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Capturing Multivalued Solutions

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CAPTURING MULTIVALUED SOLUTIONS

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Abstract

Multivalued solutions with a limited number of branches of the inviscid Burgers equation can be obtained by solving closed systems of moment equations. For this purpose, a suitable concept of *entropy* multivalued solutions with K branches is introduced.

1. Introduction

It is a classical idea to solve, at least approximately, kinetic equations, set in a phase space (t, x, v), with the help of finite systems of moment equations set on the reduced space (t, x). A well known example is Grad's closure of the Boltzmann equation. There has been a new interest for this approach in the recent years. Let us quote in particular Levermore's work on the Grad approximation [Le]. In the present paper, we consider an academic problem, that can be seen as a model for realistic applications such as multiple arrival times in ray tracing for geophysical problems [TS], [EFO] or multiple beams in optics or plasma physics [FF], [Co]. We are interested in the multivalued solutions of the inviscid Burgers equation

$$\partial_t u + \partial_x (\frac{u^2}{2}) = 0, \tag{1}$$

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where the initial condition is a function $u_0(x)$, supposed to be valued in some bounded interval $[0, L]$ for simplicity, or, equivalently, in the solution $f(t, x, v)$ of the free transport equation

$$\partial_t f + v\partial_x f = 0, \tag{2}$$

with

$$f(0, x, v) = f_0(x, v) = H(u_0(x) - v)H(v), \tag{3}$$

where H denotes the Heaviside function. The exact solution is very simple

$$f(t, x, v) = f_0(x - tv, v) = H(u_0(x - tv) - v)H(v) \tag{4}$$

and is the characteristic function of a domain $D(t)$ of the plane (x, v) included in the slab $0 \leq v \leq L$. If u_0 is given in $C^1_c(R)$, then there is a finite time $T^* = T^*(u_0) < +\infty$ such that, for $0 < t < T^*$,

$$f(t, x, v) = H(u(t, x) - v)H(v)$$

where $u(t, x)$ is the unique smooth singlevalued solution to (1). For larger values of t, the upper boundary of $D(t)$ is a curve with many projections onto the real axis and can be seen as the 'graph' of the multivalued solution to the Burgers equation corresponding to the initial condition u_0. The number of branches of this multivalued solution can grow in time and is limited by the number of extremal points of u_0. We are interested in finding these branches without working in the phase space (t, x, v). It is worth considering a slightly more general framework when $f_0(x, v)$ is the characteristic function of a domain D_0 contained in the slab $0 \leq v \leq L$, not necessarily limited by the x-axis and the graph of a singlevalued function. For instance, we can consider D_0 to be the circle of center $(0, L/2)$ and radius $L/2$... An elementary but key observation is that, if we *a priori* know, on a given time interval $[0, T]$, an upper bound $K \leq 1$ for the number of branches, then it is theoretically possible to recover the entire solution by solving a closed system of K moment equations. More precisely, the moments

$$m_k(t, x) = \int_0^L v^k f(t, x, v) dv, \quad k = 0, 1, 2, ... \tag{5}$$

satisfy

$$\partial_t m_k + \partial_x m_{k+1} = 0. \tag{6}$$

This system can be closed at order $k = K-1$ since, for every (t,x), the knowledge of the K first moments is sufficient to determine the K branches of the solution and then, express $m_K(t,x)$ as a function of $m_0(t,x), ..., m_{K-1}(t,x)$. Let us consider a simple example, when the solution has two branches

$$f(t,x,v) = H(b(t,x)-v) - H(a(t,x)-v)$$

where $0 \le a(t,x) \le b(t,x) \le L$ are smooth functions. Then, for all (t,x),

$$m_0 = b-a, \;\; m_1 = \frac{1}{2}(b^2-a^2)$$

which immediately leads to

$$a = \frac{m_1}{m_0} - \frac{m_0}{2}, \;\; b = \frac{m_1}{m_0} + \frac{m_0}{2},$$

