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# ON THE MODEL OF PRESSURELESS GASES WITH STICKY PARTICLES

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## Abstract

Global existence of measure solutions to the equations of pressureless gases with sticky particles is shown in one space variable. In higher dimension, the equations are related to the general Hamilton-Jacobi equation.

## Introduction

There has been a recent interest for the model of pressureless gases with sticky particles. This model can be described at a discrete level by a finite collection of particles that get stuck together right after they collide with conservation of momentum. At a continuous level, the gas can be described by a density and a velocity fiels $\rho(t,x)$, $u(t,x)$ that satisfy the mass and momentum conservation laws

$$\partial_t \rho + \nabla.(\rho u) = 0,$$

$$\partial_t \rho u_i + \nabla.(\rho u u_i) = 0, \quad \forall i = 1, ..., d.$$

This system can be formally obtained from the usual Euler equations for ideal compressible fluids by letting the pressure go to zero, or from the

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Boltzmann equation by letting the temperature go to zero. The model was introduced by Zeldovich [Ze],[SSZ] as a gravitational model. Important papers have been devoted to its geometrical [Ar] and statistical properties, for which we refer to [Si],[SAF]. Bouchut [Bo] has pointed out the mathematical difficulties to get a rigorous derivation of the model. In the present report, we discuss the existence of global solutions to the initial value problem by two very different approaches.
1) The first approach, due to the second author, starts from the elementary *discrete* sticky particle model and leads to a direct and elementary proof of convergence when the number of particles goes to $+\infty$. This provides a global existence theorem of measure solutions to the system of pressureless gas dynamics for rather general initial conditions. In addition, a useful stability result is obtained for the discrete sticky particle model.

2) The second approach, due to the first author, deals directly with the *continuous* model and provides a simple derivation of he model, at the multidimensional level, from the general Hamilton-Jacobi equation

$$\partial_t \Psi + \Theta_0(\partial_x \Psi) = 0,$$

In the special case considered in [Ze], when the initial density $\rho_0$ is constant and the initial velocity field is potential, namely $u_0 = \nabla\Theta_0$ for some scalar potential $\Theta_0$, it is confirmed that the simpler Hopf equation

$$\partial_t \Theta + \frac{1}{2}|\nabla\Theta|^2 = 0,$$

is sufficient to describe the model, as already explained by Zeldovich [Ze]. However, the proof uses a geometric result on convex hulls, for which we give only a rather naive proof that needs unclear strong smoothness assumptions on the initial data. A complete proof would probably require geometric measure arguments. This is why, we do not claim that a fully satisfactory global existence theorem comes out from this second approach. In the one dimensional case, the Hamilton Jacobi equation can also be considered in the kinetic framework of [LPT]. This leads to an alternative formulation of the sticky particle model which seems also of interest.

The paper is organized in two parts.
Part 1 is devoted to the discrete approach. The convergence of the discrete solution is proved when the number of particles goes to $+\infty$ which shows that the pressureless gas equation have global measure solutions for

general prescribed initial conditions (subsection 1.1). A stability result is obtained for the discrete solutions (subsection 1.2) and generalizations are finally considered (subsection 1.3).
Part 2 is devoted to the continuous approach. A suitable characterization of the model is first decribed (subsection 2.1). The derivation from the Hamilton-Jacobi equation is stated (subsection 2.2) and the particular case of a constant initial density is discussed (subsection 2.3). Then follows a geometric proof of the derivation (subsection 2.4). The one dimensional kinetic approach is finally introduced (subsection 2.5).

# 1. The discrete approach to the sticky particle model

At the discrete level, we consider a set of sticky particles, that is particles which go straightforward with constant velocity until they hit other particles. Then they remain together (they get stuck) in such a way that their impulsion remains constant through the shock ([Bo]). This way appears to be successful in order to prove global existence of solutions. In fact we will discretize the initial data and approximate $\rho(0,x)$ by Dirac measures. We then have to let the number of Dirac masses go to infinity. To describe the limit, we immediately see that even if the initial data are smooth, singularities (that is concentration of $\rho$) can occur in finite time (which is related to the formation of shocks in Hopf's equation), so we have to look as in [Bo] for a time continuous positive measure $\rho$ (with no additionnal regularity) and a $\rho dt$ measurable function $u$ defined $\rho dt$ almost everywhere

$$\rho(t,x) \in C([0,\infty[,\mathcal{M}_+(\mathbb{R})) \tag{1}$$

$$u(t,x) \in L^\infty([0,\infty[\times\mathbb{R},\rho dt) \tag{2}$$

**Theorem 1..1** *Let $\rho^0(x)$ be a positive Borel measure on $\mathbb{R}$, and $u^0(x)$ be a $\rho^0$- almost everywhere defined bounded function, of bounded variation on every compact interval, then there exists*

$$\rho(t,x) \in C([0,\infty[,\mathcal{M}_+(\mathbb{R})),$$

$$q(t,x) \in C([0,\infty[,\mathcal{M}_+(\mathbb{R})),$$

*with*

$$|q| \leq |u^0|_{L^\infty(\rho^0)}\rho \quad a.e.\ in\ t, \tag{3}$$

*which allows us to define* $u(t,x) \in L^\infty(\rho dt)$ *by*

$$q(t,x) = u(t,x)\rho(t,x)$$

*such that :*

$$\rho(0,x) = \rho^0(x) \quad (4)$$

$$q(0,x) = \rho^0 u^0(x), \quad (5)$$

*and*

$$\partial_t \rho + \partial_x \rho u = 0$$

$$\partial_t \rho u + \partial_x \rho u^2 = 0$$

*in the sense of the distributions.*

In the first subsection, we prove Theorem 1..1. Then in the second subsection, we prove a stability result, before investigating other systems in the last part.

## 1.1. Existence of a global solution

Let us first assume that $\rho^0$ has a compact support, in $[-A, A]$ for some $A > 0$ (in fact this is not a restriction, since the velocity $u$ will be uniformly bounded in time, see the remark at the end of the section 1.1.3.).

### 1.1.1. Discretization of the initial data

If $u^0(x)$ were continuous, we could simply discretize $\rho^0(x)$ by

$$\rho_i^n(x) = \int_{i/n}^{(i+1)/n} \rho^0(x)dx$$

where $i \in \mathbb{Z}$ and $n$ is the index of discretization. But as $u^0$ is only of bounded variation, we have to be a little more careful. First let us set $M = |u^0|_{L^\infty(\rho^0)}$.

Let $\mathcal{V}(x)$ be the total variation of $u^0$ between $-A$ and $x$. $\mathcal{V}$ is an increasing function, with $\mathcal{V}(-A) = 0$. We then define the $n^{th}$ discretization. For that, let

$$y_j^n = \sup_x \{\mathcal{V}(x) \leq \frac{j}{2^n}\mathcal{V}(A)\}$$

with $0 \leq j \leq 2^n$. We define $y_j^n$ for $2^n < j \leq 2^{n+1}$ by

$$y_j^n = -A + 2A\frac{j - 2^n}{2^{n+1}}$$

Some $y_j^n$ can be equal, so let $(\tilde{y}_l^n)_{1 \leq l \leq m(n)}$ the ordered list of $y_j^n$, with $m(n) \leq 2n$, where $m(n) = \text{Card}\,\{y_j^n, 1 \leq j \leq 2^{n+1}\}$ and where the $\tilde{y}_l^n$ are all distincts.

We approximate $(\rho^0, u^0)$ by Dirac masses ("particles") in the following way : we consider $2m(n)$ particles $P_i^n(0)$ $(1 \leq i \leq 2m(n))$ defined by

- if $i = 2l - 1$, $P_i^n(0)$ has initial position $x_i^n(0) = (\tilde{y}_l^n + \tilde{y}_{l-1}^n)/2$,

$$\text{initial mass} \quad m_i^n = \rho^0(]\tilde{y}_{l-1}^n, \tilde{y}_l^n[)$$

$$\text{initial velocity} \quad v_i^n(0) = q^0(]\tilde{y}_{l-1}^n, \tilde{y}_l^n[)/m_i^n$$

  if $m_i^n \neq 0$ and 0 else,

- if $i = 2l$, $x_i^n(0) = \tilde{y}_l^n$,

$$m_i^n = \rho^0(\{\tilde{y}_l^n\})$$

$$v_i^n(0) = q^0(\{\tilde{y}_l^n\})/m_i^n$$

  if $m_i^n \neq 0$ and 0 else.

