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Small Beam Nonparaxiality Arrests Self-Focusing of Optical Beams

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Small beam nonparaxiality arrests self-focusing of optical beams

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A new equation for self-focusing in the presence of small beam nonparaxiality is derived. Analysis of this equation shows that nonparaxiality remains small as the beam propagates. Nevertheless, nonparaxiality arrests self-focusing when the beam width becomes comparable to its wavelength. A series of focusing-defocusing cycles of decreasing magnitude follows, ending with a final defocusing stage.

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The model equation for propagation of a laser beam in a media with a Kerr nonlinearity is the nonlinear Schrödinger equation (NLS)

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi = 0 \tag{1}$$

where $\psi(x,y,z)$ is the electric field envelope, z is the distance in the direction of the beam propagation and $\Delta_\perp$ is the two dimensional Laplacian in the transverse (x,y) plane. Based on this equation Kelley predicted the possibility of catastrophic self-focusing of optical beams whose power is above a threshold value. [1]. Although this prediction was later confirmed in experiments [2], the use of NLS as a model equation for the advanced stages of self-focusing has been often criticized. The singularity formation in NLS is clearly non-physical and it implies that a description of physical self-focusing near and beyond the singularity point should include an additional stabilizing mechanism which is initially small but becomes important near the blowup point, much like the role of viscosity in shock waves formation. Beginning with Feit and Fleck [3], it was argued that the paraxial approximation used in the derivation of NLS from Helmholtz equation is inconsistent with the large focusing angles during the advanced stages of NLS self-focusing and that no singularity will form if beam nonparaxiality is included. Indeed, in the numerical simulations of Feit and Fleck [3] self-focusing is arrested before the beam diameter goes below the order of one wavelength, followed by several focusing-defocusing cycles. Similar behavior was observed in numerical simulations of a 'paraxially modified' NLS [4]. However, there was no analytical theory that explains this behavior, nor was it clear how to reconcile the 'nonparaxial' criticism with the ability of NLS theory to predict the existence and value of a critical power threshold, above which self-focusing is not compensated by diffraction.

In this letter we show that NLS and beam nonparaxiality can be combined into a single model. Our starting point is the scalar Helmholtz equation for the propagation of a laser beam of the form $E = \psi(x,y,z)exp(ikz)$ through a Kerr media:

$$\epsilon\psi_{zz} + i\psi_z + \Delta_\perp\psi + |\psi|^2\psi = 0 \ , \quad \epsilon = \left(\frac{\lambda}{4\pi r_0}\right)^2 \ , \tag{2}$$

where (x,y) and z were scaled by the initial beam radius r_0 and twice the diffraction length $4\pi r_0^2/\lambda$, respectively. Since the beam wavelength λ is much smaller than r_0

$$0 < \epsilon \ll 1 \ .$$

Therefore, the ψ_{zz} term is usually neglected (the *paraxial approximation*), in which case equation (2) reduces to (1). However, from the expression for ϵ it is clear that as the beam is self-focusing the nonparaxial term increases and it becomes comparable to the other terms when the beam width is of the order of a wavelength. In fact, the nonparaxial term has a large effect on self-focusing even when it is still small, since as solutions of NLS self-focus the Laplacian and the cubic nonlinearity balance each other almost completely. As we will see, this key observation will allow us to treat nonparaxiality as a small perturbation.

We now briefly review NLS self-focusing [5,6]. A self-focusing beam can be written as $\psi = \psi_s + \psi_{nf}$, where ψ_{nf} is the non-focusing part of the beam which is 'left behind' as the focusing part of the beam ψ_s approaches the radially symmetric asymptotic lens profile:

$$\psi_s(r,z) \sim \frac{1}{L(z)} V(\xi,\zeta)\exp(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}) \ , \tag{3}$$

$$\xi = \frac{r}{L(z)} \ , \quad \frac{d\zeta}{dz} = \frac{1}{L^2(z)} \ , r = \sqrt{x^2+y^2} \ . \tag{4}$$