thus

$$m_2 = \frac{1}{3}(b^3-a^3) = \frac{m_1^2}{m_0} + \frac{1}{12}m_0^3.$$

If we set $\rho = m_0$ et $q = m_1$, the resulting closed system is nothing but

$$\partial_t\rho + \partial_x q = 0, \;\; \partial_t q + \partial_x(\frac{q^2}{\rho} + p(\rho)) = 0, \tag{7}$$

namely the isentropic gas dynamic equations with $p(\rho) = \frac{1}{12}\rho^\gamma$ and $\gamma = 3$. This system is hyperbolic with a degeneracy at $\rho = 0$, which corresponds to the case $a = b$ when the solution is singlevalued. The goal of this paper is to build up a closure formalism for the K moment system allowing us to recover all multivalued solutions having at most K branches by solving a non linear hyperbolic system of conservation laws. Our method is very close to Levermore's work [Le], since we are going to define ad hoc 'maxwellian' functions through an entropy maximization principle. This will also lead in a natural way to a kinetic formulation for multivalued solutions with *at most* K branches, in the spirit of [LPT], [Br], [GM]. When $K = 2$ the formulation does not differ from Lions Perthame Tadmor kinetic formulation of the isentropic gas dynamics with $\gamma = 3$. We call these solutions *entropy*

$K-$ multivalued solutions : they will differ from the regular multivalued solutions as soon as the number of branches becomes larger than K, exactly as in the well known case $K = 1$, when shock waves form !

The paper is organized as follows : In section 2, we define $K-$ branch maxwellian functions as functions of the v variable that maximizes suitable entropy functions with prescribed K first moments. In section 3, we introduce a kinetic formulation and its equivalence with an hyperbolic system of K nonlinear conservation laws. In section 4, we get an existence theorem for entropy multivalued solutions by introducing a time discrete approximation, which generalizes the 'transport-collapse' method of [Br2] for $K > 1$, and showing the convergence of the approximate solutions when the time step goes to zero, by using the averaging lemmas of [LPT]. The detailed proofs will be published in a forthcoming paper [BC]. In sections 5 and 6, we discuss numerical issues and various possible extensions.

2. K branch maxwellian functions

Let us consider the class

$$C = \{f \in L^\infty([0, L]),\ \ 0 \le f(v) \le 1, p.p.\}. \tag{8}$$

Let θ be a smooth function on R with an everywhere positive K-th derivative. For any $f \in C$, we denote by $m(f) \in R^K$ the moment vector

$$m_k(f) = \int_0^L v^k f(v)dv,\ \ k = 0, 1, 2, ...$$

Then we define the set of all 'attainable' moments

$$M_K = \{m(f) = (m_0(f), ..., m_{K-1}(f)) \in R^K,\ \ f \in C\}, \tag{9}$$

which is compact and convex, and, for all $m \in M_K$,

$$C_K(m) = \{g \in C, \;\; m_k(g) = m_k, \;\; \forall k = 0, ..., K-1\}, \tag{10}$$

$$S_{\theta,K}(m) = \inf\{\int_0^L \theta(v)g(v)dv, \;\; g \in C_K(m)\}. \tag{11}$$

Our first result shows the existence and uniqueness of a function reaching this infimum.

Theorem 2..1 *For all $K \geq 1$ and $m \in M_K$, there is a unique function $v \to G_{K,m}(v)$ such that $m(G_{K,m}) = m$ and*

$$\int_0^L \theta(v)G_{K,m}(v)dv = S_{\theta,K}(m) \tag{12}$$

for all smooth functions θ with everywhere positive $K-$th derivative. Moreover, $S_{\theta,K}(m)$ is continous on M_K.