So we approximate $\rho^0$ by

$$\rho^n(0) = \sum_{i=1}^{2m(n)} m_i^n \delta_{x_i^n(0)}$$

and $q^0$ by

$$q^n(0) = \sum_{i=1}^{2m(n)} m_i^n v_i^n(0) \delta_{x_i^n(0)}$$

**Lemma 1..2**

$$\rho^n(0) \rightharpoonup \rho^0 \quad in \quad \mathcal{M}_+(\mathbb{R}) \tag{6}$$

$$q^n(0) \rightharpoonup \rho^0 u^0 \quad in \quad \mathcal{M}(\mathbb{R}) \tag{7}$$

**Proof** By construction, we have $\sup_i |x_{i+1}^n(0) - x_i^n(0)| < 2A/2^n$, which goes to zero as $n \to +\infty$, so the proof of Lemma 1..2 is obvious. □

### 1.1.2. Sticky particles dynamics

We say that $P_i(t)$ defined by $(m_i(t), x_i(t), v_i(t))$ has a sticky particles dynamics with initial conditions $(m_i, x_i, v_i)$, if

- $m_i(t) = m_i$ for all $t \geq 0$,
- $x_i(0) = x_i$ and $v_i(0) = v_i$,
- the speed of $P_i$ is constant as long as it meets no new particles : if on $[t_1, t_2]$, $x_i(t) = x_j(t)$ implies $x_j(t_1) = x_i(t_1)$ for all $j$, then $v_i(t)$ is constant on $[t_1, t_2]$,
- the speed changes only when shocks occur : if at $t_0$ there exists $j$ such that $x_j(t_0) = x_i(t_0)$ and $x_j(t) \neq x_i(t)$ for all $t < t_0$ then

$$v_i(t_0+) = \frac{\sum_{j/x_j(t_0)=x_i(t_0)} m_j v_j(t_0-)}{\sum_{j/x_j(t_0)=x_i(t_0)} m_j} \tag{8}$$

Notice that only a finite number of shocks can occur because particles remain together after a shock.

We have immediately (we have only to deal with a finite number of particles)

**Lemma 1..3** *Let $P_i(m_i(0), x_i(0), v_i(0))$ $(1 \leq i \leq n)$ be given, then there exists a unique solution $P_i(t)(m_i(t), x_i(t), v_i(t))$ of the sticky particles dynamics problem. Moreover, let*

$$\rho(t,x) = \sum_{i=1}^{n} m_i \delta_{x_i(t)} \tag{9}$$

$$q(t,x) = \sum_{i=1}^{n} m_i v_i(t) \delta_{x_i(t)} \tag{10}$$

*and $v(t,x)$ defined only in $x_i(t)$ $(1 \le i \le n)$ with*

$$v(t, x_i(t)) = v_i(t), \tag{11}$$

*then $(\rho, q, v)$ solves the system pressureless gas dynamics with initial data $\rho(0,x) = \sum m_i \delta_{x_i(0)}$ and $q(0,x) = \sum m_i v_i(0) \delta_{x_i(0)}$.*

(For the second part of this Lemma, see [Bo]).

**Remark** if $|v_i(0)| \le M$ for every $i$, then $|v_i(t)| \le M$ for every i and for all $t \ge 0$.

### 1.1.3. Limit of $\rho^n$ and $q^n$

Let $P_i^n(t)$ be the solution of the sticky particles dynamics problem with initial data $P_i^n(0)$ (defined in 1.1.2.). Let

$$\rho^n(t,x) = \sum_i m_i^n \delta_{x_i^n(t)} \tag{12}$$

and

$$q^n(t,x) = \sum_i m_i^n v_i^n(t) \delta_{x_i^n(t)}. \tag{13}$$

**Proposition 1..4** *There exist $\rho \in C([0,\infty[, \mathcal{M}_+(\mathbb{R}))$ and $q \in C([0,\infty[, \mathcal{M}(\mathbb{R}))$ such that, for a subsequence :*

$$\rho^n \to \rho \quad in \quad \mathcal{D}'(]0,\infty[\times\mathbb{R}), \tag{14}$$

$$q^n \to q \quad in \quad \mathcal{D}'(]0,\infty[\times\mathbb{R}), \tag{15}$$

*and $\partial_t \rho + \partial_x q = 0$.*

Moreover $|q| \le M\rho$, where $M = |u^0|_{L^\infty}$, so we can define by the Radon-Nikodym Theorem the velocity $u$, defined $\rho dt$ almost everywhere, and bounded by $M$, by

$$q = \rho u.$$

**Proof** Let $t > 0$ and $t' > 0$, and let $\phi \in C^1([0, +\infty[\times\mathbb{R})$. Since $|x_i^n(t') - x_i^n(t)| \leq M|t' - t|$, because the velocities are bounded by $M$,

$$| < \rho^n(t'), \phi > - < \rho^n(t), \phi > | \leq M|t' - t||\partial_x\phi|_{L^\infty}, \tag{16}$$

and similarly for $q^n$. Moreover $\rho(t, \mathbb{R}) = \rho^0(\mathbb{R}) < +\infty$ and $|q|(t, \mathbb{R}) < |u^0|_{L^\infty}\rho^0(\mathbb{R})$, so $\rho^n(t)$ and $q^n(t)$ are bounded measures, uniformly in time. The Proposition is then straightforward. By taking the limit of $\partial_t\rho^n + \partial_x q^n = 0$, we get

$$\partial_t\rho + \partial_x q = 0.$$

As $\rho$ and $q$ are continuous in time by (16), we recover the initial data $\rho^0$ and $\rho^0 u^0$:

$$\rho(0, x) = \rho^0(x) \text{ and } q(0, x) = \rho^0(x)u^0(x).$$

**Remark** Since the particles move with bounded velocities (bounded by $M = |u^0|_{L^\infty}$), we see that the values of $\rho$ and $q$ on $[-A, A]$ at a time $t$ only depend on $q^0$ and $\rho^0$ restricted to $[-A - Mt, A + Mt]$.

So in the general case ($\rho^0$ of unbounded support), the former proof can be fullfilled, provided we first localize properly the initial data.

### 1.1.4. Discontinuities of the velocity

Let $\rho^0$ be compactly supported in $[-A, A]$. We will introduce for $t > 0$ and $\varepsilon > 0$ the set $\mathcal{P}_t^\varepsilon$ of points where the velocity of the particles is "$\varepsilon-$ discontinuous" in any neighbourhood :

$$\mathcal{P}_t^\varepsilon = \{x \ / \ \forall\eta, \forall m > 0, \exists n > m \text{ and } i, j \text{ with}$$
$$x - \eta < x_i^n(t) < x_j^n(t) < x + \eta, \text{ and } |v_i^n(t) - v_j^n(t)| > \varepsilon\}.$$

The aim of this section is to prove

**Proposition 1..5** *Up to the extraction of a subsequence, the cardinal of* $\mathcal{P}_t^\varepsilon$ *is bounded by* $8A/(\varepsilon t)$ *for all* $t > 0$ *and all* $\varepsilon > 0$.

The points $y_i$ of $\mathcal{P}_t^\varepsilon$ will need a special treatment when we will take the limit of $\rho^n(v^n)^2$, because the velocity is not continuous near these points. If $\rho(t, \{y_i\}) = 0$, it is of no importance, but we have to take care of points

$y_i$ such that $\rho(t, \{y_i\}) > 0$. If almost all the mass of $\rho$ comes from particles which have the same velocity (up to $\varepsilon$), it is always possible to take the limit of $\rho^n(v^n)^2$ (it is the case when in the limit $n \to +\infty$, a Dirac mass meets a diffuse measure of different speed). But if we can find two groups of particles of non vanishing total masses and whose velocities differ from more than $\varepsilon$, it is no longer possible to take the limit of $\rho^n(v^n)^2$ (it is the case when two Dirac masses meet). These points will be called meeting points.

**Lemma 1..6** *Let $P_i(t)$ $(1 \leq i \leq n)$ be a set of particles with sticky dynamics $P_i(t) = (m_i(t), x_i(t), v_i(t))$. Let $i, j \in \{1, .., n\}$, and $t > 0$. If $x_i(t) < x_j(t)$ then*

$$v_i(t) \leq v_j(t) + \frac{2A}{t} \tag{17}$$

$$\text{and} \quad v_i(t) \geq v_j(t) - \frac{x_j(t) - x_i(t)}{t}. \tag{18}$$

**Proof** Let $I = \{i', x_{i'}(t) = x_i(t)\}$, and $J = \{j', x_{j'}(t) = x_j(t)\}$. Let $i_0 \in I$ such that $\forall i' \in I$, $x_{i'}(0) \geq x_{i_0}(0)$, and $j_0 \in J$ such that for all $j' \in J$, $x_{j'}(t) \leq x_{j_0}(t)$. Then $v_{i_0}(t)$ is decreasing and $v_{j_0}(t)$ is increasing because $i_0$ only have shocks with particles on its right, and $j_0$ with particles on its left. So $x_i(t) - tv_i(t) \geq x_{i_0}(0)$, and $x_j(t) - tv_j(t) \leq x_{j_0}(0)$. So

$$x_{i_0}(0) + tv_i(t) \leq x_i(t) \leq x_j(t) \leq x_{j_0}(0) + tv_j(t),$$

that is

$$v_j(t) - v_i(t) \geq \frac{x_{i_0}(0) - x_{j_0}(0)}{t} \geq \frac{-2A}{t},$$

so $v_i(t) \leq v_j(t) + 2A/t$.

