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Abstract. A combustion model which captures the interactions among nonlinear convection, chemical reaction and radiative heat transfer is studied. New phenomena are found with radiative heat transfer present. In particular, there is a weak detonation solution for each radiative heat loss coeffcient. The speed of the weak detonation wave decreases as the heat loss coefficient increases and the detonation wave does not exist when the heat loss coefficient exceeds a critical value, as expected physically.

We study the time-asymptotic limit of solutions of initial value problem for the same problem. We prove that the solution exists globally and the solution converges uniformly, away from the shock, to a shifted traveling wave solution as $t \to +\infty$ for certian 'compact support' initial data.

Numerical results showing convergence are presented at the end.

Key words. Weak detonation, shock wave, traveling wave, asymptotic behavior.

AMS(MOS) subject classifications. 35L65, 35B40, 35B50, 76L05, 76J10.

1 Introduction

We consider the combustion problem

$$u_t + (\frac{1}{2}u^2 - qz)_x = -\sigma(u - u_0) \tag{1}$$

$$z_x = K\varphi(u)z^\alpha, \tag{2}$$

$\varphi(u)$ has the ignition form

$$\varphi(u) = \begin{cases} 1 & u \geq u_i \\ 0 & u < u_i \end{cases}$$

where $u_i > u_0 > 0$, $0 < \alpha < 1$, $\sigma, K, q > 0$ are constants and u and z are functions of (x,t). This model describes the motion of the combustible gas in a tube. Emphasis is on the interactions among nonlinearity, chemical reaction and heat radiation during the combustion process.

When there is no radiative heat transfer, this model was proposed and studied by Majda and Rosales [8]. They computed the detonation waves predicted by the analogue of the Z-N-D (Zeldovich-von-Neumann-Doring) theory [2]: a detonation wave traveling at speed D has the internal structure of an ordinary percursor fluid dynamic shock wave traveling at speed D followed by a reaction-zone. In particular, by assuming the Z-N-D hypothesis, they found that weak detonations were impossible, also see Courant and Friedrichs [3], only strong and CJ (Chapman-Jouguet) detonation waves occurred.

Radiative heat transfer is important physically. It could cause up to 30% temperature loss during the chemical reaction [9]. Suggested by Chen [1], we include radiative heat transfer in the present model.

In Section 2, we prove that there is a weak detonation solution for (1) (2). Furthermore, the propagating speed D decreases as the heat loss coefficient increases and the detonation wave does not exist when the heat loss coefficient exceeds a critical value, which is the case physically [9]. A weak detonation is characterized by that the speed of the characteristic behind the weak detonation is subsonic [2]. It occurs because the radiation effect.

Natural questions come up: What is the stability property of the weak detonation waves? Or, if we take the approach of large-time behavior of solutions of initial value problem, what is the asymptotic limit (in case it exists)? Since (1) is no longer a conservation law, how to determine the phase shift?

In this paper, we study the initial value problem (1) (2) with the following data:

$$u(x,0) = \begin{cases} u_1(x) & -d \leq x \leq 0 \\ u_0 & \text{elsewhere} \end{cases} \tag{3}$$

$$z(+\infty, t) = 1 \tag{4}$$

where $u_1(x)$ satisfies certain conditions to be specified later.

We prove global existence of solution to the problem (1) (2) (3) (4) and that the solution's convergence to a shifted traveling wave solution. We prove that the energy is conserved 'asymptotically'. A compactness argument proves that such a shift exists.

In Section 2 we prove existence of the traveling wave solution for (1) and (2). Section 3 is the proof of the global existence for the initial value problem. Convergence is proved in Section 4. We show numerical results in Section 5.

2 Traveling Wave Solutions

A traveling wave solution is a solution of the following form

$$(u(x,t), z(x,t)) = (\psi(x-Dt), z(x-Dt))$$

where D is the speed of the traveling wave solution. Let $\xi = x - Dt$ and $(u(x,t), z(x,t)) = (\psi(\xi), z(\xi))$. Then $(\psi, z)(\xi)$ solves the following ordinary differential equations

$$-D\psi' + \psi\psi' = qz' - \sigma(\psi - u_0) \tag{5}$$

$$z' = K\varphi(\psi)z^{\alpha}. \tag{6}$$

The boundary conditions are

$$\lim_{\xi\to-\infty} (\psi, z)(\xi) = (u_0, 0) \tag{7}$$

$$(\psi, z)(\xi) = (u_0, 1),\ \xi > 0. \tag{8}$$

The result about traveling wave is the following.

Theorem 2.1 *There is a unique solution ψ, Z to problem (5) (6) (7) (8). The propagating speed D satisfies*

$$D = \frac{u_l + u_0}{2} + \frac{q}{u_l - u_0} - \sigma\frac{\int_{-l}^{0}(\psi(\xi) - u_0)d\xi}{u_l - u_0}$$

and

$$\int_{-\infty}^{+\infty} (\psi(\xi) - u_0)d\xi = \frac{q}{\sigma}$$

provided

$$u_0 + \frac{qK}{2\sigma} > D \geq u_0 + \frac{2qK}{3\sigma + \sqrt{\sigma^2 + 4qK^2\alpha}}$$

and $\alpha > \frac{1}{2}$.

Furthermore, the solution satisfies

$$\psi' \geq 0, \quad \psi'' \geq 0$$

except at the shock discontinuity. The structure of the traveling wave solution is a weak detonation wave followed by a non-reaction zone.

Proof. Let $\xi = 0$ be the shock wave location.

Solving Z from (6) in the reaction-zone where $u > u_i$, or $\phi(u) = 1$, we have

$$Z(\xi) = (K(1-\alpha)\xi + 1)^{\frac{1}{1-\alpha}}, \quad -l \leq \xi \leq 0$$

where $l = \dfrac{1}{K(1-\alpha)}$ is the length of the reaction-zone.

Combining with the boundary conditions (7) (8), we have

$$Z(\xi) = \begin{cases} 1 & x \leq 0 \\ (K(1-\alpha)\xi + 1)^{\frac{1}{1-\alpha}} & -l < \xi < 0 \\ 0 & \xi \leq -l. \end{cases} \tag{9}$$

Plug (9) into (5) to have the equation for ψ

$$(\psi - D)\psi' + \sigma(\psi - u_0) = \begin{cases} 0 & \xi \geq 0 \\ qK(K(1-\alpha)\xi + 1)^{\frac{\alpha}{1-\alpha}} & -l \leq \xi \leq 0 \\ 0 & \xi < -l. \end{cases} \tag{10}$$

The solution consists of three parts: each part is on one of $(-\infty, -l)$, $(-l, 0)$ and $(0, +\infty)$. Physically, they correspond to the non-reaction zone after the reaction-zone, reaction-zone and the non-reaction zone before reaction-zone.

Integrating equation (10) over $(-l, +\infty)$ and using (8) and (9), we have

$$D = \frac{u_l + u_0}{2} + \frac{q}{u_l - u_0} - \sigma \frac{\int_{-l}^{0} (\psi(\xi) - u_0) d\xi}{u_l - u_0}, \tag{11}$$

which is one relation between D and $u_l = \psi(-l)$.

Integrating equation (10) over $(-\infty, +\infty)$ and using (7), (8) and (9), we have

$$\frac{d}{dt} \int_{-\infty}^{+\infty} (\psi(x - Dt) - u_0) dx - q = -\sigma \int_{-\infty}^{+\infty} (\psi(x - Dt) - u_0) dx.$$

Noticing that $\int_{-\infty}^{+\infty}(\psi(x-Dt)-u_0)dx$ is independent of t, we have

$$\int_{-\infty}^{+\infty}(\psi(x)-u_0)dx=\frac{q}{\sigma}. \tag{12}$$

Condition (12) together with (11) determine D and u_l.

It can be proved easily from (11) that $D > u_0$.

In order to construct a nondecreasing ψ, we require $u_0 \le \psi(\xi) \le D$ for $\xi \le -l$. In particular

$$u_0 < u_l < D, \tag{13}$$

i.e., the speed at the end of the reaction-zone is subsonic. Condition (13) determines the nature of the solution, i.e., it must be a weak detonation wave.

On the other hand, there is a strong shock discontinuity at $\xi = 0$. Let

$$\psi(0-) = u_r. \tag{14}$$

Then

$$u_r = 2D - u_0 > D.$$

Hence, there is a point $-l < \xi_0 < 0$ such that $\psi(\xi_0) = D$. Equation (10) becomes degenerate at $\xi = \xi_0$. We will show that the solution exists and it is monotone and convex despite the degeneracy of (10) at $\xi = \xi_0$.

The proof of the theorem will be finishied following the lemmas. ■

Lemma 2.2 *The solution to problem (10) (14) is nondecreasing on* $[-l,\ \xi_0]$*, i.e.,*

$$\psi'(\xi) \ge 0, \quad -l \le \xi \le \xi_0.$$

Proof.: For $\delta > 0$ small, from equation (10) we see that

$$\psi'(\xi) > 0, \quad -l \le \xi \le -l+\delta.$$

Assume that there exist $-l \le \xi_1 < \xi_2 \le \xi_0$, such that $\psi(\xi_1) > \psi(\xi_2)$.