The resulting equation for V is

$$iV_\zeta + \Delta_\perp V - V + |V|^2 V + \frac{1}{4}\beta\xi^2 V = 0 \ , \tag{5}$$

$$\beta = -L^3 L_{zz} \ . \tag{6}$$

As the beam is focusing $\beta \searrow 0$ and V approaches the Townes soliton, which is the positive solution of

$$\Delta_\perp R - R + R^3 = 0 \ , \quad R'(0) = 0 \ ,$$

$$\lim_{\xi\to\infty} R(\xi)\xi^{1/2}e^{\xi} = A_R \cong 3.52$$

The Townes soliton has exactly the critical power for self-focusing $N_c = \int_0^\infty R^2(r)\, rdr \cong 1.86$.

When β is small, it is related to the excess power above critical of ψ_s:

$$\beta \sim \frac{N_s - N_c}{M} \ , \ \ N_s = \int |\psi_s|^2 \, r dr \ , \tag{7}$$

$$M = \frac{1}{4}\int_0^\infty r^3 R^2(r)\, dr \cong 0.55 \ .$$

Similarly, the 'energy' of ψ_s is given by

$$H_s \sim \frac{M}{2}(L^2)_{zz} = M\left(L_z^2 - \frac{\beta}{L^2}\right) \ , \tag{8}$$

$$H_s = \int |\nabla_\perp \psi_s|^2 \, rdr - \frac{1}{2}\int |\psi_s|^4 \, rdr \ , \ \ \nabla_\perp \psi = (\psi_x, \psi_y) \ .$$

The rate of power loss of ψ_s (to ψ_{nf}) is given by

$$\frac{d}{dz}N_s \sim \frac{-M\nu(\beta)}{\mathrm{L}^2} \ , \tag{9}$$

$$\nu(\beta) = \begin{cases} \dfrac{2A_R^2}{M} e^{-\pi/\sqrt{\beta}} & \beta > 0 \\ 0 & \beta \le 0 \ . \end{cases} \tag{10}$$

Returning to equation (2), if we multiply it by ψ^* and subtract the conjugate equation, we obtain an equation of power balance for nonparaxial NLS:

$$\frac{d}{dz}\int |\psi|^2 \, rdr = -2\epsilon \int Im(\psi^* \psi_{zz})\, rdr \ . \tag{11}$$

We now make the assumption (that will be justified later) that the nonparaxial term remains small compared with the other terms in (2). Therefore, the left-hand-side of (11) can still be approximated using (7) and (9):

$$\frac{d}{dz}\int |\psi|^2 \, rdr \sim M\beta_z + \frac{M}{L^2}\nu(\beta) \ . \tag{12}$$

To approximate the right-hand-side of (11) we use (3, 4) and the fact that β is small to get

$$\int Im(\psi^* \psi_{zz})\, rdr \ \sim N_c \left(\frac{1}{L^2}\right)_z \ . \tag{13}$$

Combining (11)–(13), nonparaxial self-focusing is described by

$$\beta_z = -\frac{1}{L^2}\nu(\beta) - \frac{2\epsilon N_c}{M}\left(\frac{1}{L^2}\right)_z \tag{14}$$

together with (6). Equation (14) can be also derived from a solvability condition for V.

Since β is small, power losses of ψ_s to ψ_{nf} are small compared with nonparaxial effects. Therefore, we begin the analysis by considering the adiabatic version of (14):

$$\beta' = \frac{-2\epsilon N_c}{M}\left(\frac{1}{L^2}\right)' \ , \ \ ' = \partial_z \ . \tag{15}$$

Direct integration gives

$$\beta = \beta_0 - \frac{2\epsilon N_c}{M}\frac{1}{L^2} \ , \tag{16}$$

$$\beta_0 = \beta(0) + \frac{2\epsilon N_c}{M}\frac{1}{L_0^2} \ , \ \ L_0 = L(0) \ .$$