On the other side, let $i_1 \in I$ such that $\forall i' \in I$, $x_{i'}(0) \leq x_{i_1}(0)$ and $j_1 \in J$ such that $x_{j'}(0) \geq x_{j_1}(0)$ for all $j' \in J$. $v_{i_1}(t)$ is increasing so $x_i(t) - tv_i(t) \leq x_{i_1}(0)$ and $x_j(t) - tv_j(t) \geq x_{j_1}(0)$, but $x_{i_1}(0) < x_{j_1}(0)$, so

$$x_i(t) - tv_i(t) \leq x_j(t) - tv_j(t)$$

and $v_i(t) - v_j(t) \geq (x_i(t) - x_j(t))/t$. □

**Remark** This Lemma is a discrete way to write that the total variation of $u$ is decreasing, and that the spacial derivative of $u$ is less than $1/t$, which is a well-known result for Hopf's equation.

Now let $B > 0$. Let us split $[-B, B]$ in $N$ intervals of equal length.

**Lemma 1..7** *As $n \to +\infty$, the number of intervals $I$ such that there exists two particles in $I$ whose velocities differ from more than $\varepsilon$ is bounded uniformly in $n$ by $2A/\varepsilon t$, if $N > 2B/\varepsilon t$.*

**Proof** It is a more elaborated version of Lemma 1..6. Let $I$ be an interval such that there are two particles $P_i^n$ and $P_j^n$ in $I$ with $|v_i^n(t) - v_j^n(t)| > \varepsilon$. Let us assume that $x_i^n(t) < x_j^n(t)$.

Is is impossible that $v_i^n(t) < v_j^n(t)$ because $v_j^n(t) < v_i^n(t) + (x_j(t) - x_i(t))/t < v_i^n(t) + 2B/Nt < v_i^n(t) + \varepsilon$ (by (18)) and $N > 2B/\varepsilon t$.

So $v_i^n > v_j^n$. But with the same notations as in Lemma 1..6, at a such couple $(P_i^n, P_j^n)$, one can associate the interval $[x_{i_0}(0), x_{j_0}(0)]$ whose lenght is more than $t\varepsilon$, in $[-A, A]$. At two different couples, we associate two different intervals, so the number of such couples is less than $2A/\varepsilon t$. □

Let $\varepsilon = 2^{-m}$ with $m$ an integer, let $r > 0$ be a positive rational, and $N > 2(A + Mr)/r\varepsilon$ be an integer. At $t = r$, let us consider the intervals $I_k = [-A - Mt + 2kA/N, -A - Mt + 2(k+2)A/N]$, with $0 \le k \le N(2A + 2Mr)/2A$. By Lemma 1..7, the number of such intervals in which there exist two particles whose velocities differ from more than $\varepsilon$ is bounded by $4A/\varepsilon t$. So, up to the extraction of a subsequence, one can assume that this number converges, to $\alpha$, with $\alpha \le 4A/\varepsilon t$, and that there exists exactly $\alpha$ intervals $I_{k_1}, .., I_{k_\alpha}$ such that for all $n$ large enough,

$$\{k_1, .., k_\alpha\} = \{k, \exists i, j, x_i^n(t), x_j^n(t) \in I_k, \text{ and } |v_i^n(t) - v_j^n(t)| > \varepsilon\}.$$

Up to a diagonal extraction, we can do this for all $r > 0$ and all $N$, and all $m$ ($\varepsilon = 2^{-m}$). We denote again by $P_i^n$ the extracted subsequence.

**Lemma 1..8** *For all $\varepsilon$, for all $t > 0$ and for all $N > 2(A + Mt)/t\varepsilon$, there exists $\alpha < 8A/t\varepsilon$ and $\{k_1, ..., k_\alpha\}$ such that*

$$\{k_1, ..., k_\alpha\} \supset \{k, \exists i, j, x_i^n(t), x_j^n(t) \in I_k, \textit{ and } |v_i^n(t) - v_j^n(t)| > \varepsilon\}$$

*for $n$ large enough*

**Proof** Just use that if two particles $P$ and $P'$ at time $t$ are separeted by a distance $d$ and have velocities which differ by more than $\varepsilon$, then they steem from particles at time $t' < t$ at a distance less than $d + M(t' - t)$ and which velocities differ by more than $\varepsilon/2$. □

Roughly, Lemma 1..8 says that the number of jumps of size bigger than $\varepsilon$ in the velocity $v^n$ is finite and bounded uniformly in $n$. More precisely, we deduce from Lemma 1..8 that the cardinal of $\mathcal{P}_t^\varepsilon$ is less than $< 8A/t\varepsilon$, which ends the proof of Proposition 1..5.

### 1.1.5. Meeting points

Let $\varepsilon > 0$ be fixed in this section.

**Definition 1..9** *A point $(t, x)$ is a meeting point if*

- $\rho(t, \{x\}) > 0$
- *there exists $v$ and $\varepsilon_{mass} > 0$ such that, for all $\eta > 0$ and all $N > 0$, there exists $n > N$, with*

$$\sum_{x_i^n(t) \in [x-\eta, x+\eta],\ v_i^n(t) > v+\varepsilon} m_i^n > \varepsilon_{mass} \tag{19}$$

*and*

$$\sum_{x_i^n(t) \in [x-\eta, x+\eta],\ v_i^n(t) < v-\varepsilon} m_i^n > \varepsilon_{mass}. \tag{20}$$

*We then say that $(t, x)$ has a meeting mass greater than $\varepsilon_{mass}$.*

**Proposition 1..10** *There is at most a countable set of meeting points.*

In fact, we will prove Lemma 1..11 which is a little more precise.

**Lemma 1..11** *Let $\varepsilon_{mass} > 0$ be fixed. At each meeting point $(t_0, x_0)$ of meeting mass greater than $\varepsilon_{mass}$ one can associate a square $\mathcal{R}_{t_0,x_0}$ such that there is no other meeting points of meeting mass greater than $\varepsilon_{mass}$ in $\mathcal{R}_{t_0,x_0}$.*

By letting $\varepsilon_{mass}$ to zero, we easily obtain Proposition 1..10.

Let us turn to the proof of Lemma 1..11. It is a play with barycentres. Let $(t_0, x_0)$ be a meeting point, and $v$ as in Lemma 1..11. We have by (19,20),

$$\rho(t_0, \{x_0\}) > 2\varepsilon_{mass} > 0. \tag{21}$$

There exists $\eta_1$ such that

$$\rho(t_0, [x_0 - 2\eta_1, x_0 + 2\eta_1]) < \rho(t_0, \{x_0\}) + \frac{\varepsilon_{mass}}{4}. \tag{22}$$

So, for $n$ large enough,

$$\rho(t_0, \{x_0\}) - \frac{\varepsilon_{mass}}{4} < \rho^n(t_0, [x_0 - \eta_1, x_0 + \eta_1]) < \rho(t_0, \{x_0\}) + \frac{\varepsilon_{mass}}{4}. \tag{23}$$

Now let $\eta_2 > 0$, with $\eta_2 < \eta_1/2$, and $\eta_2 < \eta_1/4|v|_{L^\infty}$. For $n$ large enough,

$$\rho(t_0, \{x_0\}) - \frac{\varepsilon_{mass}}{4} < \rho^n(t_0, [x_0 - \eta_2, x_0 + \eta_2]) < \rho(t_0, \{x_0\}) + \frac{\varepsilon_{mass}}{4}.$$

Let us assume that $(t, x)$ with $|x - x_0| \leq \eta_2$ and $t_0 \leq t \leq t_0 + 2\eta_2$ satisfies the assumption of Lemma 1..11. If $t = t_0$, as by (22),

$$\rho(t_0, [x_0 - 2\eta_1, x_0 + 2\eta_1] \setminus \{x_0\}) < \frac{\varepsilon_{mass}}{4},$$

we see that necessarly $x = x_0$. So $t_0 < t \leq t_0 + 2\eta_2$. Let $v$ be given by the definition of a meeting point, and let $\eta > 0$. Let

$$I_+ = \{i / |x_i^n(t) - x| < \eta, v_i^n(t) > v + \varepsilon\},$$

and

$$I_- = \{i / |x_i^n(t) - x| < \eta, v_i^n(t) < v - \varepsilon\}.$$

We have : for all $N$, there exists $n > N$ with

$$\sum_{i \in I_+} m_i^n > \varepsilon_{mass} \text{ and } \sum_{i \in I_-} m_i^n > \varepsilon_{mass}.$$

Let $G_r(I_+, t')$ be the barycentre of the particles $I_+$ at time $t'$. As particles $i \in I_+$ collide only with particles $j$ which are in $I_+$ between $t' = 0$ and $t' = t$, the total impulsion of the particles of $I_+$ remain constant through the shocks and the point $G_r(I_+, t')$ has a constant velocity between $t' = 0$ and $t' = t$, so

$$G_r(I_+, t_0) < x + \eta - (v + \varepsilon)(t - t_0) \tag{24}$$

and similarly

$$G_r(I_-, t_0) > x - \eta - (v - \varepsilon)(t - t_0), \tag{25}$$

so there exists $I'_+ \subset I_+$ and $I'_- \subset I_-$, with

$$\sum_{i \in I'_+} m_i^n > \frac{\varepsilon_{mass}}{2} \text{ and } x_i^n(t_0) \leq G_r(I_+, t_0) \quad \forall i \in I'_+, \tag{26}$$

and

$$\sum_{i \in I'_+} m_i^n > \frac{\varepsilon_{mass}}{2} \text{ and } x_i^n(t_0) \geq G_r(I_-, t_0) \quad \forall i \in I'_-. \tag{27}$$