And yet, $\psi(\xi_2) < D = \psi(\xi_0)$.

Therefore there exist $-l < \eta_1 \le \xi_1 < \eta_2 \le \xi_2$, local maximum and minimum points respectively, satisfying

$$\begin{aligned}\psi'(\eta_1) &= \psi'(\eta_2) = 0\\ \psi(\eta_1) &> \psi(\eta_2).\end{aligned}$$

Evaluating equation (10) at $\xi = \eta_1$ and $\xi = \eta_2$, we get a contradiction since the right hand side is nondecreasing in ξ. ■

Lemma 2.3 *The solution to problem (10) (14) is convex on* $[-l,\ \xi_0]$, *i.e.,*

$$\psi''(\xi) \geq 0, \quad -l \leq \xi \leq \xi_0$$

provided $\alpha > \frac{1}{2}$.

Proof.: Differentiating (10), we have

$$(\psi - D)\psi'' + \sigma\psi' + \psi'^2 = qK^2\alpha(k(1-\alpha)\xi + 1)^{\frac{2\alpha-1}{1-\alpha}}.$$

Noticing that the right hand side is a nondecreasing function when $\alpha > \frac{1}{2}$ and $\psi''(-l) > 0$, the lemma is proved by the same argument used in the previous lemma. Details are omitted. ■

For solution of problem (10) (14) on $[\xi_0,\ 0]$, we prove the similar results.

Lemma 2.4 *The solution to problem (10) (14) is nondecreasing on* $[\xi_0,\ 0]$ *provided*

$$D < u_0 + \frac{qK}{2\sigma}.$$

Lemma 2.5 *The solution to problem (10) (14) is convex on* $[\xi_0,\ 0]$ *provided*

$$D \geq u_0 + \frac{2qK}{3\sigma + \sqrt{\sigma^2 + 4qK^2\alpha}}.$$

Since $\psi(\xi)$ is convex on both sides of ξ_0, $\psi(\xi)$ is differentiable at ξ_0 from left and right. From (10), we get that the left and the right differential are the same. So $\psi(\xi)$ is differentiable at ξ_0. Hence ξ_0 is an ordinary point for equation (10).

The proof of the uniqueness follows from the following lemma.

Lemma 2.6 *If* ψ_1 *and* ψ_2 *are solutions to the following problems*

$$(\psi_1 - D)\psi_1' + \sigma(\psi_1 - u_0) = qK(K(1-\alpha)\xi + 1)^{\frac{\alpha}{1-\alpha}}, \quad 0 \geq \xi \geq -l \qquad (15)$$

$$\psi_1(0) = u_r \qquad (16)$$

$$(\psi_2 - D)\psi_2' + \sigma(\psi_2 - u_0) = qK(K(1-\alpha)\xi + 1)^{\frac{\alpha}{1-\alpha}}, \quad 0 \geq \xi \geq -l \qquad (17)$$

$$\psi_2(0) = u_r \qquad (18)$$

respectively, then

$$\psi_1(\xi) = \psi_2(\xi), \quad 0 \geq \xi \geq -l.$$

Proof. Subtracting (17) from (15), we have

$$(\psi_1 - D)(\psi_1 - \psi_2)' + (\psi_2' + \sigma)(\psi_1 - \psi_2) = 0. \tag{19}$$

Let us prove $\psi_1 - \psi_2 \geq 0$ first.

If not, then $\psi_1 - \psi_2$ attains its negative minimum at some point ξ_1.

i. If $\xi_1 = -l$, then $(\psi_1 - \psi_2)(\xi_1) < 0$ and $(\psi_1 - \psi_2)'(\xi_1) > 0$. Evaluating (19) at $\xi = -l$ and noticing that $\psi_1(-l) - D < 0$ and $\psi_2'(-l) + \sigma > 0$, we get a contradiction.

ii. If ξ_1 is an interior point, then

$$(\psi_1 - \psi_2)'(\xi_1) = 0, \quad (\psi_1 - \psi_2)(\xi_1) < 0.$$

Noticing that $\psi_2'(\xi_1) + \sigma > 0$, again we get a contradiction by evaluating (19) at ξ_1.

In the same way, we prove that $\psi_1 - \psi_2 \leq 0$.

The lemma then follows immediately. ■

3 Global Existence

We prove the global existence of the initial value problem (1) (2) (3) (4) by a constructive method.

Let us denote the shock wave position by $x = s(t)$. From the Rankine-Hugoniot condition and the initial condition, we have

$$\frac{ds}{dt} = \frac{1}{2}(u(s(t), t) + u_0) \tag{20}$$

$$s(0) = 0. \tag{21}$$

Writing equation (1) in characteristic form, we have

$$\frac{dx}{dt} = u(x, t)$$
$$\frac{du}{dt} = qz_x - \sigma(u - u_0).$$

Hence,

$$\frac{d^2x}{dt^2} = \frac{du}{dt} = qK\varphi(u)z^\alpha - \sigma(u - u_0).$$

Noticing that $\lim_{x\to+\infty} z(x,t) = 1$, z can be solved as

$$z(x,t) = \begin{cases} 0 & x \le s(t) - l \\ (K(1-\alpha)(x - s(t)) + 1)^{\frac{1}{1-\alpha}} & s(t) - l < x \le s(t) \\ 1 & x > s(t) \end{cases} \tag{22}$$

where $l = \dfrac{1}{K(1-\alpha)}$ is the length of the reaction-zone.

Plugging z into (1), the system is reduced to

$$u_t + uu_x + \sigma(u - u_0) = F(x - s) \tag{23}$$

where $F(x-s) = qKz^\alpha(x,t)$ is $s(t) - l < x \le s(t)$ and it is 0 elsewhere.

Consider first the following auxiliary problem.

$$u_t + uu_x + \sigma(u - u_0) \;=\; F(x - j) \tag{24}$$

$$u(x,0) \;=\; \begin{cases} u_1 & x \ge -d \\ u_0 & x < -d \end{cases} \tag{25}$$

where the initial data satisfies one of the following

$$(a)\;\; u_i \;<\; u_1 < \psi(-l),\;\; 0 < d \le l \tag{26}$$

$$(b)\;\; u_1 \;>\; \psi(0),\;\; d \ge l. \tag{27}$$

And $j \in E$,

$$E = \left\{ j : j \in C^1[0,T], j(0) = 0, j'(0) = \frac{1}{2}(u_1 + u_0), j'(t) \le D \right\}$$

or $j \in F$,

$$F = \left\{ j : j \in C^1[0,T], j(0) = 0, j'(0) = \frac{1}{2}(u_1 + u_0), j'(t) \ge D \right\}$$

respectively, where ψ is the traveling wave and D is the propagating speed.

Clearly, E and F are nonempty closed bounded subsets of $C^1[0,T]$.

We iterate on j to find a fixed point of

$$s'(t) = \frac{1}{2}(u(j(t),t) + u_0)$$

such that it is a solution of (20) and hence u is a solution of (23).

Lemma 3.1 *There exists a unique entropy solution u of problem (24) (25) (26) for all $t > 0$, which satisfies*

$$\frac{\partial u}{\partial x} \geq 0, \text{ for } x \leq j(t) \tag{28}$$

wherever u is smooth.

Proof. The solution of (24) and (25) can be constructed through its characteristic lines. Global solution exists if two characteristic lines never intersect.

Take any two characteristic lines $x_1(t)$ and $x_2(t)$ of $u(x,t)$, where $x_1(0) < x_2(0)$. We claim that the two characteristic lines $x_1(t)$ and $x_2(t)$ never intersect.

To prove the claim, suppose the contrary that at time $t = t_0 > 0$ they intersect for the first time. Then $x_1(t_0) = x_2(t_0)$ and $x_1(t) < x_2(t)$, $0 \leq t < t_0$. Hence,

$$u(x_1(t_0), t_0) = \frac{dx_1(t_0)}{dt} > \frac{dx_2(t_0)}{dt} = u(x_2(t_0), t_0). \tag{29}$$

On the other hand, from (24) and (25)

$$\begin{aligned} u(x_1(t_0), t_0) - u_0 &= (u(x_1(0), 0) - u_0)e^{-\sigma t} + \\ &+ e^{-\sigma t} \int_0^{t_0} qK(K(1-\alpha)(x_1(s) - j(s)) + 1)^{\frac{\alpha}{1-\alpha}} e^{\sigma s} ds \\ < u(x_2(t_0), t_0) - u_0 &= (u(x_2(0), 0) - u_0)e^{-\sigma t} + \\ &+ e^{-\sigma t} \int_0^{t_0} qK(K(1-\alpha)(x_2(s) - j(s)) + 1)^{\frac{\alpha}{1-\alpha}} e^{\sigma s} ds \end{aligned}$$

which contradicts equation (29).

