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Zero Viscosity Limit for Analytic Solutions of the Navier-Stokes Equation on a Half-Space I. Existence for Euler and Prandtl Equations

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Abstract

This is the first of two papers on the zero-viscosity limit for the incompressible Navier-Stokes equations in a half-space. In this paper we prove short time existence theorems for the Euler and Prandtl equations with analytic initial data in either two or three spatial dimensions. The main technical tool in this analysis is the abstract Cauchy-Kowalewski theorem. For the Euler equations, the projection method is used in the primitive variables, to which the Cauchy-Kowalewski theorem is directly applicable. For the Prandtl equations, Cauchy-Kowalewski is applicable once the diffusion operator in the vertical direction is inverted.

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1 Introduction

The zero-viscosity limit for the incompressible Navier-Stokes equations in a half-space is a challenging problem due to the formation of a boundary layer whose thickness is proportional to the square root of the viscosity. Boundary layer separation, which is difficult to control, may cause singularities in the boundary layer equations. In this and the companion paper [12], we overcome these difficulties by imposing analyticity on the initial data. Under this condition, we prove that, in the zero-viscosity limit and for a short time, the Navier-Stokes solution in a half-space goes to an Euler solution outside a boundary layer in either two or three spatial dimensions, and that it is close to a solution of the Prandtl equations within the boundary layer.

The construction of the Navier-Stokes solution is performed as a composite asymptotic expansion involving an Euler solution, a Prandtl boundary layer solution and a correction term. It follows the earlier, unpublished analysis of Asano [1], who also restricted the data to be analytic, but our work contains a considerably simplified exposition, explicit use of the Prandtl equations, and several other technical differences: Asano used a sup-estimate on the divergence-free projection operator, which we have been unable to verify. He also used high order derivative norms in the y (normal) variable, whereas we find it necessary to only use second derivatives.

An earlier attempt to analyze this problem, without the requirement of analyticity and without explicit use of the Prandtl equations, was made by Kato [8]. It was not completely successful, since it required some unverified assumptions on the Navier-Stokes solution. Analysis of the zero-viscosity limit for the Navier-Stokes solution in an unbounded domain was performed in [3, 7, 13]

In this first part, we present short-time existence results for the Euler and Prandtl equations in a half-space with analytic initial data. The main significance of the Euler result is that it is stated in terms of the function spaces used in the Navier-Stokes result of Part II [12]. For the Euler equations, of course, this is not an optimal result since analyticity is not needed for existence of a solution. Moreover a more general existence result for analytic solutions of the incompressible Euler equations was proved earlier by Bardos and Benachour [2]. The present proof is somewhat different, since it uses the projection method on the primitive variables, rather than the vorticity formulation.

For the Prandtl equations, on the other hand, our result on existence for short time and analytic initial data is the first general existence theorem for the unsteady problem. To the best of our knowledge, the only previous existence theorem for the unsteady Prandtl equations was by Oleinik [10]. For the Prandtl equations with upstream velocity prescribed at the left ($x = 0$) as well as at infinity and at $t = 0$, she proved existence for either a short time for all $x > 0$ or for a short distance and for all time, without the analyticity assumption. The proof required conditions that the prescribed horizontal velocities are all positive and strictly increasing, which are not required in our result. For a review of related mathematical results on both the steady and unsteady Prandtl equations, see [9].

In fact, we conjecture that the general initial value problem for the Prandtl equations is ill-posed in Sobolev space. Although ill-posedness has not been proved, there is some evidence in its favor: First, previous attempts to construct such solutions have failed. Second, there are numerical solutions of Prandtl that develop singularities associated with boundary layer separation in finite time [4, 5, 6]. This is not enough to show ill-posedness, however, because in these computations the singularity time is not small.