We are interested in the case where in the absence of nonparaxiality the solution will become singular. Therefore, $\beta_0 > 0$. Equation (16) shows that β becomes negative once L goes below $\sqrt{2\epsilon N_c/M\beta_0}$. To see that this is followed by the arrest of self-focusing we multiply (16) by $L'L^3$, use (6) and integrate one more time to get:

$$y'^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m) \ , \ \ y = L^2 \ , \tag{17}$$

$$y_m = \frac{M\beta_0}{-2H_0}\left(1 - \sqrt{1-4\delta}\right) \sim \frac{\epsilon N_c}{M\beta_0}(1 + O(\delta)) \ , \tag{18}$$

$$y_M = \frac{M\beta_0}{-2H_0}\left(1 + \sqrt{1-4\delta}\right) \sim \frac{M\beta_0}{-H_0}(1 + O(\delta)) \ , \tag{19}$$

$$\delta = \frac{-H_0 N_c \epsilon}{M^2\beta_0^2} \ , \ H_0 = H_s(0) - \frac{\epsilon N_c}{L_0^4} \tag{20}$$

and if ϵ is sufficiently small so that $|\delta| \ll 1$,

$$\frac{y_m}{y_M} \sim \delta \ . \tag{21}$$

If $\beta_0 > 0$ and $H_0 < 0$ then $0 < y_m < y_M$. In this case the solution of (17) is periodic, oscillating between y_m and y_M. The period of the oscillations is (17):

$$\Delta Z = \sqrt{\frac{M}{-H_0}}\int_{y_m}^{y_M}\sqrt{\frac{y}{(y_M - y)(y - y_m)}}\, dy$$

or (substituting $(y - y_m)/(y_M - y_m) = \cos^2 u$)

$$\Delta Z = 2\sqrt{\frac{My_M}{-H_0}}\, \mathrm{E}\left(1 - \frac{y_m}{y_M}\right) \ , \tag{22}$$

where E is the complete elliptic integral of the second kind. The first arrest of self-focusing occurs at $z_0 = \int_{y_m}^{L_0^2} |z_y|\, dy$. In the case of a collimated beam $L_z(0) = 0$, $L_0^2 = y_M$ and $z_0 = \Delta Z/2$. Therefore, as $\epsilon \searrow 0$, z_0 approaches the blowup point in the absence of nonparaxiality $Z_c = L_0^2/\sqrt{\beta(0)}$ [6].

When $\beta_0 > 0$ and $H_0 > 0$ equation (17) can be rewritten as

$$y'^2 = \frac{4H_0}{M}\left(|y_M| + y\right)\left(1 - \frac{y_m}{y}\right) \ .$$

Therefore, focusing is still arrested at $y = y_m$ but from then on the solution will defocus. Note that in both cases the minimal value of L is

$$L_m = y_m^{1/2} \sim \sqrt{\epsilon N_c/M\beta_0} \ , \qquad (23)$$

which in physical variables corresponds to a beam width $\sim 0.15/\sqrt{\beta_0}$ wavelengths. Even at this stage the nonparaxial term is only $O(\beta)$ compared with the other terms in (2), providing an a-posteriori justification for treating it as a small perturbation.

In the analysis up to this point we have neglected non-adiabatic effects. When these are included, after each cycle there is an overall power drop of (14):

$$\Delta\beta := \beta(z+\Delta Z) - \beta(z) = -\int_z^{z+\Delta Z} \frac{\nu(\beta)}{y}\, dz \ . \qquad (24)$$

Therefore, beam propagation is quasi-periodic if

$$\frac{|\Delta\beta|}{\beta_M} \ll 1 \ , \quad \beta_M = \beta(y_M) \ . \qquad (25)$$

In order to estimate $\Delta\beta$ we first use (6, 17) to get:

$$\beta = \frac{1}{4}y'^2 - \frac{1}{2}yy'' = \beta_M\left(1 - 2\frac{y_M/y - 1}{y_M/y_m - 1}\right) \ . \qquad (26)$$

The integral in (24) can be approximated using Laplace method [7] and (10, 26), showing that most power radiation occurs when $y \sim y_M$ and that

$$\Delta\beta \sim \frac{-2A_R^2}{M}\beta_M^{-1/4}\left(\frac{y_M}{y_m} - 1\right)^{1/2} \exp\left(\frac{-\pi}{\sqrt{\beta_M}}\right) \ . \qquad (27)$$

The overall 'energy' increase per cycle is given by (27) and

$$\Delta H := H_s(z+\Delta Z) - H_s(z) \sim -M\Delta\beta \ ,$$

which follows from (8) and the fact that $y' = 0$ when $y = y_M$.