Thus $x_1(t)$ and $x_2(t)$ never intersect.

Given any two points (x_{10}, t_0) and (x_{20}, t_0), $t_0 > 0$ and $x_{10} < x_{20}$, draw characteristic lines $x_1(t)$ and $x_2(t)$ backwards in time. Since any two characteristic lines never intersect, we have

$$x_1(t) < x_2(t), \quad 0 \leq t \leq t_0.$$

Integrating (24) along the characteristic lines, we have

$$u(x_1(t_0), t_0) < u(x_2(t_0), t_0).$$

Wherever u is smooth, we have

$$\frac{\partial u}{\partial x} \geq 0, \text{ for } t > 0.$$

This proves (28). ∎

Similarly, we have the following result in case (*b*) (27).

Lemma 3.2 *There exists a unique entropy solution u of problem (24) (25) (27) for all $t > 0$, which satisfies*

$$\frac{\partial u}{\partial x} \geq 0, \text{ for } x \leq j(t) \qquad (30)$$

wherever u is smooth.

We prove a comparison principle for solutions of (1) and (2).

In the following theorem the initial value has the following form

$$u(x,0) = \begin{cases} u_{i0}(x) & x \leq s(0) \\ u_0 & x > s_i(0). \end{cases}$$

where $u_{i0}(x) \geq 0$ is a nondecreasing function and $s_i(0)$ is the initial shock wave position.

Theorem 3.3 *(A Comparison Principle)*

Suppose that $u_1(x,t)$ and $u_2(x,t)$ are solutions of (1) and (2) with nondecreasing initial data $u_{10}(x)$ and $u_{20}(x)$ and shock wave positions $s_1(t)$ and $s_2(t)$, respectively. If

$$s_1(0) < s_2(0)$$

and

$$u_{10}(x) \geq u_{20}(x), \; x \leq s_1(0),$$

then there is a $T > 0$ such that for $0 < t < T$,

$$u_1(x,t) > u_2(x,t), \; x \leq s_1(t).$$

Proof. Since $s_1(0) < s_2(0)$, there exists a $T > 0$ such that

$$s_1(t) < s_2(t), \quad 0 < t < T.$$

From point (x, t_0), where $0 < t_0 < T$ and $x \leq s_1(t_0)$, draw characteristic lines $x_1(t)$ and $x_2(t)$ of u_1 and u_2 backwards respectively.

We claim: $u_1(x,t_0) > u_2(x,t_0)$.

Suppose the contrary that $u_1(x,t_0) \leq u_2(x,t_0)$.

Step 1. Suppose that (x, t_0) is the first intersection point of the two characteristic lines. Then, $x_1(t) > x_2(t)$, for $0 \leq t < t_0$. We also know that

$s_1(t) < s_2(t)$, for $0 \le t < t_0$. So, $x_1(t) - s_1(t) > x_2(t) - s_2(t)$ for $0 \le t < t_0$. Integrating the equations for u_1 and u_2 along their characteristic lines $x_1(t)$ and $x_2(t)$ respectively and noticing that $u_1(x_1(0),0) \ge u_2(x_2(0),0)$, we have

$$\begin{aligned} u_1(x,t_0) - u_0 &= (u_1(x_1(0),0) - u_0)e^{-\sigma t_0} + \\ &+ e^{-\sigma t_0} \int_0^{t_0} qK(K(1-\alpha)(x_1(t) - s_1(t)) + 1)^{\frac{\alpha}{1-\alpha}} e^{\sigma t} dt \\ > u_2(x,t_0) - u_0 &= (u_2(x_2(0),0) - u_0)e^{-\sigma t_0} + \\ &+ e^{-\sigma t_0} \int_0^{t_0} qK(K(1-\alpha)(x_2(t) - s_2(t)) + 1)^{\frac{\alpha}{1-\alpha}} e^{\sigma t} dt \end{aligned}$$

which contradicts our assumption that $u_1(x,t_0) \le u_2(x,t_0)$.

Thus, $u_1(x,t_0) > u_2(x,t_0)$.

Step 2. If the two characteristic lines intersect more than once, let ξ be the last one before t_0. Then $u_1(x_1(\xi),\xi) > u_2(x_2(\xi),\xi)$ and $x_1(t) - s_1(t) > x_2(t) - s_2(t)$ for $\xi < t < t_0$. Hence

$$\begin{aligned} u_1(x,t_0) - u_0 &= (u_1(x_1(\xi),\xi) - u_0)e^{-\sigma t_0} \\ &+ e^{-\sigma t_0} \int_\xi^{t_0} qK(K(1-\alpha)(x_1(t) - s_1(t)) + 1)^{\frac{\alpha}{1-\alpha}} e^{\sigma t} dt \\ > u_2(x,t_0) - u_0 &= (u_2(x_2(\xi),\xi) - u_0)e^{-\sigma t_0} \\ &+ e^{-\sigma t_0} \int_\xi^{t_0} qK(K(1-\alpha)(x_2(t) - s_2(t)) + 1)^{\frac{\alpha}{1-\alpha}} e^{\sigma t} dt \end{aligned}$$

which is again a contradiction.

Therefore

$$u_1(x,t_0) > u_2(x,t_0)$$

for all $x \le s_1(t_0)$, and $0 < t_0 < T$.

■

Using the comparison principle, we get an upper bound on the shock location in case (a) (26).

Lemma 3.4

$$\frac{1}{2}(u(j(t),t) + u_0) < D,\ t > 0. \tag{31}$$

Proof. From point $(j(t_0),t_0)$, draw a straight line with slope D. Since $j'(t) \le D$, we have

$$x(t) = j(t_0) + D(t - t_0) < j(t),\ \text{for } 0 \le t < t_0.$$

$x(t)$ can be viewed as the shock wave position of the traveling wave $\psi(x - Dt + c)$ where $c = -j(t_0) + Dt_0 > 0$. Comparing initial data of $\psi(x - Dt + c)$ and $u(x,t)$, we have

$$\psi(x+c) \geq u(x,0), \quad x + c \leq 0.$$

Applying the comparison principle in Theorem 3.3 to $\psi(x - Dt + c)$ and $u(x,t)$, we get

$$\psi(x - Dt + c) > u(x,t), \; x - Dt + c \leq 0, \text{ for } 0 \leq t \leq t_0.$$

In particular, at point $(j(t_0), t_0)$,

$$2D - u_0 = \psi(0) > u(j(t_0), t_0).$$

That proves the lemma. ■

In case (*b*) (27), we have the following lower bound on front of the reaction-zone.

Lemma 3.5

$$\frac{1}{2}(u(j(t),t) + u_0) > D, \; t > 0. \tag{32}$$

Proof. Noticing that the characteristic line with $x'(0) = \psi(-l)$ and $x(0) = -l$ never intersects $x = Dt - l$ for $t > 0$ (since $\psi(-l) < D$), the proof of the inequality is similar to the proof in the previous lemma. ■

Now let s be the solution of

$$\begin{aligned} s'(t) &= \frac{1}{2}(u(j(t),t) + u_0), \\ s(0) &= 0. \end{aligned}$$

By Lemma 3.4, $s'(t) < D$. Therefore, $s \in E$.

Define $A : E \to C^1[0,T]$ such that

$$s = Aj.$$

Clearly, $AE \subset E$, i.e. A maps E into itself and is continuous.

Similarly, $AF \subset F$.

Lemma 3.6 *The mapping A has a fixed point.*

Proof. We use Schauder's fixed point theorem to prove the argument. To this end, we show A is compact. By the Arzela-Ascoli theorem, it is enough to prove that $s'' = (Aj)''$ is bounded for all $j \in E$

To get a bound of s'', we need to estimate u_x first.

Differentiating equation (24) with respect to x and writing the equation in characteristic form, we have

$$\frac{du_x}{dt} + u_x^2 + \sigma u_x = qK^2\alpha(K(1-\alpha)(x(t) - j(t)) + 1)^{\frac{2\alpha-1}{1-\alpha}} \leq qK^2\alpha.$$

If $\frac{du_x}{dt} \geq 0$, then $0 \leq u_x \leq K\sqrt{q\alpha} = M$.

If $\frac{du_x}{dt} < 0$, then there exists $0 \leq t_0 < t$ such that

$$\int_{t_0}^{t} \frac{du_x}{dt} dt \leq 0$$

and

$$\int_{t_1}^{t} \frac{du_x}{dt} dt \geq 0$$

for any $t_1 \leq t_0$.

(i) If $t_0 > 0$, then $\frac{du_x(x(t_0), t_0)}{dt} \geq 0$ and hence $0 \leq u_x(x(t_0), t_0) \leq M$. So

$$u_x(x(t), t) = \int_{t_0}^{t} \frac{du_x}{dt} dt + u_x(x(t_0), t_0) \leq u_x(x(t_0), t_0) \leq M.$$

(ii) If $t_0 = 0$, then

$$u_x(x(t), t) = \int_{0}^{t} \frac{du_x}{dt} dt + u_x(x(0), 0) \leq u_x(x(0), 0) \leq M.$$

Therefore u_x is bounded.

