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Abstract: A numerical homogenization procedure for elliptic differential equations is presented. It is based on wavelet decompositions of discrete operators in fine- and coarse-scale components followed by the elimination of the fine-scale contributions. If the operator is in divergence form, this is preserved by this homogenization procedure. For periodic problems, results similar to classical effective coefficient theory are proved. The procedure can be applied to problems that are not cell-periodic and can be viewed as a direct solver using approximate and sparse block Gauss elimination.

1 Introduction

In many applications the problem and solution exhibit a number of different scales. In certain cases we are interested in finding the correct coarse-scale features of the solution without resolving the finer scales. The fine-scale features may be of lesser importance, or they may be prohibitively expensive to compute. However, the fine scales cannot be completely ignored because they contribute to the coarse scale solution: the high frequencies of solution may combine with the high frequencies of

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the differential operator to yield low frequency components.

The homogenization problem can be stated in various formulations. A classical formulation, see e.g. Bensoussan *et al.* [1], is the following: Consider a family of operators L_ε indexed by the small parameter ε, and for a given f, let u_ε solve the problem

$$L_\varepsilon u_\varepsilon = f. \tag{1}$$

Assume that $u_\varepsilon \to \overline{u}$, as $\varepsilon \to 0$. The homogenization problem is to find an operator $\overline{L}$ and $\overline{f}$ such that:

$$\overline{L}\overline{u} = \overline{f}. \tag{2}$$

For example, consider the following operator with oscillating coefficients

$$L_\varepsilon = \frac{d}{dx}\left(a(x/\varepsilon)\frac{d}{dx}\right),$$

where $a(x)$ is a positive 1-periodic function bounded away from zero. Then it is easy to show that u_ε converges point-wise when $\varepsilon \to 0$ and the limit $\overline{u}$ satisfies a constant coefficients equation. The coefficient is not the average of $a(x)$ over a period, but rather the harmonical average

$$a_{eff} = \left(\int_0^1 1/a(x)\, dx\right)^{-1},$$

also called the effective coefficient. The homogenized operator is

$$\overline{L} = a_{eff}\frac{d^2}{dx^2}$$

since

$$u_\varepsilon \to \overline{u} = \overline{L}^{-1} f.$$

In practice, we often need to solve the equation (1) for a small but fixed ε. Since u_ε is close to $\overline{u}$, we may solve the homogenized equation (2) instead of the original equation. The homogenized equation is usually much simpler to solve. In the case of effective coefficients, the solution of the homogenized equation contains no high frequency components and thus it is an approximation to the coarse scale behavior of u_ε.

In a very interesting paper, M. E. Brewster and G. Beylkin [3] describe a homogenization procedure based on a multi-resolution analysis (MRA) decomposition. They consider integral equations, which may arise, e.g., from the Method of Lines discretization of a PDE, and homogenize over the time-variable. In a MRA, the concept of different scales is contained in the nested spaces V_j. Homogenization is reduced to projecting the solution of the original equation from the fine resolution space V_n onto the coarse resolution space V_0. The homogenized operator, if it exists, operates on the space V_0, but in general it is not the projection of the original operator onto the coarse space.

Many classical homogenization techniques are based on the essential assumption that the coefficients are periodic on the fine scale. However, this does not hold in many applications. The analytic expansions methods require an à priori known number of scales, which again may be a serious restriction, see e.g., L. Durlofsky [7]. For two-dimensional elliptic operators with cell-periodic coefficients

$$L_\varepsilon = -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}(x/\varepsilon) \frac{\partial}{\partial x_i},$$

the homogenized operator is

$$\overline{L} = \sum_{i,j} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

The effective coefficients A_{ij} are found by computing

$$A_{ij} = \sum_k \int \left(a_{ij} - a_{ik} \frac{\partial \xi^j}{\partial x_k} \right),$$

where $\xi^j(x)$ is the solution to the cell-problem:

$$-\sum \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) = -\sum_i \frac{\partial a_{ik}}{\partial x_i}.$$

Wavelet-based homogenization can deal with both non-periodic coefficients on the fine scale and use the all the scales involved from the fine-scale space V_j to the coarse-scale space V_0.

Following the construction in [6], using the Haar system, we build a homogenized operator $\overline{L_J}$ for the discrete operator $\frac{1}{h^2}\Delta_+ \; diag(a) \; \Delta_-$. The grid-size is $h = 2^{-J}$ and

Δ_+ and Δ_- are the forward and backward (undivided) difference operators. We show that the homogenized operator has a natural structure of the form $\frac{1}{(2h)^2}\Delta_+ H \Delta_-$, where H is well approximated by a band diagonal matrix. In some cases, we prove that H equals the effective coefficients predicted by the classical homogenization theory, modulo a small error-term. In the two-dimensions, we show that our technique preserves divergence form of operators.

2 Wavelet spaces

We are concerned with a model of fine- and coarse-resolution spaces. The framework is multi-resolution analysis or wavelet formalism. In this framework, we have the concepts of fine and coarse scales together with the locality properties needed in analyzing operators with variable coefficients.

For the precise definitions of a MRA, we defer the reader to the books by I. Daubechies [4] and Y. Meyer [9]. We consider a ladder of spaces $V_J \subset V_{J+1}$ which are spanned by the dilates and integer translates of one shape-function $\varphi \in V_0$:

$$V_J = span\{\varphi_{J,k}(x) = 2^{J/2}\varphi(2^J x - k)\}.$$

The functions $\varphi_{J,k}$ form an L_2-orthonormal basis. The orthogonal complement of V_J in V_{J+1} is denoted by W_J and it is generated by another orthonormal basis $\psi_{J,k}(x) = 2^{J/2}\varphi(2^J x - k)$, where ψ is called the mother wavelet. The transformation

$$\mathcal{W}_J : V_{J+1} \to W_J \oplus V_J$$

that mapping the basis $\{\varphi_{J+1,k}\}$ into $\{\psi_{J,k}, \varphi_{J,k}\}$ is an orthogonal operator and we denote its inverse by $\mathcal{W}^\top$. The product $\mathcal{W}_J\mathcal{W}_{J-1}\cdots\mathcal{W}_0$ maps V_{J+1} into $V_0 \oplus \sum_{0\le j\le J} W_j$ is called the wavelet transform and it can be optimally implemented (called the fast wavelet transform). We denote by P_j and Q_j the L_2-projections onto V_j and W_j. If an operator L_{J+1} is acting on the space V_{J+1}, it can be decomposed into four operators $L_{J+1} = A_J + B_J + C_J + L_J$ acting on the subspaces W_J and V_J, where

$$\begin{array}{ll} A_J = Q_J L_{J+1} Q_J : W_J \to W_J & B_J = Q_J L_{J+1} P_J : V_J \to W_J \\ C_J = P_J L_{J+1} Q_J : V_J \to W_J & L_J = P_J L_{J+1} P_J : V_J \to V_J \end{array}$$

As a shorthand notation we have that

$$\mathcal{W}_J L_{J+1} \mathcal{W}_J^\top (\mathcal{W}_J U) = \begin{bmatrix} A_J & B_J \\ C_J & L_J \end{bmatrix} \begin{bmatrix} Q_J U \\ P_J U \end{bmatrix},$$

or simply

$$\mathcal{W}_J L_{J+1} \mathcal{W}_J^\top = \begin{bmatrix} A_J & B_J \\ C_J & L_J \end{bmatrix}.$$

Note that if evaluated on a basis, the operator notation becomes a legitimate block-matrix construction.

The identification of a function $f \in V_J$ with the sequence c of coefficients in the basis $\varphi_{J,K}$ is an isometry: If $f = \sum c_k \varphi_{J,k}$, then

$$||f||_{L_2} = ||c||_{l_2}.$$

Unless otherwise specified, the $||.||$ notation refers to the corresponding 2-norm (continuous or discrete). The same holds for the inner-product notation.

