UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Stiff Systems of Hyperbolic Conservation Laws. Convergence and Error Estimates

A. Kurganov
E. Tadmor

April 1996
CAM Report 96-8

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

STIFF SYSTEMS OF HYPERBOLIC CONSERVATION LAWS. CONVERGENCE AND ERROR ESTIMATES*

A. KURGANOV† AND E. TADMOR†‡

Abstract. We are concerned with 2×2 nonlinear relaxation systems of conservation laws of the form, $u_t + f(u)_x = -\frac{1}{\delta}S(u,v)$, $v_t = \frac{1}{\delta}S(u,v)$, which are coupled through the stiff source term, $\frac{1}{\delta}S(u,v)$. Such systems arise as prototype models for combustion, adsorption, etc. Here we study the convergence of $(u,v) \equiv (u^\delta, v^\delta)$ to its equilibrium state, $(\bar{u}, \bar{v})$, governed by the limiting equations, $\bar{u}_t + \bar{v}_t + f(\bar{u})_x = 0$, $S(\bar{u}, \bar{v}) = 0$. In particular, we provide sharp convergence rate estimates as the relaxation parameter $\delta \downarrow 0$. The novelty of our approach is the use of a weak $W^{-1}(L^1)$-measure of the error, which allows us to obtain sharp error estimates. It is shown that the error consists of an initial contribution of size $||S(u_0^\delta, v_0^\delta)||_{L^1}$, together with accumulated relaxation error of order $\mathcal{O}(\delta)$. The sharpness of our results is found to be in complete agreement with the numerical experiments reported in [STW].

AMS(MOS) subject classification. Primary 35L65; Secondary 58G16, 58G18.

Keywords. Conservation laws, stiff source terms, relaxation, Lip$^+$-stability, convergence rate estimates.

1. Introduction. We are concerned with one-dimensional systems of conservation laws which are coupled through a stiff source term. The purpose of this paper is to study a convergence rate of such systems to their equilibrium solutions as the stiff relaxation parameter tends to zero.

Our system takes the form

$$u_t + f(u)_x = -\frac{1}{\delta}S(u,v), \tag{1.1}$$

$$v_t = \frac{1}{\delta}S(u,v), \tag{1.2}$$

where $\delta > 0$ is the small relaxation parameter. The stiff source term, $S(u,v)$, and the convective flux, $f(u)$, are assumed smooth functions. We consider the Cauchy problem associated with (1.1)-(1.2), subject to periodic or compactly supported initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \tag{1.3}$$

Here $u(x,t) := u^\delta(x,t)$, $v(x,t) := v^\delta(x,t)$ is the unique entropy solution of (1.1)-(1.3), which can be realized as the vanishing viscosity limit, $u^\delta = \lim_{\nu\downarrow 0} u^{\delta,\nu}$, $v^\delta = \lim_{\nu\downarrow 0} v^{\delta,\nu}$, where $(u^{\delta,\nu}, v^{\delta,\nu})$ is the solution of the regularized viscosity system,

$$u_t^{\delta,\nu} + f(u^{\delta,\nu})_x = -\frac{1}{\delta}S(u^{\delta,\nu}, v^{\delta,\nu}) + \nu u_{xx}^{\delta,\nu}, \tag{1.4}$$

$$v_t^{\delta,\nu} = \frac{1}{\delta}S(u^{\delta,\nu}, v^{\delta,\nu}). \tag{1.5}$$

By standard arguments (which we omit), this regularized system admits an unique global classical solution (see e.g., [HW],[Lu]).

* Research was supported by ONR Grant #N0014-91-J1076, NSF Grant #DMS91-03104 and GIF grant #318.195.06/93

† School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978 Israel; Email: kurganov@math.tau.ac.il

‡ Department of Mathematics, UCLA, Los-Angeles CA 90095; Email: tadmor@math.ucla.edu

Once we identify the unique entropy solution, (u^δ, v^δ), we seek its equilibrium state as $\delta \downarrow 0$, $(\bar{u}, \bar{v})$. Formally, our equilibrium solution is governed by the limit system obtained by letting $\delta \downarrow 0$ in (1.1)-(1.2),

$$(\bar{u}+\bar{v})_t + f(\bar{u})_x = 0, \tag{1.6}$$
$$S(\bar{u},\bar{v}) = 0. \tag{1.7}$$

To obtain the limiting equation (1.6), add (1.2) to (1.1); to obtain the constraint equation (1.7), multiply (1.2) by δ and pass to the formal limit as $\delta \to 0$.

The two main questions that we address in this paper are concerned with the convergence of the entropy solution (u^δ, v^δ) to its expected equilibrium state $(\bar{u}, \bar{v})$:
#1. *Convergence.* We prove the convergence to the expected limits,

$$\bar{u} = \lim_{\delta,\nu\downarrow 0} u^{\delta,\nu}, \qquad \bar{v} = \lim_{\delta,\nu\downarrow 0} v^{\delta,\nu}. \tag{1.8}$$

Moreover, we provide
#2. *Error Estimates.* We estimate the convergence rate as $\nu \to 0$, and in particular as $\delta \to 0$.