Now come back to estimate s'',

$$\begin{aligned} 0 < s'' &= \frac{1}{2}\frac{d}{dt}(u(j(t), t) + u_0) \\ &= \frac{1}{2}qK + \frac{1}{2}(j'(t) - u)u_x - \frac{1}{2}\sigma(u - u_0) \\ &\leq M \end{aligned}$$

where $M > 0$ is a constant.

Similarly, it can be proved that

$$-M \leq s'' < 0$$

in case (b) (27). ■

Using this fixed point, we construct the following solution of (1), (2), (3) and (4):

$$u(x,t) = \begin{cases} u_j(x,t) & x \leq s(t) \\ u_0 & x > s(t) \end{cases} \tag{33}$$

$$z(x,t) = \begin{cases} 0 & x \leq s(t) - l \\ (K(1-\alpha)(x-s(t))+1)^{\frac{1}{1-\alpha}} & s(t)-l < x \leq s(t) \\ 1 & x > s(t), \end{cases} \tag{34}$$

where u_j is the solution of (24) and (25).

It is easy to check that (u, z) is a weak solution of (1), (2), (3) and (4).

Now, we prove that the above defined solution is unique. This allows us to extend the solution to $t = +\infty$.

Lemma 3.7 *Solutions of form (33) and (34) are unique.*

Proof. Suppose that there are two solutions $u(x,t)$ and $v(x,t)$ with shock wave positions $s_1(t)$ and $s_2(t)$ respectively.

Because of the uniqueness of solutions of (24) and (25), we need only to show that $s_1(t) = s_2(t)$.

Translate s_2 along the x axis to the right of s_1 until they just touch. If the lowest touch point is $(s_1(t_0), t_0)$, then $t_0 = 0$.

Suppose for the contrary that $t_0 > 0$. Let $s_2 + \delta$ be the translation of s_2 such that $(s_1(t_0), t_0)$ is on $s_2 + \delta$. Then $\delta > 0$ and $s_1(t) < s_2(t) + \delta$ for $0 < t < t_0$.

Let

$$\begin{aligned} v_1(x,t) &= v(x-\delta, t), \\ z_1(x,t) &= z(x-\delta, t). \end{aligned}$$

Then (v_1, z_1) satisfies

$$\begin{aligned} v_{1t} + v_1 v_{1x} &= q z_{1x} \\ z_{1x} &= K\phi(v_1) z_1^{\alpha} \\ v_1(x,0) &= \begin{cases} u_1 & \delta - d \leq x \\ u_0 & x \leq \delta - d \end{cases} \\ z_1(x,0) &= 1. \end{aligned}$$

It is easy to check that the conditions for the comparison principle hold. Applying the comparison principle to u and v_1, we have $u(x, t_0) > v_1(x, t_0)$ for $x \le s_1(t_0) = s_2(t_0) + \delta$. This implies

$$\int_{-\infty}^{+\infty} (u - v_1)(x, t_0)dx = \int_{-\infty}^{s_1(t_0)} (u - v_1)(x, t_0)dx > 0,$$

On the other hand, by (1) and (3)

$$\frac{d}{dt}(e^{\sigma t} \int_{-\infty}^{+\infty} (u - v_1)(x, t)dx) = 0.$$

Hence

$$e^{\sigma t_0} \int_{-\infty}^{+\infty} (u - v_1)(x, t_0)dx = \int_{-\infty}^{+\infty} (u - v_1)(x, 0)dx = 0$$

which is a contradiction.

Thus, $t_0 = 0$. This implies that $\delta = 0$. It follows from the assumption that $s_1(t) \le s_2(t)$.

Similarly, we prove $s_2(t) \le s_1(t)$.

Hence $s_2(t) = s_1(t)$.

■

We prove the important properties of $u_x(s(t), t)$ and $s(t)$ in the next lemma.

Lemma 3.8

$$u_x(s(t), t) < \psi_x(0), \qquad (35)$$

$$\frac{d}{dt}u(s(t), t) > 0, \quad \textit{for } t \ge 0, \qquad (36)$$

$$s''(t) > 0, \quad \textit{for } t \ge 0. \qquad (37)$$

Proof. We estimate u_x first.

Differentiating equation (24) with respect to x and writing the equation in characteristic form, we have

$$\frac{du_x}{dt} + u_x^2 + \sigma u_x = qK^2\alpha(K(1-\alpha)(x(t) - s(t)) + 1)^{\frac{2\alpha-1}{1-\alpha}} = G(x(t) - s(t)).$$

Fix a t_0 and compare $u_x(x, t_0)$ with $\psi_x(x - D(t - t_0) - s(t_0))$ at $x = s(t_0)$.

If $u_x(s(t_0), t_0) > \psi_x(s(t_0), t_0)$, then $\frac{du_x(x(t), t)}{dt}|_{t_0} < \frac{d\psi_x(x_1(t), t)}{dt}|_{t_0}$, where $x(t)$ and $x_1(t)$ are characteristic of u and ψ passing point $(s(t_0), t_0)$, respectively. Furthermore, there is a (take the first one) $0 \le t_1 < t_0$ such that

$$u_x(x(t), t) - \psi_x(x_1(t), t) > 0$$

for all $t_1 < t < t_0$.

We assume $u_x(s(t_0), t_0) = \psi_x(s(t_0), t_0)$ for definitiveness by solutions of ODE's continuously depending on parameters.

On the other hand, noticing that u satisfying

$$\frac{du}{dt} + \sigma(u - u_0) = qK(K(1-\alpha)(x(t) - s(t)) + 1)^{\frac{\alpha}{1-\alpha}} = F(x(t) - s(t)) \quad (38)$$

and the comparison principle, we have that $u(x(t_0), t_0) < \psi(x_1(t_0), t_0)$. So

$$\begin{aligned} (x(t) - s(t))'|_{t_0} &= \frac{1}{2}(u(s(t), t) - u_0)|_{t_0} \\ < \quad & \frac{1}{2}(\psi(s_1(t), t) - u_0)|_{t_0} = (x_1(t) - s_1(t))'|_{t_0}. \end{aligned}$$

Hence there is a $t_2 < t_0$ such that $(x(t) - s(t)) > (x_1(t) - s_1(t))$ for $t_2 < t < t_0$.

Let $t_3 = \max\{t_1,\ t_2\}$. Integrating the equations for u_x and ψ_x , we have

$$\begin{aligned} 0 &> -(u_x - \psi_x)|_{t_3} \\ &= (u_x - \psi_x) e^{\int_{t_3}^{t} (u_x + \psi_x + \sigma) dt} \Big|_{t_3}^{t_0} \\ &= \int_{t_3}^{t_0} e^{\int_{t_3}^{t} (u_x + \psi_x + \sigma) dt} (G(x - s) - G(x_1 - s_1)) dt \\ &> 0, \end{aligned}$$

which is a contradiction.

Therefore $u_x(s(t_0), t_0) < \psi_x(0)$.

To prove (36), noticing $u(s(t), t) < \psi(0)$ and (35), we have

$$\begin{aligned} \frac{d}{dt} u(s(t), t) &= qK - (\sigma + \frac{1}{2} u_x(s(t), t))(u(s(t), t) - u_0) \\ &> 0. \end{aligned}$$

(37) is proved by its definition. ■

Now we have the following theorem.

Theorem 3.9 *Solution of (1), (2), (3) and (4) exists globally and satisfies*

$$0 \le u \le M,\ 0 \le u_x \le M,\ |u_t| \le M,$$

$$0 \le z \le 1,\ 0 \le z_x \le K,\ |z_t| \le M$$

where M depends only on q, K, u_0, σ. All the estimates involving derivatives are valid away from the shock curve $(s(t), t)$.

Furthermore, $s(t)$ satisfies

$$s'' \geq 0 (or \leq 0)$$

and

$$\lim_{t \longrightarrow +\infty} s'(t)$$

exists.

Proof. From Lemma 3.1 and Lemma 3.7, $u_j(x,t)$ and $s = s(t)$ are defined for all $t > 0$. Therefore, solution of (1), (2), (3) and (4) exists for all $t > 0$.

$u(s(t), t) \leq 2D - u_0$ by Lemma 3.4. In case (b) (27), $u(s(t), t) \leq u(s(0), 0)$. From (28), we have that $u(x,t) \leq u(s(t), t)$ for $x \leq s(t)$. Hence, u is bounded.

u_x is bounded by Lemma 3.8.

Using equation (23), we have

$$u_t = qK(K(1-\alpha)(x(t) - s(t)) + 1)^{\frac{\alpha}{1-\alpha}} - uu_x - \sigma(u - u_0),$$

from which, the bound for u_t can be obtained.

From (34), clearly, z and z_x satisfy the desired estimates.

For z_t, we have

$$z_t(x,t) = -Ks'(t)(K(1-\alpha)(x(t) - s(t)) + 1)^{\frac{\alpha}{1-\alpha}}, \; s(t) - l \leq x \leq s(t)$$

which is bounded.

