Capturing Multivalued Solutions

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Abstract

Multivalued solutions with a limited number of branches of the inviscid Burgers equation can be obtained by solving closed systems of moment equations. For this purpose, a suitable concept of entropy multivalued solutions with \( K \) branches is introduced.

1. Introduction

It is a classical idea to solve, at least approximately, kinetic equations, set in a phase space \((t, x, v)\), with the help of finite systems of moment equations set on the reduced space \((t, x)\). A well known example is Grad’s closure of the Boltzmann equation. There has been a new interest for this approach in the recent years. Let us quote in particular Levermore’s work on the Grad approximation [Le]. In the present paper, we consider an academic problem, that can be seen as a model for realistic applications such as multiple arrival times in ray tracing for geophysical problems [TS], [EFO] or multiple beams in optics or plasma physics [FF], [Co]. We are interested in the multivalued solutions of the inviscid Burgers equation

\[
\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0,
\]

\[\text{(1)}\]

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where the initial condition is a function $u_0(x)$, supposed to be valued in some bounded interval $[0, L]$ for simplicity, or, equivalently, in the solution $f(t, x, v)$ of the free transport equation

$$\partial_t f + v \partial_x f = 0,$$

with

$$f(0, x, v) = f_0(x, v) = H(u_0(x) - v)H(v),$$

where $H$ denotes the Heaviside function. The exact solution is very simple

$$f(t, x, v) = f_0(x - tv, v) = H(u_0(x - tv) - v)H(v)$$

and is the characteristic function of a domain $D(t)$ of the plane $(x, v)$ included in the slab $0 \leq v \leq L$. If $u_0$ is given in $C^1(\bar{R})$, then there is a finite time $T^* = T^*(u_0) < +\infty$ such that, for $0 < t < T^*$,

$$f(t, x, v) = H(u(t, x) - v)H(v)$$

where $u(t, x)$ is the unique smooth singlevalued solution to (1). For larger values of $t$, the upper boundary of $D(t)$ is a curve with many projections onto the real axis and can be seen as the 'graph' of the multivalued solution to the Burgers equation corresponding to the initial condition $u_0$. The number of branches of this multivalued solution can grow in time and is limited by the number of extremal points of $u_0$. We are interested in finding these branches without working in the phase space $(t, x, v)$. It is worth considering a slightly more general framework when $f_0(x, v)$ is the characteristic function of a domain $D_0$ contained in the slab $0 \leq v \leq L$, not necessarily limited by the $x$-axis and the graph of a singlevalued function. For instance, we can consider $D_0$ to be the circle of center $(0, L/2)$ and radius $L/2$... An elementary but key observation is that, if we a priori know, on a given time interval $[0, T]$, an upper bound $K \leq 1$ for the number of branches, then it is theoretically possible to recover the entire solution by solving a closed system of $K$ moment equations. More precisely, the moments

$$m_k(t, x) = \int_0^L v^k f(t, x, v)dv, \quad k = 0, 1, 2, ...$$
satisfy

$$\partial_t m_k + \partial_x m_{k+1} = 0. \quad (6)$$

This system can be closed at order $k = K - 1$ since, for every $(t, x)$, the knowledge of the $K$ first moments is sufficient to determine the $K$ branches of the solution and then, express $m_K(t, x)$ as a function of $m_0(t, x), \ldots, m_{K-1}(t, x)$.

Let us consider a simple example, when the solution has two branches

$$f(t, x, v) = H(b(t, x) - v) - H(a(t, x) - v)$$

where $0 \leq a(t, x) \leq b(t, x) \leq L$ are smooth functions. Then, for all $(t, x)$,

$$m_0 = b - a, \quad m_1 = \frac{1}{2}(b^2 - a^2)$$

which immediately leads to

$$a = \frac{m_1}{m_0} - \frac{m_0}{2}, \quad b = \frac{m_1}{m_0} + \frac{m_0}{2},$$

thus

$$m_2 = \frac{1}{3}(b^3 - a^3) = \frac{m_1^2}{m_0} + \frac{1}{12}m_0^3.$$  

If we set $\rho = m_0$ et $q = m_1$, the resulting closed system is nothing but

$$\partial_t \rho + \partial_x q = 0, \quad \partial_t q + \partial_x \left( \frac{q^2}{\rho} + p(\rho) \right) = 0, \quad (7)$$

namely the isentropic gas dynamic equations with $p(\rho) = \frac{1}{12} \rho^\gamma$ and $\gamma = 3$.