Our results are proven in the simplest multi-resolution analysis, the Haar system. The shape function is the indicator function of the interval [0 1] and the mother wavelet is

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ -1, & \text{if } 1/2 \le x \le 1/ \\ 0, & \text{otherwise} \end{cases}$$

The Haar system provides an orthonormal basis in both $L_2(\mathbb{R})$ and $L_2([0\ 1])$. The space V_J consists of piece-wise constant functions on a grid with step-size $h = 2^{-J}$. It is identified with l_2 (or $\mathbb{R}^{2^J}$ in the finite case).

The Haar transform from V_{J+1} to $W_J \oplus V_J$ is simply:

$$\varphi_{J,k} = \frac{1}{\sqrt{2}}(\varphi_{J+1,2k} + \varphi_{J+1,2k+1}), \quad \psi_{J,k} = \frac{1}{\sqrt{2}}(\varphi_{J+1,2k} - \varphi_{J+1,2k+1}).$$

3 Homogenization in the Haar Basis

Discretize the equation

$$\frac{d}{dx}\left(a\frac{d}{dx}u\right) = f$$

on a uniform grid with $h = 2^{-(J+1)}$ using finite volumes. Let $diag(a)$ be the diagonal matrix containing the values of $a(x)$ at the grid-points. As an operator on V_{J+1}, $diag(a)$ represents multiplication by the grid-function a. The discrete equation

$$L_{J+1}U = \frac{1}{h^2}\Delta_+ diag(a)\Delta_- U = F \tag{3}$$

is split by the natural decomposition $V_{J+1} = W_J \oplus V_J$ into

$$\frac{1}{h^2} W_J \Delta_+ diag(a) \Delta_- W_J^T W_J U = \begin{bmatrix} A_J & B_J \\ C_J & L_J \end{bmatrix} \begin{bmatrix} U_h \\ U_l \end{bmatrix} = \begin{bmatrix} F_h \\ F_l \end{bmatrix}, \tag{4}$$

where the indices h and l denote the W_J and V_J components. The equation (3) is a discretization of the continuous equation $(a(x)u')' = f$, with $U_m \approx u(x_m) = u(mh)$.

The coarse scale solution of the discrete equation (3) is the projection of U onto V_J, i.e. U_l. Eliminating U_h yields the equation for U_l:

$$(L_J - C_J A_J^{-1} B_J) U_l = F_l - C_J A_J^{-1} F_h. \tag{5}$$

The homogenized operator is the Schur complement

$$\overline{L_J} = L_J - C_J A_J^{-1} B_J. \tag{6}$$

Let us make some preliminary remarks:

- The homogenization procedure is in fact block Gaussian elimination. The idea is not new, it can be found, e.g., in odd-even reduction techniques. There is a real gain only if the homogenized operator $\overline{L_J}$ can be well approximated by a sparse matrix. It is the compression properties of wavelets that maintain the homogenization procedure efficient, similar to the case of Calderon-Zygmund operators as seen in [2, 5].
- The experience with the non-standard form representation of elliptic operators indicate that A_J has a strong diagonal dominance and thus its inversion will not be as difficult as inverting the operator L_{J+1}, see [6].
- We expect that the homogenized operator $\overline{L_J}$ will have a similar structure as the operator L_J. In fact we will see that if L_{J+1} is in divergence form,

  $$\overline{L_J} = \frac{1}{(2h)^2} \Delta_+ H \Delta_-,$$

  where H is a strongly diagonal dominant matrix. We will call H the homogenized coefficient matrix.

- The homogenization procedure can be applied recursively. If we have the equation

$$\overline{L_{j+1}}u_{j+1} = f_{j+1}$$

that produces the solution on the scale $j+1$, i.e. $u_{j+1} = P_{j+1}U$, then we homogenize the operator $\overline{L_{j+1}}$. This means that we produce the operator $\overline{L_j}$ on V_j, and the right-hand side f_j such that the solution of the homogenized equation

$$\overline{L_j}u_j = f_j.$$

is $u_j = P_j u_{j+1} = P_j U$.

- If the homogenized operator has a rapid decay away from the diagonal, then it can be well approximated by a band-diagonal operator. The same applies for the matrix H.

The structure of the homogenized operator is given by the decomposition of the discrete operators Δ_+, $\Delta_- = \Delta_+^\top$ and $diag(a)$.

3.1 Point-wise multiplication operator

We first examine the multiplication-by-functions operator $diag(a)$. The following lemma is obvious:

Lemma 1 *If φ is the Haar system's shape function and ψ the mother wavelet, then*

$$\psi_{j,k}\psi_{j,l} = 2^{j/2}\delta_{k,l}\varphi_{j,k}, \quad \psi_{j,k}\varphi_{j,l} = 2^{j/2}\delta_{k,l}\psi_{j,k}, \quad \varphi_{j,k}\varphi_{j,l} = 2^{j/2}\delta_{k,l}\varphi_{j,k}.$$

For $v \in V_{J+1}$, we use the notation $\mathcal{W}_J v = [\tilde{v}\ \overline{v}]^\top$, with $\tilde{v} \in W_J$ and $\overline{v} \in V_J$. Let $a \odot v$ denote the component-wise multiplication of vectors. We have the following point-wise multiplication rule:

Lemma 2

$$\mathcal{W}_J(a \odot v) = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{a} \odot \overline{v} + \overline{a} \odot \tilde{v} \\ \\ \tilde{a} \odot \tilde{v} + \overline{a} \odot \overline{v} \end{bmatrix}$$

Proof: Set $a = \sum a_{J+1,k}\varphi_{J+1,k}$ and $v = \sum v_{J+1,k}\varphi_{J+1,k}$. Using Lemma 1, we have

$$\begin{aligned} av &= \sum_{k,l} a_{J+1,k} v_{J+1,l} \varphi_{J+1,k} \varphi_{J+1,l} \\ &= 2^{(J+1)/2} \sum_k a_{J+1,k} v_{J+1,k} \varphi_{J+1,k}. \end{aligned}$$

Thus point-wise multiplication of functions is the equivalent to component-wise multiplication of the coefficients. Then we have:

$$\begin{aligned} av &= \left(\sum_k \tilde{a}_k \psi_{J,k} + \overline{a}_k \varphi_{J,k} \right) \left(\sum_k \tilde{v}_k \psi_{J,k} + \overline{v}_k \varphi_{J,k} \right) \\ &= 2^{J/2} \left(\sum_k (\tilde{a}_k \overline{v}_k + \overline{a}_k \tilde{v}_k) \psi_{J,k} + \sum_k (\tilde{a}_k \tilde{v}_k + \overline{a}_k \overline{v}_k) \varphi_{J,k} \right) \end{aligned}$$

which proves the statement.□

The high frequency components of a and v interact and contribute to the low frequency part of the product av. This is modeled in the Haar basis by correcting the product $\overline{a} \odot \overline{v}$ of the coarse scale coefficients with the fine scale contribution $\tilde{a} \odot \tilde{v}$.