By definition of s and Lemma 3.1, we have

$$\begin{aligned} s'(t) &= \frac{1}{2}(u(s(t), t) + u_0) < D \\ s''(t) &> 0. \end{aligned}$$

Hence, $\lim_{t \to +\infty} s'(t)$ exists.

In case (b) (27), $\lim_{t \to +\infty} s'(t)$ exists for similar reasons. ∎

4 Convergence to the Traveling Wave

In this section we prove that the solution of the problem (1), (2), (3) and (4) converges to a shifted traveling wave. It is difficulty to determine the shift since equation (1) is not a conservation law now. Instead, we prove that the energy is conserved 'asymptotically'. We will show that the shift is finite by a compactness argument.

Lemma 4.1 *If $u(x,t)$ is solution of (1), (2), (3) and (4), then*

$$\frac{d}{dt}(e^{\sigma t}\int_{-\infty}^{+\infty}(u(x,t)-\psi(x-Dt))dx)=0. \tag{39}$$

Or

$$\int_{-\infty}^{+\infty}(u(x,t)-\psi(x-Dt))dx=e^{-\sigma t}\int_{-\infty}^{+\infty}(u(x,0)-\psi(x))dx.$$

Proof. The conclusion was proved in Lemma 3.1. ∎

First, we prove the following asymptotic property of $s'(t)$.

Lemma 4.2 *If $u(x,t)$ is the solution of (1), (2), (3) and (4), then*

$$\lim_{t\to+\infty} s'(t)=D.$$

Proof. From the previous section, we know that $0\le s'(t)\le D$ and $s''(t)>0$. Hence $\lim_{t\to+\infty} s'(t)$ exists.

If the conclusion does not hold, then there is a $D'<D$ such that

$$\lim_{t\to+\infty} s'(t)=D'$$

and $s'(t)<D'$ for all $t>0$.

For $t_0>0$, let

$$s_1(t)=D(t-t_0)+s(t_0),$$

then

$$s_1(t)<s(t) \ \text{ for } 0<t<t_0.$$

Applying the comparison principle Theorem 3.3 to $\psi(x-D(t-t_0)-s(t_0))$ and $u(x,t)$, we see that

$$\psi(x-s(t_0))>u(x,t_0) \ \text{ for } x\le s_1(t_0)=s(t_0),$$

and hence

$$\int_{-\infty}^{+\infty}(\psi(x-s(t_0))-u(x,t_0))dx=\int_{-\infty}^{s_1(t_0)}(\psi(x-s(t_0))-u(x,t_0))dx=\delta>0.$$

Using the asymptotically conservative property in Lemma 4.1 and letting $t_0\to+\infty$, we get a contradiction.

Similarly we can prove the same conclusion for case (b) (27). ∎

In the next lemma, we find a lower bound of $s(t)$ in case (a) (26).

Lemma 4.3 *If $u(x,t)$ is the solution of (1) (2), then there exists $C > 0$ such that $Dt - C \leq s(t) \leq Dt$, for all $t \geq 0$.*

Furthermore

$$\lim_{t\to+\infty} (s(t) - Dt + C) = 0 \tag{40}$$

provided $\sigma < qK$.

Proof. We prove the boundedness first.

Let $s_1(t) = Dt$. Then $s(t) \leq s_1(t)$ for all $t \geq 0$ since $s(0) = s_1(0)$ and $s'(t) < s_1'(t)$ for all $t \geq 0$.

We prove the other inequality by contradiction. If the conclusion were false, then for each $n > 0$, there is a $t_1 = t_1(n) > 0$ such that $s_1(t_1) = s(t_1)+n$. Since $(s''(t) - s_1''(t)) \geq 0$, we have

$$(s'(t) - s_1'(t)) \geq (s'(0) - s_1'(0)) = \frac{1}{2}(u_1 + u_0) - D, \quad \text{for } t > 0.$$

Hence

$$t_1 > \frac{n}{s_1'(0) - s'(0)} = \frac{n}{D - \frac{1}{2}(u_1+u_0)} > 0. \tag{41}$$

Let $x(t)$ be the characteristic passing $(s(t_1), t_1)$. Since $x'(t_1) \to 2D - u_0 > D$ for t_1 large, then there is a $t_0 < t_1$ such that $x(t_0) = s_1(t_0) - n$ and

$$s_1(t) - n < x(t) \quad \text{for } 0 \leq t < t_0.$$

Hence

$$\begin{aligned} 0 &= x(t_0) - s_1(t_0) + n \\ &= (x(0) - s_1(0) + n) + (x'(0) - s_1'(0))t_0 + (x'' - s_1'')(\xi)\frac{t_0^2}{2} \end{aligned}$$

for some $t_0 > \xi > 0$, where both $x(0) - s_1(0)$ and $x'(0) - s_1'(0)$ are bounded independent of n.

(i) If $x''(\xi) \geq \delta > 0$, then there is no solution for t_0 in the above equation for n large enough.

(ii) If $x''(\xi) \leq -\delta < 0$, then $t_0 = O(\sqrt{\frac{n}{\delta}})$, which contradicts with (41).

(iii) If there is a subsequence $\xi \to +\infty$ as $n \to +\infty$ such that $x''(\xi) \to 0$. Then

$$\begin{aligned} 0 &= (x - s_1 + n)(t_1) \\ &= (x - s_1 + n)(t_0) + (x' - s_1')(t_0)(t_1 - t_0) + (x'' - s_1'')(\xi_1)\frac{(t_1 - t_0)^2}{2} \\ &= (x' - s_1')(t_0)(t_1 - t_0) + (x'' - s_1'')(\xi_1)\frac{(t_1 - t_0)^2}{2}, \end{aligned}$$

for some $t_1 > \xi_1 > t_0$.

If $x'(t_0) - s_1'(t_0) \to 0$ or $t_1 - t_0 \to +\infty$, then $x''(\xi_1) \to 0$ as $n \to +\infty$. On the other hand, it can be proved (in the following Lemma) that there is at most one value of t such that $x''(t) = 0$. Hence

$$x'(t_0) - s_1'(t_0) \le -\delta' < 0$$

and $t_1 - t_0$ bounded.

Let $x_1(t)$ be the characteristic of $\psi(x - Dt + n)$ passing $(s(t_1), t_1)$. Since $x'(t_1) \to x_1'(t_1) > D$ for t_1 large, there exists $t_0 < \tau_1 < t_1$ and $\delta_1 > 0$ such that

$$x''(\tau_1) > x_1''(\tau_1) + \delta_1$$

where $\delta_1 = \dfrac{\delta'}{t_1 - t_0} > 0$.

There are two cases.

(A) If $(x - s)(\tau_1) > (x_1 - s_1 + n)(\tau_1) + \delta_1'$, then there is $\tau_1 < \tau_2 < t_1$ such that

$$\begin{aligned} 0 < \delta' &< (s_1 - n - s)(\tau_1) - (x_1 - x)(\tau_1) \\ &= (s_1 - n - s)(t_1) - (x_1 - x)(t_1) + \\ &+ (s_1 - s)'(t_1)(\tau_1 - t_1) + \frac{1}{2}(s_1 - s)''(\tau_2)(t_1 - \tau_1)^2 - \\ &- (x_1 - x)'(t_1)(\tau_1 - t_1) - \frac{1}{2}(x_1 - x)''(\tau_2)(t_1 - \tau_1)^2. \end{aligned}$$

Hence

$$(x - s)''(\tau_2) > (x_1 - s_1)''(\tau_2) + \delta_2$$

where

$$\delta_2 = \frac{2\delta_1}{t_1 - \tau_1} + 2\frac{(x_1 - x)'(t_1) - (s_1 - s)'(t_1)}{t_1 - \tau_1} > \frac{2\delta_1'}{(t_1 - \tau_1)^2} > 0.$$

The reason is the following

$$\begin{aligned}(x(t)-s(t))'|_{t_1} &= \frac{1}{2}(u(s(t),t)-u_0)|_{t_1}\\ &< \frac{1}{2}(\psi(s_1(t),t)-u_0)|_{t_1} = (x_1(t)-s_1(t))'|_{t_1}.\end{aligned}$$

Therefore

$$x''(\tau_2) > x_1''(\tau_2) + \delta_2.$$

(B) If $x'(\tau_1) < x_1'(\tau_1) - \delta_1'$, then there is $\tau_1 < \tau_2 < t_1$ such that

$$x''(\tau_2) > x_1''(\tau_2) + \delta_2.$$

In both cases, we find a point τ_2 which is closer to t_1 than τ_1 such that $x''(\tau_2)$ is bigger than $x_1''(\tau_2)$. We continue this process until $\tau_n \to t_1$ as $n \to +\infty$ which is a contradiction since $x''(t_1) \to x_1''(t_1)$.

Therefore, there exists $C > 0$, such that $s_1(t) < s(t) + C$ for all $t > 0$.