For $\eta$ small enough, we have from (24, 25),

$$G_r(I_-, t_0) - G_r(I_+, t_0) > -2\eta + 2\varepsilon(t - t_0) > \varepsilon(t - t_0).$$

So we get from (26,27),

$$\rho^n(t_0, [x_0 - \eta_1, x_0 + \eta_1] \setminus [x_0 - \frac{\varepsilon}{2}(t - t_0), x_0 + \frac{\varepsilon}{2}(t - t_0)]) > \frac{\varepsilon_{mass}}{2} \tag{28}$$

which is in contradiction with

$$\rho(t, [x_0 - 2\eta_1, x_0 + 2\eta_1] \setminus [x_0 - \frac{\varepsilon}{4}(t - t_0), x_0 + \frac{\varepsilon}{4}(t - t_0)]) < \frac{\varepsilon_{mass}}{4}. \tag{29}$$

for $n$ large enough. □

### 1.1.6. End of the proof of Theorem 0.1

It remains to study $\rho^n(v^n)^2$. For that, let $\phi \in \mathcal{D}(\mathbb{R}_+^* \times \mathbb{R})$. We want to take the limit of

$$\int \rho^n(t, x) dt (v^n)^2(t, x) \phi(t, x) = \int dt \int \rho^n(t, x)(v^n)^2(t, x)\phi(t, x), \tag{30}$$

for the whole sequence $v^n$, so we will no more extract subsequences of $v^n$ in this section. Let $\varepsilon > 0$ be fixed till the end of the section.

As $|\int \rho^n(t,x)(v^n)^2(t,x)\phi(t,x)| \leq |\phi|_{L^\infty}\rho^0([-A,A])M^2$, it is sufficient to take the limit of $|\int \rho^n(t,x)(v^n)^2(t,x)\phi(t,x)|$ at times $t$ such that there is no meeting point at this time, by Proposition 1..10 and by Lebesgue Theorem. So let $t$ be fixed in what follows.

We will at first deal with the points of $\mathcal{P}_t^\varepsilon$. The set $\mathcal{P}_t^\varepsilon$ has a finite number of elements $y_1, ..., y_p$, with $p < 8A/t\varepsilon$.

— If $\rho(t,\{y_i\}) = 0$, then there exists an interval $I_i = [y_i - \delta_i, y_i + \delta_i]$ with $\rho(t,[y_i - 2\delta_i, y_i + 2\delta_i]) < \varepsilon/2p$. We have

$$\limsup_{n\to\infty} \rho^n(t, I_i) < \frac{\varepsilon}{p},$$

so if $\tilde{\phi} \in \mathcal{D}(I_i)$,

$$|\int \rho^n(t)(v^n)^2(t,x)\tilde{\phi}(x)| < \frac{\varepsilon}{p}M^2$$

for $n$ large enough, and

$$|\int \rho^n(t)(v^n)^2\tilde{\phi} - \int \rho(t)u^2\tilde{\phi}| \leq \frac{2\varepsilon}{p}M^2. \quad (31)$$

— If $\rho(t,\{y_i\}) \neq 0$ , let $\varepsilon' < \varepsilon/100(M+1)$, and $I_i$ be an interval centered on $y_i$, with

$$\rho(t, I_i) < \rho(t,\{y_i\})(1+\varepsilon'). \quad (32)$$

Let $\psi \in \mathcal{D}(I_i)$ be a positive function, less or equal 1, with $\psi = 1$ on a neighbourhood $[y_i - \eta', y_i + \eta']$ of $y_i$. Let $v^n = \int q^n(t)\psi / \int \rho^n(t)\psi$ ($\int \rho^n(t)\psi \neq 0$ for $n$ large enough), and $v^n \to v = \int q(t)\psi / \int \rho(t)\psi$ as $n \to \infty$. Notice that $v$ depends on $I_i$, on $\psi$ and on $\varepsilon$, and is not directly linked to $u$.

Let us prove that for $\eta$ small enough and $n$ large enough,

$$\sum_{x_j^n \in [y_i - \eta, y_i + \eta],\ |v_j^n - v| \geq 3\varepsilon} m_i^n < \varepsilon\rho(t,\{y_i\}). \quad (33)$$

For that, let

$$\alpha = \lim_{\eta\to 0}\limsup_{n\to+\infty} \sum_{|x_i^n(t) - x(t)| \leq \eta,\, v_i^n(t) \geq v + 3\varepsilon} m_i^n. \quad (34)$$

If $\alpha > 0$, then there exists $\eta < \eta'$ such that for all $N$ there exists $n > N$ with

$$\sum_{|x_i^n(t) - x(t)| \leq \eta,\, v_i^n(t) \geq v + 3\varepsilon} m_i^n > \frac{\alpha}{2}. \quad (35)$$

But, for $n$ large enough, $v_n < v + \varepsilon$, so let

$$I_- = \{i / x_i^n \in [y_i - \eta, y_i + \eta], v_i^n < v + \varepsilon\}.$$

We have, by (32), $\sum_{i \in I_-} m_i^n > \delta \rho(t, \{y_i\})$, for some $\delta > 0$, which implies (with (35)) that $(t, x)$ is a meeting point, which is not the case, so $\alpha = 0$, and similarly for the velocities less than $v - 3\varepsilon$, which gives (33).

Let us bound now $u(t, y_i) - u$. For that, let $\phi$ be a positive continuous function, bounded by 1, with support in $I_i$, and with $\phi(y_i) = 1$.

$$(v - 3\varepsilon) \int \rho^n(t)\phi - M\varepsilon\rho(t, \{y_i\}) < \int q^n(t)\phi < (v + 3\varepsilon) \int \rho^n(t)\phi + M\varepsilon\rho(t, \{y_i\}),$$

by (32) and (33). By letting $n \to \infty$, one obtains

$$(v - 3\varepsilon - M\varepsilon) \int \rho(t)\phi < \int q(t)\phi < (v + 3\varepsilon + M\varepsilon) \int \rho(t)\phi,$$

so

$$v - 3\varepsilon - M\varepsilon \le u(t, y_i) < v + 3\varepsilon + M\varepsilon. \tag{36}$$

Now we can bound $|\int \rho^n (v^n)^2 \tilde{\phi} - \int \rho u^2 \tilde{\phi}|$, where $\tilde{\phi} \in \mathcal{D}(I_i)$:

$$\begin{aligned} |\int \rho^n (v^n)^2 \tilde{\phi} - v^2 \int \rho^n \tilde{\phi}| \quad &\le \sum_{i / x_i^n \in [x_i - \eta, x_i + \eta], |v_i^n - v| \le 3\varepsilon} |m_i^n \tilde{\phi}(x_i^n)(v_i^{n\,2} - v^2)| \\ &+ \sum_{i / x_i^n \in [x_i - \eta, x_i + \eta], |v_i^n - v| > 3\varepsilon} |m_i^n \tilde{\phi}(x_i^n)(v_i^{n\,2} - v^2)| \\ &\le 6\varepsilon M \rho(t, \{y_i\})(1 + \varepsilon')|\tilde{\phi}|_{L^\infty} + 4M^2 \varepsilon \rho(t, \{y_i\})|\tilde{\phi}|_{L^\infty} \end{aligned}$$

(by (32) and (33)), and

$$v^2 \int \rho^n \tilde{\phi} \to v^2 \int \rho \tilde{\phi}$$

and (by (36))

$$\begin{aligned} |\int \rho v^2 \tilde{\phi} - \int \rho u^2 \tilde{\phi}| &\le \rho(t, \{y_i\})|v^2 - u^2(t, y_i)||\tilde{\phi}(t, y_i)| + \varepsilon' \rho(t, \{y_i\}) 2M^2 |\tilde{\phi}|_{L^\infty} \\ &\le \rho(t, \{y_i\})|\tilde{\phi}|_{L^\infty}(2\varepsilon M^2 + 6\varepsilon M + 2\varepsilon M^2). \end{aligned}$$

So

$$|\int \rho^n (u^n)^2 \tilde{\phi} - \int \rho u^2| \le \varepsilon \rho(t, \{y_i\})|\tilde{\phi}|_{L^\infty} C(M) \tag{37}$$

(where $C$ is a constant which depends only on $M$), which ends the study of the points of $\mathcal{P}_t^\varepsilon$.

Now let us look at $\tilde{I} = [-A - Mt, A + Mt] \setminus (\cup_i \dot{I}_i/2)$ where $\dot{I}_i/2$ is the open interval with the same middle as $I_i$, previously defined, and of half lenght. Let $N > 0$. $\tilde{I}$ is a union of closed intervals $J_1, ...J_q$. Let us consider $J_1 = [a, b]$ to fix the ideas, and split it in $2N - 1$ parts, $J_{1,k}^N = [a + k(b - a)/2N, a + (k+2)(b-a)/2N]$, of equal lenghts $(b-a)/N$. Suppose that for all $N$ and for all $m > 0$ there exists $n > m$ and $i, j, k$ such that $x_i^n, x_j^n \in J_{1,k}^N$, and $|v_i^n - v_j^n| > \varepsilon$. Then there exists a sequence $x_{i(n)}^{\sigma(n)}, x_{j(n)}^{\sigma(n)}$ for some non decreasing function $\sigma$ such that

$$|v_{i(n)}^{\sigma(n)} - v_{j(n)}^{\sigma(n)}| > \varepsilon,$$

and

$$|x_{i(n)}^{\sigma(n)} - x_{j(n)}^{\sigma(n)}| \to 0,$$

and $x_{i(n)}^{\sigma(n)} \to x$ as $n \to \infty$.