The main technical tool of our analysis is the abstract Cauchy-Kowalewski Theorem (ACK), the optimal form of which is due to Safonov [11]. This theorem, which is for systems that are first order in some sense, is directly applicable to the Euler equations. Since the Prandtl equations are diffusive rather than first order, the classical Cauchy-Kowalewski Theorem cannot be applied, and it may at first seem surprising that the ACK Theorem is applicable to them. As pointed out by Asano [1], however, the ACK Theorem may be used for a nonlinear diffusion equation after inversion of the diffusion operator. We show below that this strategy works for the Prandtl equations, and in Part II, we shall also apply it to the Navier-Stokes equations. The main simplification of this analytic method over Sobolev methods is that it uses Cauchy estimates to bound derivatives rather than energy estimates.

In Section 2 the Euler and Prandtl equations are stated and a number of function spaces and norms are defined. The abstract Cauchy-Kowalewski Theorem is formulated in Section 3. The existence theorem for the Euler equations is stated and proved in Section 4, which includes a convenient formulation and some useful bounds for the projection method. The existence theorem for the Prandtl equations is stated and proved in Section 5, using properties of the heat operator, which are proved in an appendix. The analysis for Prandtl is completely independent of that for Euler. Some concluding remarks are made in Section 6.

For convenience, the formulation and analysis will often be written in 2D, but the extension to 3D is straightforward. Key points in the 3D extension will be noted and the main results will be stated for 2D and 3D.

Part of this work has been done while Marco Sammartino was visiting the Mathematics Department of UCLA. He wishes to express his gratitude for the warm hospitality that he received.

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2 Statement of the problem and notation

2.1 Euler Equations

The Euler equations for a velocity field $\boldsymbol{u}^{E} = (u^E, v^E)$ are

$$\partial_t \boldsymbol{u}^{E} + \boldsymbol{u}^{E} \cdot \nabla \boldsymbol{u}^{E} + \nabla p^E = 0 \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u}^{E} = 0 \tag{2.2}$$

$$\gamma_n \boldsymbol{u}^{E} \equiv v^E(x, y=0, t) = 0 \tag{2.3}$$

$$\boldsymbol{u}^{E}(x, y, t=0) = \boldsymbol{u}_0^{E}(x, y) \ . \tag{2.4}$$

Here $\boldsymbol{u}^{E}$ depends on the variables (x, y, t), where x is the transversal variable going from $-\infty$ to ∞, y is the normal variable going from 0 to ∞, and t is the time. The operator γ_n acting on vectorial functions gives the normal component calculated at the boundary. In the rest of this paper we shall also use the trace operator γ, defined by

$$\gamma \boldsymbol{u} = (u(x, y=0, t), v(x, y=0, t)) \ . \tag{2.5}$$

In Section 3 we shall prove that under suitable hypotheses for the initial condition $\boldsymbol{u}_0^{E}$ (essentially analyticity in x and y) Euler equations admit a unique solution. Although stated here for 2D, the analysis works equally well in 3D. The existence result is only for a short time.

2.2 Prandtl equations

The Euler equations are a particular case of the Navier-Stokes (N-S) equations, when the fluid has zero viscosity. Therefore, the Navier-Stokes solution for small viscosity ν is expected to be well approximated by an Euler solution, at least away from boundaries, which is confirmed by numerical and experimental observations. An analysis of the short time, spatially global behavior (in presence of a boundary) of N-S equations will be the subject of part II of this work [12].

In the vicinity of the boundaries, on the other hand, the effect of viscosity is $O(1)$ even as the viscosity goes to zero. The no-slip condition causes the creation of vorticity; moreover in a small layer there is an adjustment of the flow to the outer (inviscid) flow. Due to the resulting rapid variation of the fluid velocity, the velocity depends on a scaled normal variable $Y = y/\varepsilon$ in which $\varepsilon = \sqrt{\nu}$. Also the vertical velocity is of size ε. The resulting equations governing the velocity field $\boldsymbol{u}^P = (u^P, \varepsilon v^P)$ are Prandtl equations:

$$(\partial_t - \partial_{YY}) u^P + u^P \partial_x u^P + v^P \partial_Y u^P + \partial_x p^P = 0 \tag{2.6}$$
$$\partial_Y p^P = 0 \tag{2.7}$$
$$\partial_x u^P + \partial_Y v^P = 0 \tag{2.8}$$
$$u^P (x, Y = 0, t) = 0 \tag{2.9}$$
$$u^P (x, Y \to \infty) \longrightarrow u^E (x, y = 0) \tag{2.10}$$
$$u^P (x, Y, t = 0) = u_0^P (x, Y) \ . \tag{2.11}$$