The structure of the pointwise multiplication operator is given by the following statement:

Proposition 1 *If $\mathcal{W}_J a = [\tilde{a}\ \overline{a}]^\top$, then*

$$\mathcal{W}_J \; diag(a) \; \mathcal{W}_J^\top = \frac{1}{\sqrt{2}} \begin{bmatrix} diag(\overline{a}) & diag(\tilde{a}) \\ diag(\tilde{a}) & diag(\overline{a}) \end{bmatrix}.$$

The matrix $diag(a)$ is the point-wise multiplication operator. We have the following amusing result:

Proposition 2 *Let $M_{J+1} = diag(a)$ be the multiplication-by-function operator on V_{J+1}. The coarse-grid projection $P_J M_{J+1} P_J$ is multiplication by the arithmetical averages $(a_{2k} + a_{2k+1})/2$ The homogenized operator $\overline{M_J}$ is multiplication by the harmonical averages $\alpha_k = 2a_{2k}a_{2k+1}/(a_{2k} + a_{2k+1})$.*

Proof: The coarse grid projection of $diag(a)$ is $1/\sqrt{2}\ diag(\overline{a})$, which is, in each component, the arithmetical average $(a_{2k} + a_{2k+1})/2$ of the corresponding fine-grid values. The homogenized operator is

$$\frac{1}{\sqrt{2}}\left(diag(\overline{a}) - diag(\tilde{a})\ diag(\overline{a})^{-1}\ diag(\tilde{a})\right).$$

Component-wise this yields,

$$\frac{1}{\sqrt{2}}\frac{(\overline{a}_k - \tilde{a}_k)(\overline{a}_k + \tilde{a}_k)}{\overline{a}_k} = \frac{2a_{2k}a_{2k+1}}{a_{2k} + a_{2k+1}} = \alpha_k,$$

which is the harmonical cell-average of the corresponding fine-grid values. $\square$

3.2 Decomposition of $\Delta_+ a \Delta_-$

We start by computing the decomposition $\mathcal{W}_J \Delta_+ \mathcal{W}_J^\top$, testing Δ_+ on the basis functions of $W_J \oplus V_J$:

$$\begin{aligned}
\Delta_+ \psi_{J,k} &= \frac{1}{\sqrt{2}}\Delta_+\left(\varphi_{J+1,2k} - \varphi_{J+1,2k+1}\right) \\
&= \frac{1}{\sqrt{2}}\left(-\varphi_{J+1,2k} + 2\varphi_{J+1,2k+1} - \varphi_{J+1,2k+2}\right) \\
&= \frac{1}{2}\left(-3\psi_{J,k} - \psi_{J,k+1} + \varphi_{J,k} - \varphi_{J,k+1}\right).
\end{aligned}$$

Then we have

$$\langle \Delta_+ \psi_{J,k}, \psi_{J,l}\rangle = \frac{1}{2}\left(-3\delta_{l,k} - \delta_{l,k+1}\right), \quad \langle \Delta_+ \psi_{J,k}, \varphi_{J,l}\rangle = \frac{1}{2}\left(-\delta_{l,k} + \delta_{l,k+1}\right)$$

Similar computations yield

$$\Delta_+ \varphi_{J,k} = \frac{1}{2}\left(-\psi_{J,k} + \psi_{J,k+1} - \varphi_{J,k} + \varphi_{J,k+1}\right)$$

and then

$$\langle \Delta_+ \varphi_{J,k}, \psi_{J,l}\rangle = \frac{1}{2}\left(\delta_{l,k} - \delta_{l,k+1}\right), \quad \langle \Delta_+ \varphi_{J,k}, \varphi_{J,l}\rangle = \frac{1}{2}\left(-\delta_{l,k} + \delta_{l,k+1}\right)$$

Let S_n be the shift matrix S_n defined by $S_{k,l}^{(n)} = \delta_{k+n,l}$, which is the projection of the shift operator $Sf(x) = f(x + nh)$. We have the following proposition:

Proposition 3 *The decomposition of $\frac{1}{h}\Delta_+$ in the Haar system is*

$$\mathcal{W}_J \frac{1}{h}\Delta_+ \mathcal{W}_J^\top = \frac{1}{2h}\begin{bmatrix} M & -\Delta_+ \\ \Delta_+ & \Delta_+ \end{bmatrix},$$

where $M = -3I - S_1$.

Obviously, the structure is repeated at each level j. Since $\Delta_- = \Delta_+^\top$, we have that

$$\mathcal{W}_J \frac{1}{h}\Delta_- \mathcal{W}_J^\top = \frac{1}{2h}\begin{bmatrix} -M^\top & \Delta_- \\ -\Delta_- & \Delta_- \end{bmatrix}.$$

Dropping the *diag* notation in Proposition 1, we have that

$$\mathcal{W}_J \Delta_+ \; a \; \Delta_- \mathcal{W}^\top = \frac{1}{4\sqrt{2}}\begin{bmatrix} A_J & (M - \Delta_+)(\overline{a} + \tilde{a})\Delta_- \\ -\Delta_+(\overline{a} + \tilde{a})(\Delta_- + M^\top) & 2\Delta_+(\overline{a} + \tilde{a})\Delta_- \end{bmatrix},$$

where $A_J = \Delta_+\overline{a}\Delta_- - M\overline{a}M^\top + \Delta_+\tilde{a}M^\top - M\tilde{a}\Delta_-$.

3.3 Boundary conditions

The notations Δ_+, Δ_-, and S_n for the discrete difference operators and their corresponding matrices. They can describe periodic, Neumann or Dirichlet boundary conditions. They can also operate (as infinite matrices) on infinite sequences arising from discretizing problems on the whole real axis.

The derivation of the decompositions of the L_{J+1} and the homogenized coefficient matrix H are formally the same. However, in the periodic case, the operator L_{J+1} is singular and it is not trivial that A_J is invertible.

In the periodic case, the matrices Δ_+ and Δ_- are circulant. This property is preserved by the transform W_J. If we define the shift matrices $S_{\pm 1}$ as circulant matrices, then M is also circulant, and thus all the matrices corresponding to the level J have the same property. In the infinite case, Δ_+, Δ_-, and S_n are trivially circulant.

With periodic boundary conditions, it is easy to show that L_{J+1} has a 1-dimensional null-space spanned by the constant grid-functions: Since Δ_+ vanishes only on constants, the ellipticity condition $a > 0$ implies that any non-constant zero-eigenfunction v must satisfy $a\Delta_- v = \textit{constant}$. It follows that v is monotone, which contradicts periodicity.

The null-space of L_{J+1} is transformed by $\mathcal{W}_J$ into the one-dimensional space $\mathcal{N}$ spanned by $[0\ v^\top]^\top$, where v is a constant grid-function. The quadratic form $x^\top \mathcal{W}_J L_{J+1} \mathcal{W}_J^\top x$ is positive whenever $x \notin \mathcal{N}$. In particular, putting $x = [y^\top\ 0]^\top$, we have

$$[y^\top\ 0]^\top \begin{bmatrix} A_J & B_J \\ C_J & L_J \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = y^\top A_J y > 0$$

for any $y \neq 0$. This proves that A_J is positive definite and therefore invertible and thus the homogenized operator $\overline{L_J}$ is well-defined even with periodic boundary conditions.

Both the equations $L_{J+1} u_{J+1} = f_{J+1}$ and $\overline{L_J} u_J = f_J$ need extra conditions. If we decide e.g., to fix a boundary value, we can eliminate a row from both systems. This elimination can be done after the homogenized operator is produced. Thus we need not track the effect of the boundary condition through the homogenization process.

Other type of conditions, such as Dirichlet boundary conditions in the non-periodic case or integral conditions can be handled in a similar fashion.

3.4 The homogenized coefficient matrix

Let us consider a discretization on the whole real axis, i.e., the case where the matrices Δ_+, Δ_-, etc, are infinite. The coarse-scale projection of L_{J+1} is

$$L_J = \sqrt{2}\frac{1}{(2h)^2}\Delta_+(\overline{a} + \widetilde{a})\Delta_-.$$

L_J is the "wrong" operator for an obvious reason: the average coefficient is obtained using only the even components of the the fine-scale coefficient $a_k = a(x_k)$ and thus L_J is insensitive to variations of the odd-components in the original problem. Even if the fine-scale is not present in $a(x)$, i.e. $\widetilde{a} = 0$, L_J still has the wrong coefficient

$$\sqrt{2}\overline{a}_k = a_{2k} + a_{2k+1} = 2a_{2k}.$$

We build the homogenized operator as the Schur complement $L_J - C_J A_J^{-1} B_J$.

$$\overline{L_J} = L_J + \frac{1}{\sqrt{2}} \frac{1}{(2h)^2} \Delta_+ (\overline{a} + \tilde{a})(\Delta_- + M^\top) A_J^{-1} (M - \Delta_+)(\overline{a} + \tilde{a}) \Delta_-$$

Proposition 4 *The operator $\overline{L_J}$ has a natural structure $\frac{1}{(2h)^2} \Delta_+ H \Delta_-$.*

$$H = \sqrt{2}(\overline{a} + \tilde{a}) + \frac{1}{\sqrt{2}} (\overline{a} + \tilde{a})(\Delta_- + M^\top) A_J^{-1} (M - \Delta_+)(\overline{a} + \tilde{a}) \qquad (7)$$

Definition 1 *We call H the* homogenized coefficient matrix *of the operator $\overline{L_J}$:*

The natural question to ask is if there is any connection between the homogenized coefficient matrix H and the classical homogenized equations.