Taking infimum of such C, (40) can be proved by boundedness and monotonicity properties of $s(t) - Dt$.

In case (b), (40) can be proved similarly. ∎

Lemma 4.4 *Let $x(t)$ be a characteristic of solution $u(x,t)$, then there is at most one $t > 0$ such that $x''(t) = 0$ if $\sigma < qK$.*

Proof. Let

$$f(t) = qK((1-\alpha)K(x(t)-s(t))+1)^{\frac{\alpha}{1-\alpha}}$$

and

$$g(t) = \sigma(u(x(t),t)-u_0).$$

Then

$$x''(t) = f(t) - g(t).$$

If there were two zeros, t_1 and t_2, of $f(t) - g(t)$, then there exists ξ, which is in between t_1 and t_2, such that

$$g'(\xi) = f'(\xi) > 0.$$

Checking derivative of $g(t)$, we have

$$g'(t) = \sigma\frac{du}{dt} = \sigma(f(t)-g(t)).$$

Hence

$$\sigma(f(\xi) - g(\xi)) = g'(\xi) > 0.$$

Assume that t_0 is the time $x(t)$ enters $s(t)$. Then $f(t_0) - g(t_0) = x''(t_0) > 0$ by choosing $\sigma < qK$.

(i) If $f(0) < g(0)$, then there must be at least three zeros of $f(t) - g(t)$. In particular, there is a point ξ_1 such that

$$0 < f'(\xi_1) = g'(\xi_1) = \sigma(f(\xi_1) - g(\xi_1)) < 0$$

which is a contradiction.

(ii) If $f(0) \geq g(0)$, we arrive at the same conclusion similarly.

So there cannot be more than one zeros of $f(t) - g(t)$.

■

We prove that the solutions are asymptotically functions of one variable $x - Dt + C$.

Lemma 4.5 *There are Lipschitz functions $u_\infty(\xi)$ and $z_\infty(\xi)$ such that*

$$\lim_{t \to +\infty} \sup_{x - Dt + C \leq 0} |(u, z)(x, t) - (u_\infty, z_\infty)(x - Dt + C)| = 0.$$

Proof. Let

$$\begin{aligned} \xi &= x - Dt + C \\ u(x,t) &= U(\xi, t) \\ z(x,t) &= Z(\xi, t) \\ E(t) &= s(t) - Dt + C. \end{aligned}$$

Then $E(t)$ decreases to 0 and $s'(t)$ increases to D as $t \to +\infty$. Since $E(t) > 0 \geq \xi$, we have $s(t) > x$. By(34),

$$Z(\xi, t) = z(x, t) = (K(1-\alpha)(x - s(t)) + 1)^{\frac{1}{1-\alpha}} = (K(1-\alpha)\xi + 1)^{\frac{1}{1-\alpha}} + o(1)$$

where $o(1) \to 0$ as $t \to +\infty$.

Letting

$$z_\infty(\xi) = (K(1-\alpha)\xi + 1)^{\frac{1}{1-\alpha}}, \ -l \leq \xi \leq 0,$$

we have

$$\lim_{t \to +\infty} \sup_{\xi \leq 0} |Z(\xi, t) - z_\infty(\xi)| = 0.$$

We now look at the characteristic lines of $U(\xi, t)$ given by

$$\frac{d\xi}{dt} = \frac{dx}{dt} - D = U - D$$

and

$$\begin{aligned}\frac{d^2\xi}{dt^2} &= \frac{d^2x}{dt^2} = \frac{du}{dt} - \sigma(u-u_0) = qz_x \\ &= qK(K(1-\alpha)(x-s(t))+1)^{\frac{\alpha}{1-\alpha}} - \sigma(u-u_0).\end{aligned}$$

It is equivalent to

$$\frac{d^2\xi}{dt^2} + \sigma\frac{d\xi}{dt} = qK(K(1-\alpha)\xi+1)^{\frac{\alpha}{1-\alpha}} - \sigma(D-u_0) + o(1)$$

or

$$\begin{aligned}\frac{d\xi}{dt} &= e^{-\sigma t}\frac{d\xi}{dt}|_{t=0} + o(1)(1-e^{-\sigma t}) \\ &+ \int_0^t (qK(K(1-\alpha)\xi+1)^{\frac{\alpha}{1-\alpha}} - \sigma(D-u_0))e^{\sigma(s-t)}ds \\ &= e^{-\sigma t}\frac{d\xi}{dt}|_{t=0} + o(1)(1-e^{-\sigma t}) \\ &+ (qK(K(1-\alpha)\xi+1)^{\frac{\alpha}{1-\alpha}} - \sigma(D-u_0))(1-e^{-\sigma t}).\end{aligned}$$

Therefore $\frac{d\xi}{dt}$ and hence $U(\xi,t) = u(x,t)$ is a function of ξ as $t\to+\infty$.

■

Lemma 4.6

$$\lim_{t\to+\infty}(u,z)(\xi,t) = (u_0,1) = (u_\infty(\xi), z_\infty(\xi)),$$

where $\xi = x - Dt + C > 0$.

Proof. If $\xi > 0$, i.e. $x - Dt + C > 0$, there is a $T > 0$, such that for $t > T$,

$$0 < E(t) = s(t) - Dt + C < x - Dt + C.$$

That is,

$$x - s(t) > 0.$$

From (34) and (33), we have that

$$z(x,t) = 1, u(x,t) = u_0$$

provided $t > T$.

Evaluating (u_∞, z_∞) at $\xi > 0$, we have

$$(u_\infty, z_\infty)(\xi) = (u_0, 1).$$

■

Lemma 4.7 *$(u_\infty, z_\infty)(\xi)$ is the traveling wave $(\psi, z)(\xi)$ of (1) and (2), i.e.*

$$(u_\infty, z_\infty)(\xi) = (\psi, z)(\xi), \quad \textit{where}\ \xi = x - Dt + C.$$

Proof. It is easy to prove that $(u_\infty, z_\infty)(x - Dt + C)$ is a weak solution of (1) and (2).

We now show that $(u_\infty, z_\infty)(x - Dt + C)$ satisfies (5), (6), (7) and (8) which are equations for the traveling wave solution.

Clearly, $z_\infty(\xi)$ satisfies (6) and is smooth for $\xi \leq 0$. By Lemma 4.5 $(u_\infty, z_\infty)(\xi)$ is Lipschitz for $\xi \leq 0$, and hence absolutely continuous. Furthermore, (u_∞, z_∞) satisfies (5) almost everywhere. Using Lemma 4.6, we get

$$(u_\infty, z_\infty)(x - Dt + C) = (u_0, 1) = (\psi, z)(x - Dt + C),\ x - Dt + C > 0.$$

That is, (u_∞, z_∞) satisfies (8).

Using Lemma 4.3, we have

$$u_0 < u(x,t) < \psi(x - Dt + C),\ x - Dt + C < 0.$$

Noticing that $\psi(-\infty) = u_0$, we get

$$\lim_{x-Dt+C \longrightarrow -\infty} u_\infty(x - Dt + C) = u_0.$$

From Lemma 4.5,

$$\lim_{x-Dt+C \longrightarrow -\infty} z_\infty(x - Dt + C) = 0.$$

Hence (7) is satisfied.

Using the uniqueness of solution of (5), (6), (7) and (8),
$(u_\infty, z_\infty)(x - Dt + C) = (\psi, z)(x - Dt + C)$. ∎

Now we summarize our results in the following theorem.

Theorem 4.8 *The solution of (1), (2), (3) and (4) converges uniformly to the traveling wave and the shock front is asymptotically linear:*

$$\lim_{t \longrightarrow +\infty} \sup_{x-Dt+C \leq 0} |(u,z)(x,t) - (\psi,z)(x - Dt + C)| = 0,$$

$$\lim_{t \longrightarrow +\infty} (u,z)(x,t) = (u_0, 1),\ \textit{for}\ x - Dt + C > 0$$

and

$$\lim_{t \longrightarrow +\infty} (s(t) - Dt + C) = 0$$

provided $\sigma < qK$, $\alpha > \frac{1}{2}$ and

$$u_0 + \frac{qK}{2\sigma} > D \geq u_0 + \frac{2qK}{3\sigma + \sqrt{\sigma^2 + 4qK^2\alpha}}.$$

Proof. If the initial data satisfies the conditions (26), the conclusion is proved.

If the initial data satisfies the conditions (27), the conclusion can be proved similarly.

If the initial data is 'compact support' and is in between a data satisfies (27) and an initial data that satisfies (26) , then the conclusion can be proved by using the comparison principle Lemma 3.3.

Therefore, the conclusion holds for certain 'compact support' initial data. ■

We also have the convergence result in L^p norm ($p \geq 1$).

Corollary 4.9

$$\lim_{t \to +\infty} |(u,z)(x,t) - (\psi, z)(x - Dt + C)|_{L^p} = 0, \; p \geq 1.$$

Proof. We only prove the results for $p = 1$. Results for $p > 1$ are easy consequences of $p = 1$ and Theorem 4.8.