Assume $S_v \neq 0$ so that we can solve the constraint equation (1.7) and obtain its solution in the explicit form,

$$\bar{v} = v(\bar{u}). \tag{1.9}$$

Inserted into (1.6), we obtain that $\bar{u}$ is governed by the limiting equation,

$$[\bar{u} + v(\bar{u})]_t + f(\bar{u})_x = 0. \tag{1.10}$$

Equivalently, if we denote $\bar{w} = \bar{w}(\bar{u}) := \bar{u} + v(\bar{u})$, and let its inverse[1], $\bar{u} = \bar{u}(\bar{w})$, then we conclude that the limiting equation (1.10) can be rewritten as a single conservation law, expressed in terms of the combined flux $F(\bar{w}) := f(\bar{u}(\bar{w}))$,

$$\bar{w}_t + F(\bar{w})_x = 0. \tag{1.11}$$

We obtain our convergence results under the assumptions of convexity — of both $f(\cdot)$ and $F(\cdot)$, and the monotonicity of $S(u,v)$. In addition, we assume we start with "prepared" initial data, in the sense that $u_0 \equiv u_0^\delta$ and $v_0 \equiv v_0^\delta$ approach their equilibrium state (1.7), as $\delta \downarrow 0$, i.e.,

$$||S(u_0^\delta(x), v_0^\delta(x))||_{L^1(x)} \stackrel{\delta\to 0}{\longrightarrow} 0.$$

Specifically, we let $\epsilon = \epsilon(\delta) \downarrow 0$ denote the vanishing *initial error*

$$||S(u_0^\delta(x), v_0^\delta(x))||_{L^1(x)} \sim \epsilon(\delta) \downarrow 0. \tag{1.12}$$

Equipped with these assumptions, we formulate in §2 our main results, which we summarize here as

[1] The inverse exists since by our monotonicity assumption below, $v'(u) = -S_u/S_v > 0$.

THEOREM 1.1. *[**Main Theorem**]. Consider the system (1.3)-(1.5) subject to $W^2(L^1)$-"prepared" initial data, (1.12). Then $(u^{\delta,\nu}, v^{\delta,\nu})$ converges to $(\bar{u}, \bar{v})$ as $\nu \to 0$, $\delta \to 0$, and the following error estimate holds $\forall p$, $1 \le p \le \infty$,*

$$\|u^{\delta,\nu}(\cdot,t) - \bar{u}(\cdot,t)\|_{W^s(L^p(x))} \le Const_T \cdot (\epsilon(\delta) + \delta + \nu)^{\frac{1-sp}{2p}}, \quad -1 \le s \le \frac{1}{p}. \tag{1.13}$$

Thus, (1.13) reflects three sources for error accumulation: the initial error of size $\epsilon(\delta)$, the relaxation error of order δ, and the vanishing viscosity of order ν. For example, in the inviscid case ($\nu = 0$) and with "canonically prepared" initial data such that $\epsilon(\delta) \sim \delta$, we set $(s,p) = (0,1)$ in (1.13) to conclude an L^1-convergence rate of order $\mathcal{O}(\sqrt{\delta})$; in fact, in Corollary 2.3 below we extend this L^1-estimate to the v-variable, stating

$$\|u^\delta(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1} + \|v^\delta(\cdot,t) - \bar{v}(\cdot,t)\|_{L^1} = \mathcal{O}(\sqrt{\delta}). \tag{1.14}$$

The two-step proof of the main theorem is presented in §3 (– stability) and §4 (– consistency).

We close this introduction with three prototype examples.

Example 1. Combustion. We consider a combustion model, proposed by Majda in [Ma]. This model was consequently studied in e.g., [Le],[TY],[Lu]. It takes the form

$$\begin{aligned} u_t + f(u)_x &= \frac{1}{\delta}A(u)v + \nu u_{xx}, \\ v_t &= -\frac{1}{\delta}A(u)v. \end{aligned} \tag{1.15}$$

Here $u \equiv u^{\delta,\nu}$ is a lumped variable representing some features of density, velocity and temperature, while $v \equiv v^{\delta,\nu} \ge 0$ represents the mass fraction of unburnt gas in a simplified kinetics scheme, $\frac{1}{\delta}$ is the rate of reaction and the parameter $\nu > 0$ is a lumped parameter representing the effects of diffusion and heat conduction.

In this model, $S(u,v) = -A(u)v$ and our convexity and monotonicity assumptions (2.1)-(2.3) below hold, provided

$$A'(u) < 0, \ A(u) \ge \eta > 0; \quad f''(u) \ge \alpha > 0. \tag{1.16}$$

The limiting equation (1.10) in this example reads

$$\bar{u}_t + f(\bar{u})_x = 0,$$

and hence $u^{\delta,\nu} - \bar{u}$ satisfies the error estimate (1.13).

Example 2. Adsorption. We consider the following stiff system:

$$\begin{aligned} u_t + f(u)_x &= -\frac{1}{\delta}(A(u) - v), \\ v_t &= \frac{1}{\delta}(A(u) - v). \end{aligned} \tag{1.17}$$

In this example $u \equiv u^\delta$ denotes the density of some species contained in a fluid flowing through a fixed bed, and $v \equiv v^\delta$ denotes the density of the species adsorbed on the material in the bed, $\delta > 0$ is referred to as the relaxation time. Different forms of adsorption functions, $A(u)$, are discussed in [STW],[TW1],[TW2] and the references therein.