Then, by definition , $x$ is in $\mathcal{P}_t^\varepsilon$, so in $\cup_i \dot{I}_i/2$, which is impossible.

So $\exists N > 0$, $\exists m > 0$, forall $n > m$, forall $i, j, k$,

$$x_i^n, x_j^n \in J_{1,k}^N \text{ implies } |v_i^n - v_j^n| < \varepsilon. \tag{38}$$

That is : the velocities of the particles are nearly continuous on $\tilde{I}$.

Let us consider $J_{i,k}^N$ for $1 \leq k \leq N - 1$, and $1 \leq i \leq q$.

— If $\rho(t, J_{i,k}^N) = 0$, then for all $\tilde{\phi} \in \mathcal{D}(J_{i,k}^N)$,

$$\int \rho^n (v^n)^2 \tilde{\phi} \to 0$$

and

$$\int \rho u^2 \tilde{\phi} = 0,$$

so

$$|\int \rho^n (v^n)^2 \tilde{\phi} - \int \rho u^2 \tilde{\phi}| < \frac{\varepsilon}{Nq} \tag{39}$$

for $n$ large enough.

— If $\rho(t, J_{i,k}^N) \neq 0$, let $\psi_{i,k}^N \in \mathcal{D}(J_{i,k}^N)$ be a positive function, less or equal one, let

$$v = \frac{\int q(t, J_{i,k}^N)\psi_{i,k}^N(x)}{\int \rho(t, J_{i,k}^N)\psi_{i,k}^N(x)}$$

and

$$v^n = \frac{\int q^n(t, J_{i,k}^N)\psi_{i,k}^N(x)}{\int \rho^n(t, J_{i,k}^N)\psi_{i,k}^N(x)}.$$

We have that $v^n \to v$ and for $n$ large enough, as the velocities of two particles in $J_{i,k}^N$ differ at most by $\varepsilon$ (by (38)),

$$(v - 2\varepsilon)\rho^n \leq q^n \leq (v + 2\varepsilon)\rho^n.$$

So, on $J_{i,k}^N$,

$$v - 2\varepsilon \leq u \leq v + 2\varepsilon, \tag{40}$$

and, (by (38)), for $n$ large enough, for all particle $P_j^n$ such that $x_j^n(t) \in J_{i,k}^N$,

$$v - 2\varepsilon \leq v_j^n(t) \leq v + 2\varepsilon. \tag{41}$$

We will now bound $|\int \rho^n(v^n)^2\tilde{\phi} - \int \rho u^2\tilde{\phi}|$, where $\tilde{\phi} \in \mathcal{D}(J_{i,k}^N)$.

$$|\int \rho^n(v^n)^2\tilde{\phi} - \int \rho^n v^2\tilde{\phi}| \leq \int \rho^n\tilde{\phi}|(v^n)^2 - v^2| \leq 4\varepsilon M \int \rho^n\tilde{\phi},$$

by (41). But $\int \rho^n\tilde{\phi} \leq 2\rho(t, J_{i,k}^N)|\tilde{\phi}|_{L^\infty}$ for $n$ large enough, and

$$\int \rho^n v^2\tilde{\phi} \to \int \rho v^2\tilde{\phi} \text{ as } n \to \infty,$$

$$|\int \rho v^2\tilde{\phi} - \int \rho u^2\tilde{\phi}| \leq 4\varepsilon M\rho(t, J_{i,k}^N)|\tilde{\phi}|,$$

by (40), so

$$|\int \rho^n(v^n)^2\tilde{\phi} - \int \rho u^2\tilde{\phi}| \leq 16\varepsilon M|\tilde{\phi}|_{L^\infty}\rho(t, J_{i,k}^N) \tag{42}$$

for $n$ large enough.

We will now put all the convergence estimates (31,37,39,42) together to take the limit of $\rho^n v^{n2}$.

Let $\phi \in \mathcal{D}(\mathbb{R})$. We consider a partition of unity $(\phi_i)$ with support in the intervals $I_i$ and $J^N_{i,k}$. We study $|\int \rho^n(t)(v^n)^2(t)\phi_i\phi - \int \rho(t)u^2(t)\phi_i\phi|$ with the help of the bounds above and we find :

$$\limsup_{n\to\infty} |\int \rho^n(t)(v^n)^2(t)\phi_i\phi - \int \rho(t)u^2\phi_i\phi| \le \varepsilon\rho^0([-A,A])|\phi|_{L^\infty}C'(M), \tag{43}$$

(where $C'(M)$ is a constant which depends only on $M$). We can then let $\varepsilon \to 0$ to find that

$$\lim_{n\to\infty}\int \rho^n(v^n)^2\phi = \int \rho u^2\phi. \tag{44}$$

$\partial_t(\rho u) + \partial_x(\rho u^2) = 0$ is now obvious, which ends the proof of theorem 1..1. □

**Remark** In fact, we can prove in the same way that for all $\phi \in \mathcal{D}(\mathbb{R})$, for all $\psi$ continuous function on $\mathbb{R}$,

$$\lim_{n\to\infty}\int \rho^n(t,x)\phi(t,x)\psi(v^n(t,x)) = \int \rho(t,x)\phi(t,x)\psi(u(t,x)). \tag{45}$$

In particular, the solution obtained is entropic [Bo].

## 1.2. A stability result

We will prove the following stability on sticky particles dynamics.

**Proposition 1..12** *Let $P_i(t)$ $(1 \le i \le n)$ and $P'_j(t)$ $(1 \le j \le n)$ be two sets of particles with "sticky" dynamics, such that*

$$|x'_j(0) - x_j(0)| \le \varepsilon \tag{46}$$

$$|v'_j(0) - v_j(0)| \le \varepsilon \tag{47}$$

$$m'_j = m_j \tag{48}$$

*then*

$$|x'_j(t) - x_j(t)| \le \varepsilon + \varepsilon t \tag{49}$$

First we will introduce some notations.

### 1.2.1. A barycentric Lemma

Let $P_i(t)$ $(1 \leq i \leq n)$ be a set of particles defined by their mass $m_i$, their velocity $v_i(t)$ and their position $x_i(t)$.

**Definition 1..13**

- *Free barycentre : let $J$ be a subset of $\{1,..,n\}$. We define the "free" barycentre $G_f(J,t)$ by :*
  — *its mass $m_{G,f}(J) = \sum_{j\in J} m_j$*
  — *its velocity $v_{G,f}(J,t) = (\sum_{j\in J} m_j v_j(0))/(\sum_{j\in J} m_j)$*
  — *its position*

$$x_{G,f}(J,t) = \frac{\sum_{j\in J} m_j x_j(0)}{m_{G,f}(J,t)} + t v_{G,f}(J,0)$$

- *and the "real" barycentre $G_r(J,t)$ by :*
  — *its mass $m_{G,r}(J) = m_{G,f}(J) = \sum_{j\in J} m_j$*
  — *its velocity $v_{G,r}(J,t) = (\sum_{j\in J} m_j v_j(t))/(\sum_{j\in J} m_j)$*
  — *its position $x_{G,r}(J,t) = (\sum_{j\in J} m_j x_j(t))/(\sum_{j\in J} m_j)$*

We have immediately

**Lemma 1..14** *Let $J$ be a subset of $\{1,...,n\}$, let $T > 0$*

- *if $J$ is "isolated" till $T$ that is if $\forall t \leq T$, $x_i(t) = x_j(t)$ and $j \in J$ implies $i \in J$, then $x_{G,r}(J,t) = x_{G,f}(J,t)$ for all $t \leq T$,*
- *if $J$ is isolated till $T$ and if $J' \subset J$, such that at $t = 0$, $\forall j \in J'$, $\forall i \in J \setminus J'$, $x_j(0) \leq x_i(0)$ then $x_{G,r}(J',t) \leq x_{G,f}(J',t)$.*

To prove this Lemma, just use the conservation of the impulsion through the shock.

### 1.2.2. Proof of Proposition 1..12

Let $1 \leq i \leq n$, let $t > 0$, let $I = \{i', x^n_{i'}(t) = x^n_i(t)\}$, let $J = \{j, \exists i' \in I, x^{n'}_{i'}(t) = x^{n'}_j(t)\}$. Let $J_1, .., J_p$ be the equivalence classes of $J$ with respect to : $j \equiv j'$ if and only if $x^{n'}_{j'}(t) = x^{n'}_j(t)$, such that if $p_1 < p_2$, $j \in J_{p_1}$ and $j' \in J_{p_2}$ implies $x^{n'}_j(0) < x^{n'}_{j'}(0)$.