Let $s_1(t) = Dt - C$. By the asymptotic conservative property, we have that for t large

$$\begin{aligned} & \int_{-\infty}^{s_1(t)} (\psi(x - Dt + C) - u(x,t))dx \\ = & -\int_{s_1(t)}^{s(t)} (\psi(x - Dt + C) - u(x,t))dx + o(1) \end{aligned}$$

where $o(1) \to 0$ as $t \to +\infty$. Therefore,

$$\begin{aligned} & |u(x,t) - \psi(x - Dt + C)|_{L^1} \\ = & \int_{-\infty}^{s_1(t)} (\psi(x - Dt + C) - u(x,t))dx \\ + & \int_{s_1(t)}^{s(t)} (u(x,t) - \psi(x - Dt + C))dx \\ = & \; 2\int_{s_1(t)}^{s(t)} (u(x,t) - \psi(x - Dt + C))dx + o(1) \\ \leq & \; 2(s(t) - s_1(t))(2D - u_0) + o(1) \to 0, \; t \to +\infty. \end{aligned}$$

Similar result for z directly follows from $s(t) - s_1(t) \to 0$. ■

5 Numerical Results

In this section, we present the numerical results showing convergence for certain 'compact support' initial data.

We use fractional step to treat the transport term and the reaction and radiation terms. For the transport term, we use first order upwind scheme. Since in our problem the term u is always nonnegative, the upwind scheme gives

$$u_j^{n'} = u_j^n - \frac{\Delta t}{\Delta x}(f(u_j^n) - f(u_{j-1}^n))$$

where $(j\Delta x, n\Delta t)$ is the discrete point in xt plane and $f(u) = \frac{u^2}{2}$.

We then evaluate the reaction and radiation terms.

$$u_j^{n+1} = u_j^{n'} - \Delta t\sigma(u_j^{n'} - u_0) + \frac{\Delta t}{\Delta x}q\varphi(u_j^{n'})(z_j^n - z_{j-1}^n).$$

z is updated by the motion of the front and the equation (2). Here the position where z achieves its maximum value is chosen to be the same as where u achieves its maximum value. This choice is in the same spirit of the numerical induction mechanisims introduced in Engquist and Sjogreen [4]. The purpose is to avoid spurious solutions.

We show four examples. The physical parameters are as following: $k = 3$, $q = 0.3$, $\alpha = 0.75$, $\sigma = 0.5$, $u_i = 0.1$, $u_0 = 0$ and $CFL = 0.25$. Conditions in Theorem 4.8 are satisfied.

As $t \to +\infty$, all solutions converge to the same traveling wave as predicted in Theorem 4.8.

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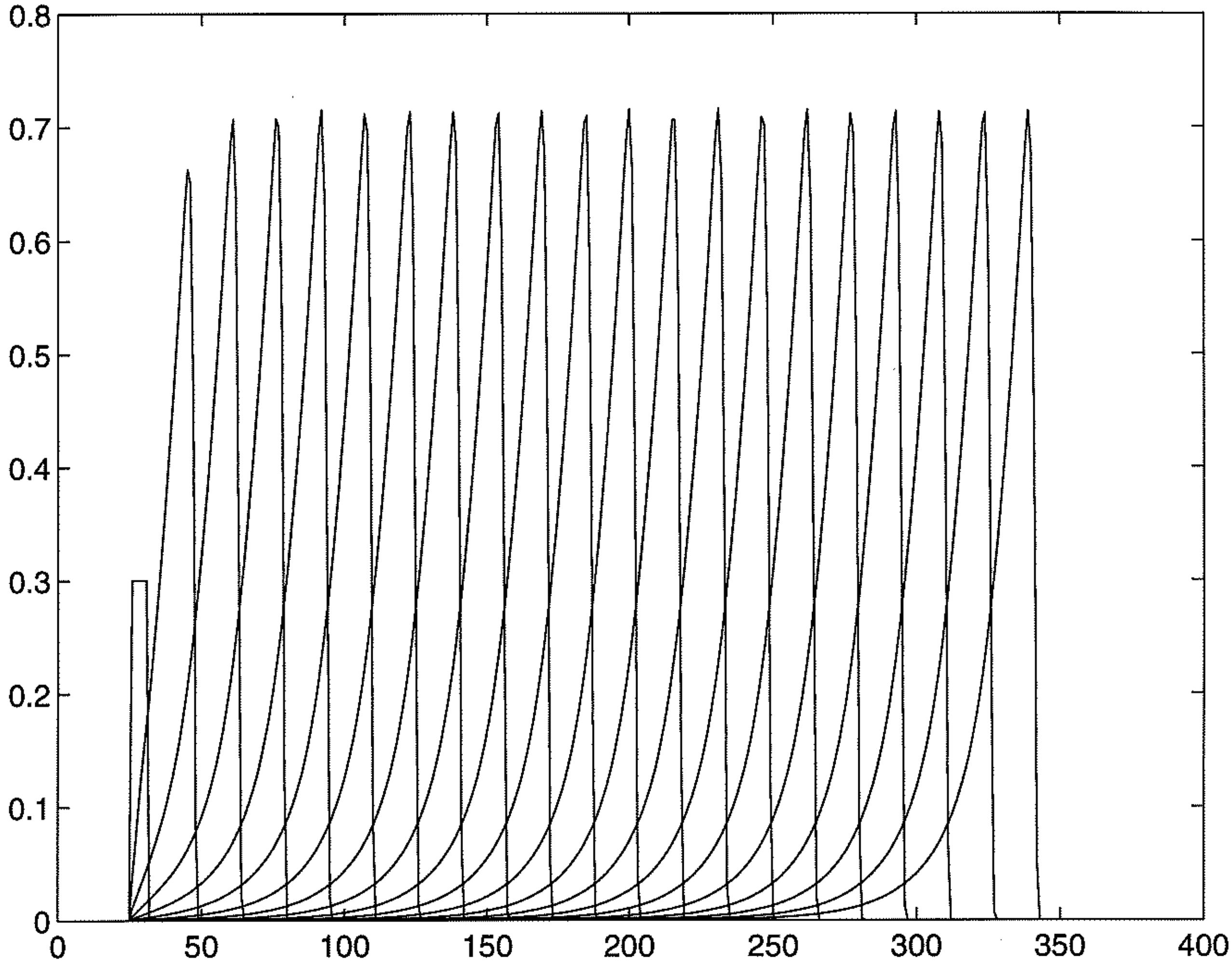

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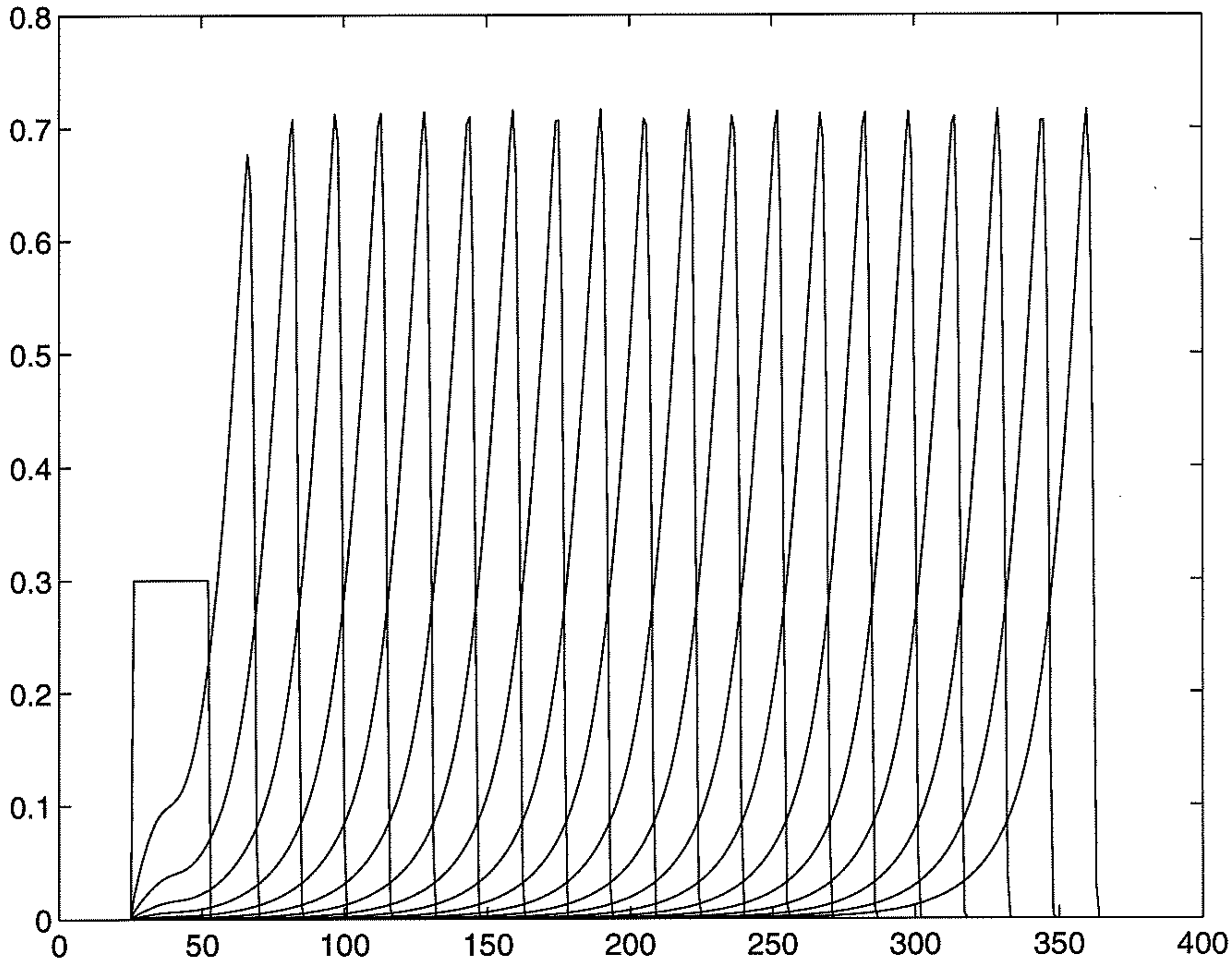