This system is hyperbolic with a degeneracy at $\rho = 0$, which corresponds to the case $a = b$ when the solution is singlevalued. The goal of this paper is to build up a closure formalism for the $K$ moment system allowing us to recover all multivalued solutions having at most $K$ branches by solving a non-linear hyperbolic system of conservation laws. Our method is very close to Levermore’s work [Le], since we are going to define ad hoc 'maxwellian' functions through an entropy maximization principle. This will also lead in a natural way to a kinetic formulation for multivalued solutions with at most $K$ branches, in the spirit of [LPT], [Br], [GM]. When $K = 2$ the formulation does not differ from Lions Perthame Tadmor kinetic formulation of the isentropic gas dynamics with $\gamma = 3$. We call these solutions entropy
$K-$ multivalued solutions: they will differ from the regular multivalued solutions as soon as the number of branches becomes larger than $K$, exactly as in the well known case $K = 1$, when shock waves form!

The paper is organized as follows: In section 2, we define $K-$ branch maxwellian functions as functions of the $v$ variable that maximizes suitable entropy functions with prescribed $K$ first moments. In section 3, we introduce a kinetic formulation and its equivalence with an hyperbolic system of $K$ nonlinear conservation laws. In section 4, we get an existence theorem for entropy multivalued solutions by introducing a time discrete approximation, which generalizes the 'transport-collapse' method of [Br2] for $K > 1$, and showing the convergence of the approximate solutions when the time step goes to zero, by using the averaging lemmas of [LPT]. The detailed proofs will be published in a forthcoming paper [BC]. In sections 5 and 6, we discuss numerical issues and various possible extensions.

2. $K$ branch maxwellian functions

Let us consider the class

$$C = \{ f \in L^\infty([0, L]), \ 0 \leq f(v) \leq 1, \text{p.p.}\}. \quad (8)$$

Let $\theta$ be a smooth function on $R$ with an everywhere positive $K$-th derivative. For any $f \in C$, we denote by $m(f) \in R^K$ the moment vector

$$m_k(f) = \int_0^L v^k f(v) dv, \ k = 0, 1, 2, ...$$

Then we define the set of all 'attainable' moments

$$M_K = \{ m(f) = (m_0(f), ..., m_{K-1}(f)) \in R^K, \ f \in C \}, \quad (9)$$

which is compact and convex, and, for all $m \in M_K$,
\[ C_K(m) = \{ g \in C, \ m_k(g) = m_k, \ \forall k = 0, \ldots, K - 1 \}, \quad (10) \]

\[ S_{\theta,K}(m) = \inf \{ \int_0^L \theta(v)g(v)dv, \ g \in C_K(m) \}. \quad (11) \]

Our first result shows the existence and uniqueness of a function reaching this infimum.

**Theorem 2.1** For all \( K \geq 1 \) and \( m \in M_K \), there is a unique function \( v \rightarrow G_{K,m}(v) \) such that \( m(G_{K,m}) = m \) and

\[ \int_0^L \theta(v)G_{K,m}(v)dv = S_{\theta,K}(m) \quad (12) \]

for all smooth functions \( \theta \) with everywhere positive \( K \)-th derivative. Moreover, \( S_{\theta,K}(m) \) is continuous on \( M_K \).

**Sketch of the proof**

The existence of an optimal function, depending on \( \theta \) and (temporarily) denoted by \( G_{\theta,K,m} \), follows from elementary weak compactness considerations. Then we use (see [BC] for more details) Rockafellar’s duality theorem and get

\[ S_{\theta,K}(m) = \inf_{g \in C} \sup_{\lambda \in \mathbb{R}^K} L_{\theta,K}(g, \lambda, m) = \sup_{\lambda \in \mathbb{R}^K} \inf_{g \in C} L_{\theta,K}(g, \lambda, m), \quad (13) \]

where

\[ L_{\theta,K}(g, \lambda, m) = \int_0^L g(v)(\theta(v) - \lambda_k v^k)dv + \lambda_k m_k, \quad (14) \]
with implicit summation performed on repeated indices \( k = 0, \ldots, K - 1 \). A straightforward calculation shows that

\[
S_{\theta,K}(m) = \sup_{\lambda \in \mathbb{R}^K} \lambda_k m_k - \Sigma_{\theta,K}(\lambda),
\]

