further decompose v^l. It is known, e.g., Bramble et al. [1] that there exists a partition $\{\theta_k^l\}_{k=1}^{N_l}$ of unity for Ω^l related to the subdomains $\{\Omega_k^l\}$ such that $\sum_{k=1}^{N_l} \theta_k^l(x) = 1$ on Ω^l and for $1 \leq k \leq N_l$,

$$\text{(6)} \qquad \operatorname{supp} \theta_k^l \subset \Omega_k^l \cup \partial\Omega, \quad 0 \leq \theta_k^l \leq 1, \quad ||\nabla \theta_k^l||_{L^\infty(\Omega_i)} \leq C\,\delta_l^{-1}.$$

Using this partition of unity, we can decompose v^l as

$$v^l = \sum_{k=1}^{N_l} \Pi_l(\theta_k^l v^l) \equiv \sum_{k=1}^{N_l} v_k^l, \quad v_k^l \in V_k^l,$$

where Π_l is the standard nodal value interpolation related to V^l. This gives

$$u = \sum_{l=0}^{L} u^l = \sum_{l=0}^{L}\sum_{k=1}^{N_l} \Pi_h v_k^l.$$

By the standard proof, cf. Xu [10], Chan and Zou [5], we have

$$\sum_{k=1}^{N_l} ||v_k^l||_{a,\Omega}^2 \le C(|v^l|_{1,\Omega}^2 + (1/\delta_l^2)||v^l||_{0,\Omega}^2).$$

Thus, we deduce from the last two relations and the Cauchy-Schwarz inequality that

$$\begin{aligned} a(u,u) &= \sum_{l=0}^{L}\sum_{k=1}^{N_l} a(u, \Pi_h v_k^l) = \sum_{l=0}^{L}\sum_{k=1}^{N_l} a(P_k^l u, v_k^l) \\ &\le \Big(\sum_{l=0}^{L}\sum_{k=1}^{N_l} a(P_k^l u, P_k^l u)\Big)^{1/2} \Big(\sum_{l=0}^{L}\sum_{k=1}^{N_l} a(v_k^l, v_k^l)\Big)^{1/2} \\ &\le C\Big(\sum_{l=0}^{L}\sum_{k=1}^{N_l} a(P_k^l u, P_k^l u)\Big)^{1/2} \Big(\sum_{l=0}^{L}(\delta_l^{-2}|v^l|_0^2 + |v^l|_1^2)\Big)^{1/2}, \end{aligned}$$

and further by using Lemma 5 and (6), we obtain

$$\begin{aligned} a(u,u) &\le C\Big(\sum_{l=0}^{L}\sum_{k=1}^{N_l} a(\tilde{P}_k^l u, u)\Big)^{1/2} \Big(\sum_{l=0}^{L}(\delta_l^{-2}|u^l|_0^2 + |u^l|_1^2)\Big)^{1/2} \\ &\le Ca(\tilde{P}u, u)^{1/2}\Big(\sum_{l=0}^{L}((h_l + h_{l-1})^2/\delta_l^2 + 1)|u^l|_1^2)\Big)^{1/2} \\ &\le C(1+L)^{1/2} a(\tilde{P}u, u)^{1/2}(|u^l|_1^2)^{1/2}, \end{aligned}$$

that implies

$$a(u,u) \le C(1+L)a(\tilde{P}u, u),$$

which concludes the proof of Theorem 5.1.

REMARK 1. *We are not able to improve the condition number bound in Therorem 5.1 to be independent of the number L of mesh levels as for structured meshes (cf. Zhang [11],Oswald [9], Bramble and Pasciak [3]). In fact, it is still an open question whether this independency of the mesh level number is true for unstructured meshes.* □

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