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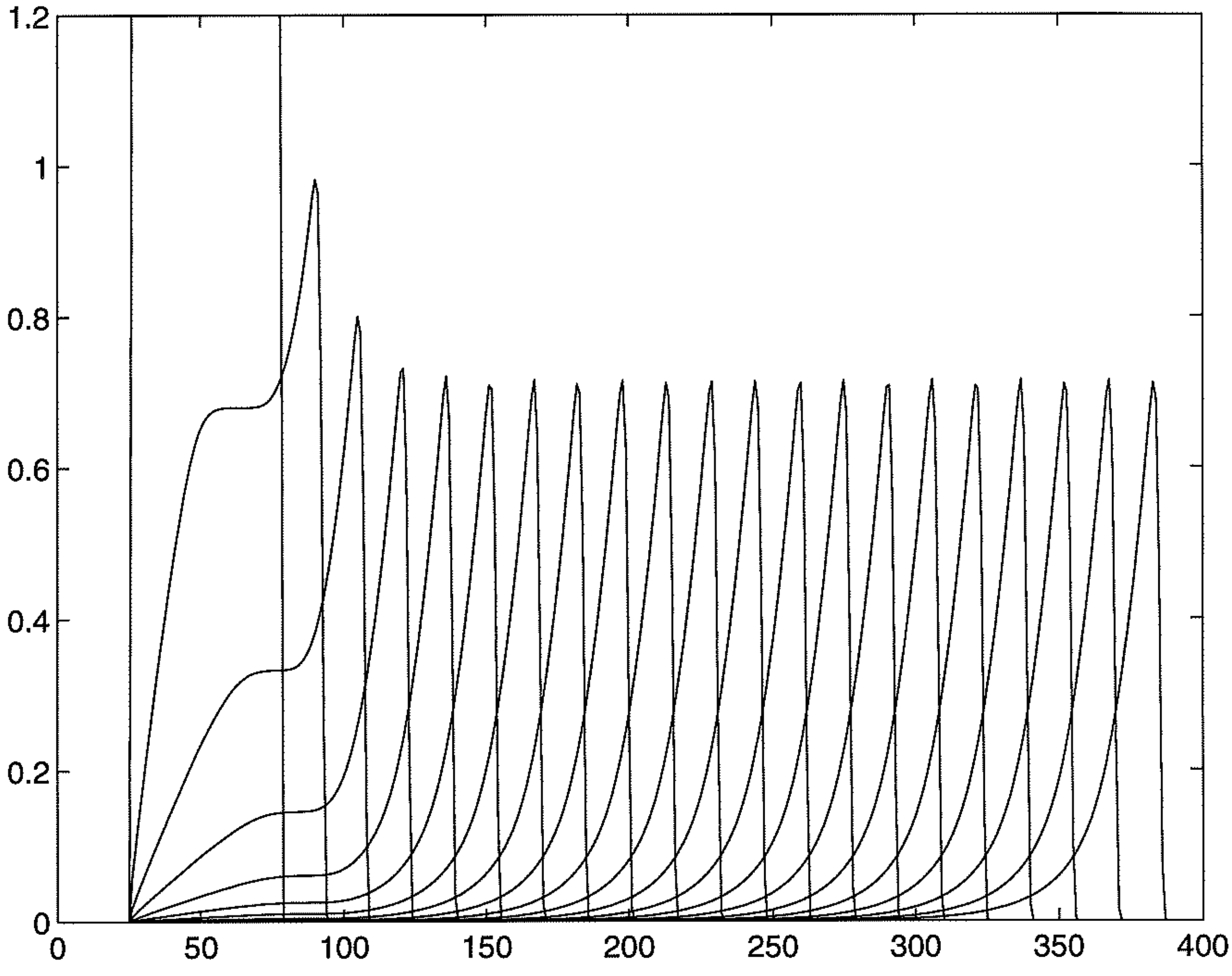
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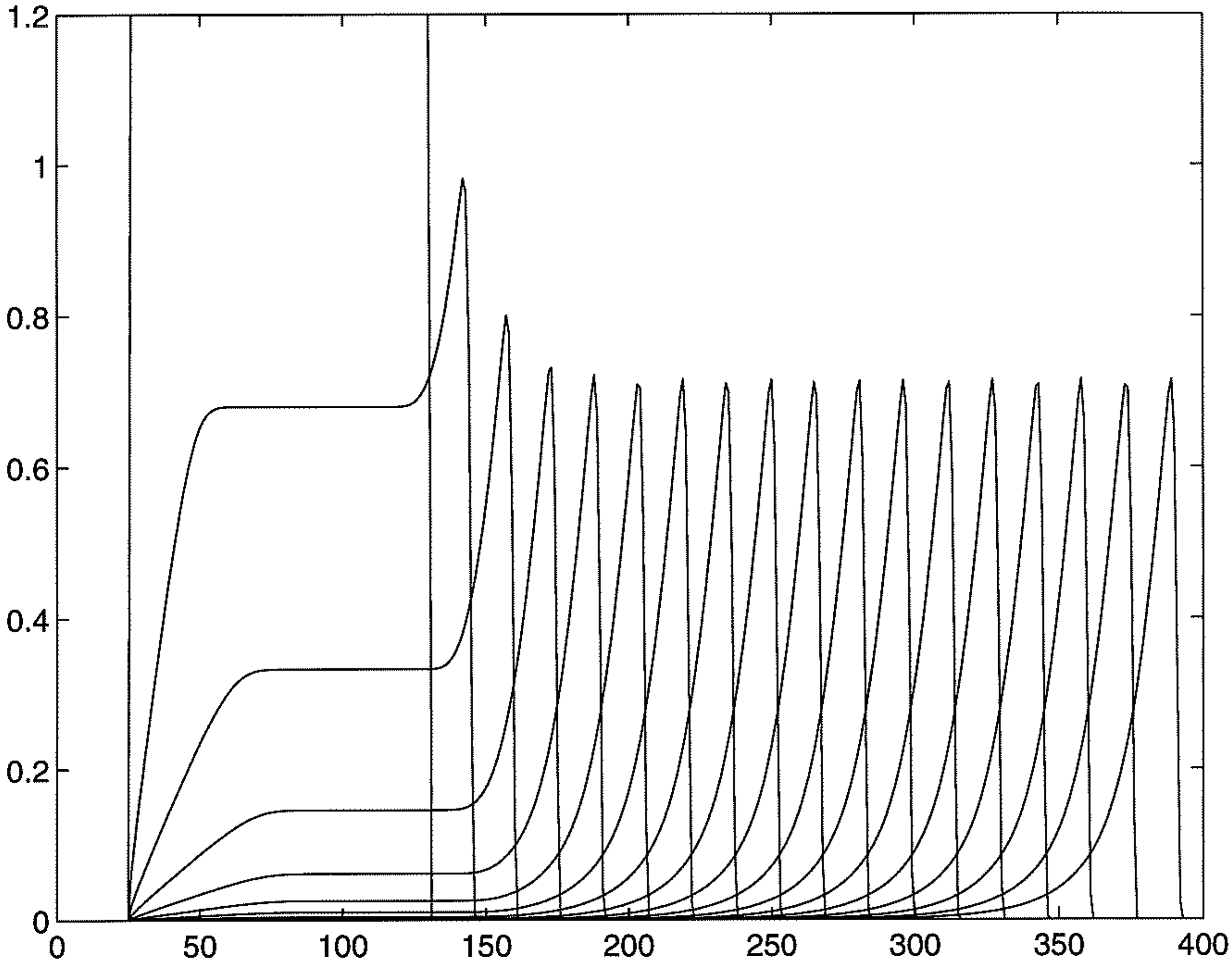

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