The source term associated with this adsorption model, $S(u,v) = A(u) - v$, yields a limiting equation of the form

$$[\bar{u} + A(\bar{u})]_t + f(\bar{u})_x = 0.$$

Under the monotonicity assumption and convexity condition (consult (2.1)-(2.3)),

$$A'(u) \geq 0, \quad \left[\frac{f'(u)}{1+A'(u)}\right]' \geq \alpha > 0, \tag{1.18}$$

we conclude the error estimate (1.13) with $\nu = 0$. In particular, for "canonically prepared" initial data such that $||A(u_0^\delta) - v_0^\delta||_{L^1} = \mathcal{O}(\delta)$, (1.14) yields a convergence rate of order $\mathcal{O}(\sqrt{\delta})$.

In this context it is interesting to contrast our above error estimates with those of [STW]. Schroll et. al. in [STW], studied the error estimates for the adsorption model (1.17) subject to "canonically prepared" initial data, $||A(u_0^\delta) - v_0^\delta||_{L^1} = \mathcal{O}(\delta)$, and concluded an L^1-convergence rate of order $\mathcal{O}(\delta^{\frac{1}{3}})$. Their reported numerical experiments, however, indicate a faster convergence rate of order $\mathcal{O}(\sqrt{\delta})$. Our results, consult e.g. (1.14), apply to their numerical experiments, and confirm this optimal $\mathcal{O}(\sqrt{\delta})$ convergence rate. It should be pointed that the $\mathcal{O}(\delta^{\frac{1}{3}})$ error estimate in [STW] was derived by interpolation between L^2- and L^1-error bounds. It is here that we take advantage of our *sharper* interpolation between the *weaker* $\mathcal{O}(\delta)$ *Lip'*- and the $\mathcal{O}(1)$ BV-bounds. This enables us to improve over [STW] in both — simplicity and generality, and conclude with the sharper estimate of order $\mathcal{O}(\sqrt{\delta})$.

Example 3. Relaxation. Let us consider the following semilinear stiff system (see e.g. [JX],[Li]):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + au_x &= \frac{1}{\delta}S(u,v), \end{aligned} \tag{1.19}$$

where $S(u,v) := f(u) - v$ and a is given positive number. The limiting equation, with $v(u) = f(u)$, is then

$$\bar{u}_t + f(\bar{u})_x = 0.$$

To study this system we rewrite it in the form of (1.1)-(1.2) by means of two changes of variables. First, we define the characteristic variables $w := \sqrt{a}\,u + v$, $z := \sqrt{a}\,u - v$. The system (1.19) then takes the form:

$$\begin{aligned} z_t - \sqrt{a}\,z_x &= -\frac{1}{\delta}S(z,w), \\ w_t + \sqrt{a}\,w_x &= \frac{1}{\delta}S(z,w), \end{aligned} \tag{1.20}$$

with $S(z,w) = S(u(z,w), v(z,w))$. Next, we make the second change of variables, $x' := x - \sqrt{a}\,t$, obtaining

$$\begin{aligned} z_t - 2\sqrt{a}\,z_{x'} &= -\frac{1}{\delta}S(z,w), \\ w_t &= \frac{1}{\delta}S(z,w). \end{aligned} \tag{1.21}$$

In this model, the flux is linear and hence our first convexity assumption, (2.2), holds. The second one, (2.3), is satisfied for convex f's. In addition, the monotonicity of S, $S_z \geq 0, S_w \leq -\eta < 0$, amounts (in terms of S_u and S_v) to the inequalities

$$S_v \leq -\eta < 0, \quad S_v\sqrt{a} \leq S_u \leq -S_v\sqrt{a}.$$

Thus, $S(u,v) = f(u) - v$ should satisfy Liu's *sub-characteristic* condition (e.g. [Li]),

$$-\sqrt{a} \leq f'(u) \leq \sqrt{a}.$$

In this case, our main theorem with $p = 1$ for example, yields

$$||u^\delta - \bar{u}||_{W^s(L^1)} = Const \cdot \left(||f(u_0^\delta) - v_0^\delta||_{L^1} + \delta\right)^{\frac{1-s}{2}}, \quad -1 \leq s \leq 1.$$

2. Statement of Main Results. We seek the behavior of the solution of regularized system (1.4)-(1.5) towards the limit solution as $\delta \to 0$, as well as $\nu \to 0$.
Throughout this section we make the following two main assumptions.
(1). Monotonicity: $S(u,v)$ is monotonic with respect to u and strictly monotonic with respect to v,

$$S_u(u,v) \geq 0, \quad S_v(u,v) \leq -\eta < 0. \tag{2.1}$$

(2). Convexity: $f(\cdot)$ is convex and $F(\cdot)$ is strictly convex function,

$$f''(u) \geq 0, \tag{2.2}$$

$$F''(w) \geq \alpha > 0 \iff \left(\frac{f'(\bar{u})}{1 + v'(\bar{u})}\right)' \geq \alpha > 0. \tag{2.3}$$

Remark. Our first assumption of monotonicity guarantees, by classical maximum principle, see e.g. [PW], the L^∞-boundedness of $(u^{\delta,\nu}, v^{\delta,\nu})$ (proof is left to the reader).