(15)

where

\[
\Sigma_{\theta,K}(\lambda) = \int_0^L \max(0, -\theta(v) + \lambda_k v^k) dv.
\]

(16)

Moreover, there is \( \lambda \in \mathbb{R}^K \) such that \((G_{\theta,K,m}, \lambda)\) satisfies the saddle-point condition

\[
G_{\theta,K,m}(v) = H(-\theta(v) + \lambda_k v^k),
\]

(17)

where \( H \) denotes the Heaviside function. Since the \( k \)-th derivative of \( \theta \) is positive,

\[
v \to -\theta(v) + \sum_{k=0}^{K-1} \lambda_k v^k
\]

has at most \( K \) zeros on the real line and goes to \(-\infty\) when \(|v| \to +\infty\). Thus, on \([0, L]\), \( v \to 1 - G_{\theta,K,m}(v) \) is the characteristic function of at most \( K \) disjoint interval \([b_{k-1}, a_k] \), \( k = 0, \ldots, K - 1 \), with

\[
0 = b_{-1} \leq a_k \leq b_k \leq a_{k+1} \leq a_{K-1} = L.
\]

Then, \( G_{\theta,K,m}(v) \) is entirely determined by its \( K \) first moments

\[
\int_0^L v^k G_{\theta,K,m}(v) dv = m_k, \quad k = 0, \ldots, K - 1,
\]

which algebraically reads

\[
\sum_{r=0}^{K-1} \frac{1}{k+1} (b_k^{k+1} - a_r^{k+1}) = m_k.
\]

(16)

Thus \( G_{\theta,K,m} \) is unique, does not depend on \( \theta \) and can be now denoted by \( G_{K,m} \).
3. Entropy $K-$ multivalued solutions

Following, we define an entropy $K-$ multivalued solution to be any measurable function $f(t, x, v)$ on $\mathbb{R}_+ \times \mathbb{R} \times [0, L]$ valued in $[0, 1]$ such that

$$\partial_t f + v \partial_x f + (-\partial_v)^K \mu = 0, \tag{18}$$

for some nonnegative measure $\mu(t, x, dv)$, subject to

$$f(t, x, v) = G_{K, m(f(t, x, \cdot))}(v), \; p.p. \tag{19}$$

This formulation can already be found [LPT], [LPT2], when $K = 1$ and $K = 2$, with a clear connection with the Burgers equation only in the case $K = 1$. In the same way as Lions, Perthame and Tadmor, we get

**Theorem 3.1** $f(t, x, v)$ is an entropy $K-$ multivalued solution if and only if, for every smooth function $\theta$, the distribution

$$\partial_t \int_0^L \theta(v) f(t, x, v) dv + \partial_x \int_0^L v \theta(v) f(t, x, v) dv \tag{20}$$

is nonpositive if the $K-$th derivative of $\theta$ is everywhere positive and null if this derivative is identically zero. Moreover, the moments

$$m_k(t, x, v) = m_k(f(t, x, \cdot)), \; k = 0, ..., K - 1$$

are solutions to the non linear hyperbolic system of conservation laws obtained from (6) by closing

$$m_K = S_{\theta, K}(m_0, ..., m_{K-1}) \tag{21}$$

with $\theta(v) = v^K$. 


Remark 1

The hyperbolicity property comes from the fact that, for all smooth function \( \theta \) with positive \( K \)-th derivative, \( m \in M_K \rightarrow S_{K,\theta}(m) \) is a convex entropy for the system.

Remark 2

Clearly Kruzhkov entropy solutions correspond to the case \( K = 1 \), as shown in [LPT] and isentropic gas dynamics with \( \gamma = 3 \) corresponds to \( K = 2 \) as in [LPT2]. Any 'classical' multivalued solution with \( K \) branches to the Burgers equation is a trivial solution to (18) (19). If, after some time, new branches develop, then this multivalued solution will differ from the entropy \( K \)-multivalued solution, just as in the well known case \( K = 1 \), when shocks develop [Br2], [LPT].