Sketch of the proof

The existence of an optimal function, depending on θ and (temporarily) denoted by $G_{\theta,K,m}$, follows from elementary weak compactness considerations. Then we use (see [BC] for more details) Rockafellar's duality theorem and get

$$S_{\theta,K}(m) = \inf_{g \in C} \sup_{\lambda \in R^K} L_{\theta,K}(g, \lambda, m) = \sup_{\lambda \in R^K} \inf_{g \in C} L_{\theta,K}(g, \lambda, m), \tag{13}$$

where

$$L_{\theta,K}(g, \lambda, m) = \int_0^L g(v)(\theta(v) - \lambda_k v^k)dv + \lambda_k m_k, \tag{14}$$

with implicit summation performed on repeated indices $k = 0, ..., K-1$. A straightforward calculation shows that

$$S_{\theta,K}(m) = \sup_{\lambda \in R^K} \lambda_k m_k - \Sigma_{\theta,K}(\lambda), \tag{15}$$

where

$$\Sigma_{\theta,K}(\lambda) = \int_0^L \max(0, -\theta(v) + \lambda_k v^k) dv. \tag{16}$$

Moreover, there is $\lambda \in R^K$ such that $(G_{\theta,K,m}, \lambda)$ satisfies the saddle-point condition

$$G_{\theta,K,m}(v) = H(-\theta(v) + \lambda_k v^k), \tag{17}$$

where H denotes the Heaviside function. Since the $k-$th derivative of θ is positive,

$$v \to -\theta(v) + \sum_{k=0}^{K-1} \lambda_k v^k$$

has at most K zeros on the real line and goes to $-\infty$ when $|v| \to +\infty$. Thus, on $[0, L]$, $v \to 1 - G_{\theta,K,m}(v)$ is the characteristic function of at most K disjoint interval $[b_{k-1}, a_k]$, $k = 0, ..., K-1$, with

$$0 = b_{-1} \leq a_k \leq b_k \leq a_{k+1} \leq a_{K-1} = L.$$

Then, $G_{\theta,K,m}(v)$ is entirely determined by its K first moments

$$\int_0^L v^k G_{\theta,K,m}(v) dv = m_k, \quad k = 0, ..., K-1,$$

which algebraically reads

$$\sum_{r=0}^{K-1} \frac{1}{k+1}(b_r^{k+1} - a_r^{k+1}) = m_k.$$

Thus $G_{\theta,K,m}$ is unique, does not depend on θ and can be now denoted by $G_{K,m}$.

3. Entropy $K-$ multivalued solutions

Following, we define an entropy $K-$ multivalued solution to be any measurable function $f(t,x,v)$ on $R_+ \times R \times [0,L]$ valued in $[0,1]$ such that

$$\partial_t f + v\partial_x f + (-\partial_v)^K \mu = 0, \tag{18}$$

for some nonnegative measure $\mu(t,x,dv)$, subject to

$$f(t,x,v) = G_{K,m(f(t,x,.))}(v), \;\; p.p. \tag{19}$$

This formulation can already be found [LPT], [LPT2], when $K=1$ and $K=2$, with a clear connection with the Burgers equation only in the case $K=1$. In the same way as Lions, Perthame and Tadmor, we get

Theorem 3..1 *$f(t,x,v)$ is an entropy $K-$ multivalued solution if and only if, for every smooth function θ, the distribution*

$$\partial_t \int_0^L \theta(v) f(t,x,v)dv + \partial_x \int_0^L v\theta(v) f(t,x,v)dv \tag{20}$$

is nonpositive if the $K-th$ derivative of θ is everywhere positive and null if this derivative is identically zero. Moreover, the moments

$$m_k(t,x,v) = m_k(f(t,x,.)), \;\; k=0,...,K-1$$

are solutions to the non linear hyperbolic system of conservation laws obtained from (6) by closing

$$m_K = S_{\theta,K}(m_0,...,m_{K-1}) \tag{21}$$

with $\theta(v) = v^K$.

Remark 1

The hyperbolicity property comes from the fact that, for all smooth function θ with positive $K-$th derivative, $m \in M_K \to S_{K,\theta}(m)$ is a convex entropy for the system.