Equipped with the two assumptions above, we now turn to the main result of this paper. To this end, our error estimate is formulated in terms of the *weak Lip'*-(semi)norm, $||\cdot||_{Lip'}$[2]. As we shall see, such weak (semi)norm has the advantage of providing us with *sharp* error estimates, which in turn, will be converted into strong ones.

THEOREM 2.1. *Consider the system (1.3)-(1.5) subject to $W^2(L^1)$-"prepared" initial data, (1.12). Then $(u^{\delta,\epsilon}, v^{\delta,\epsilon})$ converges to $(\bar{u},\bar{v})$ as $\delta \to 0$, $\nu \to 0$, and the following error estimate holds,*

$$||u^{\delta,\nu}(\cdot,T) - \bar{u}(\cdot,T)||_{Lip'(x)} \leq Const_T \cdot (\epsilon(\delta) + \delta + \nu). \tag{2.4}$$

Let us consider the particular inviscid case, $\nu = 0$. Then the entropy solution of the stiff system (1.1)-(1.2), (u^δ, v^δ), converges as $\delta \to 0$ to its equilibrium solution, $(\bar{u},\bar{v})$, and we obtain the asserted convergence rate, in terms of the initial error $\epsilon(\delta)$, and the vanishing relaxation parameter δ,

$$||u^\delta(\cdot,T) - \bar{u}(\cdot,T)||_{Lip'(x)} \leq Const_T \cdot ((\epsilon(\delta) + \delta). \tag{2.5}$$

[2] Here, as usual, $||\phi||_{Lip'} := \sup_{\psi}[(\phi - \hat{\phi}_0, \psi)/||\psi||_{W^{1,1}}]$, $\hat{\phi}_0 := \int_{supp\phi} \phi$.

Remarks. 1. Our assumption of "prepared" initial data means that at the initial moment, $||S(u_0^\delta, v_0^\delta)||_{L^1} \stackrel{\delta\to 0}{\longrightarrow} 0$. In §4 we will show that, in fact, $||S(u^{\delta,\nu}, v^{\delta,\nu})||_{L^1} \stackrel{\delta\to 0}{\longrightarrow} 0$ for all $t > 0$.

2. What about "non-prepared" initial data? in this case the initial layer formed persists in time, i.e., the initial error propagates and prevents convergence of $u^{\delta,\nu}, v^{\delta,\nu}$ to their equilibrium state.

The proof of the main theorem will be given in §3 and §4. To obtain this result we utilize the framework of Tadmor and Nessyahu [[Ta],[NT]]. To this end, we need the two ingredients of consistency and stability. Here, *consistency* – evaluated in the *Lip*′-norm, measures by how much the approximate pair $(u^{\delta,\nu}, v(u^{\delta,\nu}))$ fails to satisfy the limiting equation (1.10); and *stability* requires the Lip^+-stability[3] of $u^{\delta,\nu}$, that is, we seek a One-Sided Lipschitz Continuity (OSLC) of the viscosity solution, $u^{\delta,\nu}$,

$$\sup_x [u_x^{\delta,\nu}(x,t)]_+ \le C_t \cdot \sup_x [u_x^{\delta,\nu}(x,0)]_+. \tag{2.6}$$

By interpolation between the (weak) *Lip*′-error estimate (2.4) and the (strong) *BV*-boundedness of the error (— which follows from the Lip^+-boundedness due to (2.6)), we are able to convert the weak error estimate stated in Theorem 2.1 into a *strong* one. As in [NT], we conclude

COROLLARY 2.2. *[**Global estimate**]. Consider the inviscid problem (1.1)-(1.3),(1.12). Then the following convergence rate estimate holds,*

$$||u^\delta(x,T) - \bar{u}(x,T)||_{L^p} \le Const_T \cdot (\epsilon(\delta) + \delta)^{\frac{1}{2p}}, \qquad 1 \le p \le \infty. \tag{2.7}$$

Remark. The above-mentioned L^p-estimates in (2.7) are, in fact, particular cases of the more general error estimate in the $W^s(L^p)$-norm

$$||u^\delta(x,T) - \bar{u}(x,T)||_{W^s(L^p)} \le Const_T \cdot (\epsilon(\delta) + \delta)^{\frac{1-sp}{2p}}, \qquad -1 \le s \le \frac{1}{p}. \tag{2.8}$$

The special cases, $(s,p) = (-1,1)$ and $s = 0$ correspond, respectively, to the weak *Lip*′-estimate (Theorem 2.1), and the global L^p-estimate (Corollary 2.2).

Taking $p = 1$ in (2.7) we obtain, in particular, the L^1-error estimate which reads

$$||u^\delta(x,T) - \bar{u}(x,T)||_{L^1} \le Const_T \cdot \sqrt{\epsilon(\delta) + \delta}. \tag{2.9}$$

In this L^1-framework, we are able to extend the last estimate and obtain the same $\mathcal{O}(\sqrt{\epsilon(\delta)+\delta})$ convergence rate of v^δ towards $\bar{v}$. This brings us to

COROLLARY 2.3. *[L^1**-error estimate**]. Consider the system (1.1)-(1.3) subject to "prepared" initial data, (1.12). Then we have*

$$||u^\delta(x,T) - \bar{u}(x,T)||_{L^1} + ||v^\delta(x,T) - \bar{v}(x,T)||_{L^1} \le Const_T \cdot \sqrt{\epsilon(\delta) + \delta}. \tag{2.10}$$

[3] Here $||\phi||_{Lip^+} := ess \sup_{x\neq y} \left[\frac{\phi(x)-\phi(y)}{x-y}\right]_+$, where, as usual, $(\cdot)_+$ denotes the "positive part of". For convenience we shall use the equivalent definition of the Lip^+ norm — $||\phi||_{Lip^+} := \sup_x [\phi'(x)]_+$, where the derivative of ϕ is taken in the distribution sense.