4. Existence of solutions and time discetization

We introduce the following time discrete scheme where \( \Delta t > 0 \) denotes the time step and, for \( n = 0, 2, \ldots \), \( f((n + 1)\Delta t, x, v) \) is approximated by

\[
  f_{n+1}(x,v) = G_K(m_n(x)), \quad m_n(x) = \int_0^L f_n(x - v\Delta t, v) dv. \tag{22}
\]

In the special case \( K = 1 \), we recover the 'transport collapse' method of [Br2]. By using averaging lemmas, as in [LPT], one shows [BC]

**Theorem 4.1** For all initial condition \( f_0(x,v) \), measurable on \( \mathbb{R} \times [0,L] \) and valued in \([0,1] \), there exists a sequence of time steps \( \Delta t \to 0 \) and a \( K \)-multivalued entropy solution \( f \) such that the approximate solution of (22) converges to \( f \).
Remarks

1) The detailed proof is given in [BC].
2) An analogous result can be obtained from the "BGK" approximation, as in [LPT].
3) As long as the 'classical' multivalued solutions has no more than \( K \) branches, the semi-discrete scheme provides the exact solution. (The same phenomenon was already pointed out in [Br2] when \( K = 1 \).)
4) The uniqueness problem is open.

5. More general equations and numerical experiments

In this section, let us consider a more general transport equation

\[
\partial_t f + \partial_x (f \partial_v H) - \partial_v (f \partial_x H) = 0,
\]

where the unknown function \( f(t, x, v) \geq 0 \) is the density function of particles in the phase space \((x, v) \in \mathbb{R} \times \mathbb{R}\) and the Hamiltonian \( H(t, x, v) \) is given. This equation describes the evolution of \( f \) when the particle trajectories solve the first order differential system of Hamiltonian type

\[
\frac{dx}{dt} = \partial_v H(t, x, v), \quad \frac{dv}{dt} = -\partial_x H(t, x, v).
\]

The solution \( f \) is constant along each of these trajectories. Our purpose is to solve numerically the initial value problem, where \( f(t = 0, x, v) = f_0(x, v) \) is prescribed, by gridding only the \( x - \) space. Let us assume that the Hamiltonian has the special form

\[
H(t, x, v) = \frac{1}{2}v^2 + \Phi(t, x),
\]

where \( \Phi \) is the potential, so that the Liouville equation reads as the classical Vlasov equation

\[
\partial_t f + v \partial_x f - \partial_x \Phi(t, x) \partial_v f = 0.
\]

Then the moment equations are

\[
\partial_t m_k + \partial_x m_{k+1} + \partial_x \Phi(t, x)m_{k-1} = 0,
\]
for \( k = 0, 1, 2, \ldots \), with the conventional notation \( m_{-1} = 0 \).

Numerically, we consider the very simple case when the potential is

\[
\Phi(t, x) = \Phi(x) = \frac{1}{2} |x|^2,
\]

so that the Liouville equation reads as

\[
\partial_t f + v \partial_x f - x \partial_v f = 0. \tag{28}
\]

and describes a rigid rotation in the phase space at angular speed 1. The initial condition is chosen as the characteristic function of the square \([-0.75, -0.25] \times [0.25, 0.75]\). The exact solution at time \( t = \pi / 2 \) is given by the characteristic function of the symmetric square \([0.25, 0.75] \times [0.25, 0.75]\). To compute the solution, we use the 2-th moment closure and reduce the Liouville equation to the \( 2 \times 2 \) system of conservation laws

\[
\partial_t \rho + \partial_x q = 0, \quad \partial_t q + \partial_x \left( \frac{q^2}{\rho} + p(\rho) \right) + x \rho = 0 \tag{29}
\]

with \( p(\rho) = \frac{1}{12} \rho^3 \). Then we discretize in space time on a uniform grid with the simplest first order upwind scheme. Some troubles can be noticed when \( \rho \) approaches 0, which makes sense since then the eigenvalues of the system merge and the source term, namely \( x \rho \), generates instabilities. (As a matter of fact, this problem does not occur when there is no source terms, as in the case of the free transport equation.) So, we introduce a cutoff parameter \( \epsilon > 0 \) and modify the pressure law by setting

\[
p(\rho) = \frac{1}{12} \rho \epsilon^2, \quad \forall \rho \in [0, \epsilon].
\]

We believe that the extension of our method to more realistic problems requires such a regularization to deal with possible changes in the number of branches. A theoretical device is shortly described in the next subsection.