The analysis up to this point suggests that beam power will eventually go below critical, at which point the final defocusing stage will begin. In fact, the last defocusing stage will begin much earlier when the power is still above critical, since $N_s > N_c$ is only the necessary condition for blowup whose physical interpretation is that for blowup to occur the Kerr nonlinearity should be stronger than radial dispersion. However, for a defocusing beam to refocus, the focusing nonlinearity should overcome both radial dispersion and beam divergence, which will only occur when $H_s < 0$.

To solve equations (6) and (14) numerically we define $A = 1/L$ and use Runge-Kutta methods to integrate an equivalent system of equations:

$$\beta_\zeta = -\nu(\beta) - \frac{2\epsilon N_c}{M}(A^2)_\zeta \ , \quad A_{\zeta\zeta} = A\beta \ , \quad z_\zeta = A^{-2} \ .$$

In the following simulations the parameters used are $\beta(0) = 0.1$, $L(0) = 1$ and $L'(0) = 0$. Self-focusing arrest due to small nonparaxiality is seen in Figure 1. As expected, as $\epsilon \searrow 0$ the minimal value of L decreases (23) and the location of the first arrest approaches Z_c. The difference between adiabatic (15) and non-adiabatic (14) nonparaxial self-focusing increases as $\epsilon \searrow 0$ (Figure 2): Beam propagation is quasi-periodic when $\epsilon = 0.01$ but not when $\epsilon = 0.0001$, in agreement with (25) and $\Delta\beta \sim \epsilon^{-1/2}$ (20,21,27). The evolution of L, β and H for these two cases is seen in Figures 3 and 4 for $\epsilon = 0.01$ and 0.0001, respectively. In both cases, with each focusing-defocusing cycle the maximum of β is decreasing, the minimum of H is increasing and the extreme values of L are higher. In the case of stronger nonparaxiality (Figure 3) the focusing-defocusing cycles are almost periodic and less intense. Overall changes in β and H between iterations are small, resulting in a large number of cycles before the final defocusing stage. However, only two focusing cycles are observed in the case of very weak nonparaxiality (Figure 4) after which the beam will defocus without focusing again. Although at this point beam power is still above critical, it is not strong enough to overcome both beam divergence and diffraction (i.e. H_s remains positive).

The results in this letter are in qualitative agreement with previous numerical studies: Self-focusing arrest due to beam nonparaxiality followed by focusing-defocusing cycles with decreasing intensity were observed by Feit and Fleck in numerical simulations of the scalar Helmholtz equation [3] and by Soto-Crespo and Akhmediev in simulations of a 'paraxially modified' NLS [4]. Simulation results in [3] also show abrupt power loss at the self-foci and more gradual power loss in between that eventually lead to cessation of self-focusing. While the gradual power loss agrees with the first term on the right-hand-side of (14) (which peaks when $y \sim y_M$), the abrupt power loss in [3] has to do with the way that back-scattering is incorporated into the numerical model which "simply removes power that cannot propagate in the forward direction without accounting explicitly for where it goes" [3]. In our model we have implicitly assumed that back-scattering is negligible when we represented the solution using only its forward propagating component (3). Our model also does not included power loss due to the vectorial nature of Helmholtz's equations in physical self-focusing. Lax, Louisell and McKnight have shown that NLS is only the leading order equation for the transverse component of Helmholtz's equations and that the solution also has an $O(\epsilon^2/L^2)$ axial component [8]. Therefore, self-focusing is accompanied by power transfer from ψ_s to the axial component. Indeed, recent numerical simulations suggest that self-focusing is arrested in the vectorial case [9].