In particular, for "canonically prepared" initial data, $||S(u_0^\delta, v_0^\delta)||_{L^1} = \epsilon(\delta) \sim \delta$, *we obtain a convergence rate of order* $\sqrt{\delta}$,

$$||u^\delta(x,T) - \bar{u}(x,T)||_{L^1} + ||v^\delta(x,T) - \bar{v}(x,T)||_{L^1} \leq Const_T \cdot \sqrt{\delta}. \tag{2.11}$$

Proof. We first note that due to the strict monotonicity of $S(u,v)$ w.r.t. its second argument and the L^∞-bound of $u^\delta, v^\delta, \bar{u}$ and $\bar{v}$, we have

$$\begin{aligned} |v^\delta - \bar{v}| &= |v^\delta - v(u^\delta) + v(u^\delta) - \bar{v}| \leq |v^\delta - v(v^\delta)| + |v(v^\delta) - \bar{v}| = \\ & |v'(\tilde{u})| \cdot |u^\delta - \bar{u}| + \left| \frac{S(u^\delta, v^\delta) - S(u^\delta, v(u^\delta))}{S_v(u^\delta, \tilde{v})} \right| \sim |u^\delta - \bar{u}| + |S(u^\delta, v^\delta)|. \end{aligned}$$

Here $\tilde{u}$ and $\tilde{v}$ are appropriate midvalues, $\tilde{u} = \theta_1 u^\delta + (1-\theta_1)\bar{u}$, $\tilde{v} = \theta_2 v^\delta + (1-\theta_2)\bar{v}$. And we now obtain the desired estimate, (2.10),

$$\begin{aligned} ||v^\delta(x,T) - \bar{v}(x,T)||_{L^1} &\leq Const_T \cdot (||u^\delta(x,T) - \bar{u}(x,T)||_{L^1} + \\ & +||S(u^\delta(x,T), v^\delta(x,T)||_{L^1}) = \\ &= \mathcal{O}(\sqrt{\epsilon(\delta) + \delta}) + \mathcal{O}(\epsilon(\delta) + \delta) = \mathcal{O}(\sqrt{\epsilon(\delta) + \delta}). \end{aligned} \tag{2.12}$$

Indeed, the first $\mathcal{O}(\sqrt{\bullet})$-upperbound on the right is due to (2.9), the second upperbound, $||S(u^\delta(x,T), v^\delta(x,T))||_{L^1} = \mathcal{O}(\epsilon(\delta) + \delta)$, is outlined in §4 below. ∎

Finally, arguing along the lines of [3; Corollary 2.4], we also obtain the *pointwise* convergence towards the equilibrium solution away from discontinuities.

COROLLARY 2.4. ***[Local estimate]****. Consider the inviscid problem (1.1)-(1.3),(1.12). Then the following estimate holds,*

$$|u^\delta(x,T) - \bar{u}(x,T)| \leq Const_{x,T} \cdot (\epsilon(\delta) + \delta)^{\frac{1}{3}} \ . \tag{2.13}$$

Here, $Const_{x,T}$ *is a constant which measures the local smoothness of* $u(\cdot, T)$ *in the small neighborhood of* x, $Const_{x,T} \sim 1 + \max_{|y-x| < \sqrt[3]{\delta}} |\bar{u}_x(y,T)|$.

3. ***Lip***$^+$**- stability estimate.** We now turn to the proof of our main theorem. We begin with the *Lip*$^+$-stability of the solution of (1.4)-(1.5).

ASSERTION 3.1. *Consider the system (1.4),(1.5) subject to Lip*$^+$*-bounded initial data (1.3). Then there exist a constant (which may depend on the initial data) such that*

$$||u^{\delta,\nu}(\cdot, T)||_{Lip^+(x)} \leq Const. \tag{3.1}$$

Remark. We highlight the fact that our proof below is independent of whether the initial data are "prepared" or not.