If $p = 1$, either $I = J$, and thus as $G_r(P, I, t) = G_f(P, I, t)$ because $I$ is isolated and the same for $P'$, and as

$$|G_f(P', I, t) - G_f(P, I, t)| \leq \varepsilon + \varepsilon t$$

from the assumptions, we have

$$|x^n_i - x^{n'}_i| = |G_r(P, I, t) - G_r(P', I, t)| \leq \varepsilon + \varepsilon t$$

Or $I \neq J$. We then permute $P$ and $P'$ (that is we consider $I' = \{i', x^{n'}_i(t) = x^{n'}_{i'}(t)\} = J$, and $J' = \{j, \exists i' \in I, x^n_{i'}(t) = x^n_j(t)\}$, and $J'_1, ..J'_{p'}$). We then have $p' \neq 1$, and we can follow the proof exactly as in the case $p \neq 1$.

If $p \neq 1$ : $I \not\subset J_1$. Let $J_d = J_1 \cap I$ and $J_g = J_1 - J_d$. $J_g$ can be void, but for all $i, j$, $i \in J_g$ and $j \in J_d$ implies $x^{n'}_i(t) < x^{n'}_j(t)$.

$$x_{G,r}(P, J_d, t) = x_{G,r}(P, I, t) = x^n_i(t)$$

and

$$x_{G,f}(P, J_d, t) \geq x_{G,r}(P, J_d, t) = x^n_i(t).$$

by Lemma 1..14. On the otherside,

$$x_{G,r}(P', J_d, t) \geq x_{G,f}(P', J_d, t),$$

by the same Lemma, but

$$|x_{G,f}(P, J_d, t) - x_{G,f}(P', J_d, t)| \leq \varepsilon + \varepsilon t,$$

so

$$x_{G,r}(P', J_d, t) \geq x^n_i(t) - \varepsilon - \varepsilon t$$

and by Lemma 1..14

$$x_{G,r}(P', J_1, t) \geq x^n_i(t) - \varepsilon - \varepsilon t.$$

By similar arguments, one has

$$x_{G,r}(P', J_p, t) \leq x_i^n(t) + \varepsilon + \varepsilon t.$$

Now

$$x_{G,r}(P', J_1, t) \leq x_i^{n'}(t) \leq x_{G,r}(P', J_p, t),$$

so

$$|x_i^n(t) - x_i^{n'}(t)| \leq \varepsilon + \varepsilon t$$

which ends the proof. □

### 1.2.3. Rate of convergence of the discretization

Let us go back to the discretization of section 1.1.1.. We again assume that $\rho^0$ has its support in $[-A, A]$, for some $A > 0$. When we go from the $n^{th}$ discretization to the $n'^{th}$ (with $n' > n$), we go from $m(n)$ particles to $m(n')$ particles. So we want to link each particles of the $n'^{th}$ discretization to a particle of the $n^{th}$, which has nearby characteristics. Of course the correspondance will not be injective.

So let $P_i^n(m_i^n, x_i^n, v_i^n)$ be a particle of the $n^{th}$ discretization. Then (with the notations of 1.1.1.), at $t = 0$, we have either $x_i^n = y_j^n$ for some $j \leq 2^{n+1}$ or $x_i^n = (y_j^n + y_{j-1}^n)/2$ for some $j \leq 2^{n+1}$.

In the first case ($x_i^n = y_j^n$), if $n' > n$, there exists $j'$ such that $y_{j'}^{n'} = y_j^n$ and their exists a particle $P_{i'}^{n'}$ with initial position $y_j^n$. It is easy to see that $v_i^n = v_{i'}^{n'}$ and $m_i^n = m_{i'}^{n'}$. This particle is unique and we set

$$P_{i'}^{n' \to n} = P_i^n.$$

In the second case, there exists $j_1$ and $j_2$ such that $y_{j_1}^{n'} = y_{j-1}^n$ and $y_{j_2}^{n'} = y_j^n$ by construction. Forall particles $P_{i'}^{n'}$ such that $y_{j_1}^{n'} < x_{i'}^{n'} < y_{j_2}^{n'}$, we set

$$P_{i'}^{n' \to n} = P_i^n$$

We have

$$\sum_{i'/P_{i'}^{n' \to n} = P_i^n} m_{i'}^{n'} = m_i^n \tag{50}$$

$$|x_{i'}^{n'} - x_i^n| \leq \frac{2A}{2^n} \tag{51}$$

$$|v_{i'}^{n'} - v_i^n| \leq \frac{2\mathcal{V}(A)}{2^n} \tag{52}$$

because the total variation of $u^0$ between $y_{j-1}^n$ and $y_j^n$ is bounded by $\mathcal{V}(A)/2^n$.

So at each particle of the $n'^{th}$ discretization, we can associate a particle of the $n^{th}$ discretization satisfying (50,51,52).

So by Proposition 1..12, we have for all time $t$,

$$|x_{i'}^{n'}(t) - x_i^n(t)| \leq \frac{2A}{2^n} + \frac{2\mathcal{V}(A)}{2^n}t, \tag{53}$$

which gives the rate of convergence of the approximation of $\rho$ by $\rho^n$ : for all derivable function $\phi$, if $n' > n$,

$$|\int \rho^{n'}\phi - \int \rho^n\phi| \leq |\phi'|_{L^\infty}(\frac{2A}{2^n} + \frac{2\mathcal{V}(A)}{2^n}t), \tag{54}$$

and by taking the limit $n \to \infty$,

$$|\int \rho\phi - \int \rho^n\phi| \leq |\phi'|_{L^\infty}(\frac{2A}{2^n} + \frac{2\mathcal{V}(A)}{2^n}t). \tag{55}$$

There is a similar convergence estimate for $q^n$ and $q$.

## 1.3. Extension to other systems

The same methods lead to the global existence of a "measure-valued" solution for two related systems : relativistic sticky particles and charged sticky particles.

### 1.3.1. Relativistic particles

A Theorem similar to Theorem 1..1 holds for the following system

$$\partial_t\rho + \partial_x\rho v = 0 \tag{56}$$

$$\partial_t \beta \rho v + \partial_x \rho v^2 = 0 \tag{57}$$

where $\beta = 1/\sqrt{1-(v/c)^2}$ ( $c$ being the speed of the light).

In fact, we discretize the initial data in the same way, and we define "relativistic sticky particles dynamics" in an obvious way. The two key Propositions (1..5, 1..10) still hold. The corresponding Lemmas are just a little more complicated because of the difference between impulsion and $\rho v$.

### 1.3.2. Charged particles

We consider the coupling between zero pressure gas dynamics and a Poisson equation

$$\partial_t \rho + \partial_x \rho v = 0 \tag{58}$$

$$\partial_t \rho v + \partial_x \rho v^2 = E \tag{59}$$

$$\partial_x E = \rho - \tilde{\rho} \tag{60}$$

with $\tilde{\rho}(x) \in L^1(\mathbb{R})$, and with suitable boundary conditions on $E$ like

- in the periodic case $\int E = 0$,
- on $\mathbb{R}$, $\lim_{x \to -\infty} E(x) = 0$, or $E(0) = 0$.

A Theorem similar to Theorem 1..1 holds provided that $\rho^0(\mathbb{R}) < \infty$.

To define the sticky particles dynamics, we have to take care of $E$, which is not properly defined where there is a particle $P_i$. So we set

$$E(x_i) = \frac{1}{2}[E(x_i-) + E(x_i+)], \tag{61}$$

which is just to say that $P_i$ does not interact with particles which are at the same place. The main point is then that

$$|E|_{L^\infty} \leq \rho^0(\mathbb{R}) + \int_{\mathbb{R}} |\tilde{\rho}| < +\infty, \tag{62}$$

so the electric field is uniformly bounded and can be treated as a perturbation. A Lemma similar to Lemma 1..6 holds (the bounds being differents and more complicated), and Proposition 1..5 is true. It is easy to see that

Proposition 1..10 holds too. Moreover there is no difficulties in taking the limit of the electric field. So we can adapt the proof of Theorem 1..1 to this case.

# 2. The continuous approach

## 2.1. A characterization of the continuous sticky particle model

Let us label the particles by $m \in R^d$, with position $X(t,m)$ at time $t \geq 0$, initial position $X_0(m)$ and initial velocity $V_0(m)$. We assume velocities $\partial_t X(t,m)$ to be uniformly bounded. The density and momentum fields $\rho(t,.) \geq 0$, $q(t,.) \in R^d$ are defined as Radon measures by :

$$\int_{R^d} f(x)\rho(t,dx) = \int_{R^d} f(X(t,m))dm, \quad \forall f \in C_c^0(R^d). \tag{63}$$

$$\int f(x)q(t,dx) = \int \partial_t X(t,m) f(X(t,m))dm. \tag{64}$$

The vector measure $q(t,.)$ is absolutely continuous with respect to $\rho(t,.)$ and can be written as $q(t,.) = u(t,.)\rho(t,.)$, where $u(t,.) \in L^\infty(R^d, \rho(t,.))$ can be seen as the mean velocity field of the gas.