Remark 2

Clearly Kruzhkov entropy solutions correspond to the case $K = 1$, as shown in [LPT] and isentropic gas dynamics with $\gamma = 3$ corresponds to $K = 2$ as in [LPT2]. Any 'classical' multivalued solution with K branches to the Burgers equation is a trivial solution to (18) (19). If, after some time, new branches develop, then this multivalued solution will differ from the entropy $K-$ multivalued solution, just as in the well known case $K = 1$, when shocks develop [Br2], [LPT].

4. Existence of solutions and time discetization

We introduce the following time discrete scheme where $\Delta t > 0$ denotes the time step and, for $n = 0, 2, ...,$ $f((n+1)\Delta t, x, v)$ is approximated by

$$f_{n+1}(x, v) = G_K(m_n(x)), \quad m_n(x) = \int_0^L f_n(x - v\Delta t, v)dv. \qquad (22)$$

In the special case $K = 1$, we recover the 'transport collapse' method of [Br2]. By using averaging lemmas, as in [LPT], one shows [BC]

Theorem 4..1 *For all initial condition $f_0(x, v)$, measurable on $R \times [0, L]$ and valued in $[0, 1]$, there exists a sequence of time steps $\Delta t \to 0$ and a $K-$ multivalued entropy solution f such that the approximate solution of (22) converges to f.*

Remarks

1) The detailed proof is given in [BC].
2) An analogous result can be obtained from the "BGK" approximation, as in [LPT].
3) As long as the 'classical' multivalued solutions has no more than K branches, the semi-discrete scheme provides the exact solution. (The same phenomenon was already pointed out in [Br2] when $K = 1$.)
4) The uniqueness problem is open.

5. More general equations and numerical experiments

In this section, let us consider a more general transport equation

$$\partial_t f + \partial_x(f\partial_v H) - \partial_v(f\partial_x H) = 0, \tag{23}$$

where the unknown function $f(t,x,v) \geq 0$ is the density function of particles in the phase space $(x,v) \in R \times R$ and the Hamiltonian $H(t,x,v)$ is given. This equation describes the evolution of f when the particle trajectories solve the first order differential system of Hamiltonian type

$$\frac{dx}{dt} = \partial_v H(t,x,v), \quad \frac{dv}{dt} = -\partial_x H(t,x,v). \tag{24}$$

The solution f is constant along each of these trajectories. Our purpose is to solve numerically the initial value problem, where $f(t=0,x,v) = f_0(x,v)$ is prescribed, by gridding only the $x-$ space. Let us assume that the Hamiltonian has the special form

$$H(t,x,v) = \frac{1}{2}v^2 + \Phi(t,x), \tag{25}$$

where Φ is the potential, so that the Liouville equation reads as the classical Vlasov equation

$$\partial_t f + v\partial_x f - \partial_x\Phi(t,x)\partial_v f = 0. \tag{26}$$

Then the moment equations are

$$\partial_t m_k + \partial_x m_{k+1} + k\partial_x\Phi(t,x)m_{k-1} = 0, \tag{27}$$

for $k = 0, 1, 2, \ldots$, with the conventional notation $m_{-1} = 0$.

Numerically, we consider the very simple case when the potential is

$$\Phi(t, x) = \Phi(x) = \frac{1}{2}|x|^2,$$

so that the Liouville equation reads as

$$\partial_t f + v\partial_x f - x\partial_v f = 0. \tag{28}$$

and describes a rigid rotation in the phase space at angular speed 1. The initial condition is chosen as the characteristic function of the square $[-0.75, -0.25] \times [0.25, 0.75]$. The exact solution at time $t = \pi/2$ is given by the characteristic function of the symmetric square $[0.25, 0.75] \times [0.25, 0.75]$. To compute the solution, we use the 2-th moment closure and reduce the Liouville equation to the 2×2 system of conservation laws

$$\partial_t \rho + \partial_x q = 0, \quad \partial_t q + \partial_x(\frac{q^2}{\rho} + p(\rho)) + x\rho = 0 \tag{29}$$