Proof. The proof is based on the maximum principle for $(u_x^{\delta,\nu})_+$.
Differentiation of (1.4) and (1.5) with respect to x implies

$$(u_x^{\delta,\nu})_t + f''(u^{\delta,\nu})(u_x^{\delta,\nu})^2 + f'(u^{\delta,\nu})(u_x^{\delta,\nu})_x = -\frac{1}{\delta}[S_u u_x^{\delta,\nu} + S_v v_x^{\delta,\nu}] + \nu(u_x^{\delta,\nu})_{xx}, \tag{3.2}$$

$$(v_x^{\delta,\nu})_t = \frac{1}{\delta}[S_u u_x^{\delta,\nu} + S_v v_x^{\delta,\nu}]. \tag{3.3}$$

We now multiply (3.2) by $\frac{1+sgn(u_x^{\delta,\nu})}{2}$; using the monotonicity of $S(u,v)$ and convexity of $f(u)$ we obtain the following inequalities:

$$\begin{aligned} [(u_x^{\delta,\nu})_+]_t \;+\; & f'(u^{\delta,\nu})\cdot[(u_x^{\delta,\nu})_+]_x \le \\ & -\frac{1}{\delta}\left[S_u(u_x^{\delta,\nu})_+ + S_v v_x^{\delta,\nu}\left(\frac{1+sgn(u_x^{\delta,\nu})}{2}\right)\right] + \nu[(u_x^{\delta,\nu})_+]_{xx}, \end{aligned} \tag{3.4}$$

$$(v_x^{\delta,\nu})_t \;\le\; \frac{1}{\delta}[S_u(u_x^{\delta,\nu})_+ + S_v v_x^{\delta,\nu}]. \tag{3.5}$$

Solving the second inequality we find (with $S_v(\tau) := S_v(x,\tau) \equiv S_v(u^{\delta,\nu}(x,\tau), v^{\delta,\nu}(x,\tau))$ and $B(t) := \int_0^t S_v(\tau)d\tau$) that

$$v_x^{\delta,\nu}(t) \le e^{\frac{B(t)}{\delta}} v_x^{\delta,\nu}(0) + \frac{1}{\delta}\int_0^t e^{\frac{B(t)-B(\tau)}{\delta}} S_u(\tau)(u_x^{\delta,\nu}(\tau))_+ d\tau. \tag{3.6}$$

Plugging this into (3.4) and denoting $m(t) = \max_x (u_x^{\delta,\nu}(x,t))_+$, we end up with

$$\dot m(t) \le -\frac{S_u(t)}{\delta} m(t) - \frac{S_v(t)}{\delta} e^{\frac{B(t)}{\delta}}(v_x^{\delta,\nu}(0))_+ - \frac{S_v(t)}{\delta^2}\int_0^t e^{\frac{B(t)-B(\tau)}{\delta}} S_u(\tau)m(\tau)d\tau. \tag{3.7}$$

The first and the third terms in the RHS of (3.7) add up to a perfect derivative, so that

$$\dot m(t) \le \left(-e^{\frac{B(t)}{\delta}}(v_x^{\delta,\nu}(0))_+\right)_t - \frac{1}{\delta}\left(\int_0^t e^{\frac{B(t)-B(\tau)}{\delta}} S_u(\tau)m(\tau)d\tau\right)_t. \tag{3.8}$$

Integration of (3.8) over $(0,T)$ yields

$$m(T) \le m(0) + (v_x^{\delta,\nu}(0))_+[1 - e^{\frac{B(T)}{\delta}}] - \frac{1}{\delta}\int_0^T e^{\frac{B(T)-B(\tau)}{\delta}} S_u(\tau)m(\tau)d\tau. \tag{3.9}$$

In view of the positivity of S_u we obtain that

$$(u_x^{\delta,\nu}(x,T))_+ \le (u_x^{\delta,\nu}(x,0))_+ + (v_x^{\delta,\nu}(x,0))_+;$$

and the assertion follows with $Const = ||u^{\delta,\nu}(\cdot,0)||_{Lip^+(x)} + ||v^{\delta,\nu}(\cdot,0)||_{Lip^+(x)}$. ■

We close this section by noting that the proof of Assertion 3.1 is based on the straightforward, formal maximum principle for the positive part of $u^{\delta,\nu}$; alternatively, it could be justified, for example, by L^p iterations in (3.4).

4. *Lip'*- consistency and proof of the main result. In this section we prove the promised error estimate (2.4) in the *Lip'*-norm. According to the results of [Ta],[NT], the error $||u^{\delta,\nu} - \bar u||_{Lip'}$ is upper bounded by the truncation error,

$$\left|\left|[u^{\delta,\nu} + v(u^{\delta,\nu})]_t + f(u^{\delta,\nu})_x\right|\right|_{Lip'(x,t)}. \tag{4.1}$$

This quantity measures by how much $u^{\delta,\nu}$ fails to satisfy the limiting equation, (1.10). To complete this proof we have to show, therefore, that the truncation error is of order $\mathcal{O}(\epsilon(\delta) + \delta + \nu)$. We proceed as follows.

Adding the two components of the regularized system (1.5) to (1.4), we obtain that

$$u_t^{\delta,\nu} + v_t^{\delta,\nu} + f(u^{\delta,\nu})_x = \nu u_{xx}^{\delta,\nu},$$

which we rewrite as

$$[u^{\delta,\nu} + v(u^{\delta,\nu})]_t + f(u^{\delta,\nu})_x = u_t^{\delta,\nu} + v_t^{\delta,\nu} + f(u^{\delta,\nu})_x + [v(u^{\delta,\nu}) - v^{\delta,\nu}]_t = \nu u_{xx}^{\delta,\nu} + [v(u^{\delta,\nu}) - v^{\delta,\nu}]_t.$$