We subsequently show two computations with 100 (resp. 200) grid points along the interval \([-1, +1]\), 157 (resp. 314) time steps for the time interval \([0, \pi]\) and we set \( \epsilon = 0.01 \) in both cases.
Regularization of the moment equations

At a theoretical level, there is a rather simple regularization technique that we may consider. Let us fix $\epsilon > 0$ and introduce a mollification $h_\epsilon$ of the real function $r \to \max(r, 0)$, typically

$$h_\epsilon(r) = \frac{1}{2}(r + (r^2 + \epsilon^2)^{1/2}).$$

Then, we consider the polar function

$$h_\epsilon^*(s) = \sup_{r \in \mathbb{R}}(rs - h_\epsilon(r))$$

which is smooth in $]0, 1[$, infinite outside and goes to zero uniformly on any compact subset of $]0, 1[$ when $\epsilon \to 0$. Now we get a smooth penalty for the constraint $0 \leq f(v) \leq 1$ in the entropy maximization principle, by setting, for a fixed function $\theta$, for instance $\theta(v) = v^K$,

$$S_{\theta,K}(m) = \inf\{\int_0^L [h_\epsilon^*(g(v)) + \theta(v)g(v)]dv, \ g \in L^\infty([0, L]), \ m(g) = m\}. \tag{30}$$

A straightforward computation shows that the corresponding regularized $K$–th maxwellian function is of the form

$$G_{\theta,K,m}^\epsilon(v) = h_\epsilon'(\theta(v) + \lambda_k v^K), \tag{31}$$

Here $h_\epsilon'$ can be seen as a mollified Heaviside function. The interest of this theoretical approach is that, by closing up the moment equations with

$$m_K = \int_0^Lv^KG_{\theta,K,\{m_0,\ldots,m_{K-1}\}}^\epsilon(v)dv, \tag{32}$$

we automatically enforce the hyperbolicity of the resulting system. Notice that, because of the regularization, the maxwellian functions are no longer $\theta$– independent.
6. Extension to delta functions

Let us consider a 'classical' multivalued solution of the Burgers equation. Instead of recovering it by using characteristic functions (valued in [0, 1]) as we have done, we may try to use finite nonnegative sums of delta functions as

\[ f(t, x, v) = \sum_{k=0}^{K/2-1} \rho_k(t, x) \delta(v - u_k(t, x)) \]  

(33)

(where \( K \) is a given even positive integer), subject to be (measure) solutions of the free transport equation (2). Actually, this is a much more realistic point of view for applications to geophysics. Then, it is possible to introduce an entropy maximization principle very similar to the previous one. Indeed, we can define

\[ S_{\theta,K}(m) = \inf \{ \int_0^L \theta(v) g(v) dv, \ g \geq 0, \ m(g) = m \}. \]  

(34)

In other words, we drop the limitation \( g(v) \leq 1 \) and keep \( g(v) \geq 0 \). Then, again by using duality arguments, we obtain that, for each attainable set of moments \( (m_0, ..., m_{K-1}) \), and each smooth function \( \theta \) with positive \( K \)-th derivative, there is a unique 'maxwellian' function, independent of \( \theta \), of the form

\[ G^{K,m}(v) = \sum_{k=0}^{K/2-1} \alpha_k \delta(v - w_k). \]  

(35)

This leads to a related kinetic formulation of the Burgers equation, where \( f \) satisfies (18), as earlier, but now subject to (33). Unfortunately, the averaging lemma analysis of [LPT] is no longer adequate to get an existence theorem. As a matter of fact, even for the case \( K = 2 \), which corresponds to the model of pressureless gases with sticky particles, a global existence theorem is not a trivial issue [BG].
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