¿From definitions (63), (64), we get the continuity equation

$$\partial_t \rho + \nabla.(\rho u) = 0 \tag{65}$$

in the distributional sense. Let us now define another vector measure $q_s(t,.) = u_s(t,.)\rho(t,.)$ defined by :

$$\int f(x)q_s(t,dx) = \int V_0(m) f(X(t,m))dm, \quad \forall f \in C_c^0(R^d). \tag{66}$$

It gives at $(t,x)$ the total initial momentum of all particles located at $x$ at time $t$. A crucial observation is that the sticky particle model is characterized by :

$$\partial_t X(t,m) = u_s(t, X(t,m)), \;\; a.e., \tag{67}$$

which means that the particles having the same position moves together at the same speed and their total momentum is the sum of their initial momentum.

In the particular case $d = 1$, it is not a restriction to assume that $X_0(m)$ is a monotone non decreasing function and formula (67) is strongly reminiscent of the Lagrangian description (including the Rankine-Hugoniot jump conditions) of the non-linear scalar conservation law

$$\partial_t M + \partial_x(\Phi_0(M)) = 0,$$

where $\Phi_0(m)$ is an integral of $V_0(m)$ and the initial data $x \rightarrow M_0(x)$ is the reciprocal function of $m \rightarrow X_0(m)$. The unique entropy solution of this equation is a non decreasing function of $x$ and its reciprocal function $X(t,m)$ precisely obeys (67). This will be shown in the next sections, in a more general multidimensional setting.

## 2.2. Reduction to the general Hamilton-Jacobi equation

### Assumptions on the data

We assume, in an essential way, that both $X_0$ et $V_0$ are maps with potentials, namely $X_0 = \nabla\Phi_0$, $V_0 = \nabla\Theta_0$, and $\Phi_0$ is the dual convex function of a l.s.c. convex function $\Psi_0$ so that

$$\Phi_0(m) = \sup_{x \in R^d} (x.m - \Psi_0(x)). \tag{68}$$

$\Phi_0$ and $\Theta_0$. The assumption on $V_0$ is restrictive, except in the one dimensional case. However, it is always possible (up to technical assumptions) to consider a given initial density field $\rho(0,.) \geq 0$ as the image measure of the Lebesgue measure $dm$ by a map with convex potential $\Phi_0$, characterized as the dual convex function of the solution $\Psi_0$ (in a suitable sense) of the

Monge-Ampère equation $detD^2\Psi_0 = \rho(0,.)$. For more details, we refer to [Br],[Ca],[Mc]. For simplicity, it is assumed that $\Theta$ is bounded and

$$\Phi_0(m) = \frac{1}{2}|m|^2$$

for large $|m|$, which implies that $\rho_0(x)$ is constant and $u_0(x)$ vanishes, when $|x| \to +\infty$.

### Remark

The case of a finite number of particles can be recovered when $\Phi_0$ and $\Theta_0$ are piecewise linear on simplicial cells $C_\alpha$, each of them corresponding to a particle with initial position and velocity $X_{\alpha 0}$, $V_{\alpha 0}$. Then, assuming that $V_0$ and $X_0$ are maps with potential requires the following compatibilty condition : for each pair of neighbouring cells $C_\alpha$, $C_\beta$, $X_{\alpha 0} - X_{\beta 0}$ and $V_{\alpha 0} - V_{\beta 0}$ must be colinear, which is a collision condition. This a severe restriction on the data (except in the one dimensional case !).

### The Hamilton-Jacobi equation

Let $\Psi = \Psi(t,x)$ be the unique viscosity solution [Li] satisfying $\Psi(0,.) = \Psi_0$ of the Hamilton-Jacobi equation

$$\partial_t \Psi + \Theta_0(\nabla\Psi) = 0, \tag{69}$$

By the Hopf formula [BE]

$$\Psi(t,x) = \sup_{m \in R^d}(x.m - \Phi_0(m) - t\Theta_0(m)). \tag{70}$$

Let us set

$$X(t,m) = \nabla \sup_{x \in R^d}(x.m - \Psi(t,x)), \tag{71}$$

which, geometrically speaking, means that $X(t,.)$ is the gradient of the convex hull of function $\Phi_0 + t\Theta_0$. Under regularity assumptions on $\Phi_0$ and $\Theta$ that will be detailed later, we get

**Theorem 2..1** *If the map $X$ is defined by (70),(71), we recover the sticky particle model (67) and the fields $\rho$ et $u$ satisfy in the distributional sense*

$$\partial_t \rho + \nabla.(\rho u) = 0, \tag{72}$$

$$\partial_t \rho u_i + \nabla.(\rho u u_i) = 0, \quad \forall i = 1, ..., d. \tag{73}$$

## 2.3. The case of constant initial density and the Hopf equation

Our result envolves a general Hamilton-Jacobi equation (69). However, as indicated by Zeldovich [Ze], it is possible to reduce the sticky particle model to the special Hopf equation

$$\partial_t \Theta + \frac{1}{2}|\nabla\Theta|^2 = 0, \tag{74}$$

in the particular case when the initial density $\rho(0,.)$ is constant, namely

$$X_0(m) = m, \quad \Phi_0(m) = \frac{1}{2}|m|^2. \tag{75}$$

Indeed, we get from (70)

$$\Psi(t,x) = \sup_{m\in R^d}(x.m - \frac{|m|^2}{2} - t\Theta_0(m)), \tag{76}$$

thus

$$\frac{1}{t}(\frac{|x|^2}{2} - \Psi(t,x)) = \inf_{m\in R^d}(\frac{|x-m|^2}{2t} + \Theta_0(m)).$$

According to the second Hopf formula (see [BE]), the right hand is precisely the unique viscosity solution $\Theta(t,m)$ of (74) such that $\Theta(0,m) = \Theta_0(m)$. A further calculation shows that, as a matter of fact, $\nabla\Theta(t,x)$ is nothing else than the velocity field $u(t,x)$.

## 2.4. Geometric proof in the multidimensional case

In the multidimensional case, we use a geometrical proof based on the following lemma on convex hulls.

**Lemma 2..2** *Let $]a,b[$ be a finite interval and $(t,m) \in [a,b] \times R^d \to h(t,m)$ be a bounded sufficiently smooth function. Let $H(t,m) = \frac{|m|^2}{2} + h(t,m)$ and $X(t,.) = \nabla conv(H(t,.))$ (where conv denotes a convex hull). Then, there is a Borel vector function $u(t,x)$ such that*

$$\partial_t X(t,m) = u(t, X(t,m)), \quad p.p. \tag{77}$$

*and, for any test function $f(t,x)$,*

$$\int u(t,X(t,m))f(t,X(t,m))dmdt = \int \partial_t \nabla H(t,m) f(t,X(t,m))dmdt. \tag{78}$$

Let us first show Theorem 2..1 from Lemma 2..2.

We have $X(t,.) = \nabla conv(H(t,.))$ where

$$H(t,m) = \Phi_0(m) + t\Theta_0(m).$$

Thus

$$\partial_t \nabla H(t,m) = \nabla\Theta_0(m) = V_0(m).$$

Lemma 2..2 shows that there exists $u$ such that

$$\partial_t X(t,m) = u(t, X(t,m))$$

and, for all test function $f$,

$$\int u(t, X(t,m)) f(t, X(t,m)) dm dt = \int V_0(m) f(t, X(t,m)) dm dt.$$

By definition (63),(66), this is equivalent to

$$\int f(t,x) u(t,x) \rho(t,dx) dt = \int f(t,x) u_s(t,x) \rho(t,dx) dt.$$

Thus, the fields $u$ and $u_s$ are $\rho-$ a.e. identical. This shows that we have recovered the sticky particle model, defined by (67). Moreover,

$$\int \partial_t f(t,x) u(t,x) \rho(t,dx) dt = \int \partial_t f(t, X(t,m)) V_0(m) dt dm$$

$$= \int \frac{d}{dt} f(t, X(t,m)) V_0(m) dt dm - \int (\nabla f(t, X(t,m)).\partial_t X(t,m)) V_0(m) dt dm$$

$$= 0 - \int (\nabla f(t, X(t,m)).u(t, X(t,m))) V_0(m) dt dm$$

$$= - \int (\nabla f(t,x).u(t,x)) u_s(t,x) \rho(t,dx) dt$$

(by definition of $u_s$)

$$= - \int (\nabla f(t,x).u(t,x)) u(t,x) \rho(t,dx) dt$$

(as just shown), which completes the proof of Theorem 2..1. Let us now prove the geometric lemma 2..2.