Although more accurate models should include vectorial effects and back scattering, our analysis shows that both effects will remain small ($O(\beta)$) even when L assumes its smallest value and that self-focusing would still be arrested when $L \sim L_m$. Since both effects will lead to additional power losses (peaking when $L \sim L_m$), the number of focusing-defocusing cycles will be smaller. In our model the exponentially small power loss term plays an important role, providing the only mechanism for the decay of the oscillations. However, its effect will be probably negligible once these additional effects are included.

We have seen that small nonparaxiality has a large effect on self-focusing. However, there is very little difference between self-focusing in NLS and in Helmholtz during the first focusing cycle until the arrest at $L = L_m$. For that reason, NLS may still serve as the model equation for self-focusing in the prefocal region even though nonparaxiality is neglected.

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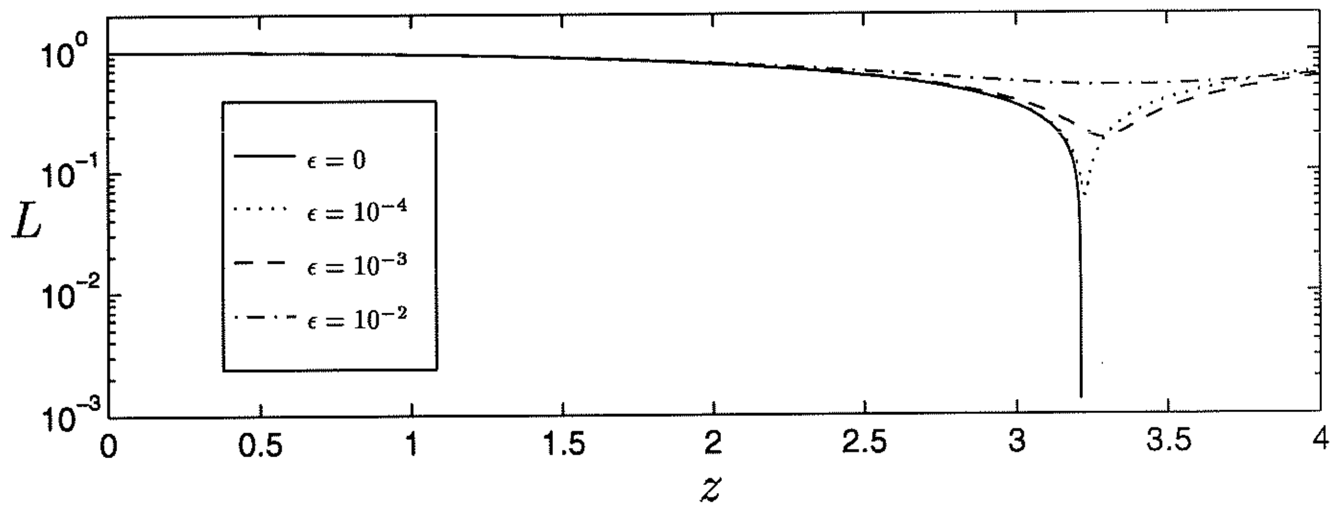

FIG. 1. $L(z)$ for various values of beam nonparaxiality ϵ. The parameters used are $\beta(0) = 0.1$, $L(0) = 1$ and $L'(0) = 0$.