Proposition 2 gives that the Schur complement of the diagonal matrix a is the diagonal matrix α containing the harmonical averages of neighboring values. This would then agree exactly with the classical homogenization theory, if the Schur complement of $\Delta_+ a \Delta_-$ could be expressed in terms of the Schur complement of the middle factor a. Unfortunately, this is not the case, so we have to use the form given in (7).

We look at the extreme case when $a(x) = \overline{a} + \tilde{a}(x)$ is the sum of a constant and the highest frequency represented on the grid, i.e., $a(x_m) = \overline{a} + |\tilde{a}| (-1)^m$. We have that $\overline{a}$ and $\tilde{a}$ are represented as constant vectors in the bases of V_J and W_J. The fact that $a(x) > 0$ implies $|\tilde{a}| < \overline{a}$.

Since $\overline{a}$ and $\tilde{a}$ are constant vectors, we have

$$H = \sqrt{2}(\overline{a} + \tilde{a}) + \frac{1}{\sqrt{2}} (\overline{a} + \tilde{a})^2 (\Delta_- + M^\top) A_J^{-1} (M - \Delta_+),$$

where

$$A_J = \overline{a}(\Delta_+ \Delta_- - M M^\top) + \tilde{a}(\Delta_+ M^\top - M \Delta_-).$$

Simple computations yield

$$\Delta_- + M^\top = -2(I + S_{-1}), \quad M - \Delta_+ = -2(I + S_1)$$

and then

$$\Delta_+ \Delta_- - M M^\top = -2(S_{-1} + 6I + S_1), \quad \Delta_+ M^\top - M \Delta_- = 2 \Delta_+ \Delta_-.$$

The homogenized coefficient matrix defined by (7) is

$$H = \sqrt{2}(\overline{a} + \widetilde{a}) \left(I - (\overline{a} + \widetilde{a})(I + S_{-1})(\overline{a}(S_{-1} + 6I + S_1) - \widetilde{a}\Delta_+\Delta_-)^{-1}(I + S_1) \right). \tag{8}$$

Classical homogenization theory yields the effective equation $\alpha\frac{d^2}{dx^2}$ where the effective coefficient is given by the harmonical average:

$$\alpha = \left(\frac{1}{2h} \int_0^{2h} a(x)dx \right)^{-1} = 2 \left(\frac{1}{a_0} + \frac{1}{a_1} \right)^{-1}. \tag{9}$$

In the rest of this section, we will be looking only at the coarse grid function space V_J. For simplicity, we will let $h = 2^{-J}$ denote the grid-size of V_J.

The following theorem shows that the numerically homogenized operator $\frac{1}{h^2}\Delta_+ H \Delta_-$ equals the discrete form $\alpha\frac{1}{h^2}\Delta_+\Delta_-$ of the classically homogenized equation, apart from a second-order term in h.

Theorem 1 *Let $a(x) = \overline{a} + \widetilde{a} \in V_{J+1}$ be such that $\overline{a} \in V_0$ is a constant and the oscillatory part $\widetilde{a} \in W_J$ has constant amplitude and satisfies the condition $|\widetilde{a}| < \overline{a}$. Let $L_{J+1} = \frac{1}{(h/2)^2}\Delta_+ a \Delta_-$ and α be the harmonical average in (9). Then there exists a constant C independent of the grid-size h such that if v is the discretization of a function $v(x)$ with a continuous and bounded fourth derivative, then*

$$\left\| \overline{L_J} v - \alpha \frac{1}{h^2} \Delta_+ \Delta_- v \right\|_\infty \leq C h^2 \left\| v^{(4)} \right\|_{L_\infty}.$$

Proof: Let us show first that the high-frequency operator A_J is invertible by showing that it is diagonal dominant. We have

$$A_J = -\overline{a}(S_{-1} + 6I + S_1) + \widetilde{a}\Delta_+\Delta_- = (\widetilde{a} - \overline{a})(S_{-1} + S_1) - (6\overline{a} + 2\widetilde{a})I,$$

and the tridiagonal structure of A_J is clearly seen. The ellipticity of L_{J+1} implies that $0 \leq |\widetilde{a}| < \overline{a}$. The diagonal entries of A_J are larger then $4\overline{a}$ while the sum of the off-diagonal terms is smaller then the same amount. The diagonal dominance of A_J gives a rapid decay of the entries of A_J^{-1} away from the diagonal. Indeed, we have

$$A_J = -(6\overline{a} + 2\widetilde{a}) \left(I - q(S_1 + S_{-1}) \right), \quad q = \frac{\widetilde{a} - \overline{a}}{6\overline{a} + 2\widetilde{a}}. \tag{10}$$

Since $|q| < 1/2$, the Neumann expansion for A_J^{-1} is convergent and (10) reveals the size of the off-diagonal entries:

$$A_J^{-1} = -\frac{1}{6\overline{a} + 2\widetilde{a}} \left(I + \sum_k q^k (S_1 + S_{-1})^k \right). \tag{11}$$

Next we compute the row-sum of H. Note first that since A_J is circulant, it has an eigenvector $c = [..1\ 1 \ldots 1..]^{\mathsf{T}}$ and the corresponding eigenvalue is $8\overline{a}$. A_J^{-1} shares the eigenvector c, which shows that all its row-sums are $1/(8\overline{a})$. Note that c is also an eigenvector of $I + S_{\pm 1}$ with the corresponding eigenvalue 2. Thus we have

$$Hc = \sqrt{2}(\overline{a} + \widetilde{a}) \left(1 - (\overline{a} + \widetilde{a}) \frac{4}{8\overline{a}} \right) c = \frac{\sqrt{2}}{2\overline{a}} (\overline{a} + \widetilde{a})(\overline{a} - \widetilde{a}) c = \alpha c$$

Finally we estimate $\overline{L_J} v$. Note that since $H = \sum_n \gamma_n S_n$, it commutes with $\Delta_- = I - S_{-1}$ and thus

$$\overline{L_J} = \frac{1}{h^2} \Delta_+ \Delta_- H.$$

Assuming v is a discrete smooth function, Taylor expansion around $v_j = v(x_j)$ yields

$$v = v(x_j) \begin{bmatrix} \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} + v'(x_j) \begin{bmatrix} \vdots \\ -h \\ 0 \\ h \\ \vdots \end{bmatrix} + \frac{v''(x_j)}{2} \begin{bmatrix} \vdots \\ h^2 \\ 0 \\ h^2 \\ \vdots \end{bmatrix} + \ldots \tag{12}$$

Let us estimate the j component of Hv. Applying H to the first term in (12) produces just αv_j. Due to symmetry, we have that $H = \sum_{n \geq 0} \gamma_n (S_n + S_{-n})$. Applying H to the odd-order terms of (12) shifts in the j component quantities with opposite signs and then adds them. The even-order terms contribute such that

$$(Hv)_j = \alpha v_j + v^{(2)}(x_j) \frac{h^2}{2} \sum_n \gamma_n n^2 + v^{(4)}(\xi) \frac{h^4}{4!} \sum_n \gamma_n n^4,$$

where $\xi \in \mathbb{R}$. We shall show later that the coefficients γ_n have exponential decay and thus $\sum \gamma_n n^k$ is convergent for any k. Applying $\frac{1}{h^2} \Delta_+ \Delta_-$ and comparing to $\alpha \frac{1}{h^2} \Delta_+ \Delta_-$ yields:

$$(\overline{L_J} - \alpha \frac{1}{h^2} \Delta_+ \Delta_-) v_j = C \frac{1}{h^2} \Delta_+ \Delta_- (v^{(2)}(x_j) h^2) + \ldots = C h^2 v^{(4)}(\xi) + \ldots$$

which in its turn gives the desired estimate with $C = \sum_n \gamma_n n^2 < \infty$.