with $p(\rho) = \frac{1}{12}\rho^3$. Then we discretize in space time on a uniform grid with the simplest first order upwind scheme. Some troubles can be noticed when ρ approaches 0, which makes sense since then the eigenvalues of the system merge and the source term, namely $x\rho$, generates instabilities. (As a matter of fact, this problem does not occur when there is no source terms, as in the case of the free transport equation.) So, we introduce a cutoff parameter $\epsilon > 0$ and modify the pressure law by setting

$$p(\rho) = \frac{1}{12}\rho\epsilon^2, \quad \forall \rho \in [0, \epsilon].$$

We believe that the extension of our method to more realistic problems requires such a regularization to deal with possible changes in the number of branches. A theoretical device is shortly decribed in the next subsection. We subsequently show two computations with 100 (resp. 200) grid points along the interval $[-1, +1]$, 157 (resp. 314) time steps for the time interval $[0, \pi]$ and we set $\epsilon = 0.01$ in both cases.

Regularization of the moment equations

At a theoretical level, there is a rather simple regularization technique that we may consider. Let us fix $\epsilon > 0$ and introduce a mollification h_ϵ of the real function $r \to \max(r, 0)$, typically

$$h_\epsilon(r) = \frac{1}{2}(r + (r^2 + \epsilon^2)^{1/2}).$$

Then, we consider the polar function

$$h_\epsilon^*(s) = \sup_{r \in R}(rs - h_\epsilon(r))$$

which is smooth in $]0,1[$, infinite outside and goes to zero uniformly on any compact subset of $]0,1[$ when $\epsilon \to 0$. Now we get a smooth penalty for the constraint $0 \leq f(v) \leq 1$ in the entropy maximization principle, by setting, for a fixed function θ, for instance $\theta(v) = v^K$,

$$S_{\theta,K}^\epsilon(m) = \inf\{\int_0^L [h_\epsilon^*(g(v)) + \theta(v)g(v)]dv, \;\; g \in L^\infty([0,L]), \;\; m(g) = m\}. \tag{30}$$

A straightforward computation shows that the corresponding regularized $K-$th maxwellian function is of the form

$$G_{\theta,K,m}^\epsilon(v) = h_\epsilon'(-\theta(v) + \lambda_k v^k), \tag{31}$$

Here h_ϵ' can be seen as a mollified Heaviside function. The interest of this theoretical approach is that, by closing up the moment equations with

$$m_K = \int_0^L v^K G_{\theta,K,(m_0,\ldots,m_{K-1})}^\epsilon(v)dv, \tag{32}$$

we automatically enforce the hyperbolicity of the resulting system. Notice that, because of the regularization, the maxwellian functions are no longer $\theta-$ independent.

6. Extension to delta functions

Let us consider a 'classical' multivalued solution of the Burgers equation. Instead of recovering it by using characteristic functions (valued in $[0,1]$) as we have done, we may try to use finite nonnegative sums of delta functions as

$$f(t,x,v) = \sum_{k=0}^{K/2-1} \rho_k(t,x)\delta(v - u_k(t,x)) \tag{33}$$

(where K is a given even positive integer), subject to be (measure) solutions of the free transport equation (2). Actually, this is a much more realistic point of view for applications to geophysics. Then, it is possible to introduce an entropy maximization principle very similar to the previous one. Indeed, we can define

$$S_{\theta,K}(m) = \inf\{\int_0^L \theta(v)g(v)dv, \;\; g \geq 0, \;\; m(g) = m\}. \tag{34}$$

In other words, we drop the limitation $g(v) \leq 1$ and keep $g(v) \geq 0$. Then, again by using duality arguments, we obtain that, for each attainable set of moments $(m_0, ..., m_{K-1})$, and each smooth function θ with positive $K-$th derivative, there is a unique 'maxwellian' function, independent of θ, of the form

$$G^{K,m}(v) = \sum_{k=0}^{K/2-1} \alpha_k \delta(v - w_k). \tag{35}$$

This leads to a related kinetic formulation of the Burgers equation, where f satisfies (18), as earlier, but now subject to (33). Unfortunately, the averaging lemma analysis of [LPT] is no longer adequate to get an existence theorem. As a matter of fact, even for the case $K = 2$, which corresponds to the model of pressureless gases with sticky particles, a global existence theorem is not a trivial issue [BG].