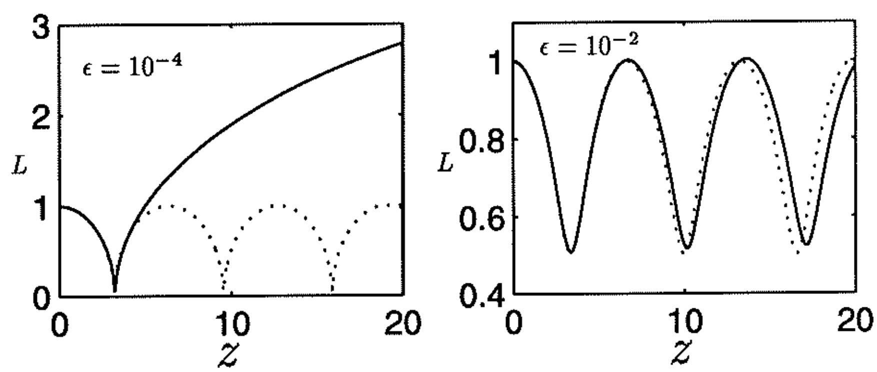

FIG. 2. Adiabatic (eq. (15), dotted line) and non-adiabatic (eq. (14), solid line) nonparaxial self-focusing. The parameters are as in Figure 1.

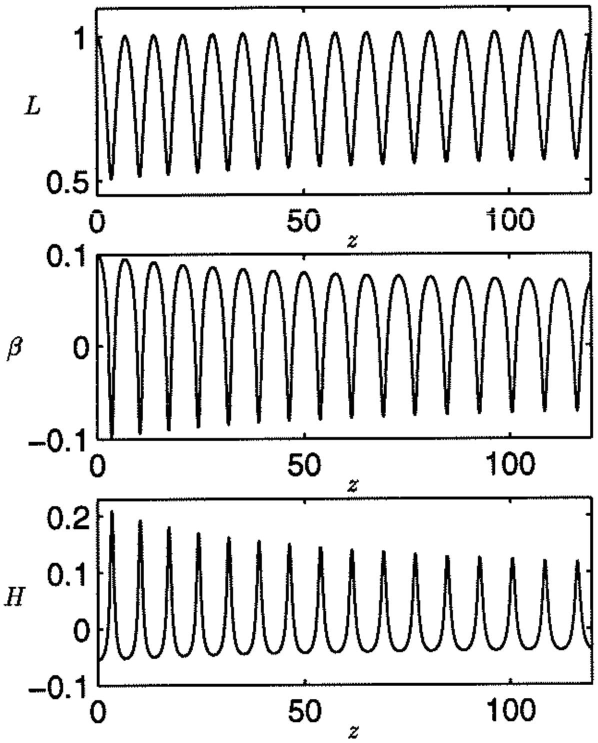

FIG. 3. Nonparaxial self-focusing (eq. 14) for $\epsilon = 0.01$. The other parameters are as in Figure 1.

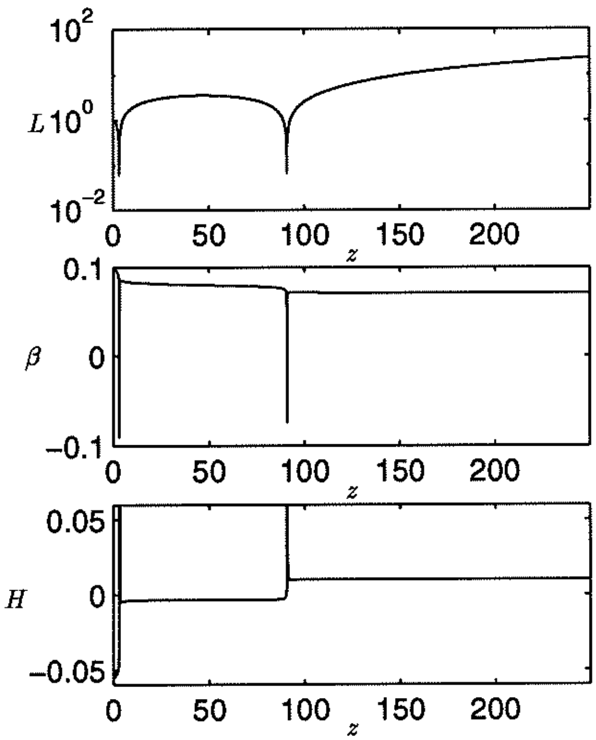

FIG. 4. Nonparaxial self-focusing (eq. 14) for $\epsilon = 0.0001$. The other parameters are as in Figure 1.