Thus, the RHS of the last equality tells us that the truncation error in (4.1) does not exceed

$$\begin{aligned} ||\nu u_{xx}^{\delta,\nu} + [v(u^{\delta,\nu}) - v^{\delta,\nu}]_t||_{Lip'(x,t)} &\leq \\ &\leq Const_T \cdot \Big[\nu||u_x^{\delta,\nu}||_{L^1(x,t)} + ||v(u^{\delta,\nu}) - v^{\delta,\nu}||_{L^1(x,t)}\Big] \\ &=: Const_T \cdot \Big[\, I + II \,\Big]. \end{aligned} \tag{4.2}$$

We proceed with estimating the two terms on the right. First, since $u^{\delta,\nu}$ is Lip^+-bounded, (3.1), it has a bounded variation, $||u_x^{\delta,\nu}||_{L^1(x,t)} \leq C_K$ (with C_K may depend on the Lip^+-bound, K, and the finite support of $u^{\delta,\nu}$), and therefore $I \leq \mathcal{O}(\nu)$. Next, we find that the second term, II, is of order

$$II \equiv ||v(u^{\delta,\nu}) - v^{\delta,\nu}||_{L^1(x,t)} \sim ||S(u^{\delta,\nu}, v^{\delta,\nu})||_{L^1(x,t)}\,. \tag{4.3}$$

Indeed, since $0 < \eta \leq -S_v \leq Const,$ we have

$$\frac{1}{\eta} \leq \frac{|v(u^{\delta,\nu}) - v^{\delta,\nu}|}{|S(u^{\delta,\nu}, v(u^{\delta,\nu})) - S(u^{\delta,\nu}, v^{\delta,\nu})|} \leq Const,$$

and hence, $|v(u^{\delta,\nu}) - v^{\delta,\nu}| \sim |S(u^{\delta,\nu}, v(u^{\delta,\nu})) - S(u^{\delta,\nu}, v^{\delta,\nu})| = |S(u^{\delta,\nu}, v^{\delta,\nu})|$, and (4.3) follows. Returning to (4.2) we find that

$$\begin{aligned} ||\nu u_{xx}^{\delta,\nu} + [v(u^{\delta,\nu}) - v^{\delta,\nu}]_t||_{Lip'(x,t)} &\leq Const_T \cdot \Big[\, I + II \,\Big] \leq \\ &\leq Const_T \cdot \Big[\nu + ||S(u^{\delta,\nu}, v^{\delta,\nu})||_{L^1(x,t)}\Big]. \end{aligned} \tag{4.4}$$

To conclude with the promised $\mathcal{O}(\epsilon(\delta)+\delta+\nu)$-bound, it remains to prove that $||S(u^{\delta,\nu}, v^{\delta,\nu})||_{L^1(x,t)}$ — or utilizing (4.3), that $\delta||v_t^{\delta,\nu}(\cdot,t)||_{L^1(x)}$, is of order $\mathcal{O}(\epsilon(\delta)+\delta)$,

$$||S(u^{\delta,\nu}(\cdot,t), v^{\delta,\nu}(\cdot,t))||_{L^1(x)} \equiv \delta||v_t^{\delta,\nu}(\cdot,t)||_{L^1(x)} = \mathcal{O}(\epsilon(\delta)+\delta). \tag{4.5}$$

To achieve such an estimate, we differentiate (1.4) with respect to t, multiply by $sgn(u_t^{\delta,\nu})$, and obtain

$$\begin{aligned} |u_t^{\delta,\nu}|_t + (f'(u^{\delta,\nu})u_t^{\delta,\nu})_x sgn(u_t^{\delta,\nu}) &= -\frac{1}{\delta}\Big(S_u|u_t^{\delta,\nu}| + S_v|v_t^{\delta,\nu}|sgn(u_t^{\delta,\nu})sgn(v_t^{\delta,\nu})\Big) + \\ &\quad + \epsilon(u_t^{\delta,\nu})_{xx} sgn(u_t^{\delta,\nu}). \end{aligned} \tag{4.6}$$

The same treatment of equation (1.5) yields

$$|v_t^{\delta,\nu}|_t = \frac{1}{\delta}\Big(S_u|u_t^{\delta,\nu}|sgn(u_t^{\delta,\nu})sgn(v_t^{\delta,\nu}) + S_v|v_t^{\delta,\nu}|\Big). \tag{4.7}$$

Next, we integrate these equations with respect to x,

$$\frac{d}{dt}||u_t^{\delta,\nu}||_{L^1(x)} \leq -\frac{1}{\delta}\left(\int\limits_x S_u|u_t^{\delta,\nu}|dx + \int\limits_x S_v|v_t^{\delta,\nu}|sgn(u_t^{\delta,\nu})sgn(v_t^{\delta,\nu})dx\right), \tag{4.8}$$

$$\frac{d}{dt}||v_t^{\delta,\nu}||_{L^1(x)} \leq \frac{1}{\delta}\left(\int\limits_x S_u|u_t^{\delta,\nu}|sgn(u_t^{\delta,\nu})sgn(v_t^{\delta,\nu})dx + \int\limits_x S_v|v_t^{\delta,\nu}|dx\right). \tag{4.9}$$