## Proof of the geometric lemma

We believe that the lemma can be proved without severe restriction on $H(t,m)$ by properly using the tools of geometric measure theory. Here, we limit ourself to a very naive proof, where the structure of the convex hull of $H(t,.)$ is *a priori* supposed to be simple enough for most $t$, in a sense that will be made precise soon. Since, in the case we are interested,

$H(t,m) = \Phi_0(m) + t\Theta_0(m)$, this only affects the choice of data $\Phi_0$ and $\Theta_0$ in some generic class of smooth functions, as it is usual in singularity theory [Ar], that we will not even try to characterize. Let us consider the set

$$U = \{(t,m) \in ]a,b[\times R^d\,;\ \ H(t,m) > conv(H(t,m))\}.$$

and the interior $W$ of $R^d - U$. Since $H(t,m) = \frac{|m|^2}{2} + h(t,m)$ where $h$ is bounded and continuous, $U$ is a bounded open set. Each slice $U(t) = \{m,\ \ (t,m) \in U\}$ is either empty or is a countable union of disjoint open (not necessarily connected) subsets $U_k(t)$, $k$ ranging from 1 to $K(t) \leq +\infty$, such that in each $U_k(t)$

$$\nabla conv(H(t,m)) = p_k(t)$$

and the $p_k(t) \in R^d$ are distinct.
We suppose that $h$ is smooth enough so that almost every time $t_0$ belongs to an open interval $I$ such that, for all $t \in I$, the slices $U(t)$ and $W(t)$ are smooth, $\partial U(t)$ is Lebesgue negligible in $R^d$, $K(t)$ is finite and constant, the $p_k(t)$ are smooth, the $U_k(t)$ have smooth boundaries, $m \to \nabla H(t,m)$ is a one-to-one map between $W(t)$ and a subset $Z(t)$ of $R^d$ that does not contain the $p_k(t)$.
For every $t$ in $I$, we set

$$u(t,x) = \frac{d}{dt}p_k(t)$$

if $x = p_k(t)$ and

$$u(t,x) = \partial_t \nabla H(t,m)$$

if $x \in Z(t)$, where $m \in W(t)$ is uniquely defined by $x = \nabla H(t,m)$.

Almost every $(t_0, m_0)$, with $t_0 \in I$, has a neighborhood $B$ such that either (case a) $B$ is contained in $W$, or (case b) $B$ is contained in $\{(t,m);\ \ t \in I,\ \ m \in U_k(t)\}$ for some $k$. In case a, we have $X(t,m) = \nabla H(t,m)$ and we immediately get

$$\partial_t X(t,m) = \partial_t \nabla H(t,m) = u(t, X(t,m))$$

for all $(t,m) \in B$. In case b, we have $X(t,m) = p_k(t)$ and, thus

$$\partial_t X(t,m) = \frac{d}{dt}p_k(t),$$

which, again, shows that

$$\partial_t X(t,m) = \partial_t \nabla H(t,m) = u(t, X(t,m))$$

for all $(t,m) \in B$. This proves the first part of Lemma 2..2. Let us now compute $u(t,x)$ more precisely. On the boundary $\partial U_k(t)$ we have

$$H(t,m) = conv(H(t,m)), \quad \nabla H(t,m) = \nabla conv(H(t,m)),$$

everywhere. For each $t \in I$, $p_k(t)$ satisfies

$$\int (p_k(t) - \nabla H(t,m)) 1_{U_k(t)}(m) dm = 0$$

Indeed, by the Green formula

$$\int_{U_k(t)} (\nabla(conv(H(t,m))) - \nabla H(t,m)) dm = 0,$$

since $H(t,m) = conv(H(t,m))$ on the boundary $\partial U_k(t)$. Thus, we deduce

$$\int (\frac{d}{dt} p_k(t) - \partial_t \nabla H(t,m)) 1_{U_k(t)}(m) dm + < \partial_t 1_{U_k(t)}(m), p_k(t) - \nabla H(t,m) >= 0,$$

where $< .,. >$ denotes the distributional bracket on $I \times R^d$. But, $\partial_t 1_{U_k(t)}(m)$ is supported by $\{(t,m), \quad t \in I, \quad m \in \partial U_k(t)\}$ where $p_k(t) - \nabla H(t,m) = \nabla(conv(H(t,m))) - \nabla H(t,m)$ vanishes. So, we obtain

$$\int (\frac{d}{dt} p_k(t) - \partial_t \nabla H(t,m)) 1_{U_k(t)}(m) dm = 0$$

which means that $\frac{d}{dt} p_k(t)$ is the mean value of $\partial_t \nabla H(t,m)$ on $U_k(t)$. Thus, for all test function $f(t,x)$ compactly supported in $I \times R^d$, we get

$$\int \partial_t X(t,m) f(t, X(t,m)) dm dt = \int_W \partial_t \nabla H(t,m) f(t, X(t,m)) dm dt$$

$$+ \sum_k \int_{m \in U_k(t)} \frac{d}{dt} p_k(t) f(t, X(t,m)) dm dt.$$

But, for all $t \in I$ and $m \in U_k(t)$, we have

$$X(t,m) = \nabla conv(H(t,m)) = p_k(t)$$

and

$$\int_{U_k(t)} (\frac{d}{dt} p_k(t) - \partial_t \nabla H(t,m)) dm = 0,$$

which implies

$$\int_{U_k} \frac{d}{dt} p_k(t) f(t, X(t,m)) dm dt = \int_{U_k} \partial_t \nabla H(t,m) f(t, X(t,m)) dm dt$$

and finally shows

$$\int \partial_t X(t,m) f(t, X(t,m)) dmdt = \int \partial_t \nabla H(t,m) f(t, X(t,m)) dmdt,$$

which completes the proof of lemma 2..2.

## 2.5. A kinetic formulation in the one dimensional case

In the one-dimensional case, we can use the kinetic formulation of scalar conservation laws as in [LPT]. Instead of dealing with Hamilton-Jacobi equation (69), we deal with the scalar conservation law

$$\partial_t M + \partial_x(\Theta_0(M)) = 0. \tag{79}$$

In this subsection, we assume that $X_0(m)$ is defined on a bounded interval $[0, L]$ (which implies that here $\rho_0$ is compactly supported in $[0, L]$ !), is non decreasing and has limit $B_0$ (resp. $A_0 \leq B_0$) when $m$ goes to $L$ (resp. 0). Then we consider the inverse function $m = M_0(x)$, extended by 0 for $x \leq A_0$ and $L$ for $x \geq B_0$, as the initial condition for (79). Notice that $M_0(x)$ is nothing but the integral from $-\infty$ to $x$ of density $\rho_0$ ! For each $t \geq 0$, the unique entropy solution $M(t,x)$, is non decreasing and there are finite numbers $A(t) \leq B(t)$ such that

$$0 < M(t,x) < L, \quad \forall x \in ]A(t), B(t)[$$

$$M(t,x) = 0, \;\; \forall x < A(t), \quad M(t,x) = L, \;\; \forall x > B(t).$$

Thus, we can define $m \in [0, L] \to X(t,m) \in [A(t), B(t)]$ as a non decresing function by setting

$$f(t,x,m) = H(M(t,x) - m)H(m), \quad \forall (x,m) \in R \times [0, L]. \tag{80}$$

where $H$ denotes the Heaviside function. Following [LPT], we know that $f$ satisfies the kinetic equation

$$\partial_t f + V_0(m) \partial_x f = \partial_m \mu, \tag{81}$$

where $V_0(m) = \Theta_0'(m)$ and $\mu = \mu(t, x, m)$ is some nonnegative measure. We believe that this formulation can lead to an alternative proof of global existence. Let us show, for example, how easily can (78) be recovered without geometric arguments. Let us consider $h(t)$ and $\zeta(x)$, two smooth nonnegative compacty supported function in $t > 0$ and $x \in R$, respectively, and $\psi(m)$ a smooth nondecreasing function on $[0, L]$. ¿From (81), we get

$$< \partial_t f + V_0(m)\partial_x f, h(t)\zeta(x)\psi(m) >\leq 0,$$

where $< .,. >$ denotes the distribution bracket, wich leads to

$$0 \geq I = I(h, \zeta, \psi) = I_1 + I_2,$$

where

$$I_1 = -\int h'(t)\zeta(x)\psi(m)H(x - X(t,m))dtdxdm,$$

$$I_2 = -\int h(t)\zeta'(x)\psi(m)V_0(m)H(x - X(t,m))dtdxdm.$$

We have

$$I_1 = -\int h'(t)(\int_{X(t,m)}^{+\infty} \zeta(y)dy)\psi(m)dtdm = -\int h(t)\zeta(X(t,m))\partial_t X(t,m)\psi(m)dtdm$$

(integrate by part in $t$) and

$$I_2 = \int h(t)\zeta(X(t,m))\psi(m)V_0(m)dtdm$$

(integrate by part in $x$). Thus we get

$$I = \int h(t)\zeta(X(t,m))\psi(m)[\partial_t X(t,m) - V_0(m)]dtdm \geq 0$$

for all non decreasing function $\psi$. Notice that, in the special case when $\psi$ is constant (that is both $\psi$ and $-\psi$ are non decreasing), $I$ vanishes.

Thus

$$\int h(t)\zeta(X(t,m))[\partial_t X(t,m) - V_0(m)]dtdm = 0,$$

which leads to (78).

### An additional remark on kinetic formulations

The kinetic formulation in the $(t, x, m)$ is not the more natural one. If we deal with the usual phase space $(t, x, v)$ it is natural to model, in the one dimensional case, the sticky particle dynamics by a phase density function $F(t, x, v) \geq 0$ subject to

$$\partial_t F + v\partial_x F + \partial_v^2 \nu = 0, \tag{82}$$

for some nonnegative measure $\nu(t, x, v)$, and

$$F(t, x, v) = \rho(t, x)\delta(v - u(t, x)). \tag{83}$$

Related kinetic models are discussed in [BC]. The relationship between the $(t, x, m)$ and the $(t, x, v)$ kinetic formulations deserves, in our opinion, further investigations.

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