It remains to be shown that the constant C is independent of the grid-size h. The expansion (11) shows that A_J^{-1} is generated infinite long stencil with exponential decay rate $\rho = |2q| < 1$. To build H from A_J^{-1}, we first apply $I+S_1$ and $I+S_{-1}$, which has the effect of adding together neighboring diagonals. Indeed, if $A_J^{-1} = \sum a_n S_n$, we have

$$(I + S_{-1})A_J^{-1}(I + S_1) = \sum a_n(I + S_{-1})S_n(I + S_1) = \sum(a_{n-1} + 2a_n + a_{n+1})S_n.$$

Since $|a_n| < K\rho^n$, the elements of the product $(I + S_{-1})A_J^{-1}(I + S_1)$ are bounded by $4K\rho^n$. H is then found by multiplication with $\sqrt{2}(\overline{a} + \widetilde{a})^2$ and the addition of a diagonal term. The decay away from the diagonal of the terms γ_n is

$$|\gamma_n| < K'\rho^n, \quad K' = 4\sqrt{2}\frac{(\overline{a} + \widetilde{a})^2}{6\overline{a} + 2\widetilde{a}} \tag{13}$$

The exponential decay in γ_n dominates the growth of n^2 and thus we find the constant C:

$$\sum_n \gamma_n n^2 < K' \sum_n \rho^n n^2 < K' \int_0^\infty \rho^x x^2 dx = \frac{2K'}{(\log \rho)^2} = C$$

□

Remark: The fact that $Hv \approx \alpha v$ for smooth functions v can be also seen from the Fourier analysis of H. By doing a discrete Fourier transform of H, we obtain a diagonal matrix $diag(\hat{g})$. The diagonal $\hat{g}$ is given by the symbol of H which is

$$\hat{g}(\omega) = \sqrt{2}(\overline{a} + \widetilde{a}) \left(1 - (\overline{a} + \widetilde{a})\frac{(\sigma + 1)^2}{\overline{a}(\sigma^2 + 6\sigma + 1) + \widetilde{a}(\sigma^2 - 2\sigma + 1)}\right), \quad \sigma = e^{2\pi i \omega/N}$$

Note that $\hat{g}$ is just the Fourier transform of any row of H. It is therefore no surprise that $\hat{g}(0) = \alpha$. It turns out that $\frac{d\hat{g}}{d\omega}(0) = 0$ which yields

$$\hat{g}(\omega/N) = \alpha + \mathcal{O}((\omega/N)^2).$$

The approximation error is indeed quadratic in ω since $\hat{g}''(0) = \pi^2\alpha(\overline{a} + \widetilde{a})/\overline{a} > 0$.

If the Fourier coefficients of v decay sufficiently fast, then we have $\hat{g}\hat{v} \approx \alpha\hat{v}$, and by the inverse transform, $Hv \approx \alpha v$. Note in Figure 1 that $\hat{g}(\omega)$ has a moderate growth even for large ω.

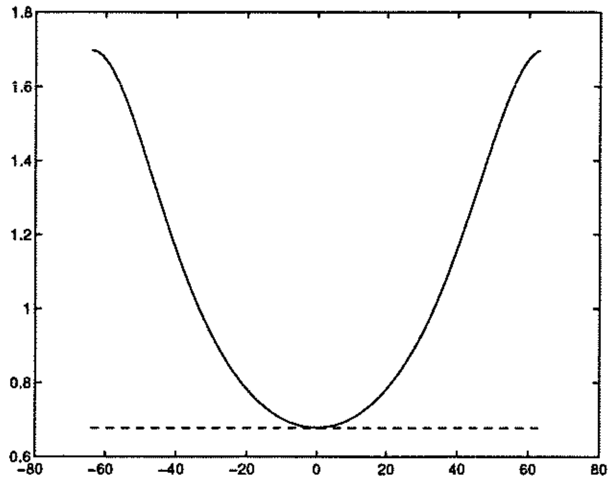

Figure 1: The Fourier components of H and αI (dashed line). $\hat{g}(\omega)$ behaves like multiplication with $\hat{g}(0) = \alpha$ for small ω. Periodic boundary conditions are assumed.

In practice, we want to approximate the homogenized operator $\overline{L_J}$ by a sparse approximation. Due to the diagonal decay, we can approximate $\overline{L_J}$ by a band-diagonal matrix $\overline{L_{J,\nu}}$ where ν is the band-width. Let us consider the operator *band* defined by

$$band(M,\nu)_{i,j} = \begin{cases} M_{i,j}, & \text{if } 2\,|i-j| \leq \nu - 1 \\ 0, & \text{otherwise} \end{cases}$$

We have in fact two obvious strategies available for producing $\overline{L_{J,\nu}}$: We can set directly $\overline{L_{J,\nu}} = band(\overline{L_J}, \nu)$ or use the homogenized coefficient form and build $\overline{L_{J,\nu}} = \frac{1}{h^2}\Delta_+ band(H, \nu - 2)\Delta_-$. Both approaches produce small perturbations of $\overline{L_J}$. However, important properties, such as divergence form, are lost in the first approach and numerical experiments show that ν needs to be rather large to compensate for this. The second approach produces $\overline{L_{J,\nu}}$ in divergence form. Moreover, the approximation error can be estimated, as in the following result:

Theorem 2 *If the conditions of Theorem 1 are valid, then*

$$||H - band(H,\nu)|| \leq C\rho^{\nu}, \quad \rho = \frac{2(\overline{a} + |\tilde{a}|)}{6\overline{a} - 2\,|\tilde{a}|} < 1.$$

If v is the discretization of a smooth function $v(x)$, then

$$\left\|(\overline{L_J} - \overline{L_{J,\nu}})v\right\|_\infty \leq C\rho^\nu \left\|v''\right\|_{L^\infty}$$

Proof: The exponential decay from the diagonal in H, given in (13), yields

$$||H - band(H,\nu)|| < K' \sum_{n=\nu}^{\infty} \rho^n = K'\rho^\nu \frac{1}{1-\rho}$$

If v is a smooth function, using the commutation property of H (and $band(H)$), we have

$$(\overline{L_J} - \overline{L_{J,\nu}})v_j = (H - band(H,\nu))\frac{1}{h^2}\Delta_+\Delta_- v_j = (H - band(H,\nu))(v''(\xi)),$$

where ξ is some point in $\mathbb{R}$. Therefore we have

$$\left|(\overline{L_J} - \overline{L_{J,\nu}})v_j\right| \leq K'\rho^\nu \frac{1}{1-\rho}\left|v''(\xi)\right|.$$

Taking the supremum over all ξ and then the maximum over all j yields the desired estimate. □

Remark: The above estimates hold also for Dirichlet boundary conditions. In the case of periodic boundary conditions, the meaning of "away from the diagonal" is different because the wrap-around effect. The diagonal band of width ν is then defined by $2(|i-j|(\mod N)) \leq \nu - 1$, where N is the size of the matrix.

3.5 Numerical experiments

We test the homogenization procedures on some examples.

- With periodic coefficients, wavelet and classical homogenization produce the same discrete solution. With non-periodic variations of the coefficients, the effective equations cannot extract the local features of the solution. Due to the localization of the wavelet basis elements, such local features are preserved by wavelet homogenization.

- Solution with several different scales. The test problem is $(a(x, x/\varepsilon_1, x/\varepsilon_2)u')' = 1$. We project the solution on spaces that resolve either both the scales $\varepsilon_1 < \varepsilon_2$, or just the finest scale ε_1.

- Comparison of the solutions of the homogenized forms using the two truncation strategies, $band(\overline{L_J}, \nu)$ and $\frac{1}{h^2}\Delta_+ band(H, \nu)\Delta_-$ with different values for ν We see that truncation of the homogenized coefficient matrix H performs much better.