Finally, we add up (4.8) and (4.9), obtaining

$$\frac{d}{dt}\left[||u_t^{\delta,\nu}||_{L^1(x)} + ||v_t^{\delta,\nu}||_{L^1(x)}\right] \leq \frac{1}{\delta}\left[\int\limits_x S_u|u_t^{\delta,\nu}|\Big(sgn(u_t^{\delta,\nu})sgn(v_t^{\delta,\nu}) - 1\Big)dx + \right.$$
$$\left. + \int\limits_x S_v|v_t^{\delta,\nu}|\Big(1 - sgn(u_t^{\delta,\nu})sgn(v_t^{\delta,\nu})\Big)dx\right] \leq 0.$$

It follows that

$$||u_t^{\delta,\nu}(\cdot,t)||_{L^1(x)} + ||v_t^{\delta,\nu}(\cdot,t)||_{L^1(x)} \leq ||u_t^{\delta,\nu}(\cdot,0)||_{L^1(x)} + ||v_t^{\delta,\nu}(\cdot,0)||_{L^1(x)}, \tag{4.10}$$

and in particular,

$$\delta||v_t^{\delta,\nu}(\cdot,t)||_{L^1(x)} \leq \delta||u_t^{\delta,\nu}(\cdot,0)||_{L^1(x)} + \delta||v_t^{\delta,\nu}(\cdot,0)||_{L^1(x)}.$$

To conclude this proof, we show that the upper bound on the right does not exceed the promised $\mathcal{O}(\epsilon(\delta)+\delta)$. Indeed, by equations (1.4),(1.5), $u_t^{\delta,\nu} = -v_t^{\delta,\nu} - f(u^{\delta,\nu})_x + \nu u_{xx}^{\delta,\nu}$, and hence

$$\delta||u_t^{\delta,\nu}(\cdot,0)||_{L^1(x)} + \delta||v_t^{\delta,\nu}(\cdot,0)||_{L^1(x)} \leq 2||S(u^{\delta,\nu}(\cdot,0), v^{\delta,\nu}(\cdot,0))||_{L^1(x)} +$$
$$+\delta||f(u^{\delta,\nu}(\cdot,0))_x||_{L^1(x)} + \delta\nu||u_{xx}^{\delta,\nu}(\cdot,0)||_{L^1(x)}.$$

The three terms on the right are upper-bounded by $\mathcal{O}(\epsilon(\delta)+\delta)$, since, by our assumption of the "prepared" initial data, (1.12), $||S(u^{\delta,\nu}(\cdot,0), v^{\delta,\nu}(\cdot,0))||_{L^1(x)} = \mathcal{O}(\epsilon(\delta))$; the BV-boundedness of $u^{\delta,\nu}$ yields $\delta||f(u^{\delta,\nu}(\cdot,0))_x||_{L^1(x)} = \mathcal{O}(\delta)$, and finally, since the initial data are assumed to be in $W^2(L^1)$, then $\delta\nu||u_{xx}^{\delta,\nu}(\cdot,0)||_{L^1(x)} = \mathcal{O}(\delta\nu) << \mathcal{O}(\delta)$. This completes the proof of Theorem 2.1. ■

Remark. We close by noting that the $W^2(L^1)$ regularity of initial data used in the last stage of the proof can be relaxed. In fact, it is sufficient to assume $||u_{0x}||_{L^1} + \nu||u_{0xx}||_{L^1} \leq Const.$

REFERENCES

[HW] JIAXIN HU AND JINGHUA WANG, *The Cauchy problem for a special system of quasilinear equations*, J. PDEs, 4 (1991), 33–44.

[JX] S. JIN AND Z. XIN, *The Relaxation Schemes for Systems of Conservation Laws in Arbitrary Space Dimensions*, Comm. Pure Appl. Math. 48 (1995), 235–276.

[Le] A. LEVY, *On Majda's model for dynamic combustion*, Commun. In PDEs, 17(3 & 4) (1992), 657–698.

[Li] T. P. LIU, *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. 108 (1987), 153–175.

[Lu] YUNG-GUANG LU, *Cauchy problem for an extended model of combustion*, Proc. Royal Soc. Ed. 120A (1992), 349–360.

[Ma] A. MAJDA, *A qualitative model for dynamic combustion*, SIAM J.Appl. Math., (1981), 70–93.

[NT] H. NESSYAHU AND E. TADMOR, *The convergence rate of approximate solutions for nonlinear scalar conservation laws*, SIAM J. Numer. Anal., 29 (1992), 1505–1519.

[PW] PROTTER AND S. WEINBERGER, *Maximum principles in DEs*, Springer Verlag, 1967.

[STW] H. J. SCHROLL, A. TVEITO AND R. WINTHER, *An error bound for a finite difference scheme applied to a stiff system of conservation laws*, preprint (1994).

[Ta] E. TADMOR, *Local error estimates for discontinuous solutions of nonlinear hyperbolic equations*, SIAM J. Numer. Anal., 28 (1991), 811–906.

[TW1] A. TVEITO AND R. WINTHER, *On the rate of convergence to equilibrium for a system of conservation laws including a relaxation term*, preprint (1994).

[TW2] A. TVEITO AND R. WINTHER, *An error estimate for a finite difference scheme approximating a hyperbolic system of conservation laws*, SIAM J. Numer. Anal., 30 (1993), 401–424.

[TY] Z. TENG AND L. YING, *Existence, uniqueness and convergence as vanishing viscosity for a reaction-diffusion-convection system*, Acta Math. Sinica (N. S.), 5 (1989), 114–135.