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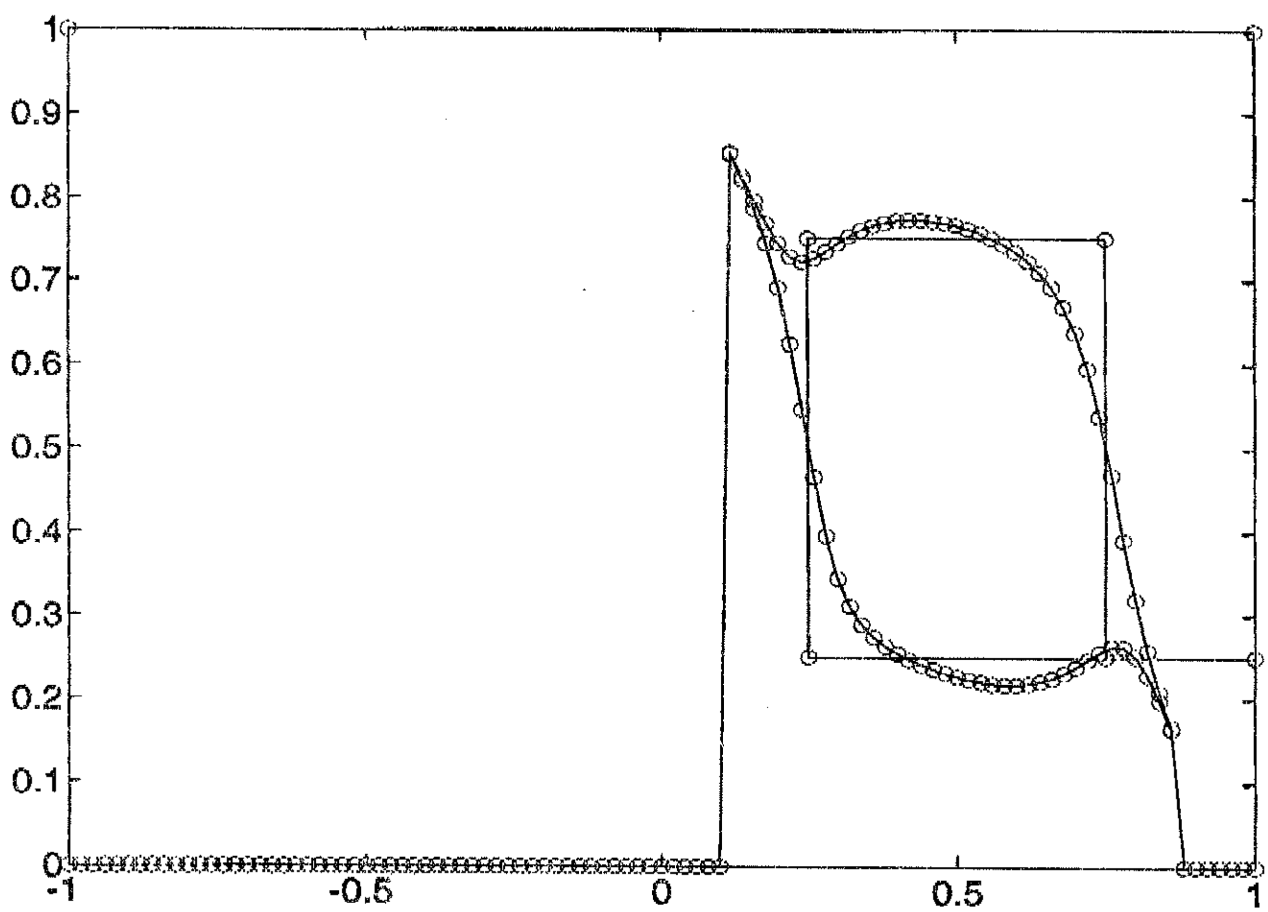