3.5.1 Non-periodic variable coefficients

First we compare the exact, classical-homogenized, and wavelet-homogenized solutions to a periodic problem. We consider the two-point boundary problem

$$-(a(x)u')' = 10, \quad u(0) = 0, \quad u'(1) = 0.$$

The exact solution solves the discrete equation $-\frac{1}{h^2}\Delta_+ a \Delta_- u = f$. We take $a(x)$ to have alternating values 1 and 100 on a fixed grid. The classical and wavelet homogenized solutions are pictured in Figure 2.

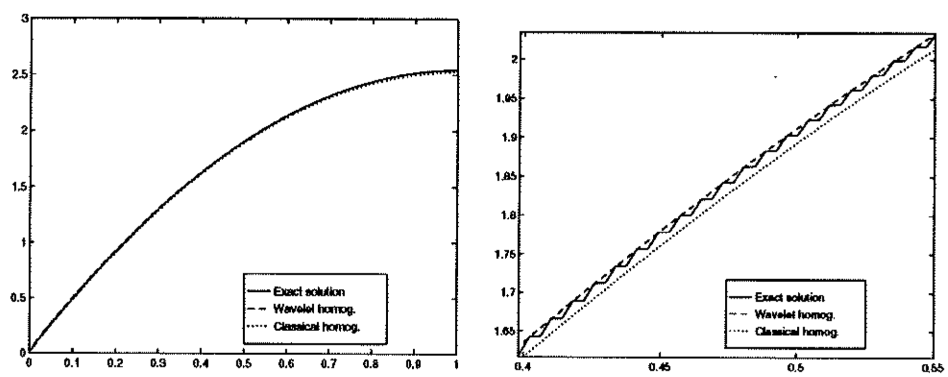

Figure 2:
Exact, classical homogenization and Haar basis homogenization solutions in the periodic case. $N = 256$ grid-points. The plot on the right is a detail of the left image.

The wavelet solution is computed using 3 levels, i.e. the coarse scale contains eight times fewer grid-points. The effective coefficient is $200/101 \approx 1.9802$ and thus classical homogenization yields the approximation

$$u_{eff} = \frac{101}{20}\left(1 - \frac{x}{2}\right)x.$$

Note the detail in Figure 2 where the wavelet based solution $\overline{u}$ is essentially a shift of u_{eff}, i.e. $\overline{u}$ contains no high frequencies.

Now we take $a(x)$ to be uniformly distributed in the interval [1 100], as plotted in Figure 3 (left). The classical homogenized coefficient (effective coefficient) is computed as

$$a_{eff} = \left(\int_0^1 \frac{1}{a(x)}\,dx\right)^{-1}.$$

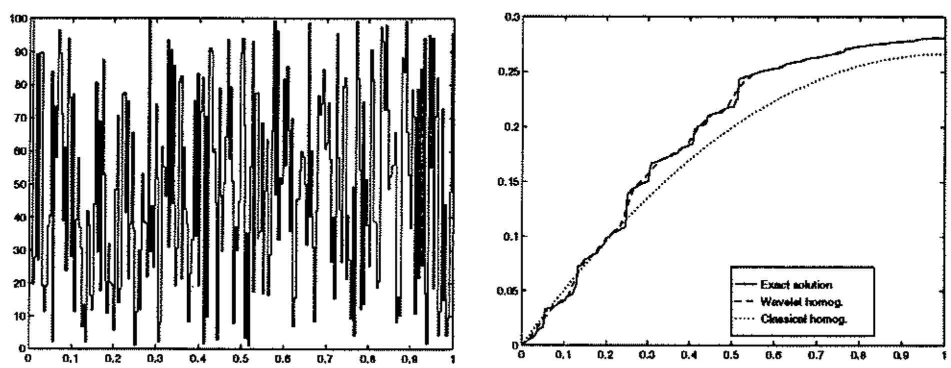

Figure 3:
Non-periodic coefficients $a(x)$ (left) and a comparison of the exact solution u, effective equations solution u_{eff} and Haar basis homogenization solutions $\overline{u}$. $N = 256$ grid-points in these plots.

Figure 3 (right) compares the exact solution u with the wavelet homogenized $\overline{u}$ and the result of classic homogenization u_{eff}, where the effective coefficient is $a_{eff} = 18.8404$. The fine grid has 256 points. Both u_{eff} and $\overline{u}$ are represented on the coarse grid using 32 points. However, the wavelet homogenized solution $\overline{u}$ captures much more coarse-scale detail then the classic solution u_{eff}.

3.5.2 Homogenization over multiple scales

We test a problem that contains three different scales: Let

$$a(x) = a(x, 2^8 x, 2^4 x) = 100 + 90\text{sign}(\cos(2^8 \pi x)) + 9\text{sign}(\cos(2^4 \pi x)).$$

The coefficient has three scales, $J = 8$, 4 and 0. The solution of the equation

$$\frac{1}{h^2} \Delta_+ \; diag(a) \; \Delta_- U = 1$$

contains all the three scales if h resolves the finest scale of $a(x)$. Put $h = 2^{-9}$ to resolve all the scales of the problem. Then we project the exact solution onto V_6.

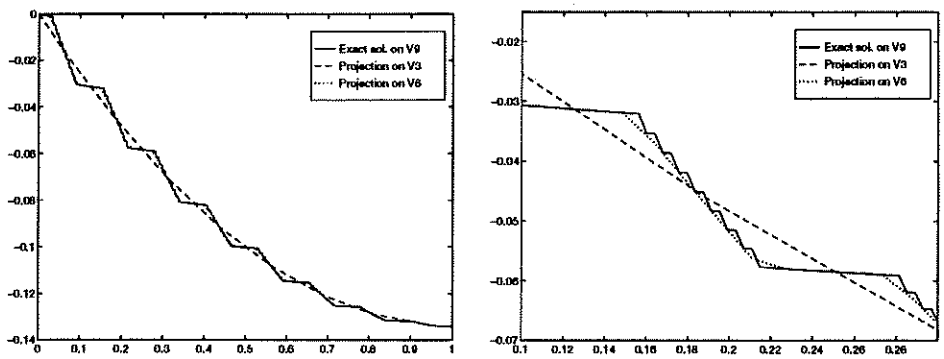

Figure 4:
Homogenization of several scales. Coefficient $a = a(x, 2^8 x, 2^4 x)$. Plot of u_9, $\overline{u_6}$ and $\overline{u_3}$ (left). Details of plots (right) shows that $\overline{u_6}$ averages the finest scale only and resolves the coarser scales. $\overline{u_3}$ resolves only the coarsest scale.

Figure 4 shows that the finest scale contribution is averaged out, but the coarser scales $J = 4$ and $J = 0$ are resolved. Projection onto V_3 averages both the finer scales $J = 8$ and $J = 4$ and the solution has the characteristics of a constant-coefficients problem.

3.5.3 Banding strategies

We test the accuracy of approximating the homogenized operator by banded matrices using the two strategies described in Section 3.4. The coefficients $a(x)$ are chosen at random, uniformly distributed in the interval (0.1 2). The boundary conditions are $u(0) = 0$ and $u'(1) = 0$.

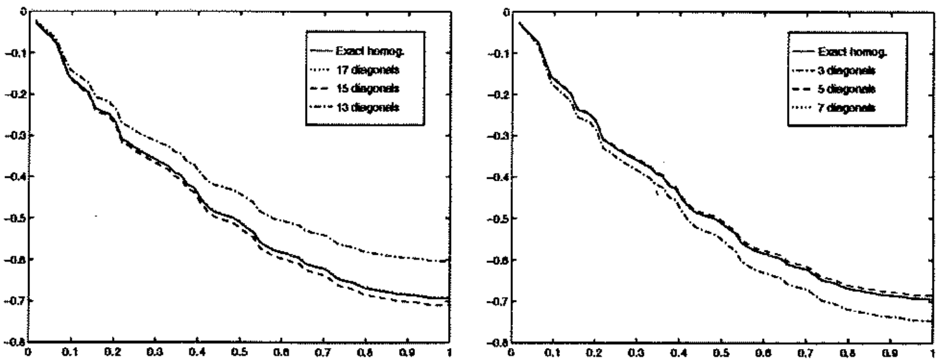

Figure 5:
The homogenized operator approximated by banded matrices. Banding the exact homogenized operator $\overline{L_J}$ (left) needs a much larger band-width ν as compared to banding the homogenized coefficient matrix H (right). 512 grid-points on the fine grid, 64 grid-points on the coarse grid.

Figure 5 (left) is the plot the solutions of $band(\overline{L_J}, \nu)u_J = 1$, with $\nu = 13, 15, 17$. To obtain even better accuracy, using the approximation of the homogenized coefficient matrix $H_\nu = band(H, \nu)$, considerably fewer diagonals are needed. Figure 5 (right) plots the solutions of $\frac{1}{(2h)^2}\Delta_+ H_\nu \Delta_- u_J = 1$ with only $\nu = 3, 5, 7$ diagonals.

4 2-D Problems

Numerical homogenization for multi-dimensional problems is of great interest since the analytic methods can only handle periodic micro-structures, see e.g., Bensoussan *et al.* [1]. The aim of this section is to show that if a 2-D fine-scale operator is in discrete divergence form

$$L_{J+1} = \frac{1}{h^2}\left(\Delta_+^x A^{(11)} \Delta_-^x + \Delta_+^x A^{(12)} \Delta_-^y + \Delta_+^y A^{(21)} \Delta_-^x + \Delta_+^y A^{(22)} \Delta_-^y\right), \tag{14}$$

then the homogenized operator $\overline{L_J}$ acting on the coarser space has the same form. As we saw in the one-dimensional case, this property is important for efficient truncation strategies.

Definition 2 *The operator L_{J+1} is called discrete elliptic if*

1. *L_{J+1} is symmetric, i.e., $A^{(i,j)} = (A^{(j,i)})^\top$.*
2. *The spectrum of L_{J+1} lies in $\{0\} \cup [\delta, +\infty)$, where $\delta > 0$, and 0 cannot be a multiple eigenvalue.*

4.1 2-D tensor product wavelet spaces

Let us make the notations precise. We consider the tensor product space $\mathbf{V}_{J+1} = V_{J+1} \otimes V_{J+1}$ generated by the canonical basis

$$\varphi_{J+1,k} \otimes \varphi_{J+1,l}(x,y) = \varphi_{J+1,k}(x)\varphi_{J+1,l}(y).$$

The coarse space is $\mathbf{V}_J = V_J \otimes V_J$ and it is generated by $\varphi_{J,k} \otimes \varphi_{J,l}$. The orthogonal complement of $\mathbf{V}_J$ in $\mathbf{V}_{J+1}$ is the wavelet space

$$\mathbf{W}_J = (W_J \otimes W_J) \oplus (V_J \otimes W_J) \oplus (W_J \otimes V_J).$$

The wavelet transform maps the standard basis of $\mathbf{V}_{J+1}$ into the union of the standard bases of $\mathbf{V}_J$ and the three components of $\mathbf{W}_J$. If L_{J+1} is the matrix of a linear operator on $\mathbf{V}_{J+1}$, then the orthogonal basis transformation $\mathcal{W}_J$ yields

$$\mathcal{W}_J L_{J+1} \mathcal{W}_J^\top = \begin{bmatrix} A_J & B_J \\ C_J & L_J \end{bmatrix}.$$

The operators A_J, B_J, C_J and L_J operate on subspaces:

$$A_J : \mathbf{W}_J \to \mathbf{W}_J, \quad B_J : \mathbf{V}_J \to \mathbf{W}_J, \quad C_J : \mathbf{W}_J \to \mathbf{V}_J, \quad L_J : \mathbf{V}_J \to \mathbf{V}_J.$$

By elimination, we have that the homogenized operator is

$$\overline{L_J} = L_J - C_J A_J^{-1} B_J.$$

Note that in the finite case, $\dim(\mathbf{W}_J) = 3\dim(\mathbf{V}_J)$.

We can continue with the decomposition of $\mathbf{V}_J = \mathbf{W}_{J-1} \oplus \mathbf{V}_{J-1}$ and obtain in this manner a multi-resolution analysis on the product space.

The product of the orthogonal transformations $\mathcal{W}_j : \mathbf{V}_{j+1} \to \mathbf{W}_j + \mathbf{V}_j$ is the (orthogonal) wavelet transform that maps $\mathbf{V}_{J+1}$ into $(\oplus_{0 \le j \le J} \mathbf{W}_j) \oplus \mathbf{V}_0$.

4.2 Invariance of divergence form

The operator Δ_+^x acts on $\mathbf{V}_{J+1}$ and is defined by $\Delta_+^x(f \otimes g) = (\Delta_+ f) \otimes g$, where Δ_+ is the 1-D forward difference operator. Δ_+^y, Δ_-^x and Δ_-^y are defined in a similar manner. We regard the operators $A^{(ij)}$ as multiplication by the discrete functions $a^{(i,j)}(x,y)$, i.e., $A^{(i,j)}(\varphi_{J+1,k} \otimes \varphi_{J+1,l}) = a^{(i,j)}(x_k, y_l)\varphi_k(x)\varphi_l(y)$. In general, $A^{(i,j)}$ can be any operator on $\mathbf{V}_{J+1}$, but then L_{J+1} is may no longer be a discretization of a differential operator.

Let us formulate the result:

Theorem 3 *Let L_{J+1} be a discrete elliptic operator in divergence form (14). Assume periodic boundary conditions in the x and y directions and let $\overline{L_J}$ be the homogenized operator using the Haar transform. Then $\overline{L_J}$ is also in divergence form.*

Proof: We begin by observing that the orthogonal transform $\mathcal{W}_J : \mathbf{V}_{J+1} \to \mathbf{W}_J + \mathbf{V}_J$ can be written as $\mathcal{W}_J = \mathcal{W}_x \mathcal{W}_y$ where $\mathcal{W}_x$ is a the corresponding 1-D transform in the x-direction, and $\mathcal{W}_y$ is defined analogously. Remark also that $\mathcal{W}_x\mathcal{W}_y = \mathcal{W}_y\mathcal{W}_x$.

Next we observe that $\Delta_+^x = \Delta_+ \otimes I$ and $\Delta_+^y = I \otimes \Delta_+$. This gives that $\Delta_+^x \mathcal{W}_y = \mathcal{W}_y \Delta_+^x$ and $\Delta_+^y \mathcal{W}_x = \mathcal{W}_x \Delta_+^y$.

The next step is to compute the decomposition of Δ_+^x in $(W_J \otimes W_J) \oplus (W_J \otimes V_J) \oplus (V_J \otimes W_J) \oplus (V_J \otimes V_J)$. Using the standard inner-product on tensor-product spaces, we apply Δ_+^x to a basis function and test it against another basis function:

$$\left\langle \Delta_+^x(f_1 \otimes g_1), f_2 \otimes g_2 \right\rangle = \langle \Delta_+ f_1, f_2 \rangle \langle g_1, g_2 \rangle,$$

where f_i, g_i can be any $\varphi_{J,k}$ or $\psi_{J,k}$. Note that the second inner-product is 0 if $g_1 \neq g_2$. The first inner-product gives the 1-D decomposition of Δ_+, as in Proposition 3. Using the notations of Proposition 3, we can synthesis the decomposition of $2\Delta_+^x$ in the following table:

	$W \times W$	$W \times V$	$V \times W$	$V \times V$
$W \times W$	$M \otimes I$		$-\Delta_+ \otimes I$	
$W \times V$		$M \otimes I$		$-\Delta_+ \otimes I$
$V \times W$	$\Delta_+ \otimes I$		$\Delta_+ \otimes I$	
$V \times V$		$\Delta_+ \otimes I$		$\Delta_+ \otimes I$

In a similar fashion, we obtain the decomposition of Δ_+^y:

$$\mathcal{W}_J\Delta_+^y\mathcal{W}_J^T = \frac{1}{2}\begin{bmatrix} I\otimes M & -I\otimes\Delta_+ & & \\ I\otimes\Delta_+ & I\otimes\Delta_+ & & \\ & & I\otimes M & -I\otimes\Delta_+ \\ & & I\otimes\Delta_+ & I\otimes\Delta_+ \end{bmatrix}$$

and

$$\mathcal{W}_J\Delta_-^x\mathcal{W}_J^T = \frac{1}{2}\begin{bmatrix} -M^T\otimes I & & \Delta_-\otimes I & \\ & -M^T\otimes I & & \Delta_-\otimes I \\ -\Delta_-\otimes I & & \Delta_-\otimes I & \\ & -\Delta_-\otimes I & & \Delta_-\otimes I \end{bmatrix}$$

The essential point is that the last block-row of the decomposition of Δ_+^x (or Δ_+^y) contains only $\Delta_+\otimes I$ (or $I\otimes\Delta_+$) entries. For the Δ_-^x (or Δ_-^y) operator, the analogous holds for the last block-column.