Figure 1

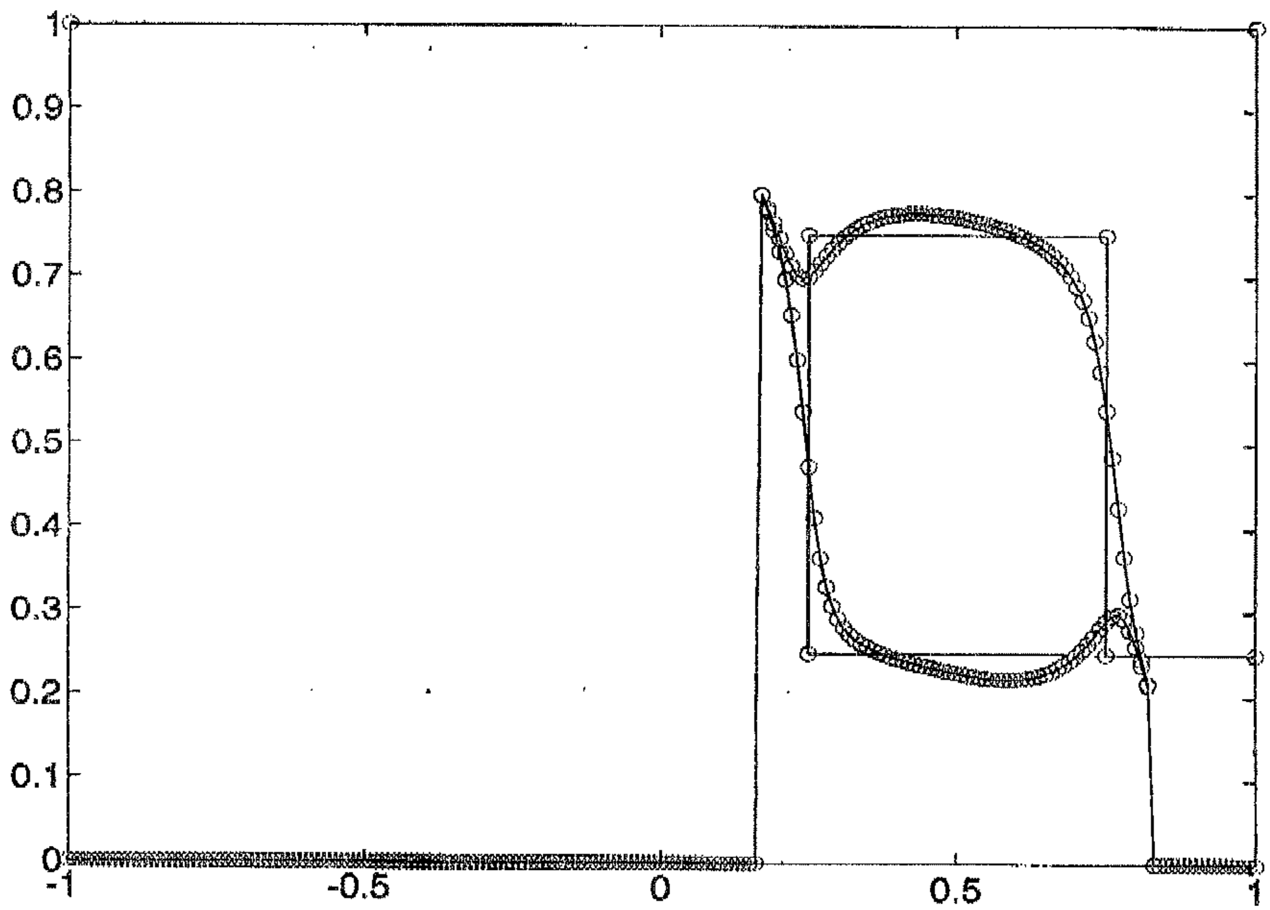

figure 2