Noting that $\Delta_+\otimes I = \Delta_+^x$, on the coarse space $\mathbf{V}_J$, we have that the decomposition of the product $\frac{1}{h^2}\Delta_+^x A^{(1,1)}\Delta_-^x$ is of the form

$$\frac{1}{4h^2}\begin{bmatrix} & & & B_1\Delta_-^x \\ & A & & B_2\Delta_-^x \\ & & & B_3\Delta_-^x \\ \Delta_+^x C_1 & \Delta_+^x C_2 & \Delta_+^x C_3 & \Delta_+^x H\Delta_-^x \end{bmatrix},$$

where A, B_i, C_i and H are some arbitrary operators. Adding the contributions of all the terms in the form (14) of L_{J+1} yields:

$$\frac{1}{4h^2}\begin{bmatrix} & & & B_1\Delta_-^x + B_1'\Delta_-^y \\ & A & & B_2\Delta_-^x + B_2'\Delta_-^y \\ & & & B_3\Delta_-^x + B_3'\Delta_-^y \\ \Delta_+^x C_1 + \Delta_+^y C_1' & \Delta_+^x C_2 + \Delta_+^y C_2' & \Delta_+^x C_3 + \Delta_+^y C_3' & D \end{bmatrix},$$

where D is in discrete divergence form.

Since L_{J+1} is elliptic, periodic boundary conditions imply it has a one-dimensional null-space, spanned by the constant functions. This null-space is mapped by the transform $\mathcal{W}_J$ into $\mathbf{V}_J$. Since the operator L_{J+1} has non-negative eigenvalues and A operates on the complement of $\mathbf{V}_J$, it follows that $v^{\mathsf{T}}Av > 0$ for any $v \neq 0$.

Therefore A is invertible and we can build the homogenized operator by block Gauss elimination. This yields

$$\overline{L_J} = \frac{1}{(2h)^2}\left(D - \sum \Delta_+^{(i)} H_{i,j} \Delta_-^{(j)}\right),$$

where $\Delta_+^{(1)}$ stands for Δ_+^x, etc. We have that $\overline{L_J}$ is in divergence form on the coarse space $\mathbf{V}_J$. □

Remark: The conservation of the divergence form of L_{J+1} under the 2-D Haar transformation has important consequences. In the multi-dimensional case, it is known that the problem

$$\nabla \cdot A_\varepsilon \nabla u^\varepsilon = f$$

admits a homogenized equation $\overline{L}\overline{u} = f$, but apart from the cell-periodic problem, there is no general algorithm for deriving the homogenized operator $\overline{L}$. In fact, the nature of $\overline{L}$ is not known and numerical homogenization can therefore be used not only as a practical tool, but also to find information about the structure homogenized operator $\overline{L}$.

4.3 A numerical example

We chose $a(x, y) = 1.01 + \text{sign}(\cos(\varepsilon\pi x))$ and solve $\nabla \cdot (a(x, y)\nabla u) = 1$. The classical homogenized equation is

$$\nabla \cdot \left(\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \nabla u\right) = 1,$$

where m_1 is the harmonical average of a in a cell with the length of a period and m_2 is the (arithmetical) average in the same cell:

$$m_1^{-1} = \frac{4}{\varepsilon^2}\int_0^{\varepsilon/2} 1/a(x, y)\, dx dy = \frac{2 \cdot 10^{-2} + 10^{-4}}{1.01} \approx 0.02, \qquad m_2 = 1.01.$$

The homogenized equation has constant coefficients but is strongly anisotropic. Figure 6 displays the exact and wavelet homogenized solutions. The domain is the unit square and there are Dirichlet boundary conditions on the coordinate axes and Neumann conditions on the other two sides.

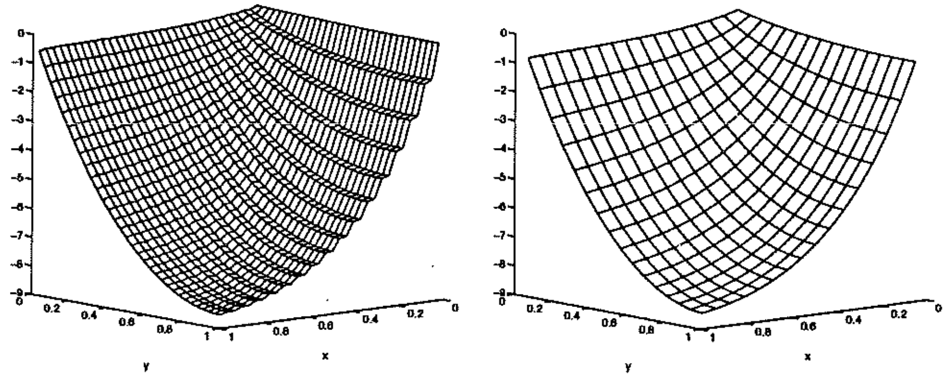

Figure 6:
Fine scale (left) and homogenized solution (right). Note that the homogenized solution captures the effect of the coarse-scale strong anisotropy, averaging only the fine-scale variations.

5 Extensions

The homogenization procedure can be carried out on coarser and coarser levels to produce a sequence of homogenized equations

$$\overline{L_J}\,\overline{u_J} = \overline{F_J}, \quad \overline{L_{J-1}}\,\overline{u_{J-1}} = \overline{F_{J-1}}, \ldots, \overline{L_0}\,\overline{u_0} = \overline{F_0}.$$

If we solve the coarse scale problem exactly, then by block back-substitution

$$\overline{u_j} = \mathcal{W}_{j-1}^T \left[\begin{array}{c} \widetilde{u_{j-1}} \\ \hline u_{j-1} \end{array} \right], \quad A_J\widetilde{u_j} = \widetilde{F_j} - B_J\overline{u_j}, \quad j = 1, 2, \ldots J$$

we produce the exact solution:

$$u_{J+1} = \mathcal{W}_J^T \left[\begin{array}{c} \widetilde{u_J} \\ \overline{u_J} \end{array} \right].$$

If no truncations are used in building the homogenized operators $\overline{L_j}$, the above strategy describes an exact, direct solver, which compares to the reduction techniques in computational linear algebra, see [8]. If truncations are used, the direct solver contains an approximation error.

The homogenization procedure can be applied recursively on any number of levels, provided that the initial operator is in discrete divergence form and is elliptic.

These two properties are sufficient for the existence of the Schur complement and they are are inherited by the homogenized operator. It is not necessary that L_{J+1} approximates a differential operator as long as it is elliptic and in divergence form.

On coarser scales, the homogenized operator resembles the inverse of a differential operator and is expected to be dense. The use of wavelets with a high number of vanishing moments, known to compress well Calderon-Zygmund operators, could have better compression effects then the Haar system, in the spirit of the ideas presented in Beylkin, Coifman and Rokhlin's work [2].

In applications, if we want to use the homogenized coefficient matrices, we would not invert A_j, but rather LU-factorize it in the prescribed bandwidth ν.

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