Equation (2.10) is the matching condition between the flow inside the boundary layer and the outer Euler flow. In Section 5 we shall prove that the Prandtl solution approaches the boundary value of the Euler solution at an exponential rate as Y goes to infinity. Equation (2.7) implies that the pressure is constant across the boundary layer; to match with the Euler pressure p^E, it must be

$$p^P = p^E (x, y = 0, t) = -\gamma(\partial_t + u^E \partial_x) u^E \ . \tag{2.12}$$

The normal component of the velocity v^P can be found, using the incompressibility condition, to be

$$v^P = -\int_0^Y \partial_x u^P(x, Y', t) dY' \ . \tag{2.13}$$

In 3D, $\partial_x u^P$ is replaced by $\nabla' \cdot \boldsymbol{u}^P$ in this integral, where ∇' is the gradient with respect to the transversal variables.

Therefore Eq. (2.6) can be considered as an equation for the transversal component u^P, with v^P given by Eq. (2.13), with boundary conditions (2.9)-(2.10), and with initial conditions (2.11). Moreover there must be compatibility between the boundary conditions and initial conditions; i.e.

$$\gamma u_0^P = 0 \tag{2.14}$$

$$u_0^P (x, Y \to \infty) - \gamma u_0^E \longrightarrow 0. \tag{2.15}$$

In this paper we shall prove the existence and the uniqueness of the solutions for equations (2.1)-(2.4) and (2.6)-(2.11). We now introduce the appropriate function spaces.

2.3 Function spaces

Let us introduce the "strip", the angular sector and the "conoid" in the complex plane

$$D(\rho) = \boldsymbol{R} \times (-\rho, \rho) = \{x \in \boldsymbol{C} \ : \Im x \in (-\rho, \rho)\} \tag{2.16}$$
$$\Sigma(\theta) = \{y \in \boldsymbol{C} \ : \Re y \geq 0 \ and \ |\Im y| \leq \Re y \tan\theta\} \tag{2.17}$$
$$\begin{aligned}\Sigma(\theta, a) = & \{y \in \boldsymbol{C} \ : 0 \leq \Re y \leq a \ and \ |\Im y| \leq \Re y \tan\theta\} \\ & \cup \{y \in \boldsymbol{C} \ : \Re y \geq a \ and \ |\Im y| \leq a \tan\theta\} \ .\end{aligned} \tag{2.18}$$

In the sequel we shall always be dealing with functions that are analytic in either the single complex variable x or the two complex variables x and y. The functions will be either L^2 in the transversal variable x and bounded in the normal variable y, or L^2 in both the transversal and normal variable. Next introduce the paths along which the L^2 integration is performed:

$$\Gamma(b) = \{x \in \boldsymbol{C} \ : \Im x = b\} \tag{2.19}$$
$$\begin{aligned}\Gamma(\theta', a) = & \{y \in \boldsymbol{C} \ : 0 \leq \Re y \leq a \ and \ \Im y = \Re y \tan\theta'\} \\ & \cup \{y \in \boldsymbol{C} \ : \Re y \geq a \ and \ \Im y = a \tan\theta'\} \ .\end{aligned} \tag{2.20}$$

Some of the norms below are defined in terms of the unscaled variable y; while others used the scaled variable $Y = y/\varepsilon$. Throughout this paper, the values of the angle θ and the parameter l counting the number of derivatives will always be restricted to

$$0 < \theta < \pi/4$$
$$4 \leq l.$$

We have not attempted to make an optimal choice of l. Given a Banach scale $\{X_\rho\}_{0\leq\rho\leq\rho_0}$ we define $B^k_\beta(A, X_\rho)$ as the space of all C^k functions from A to X_ρ with the norm

$$|f|_{k,\rho,\beta} = \sum_{j=0}^{k} \sup_{t\in A} |\partial_t^j f(t)|_{\rho-\beta t}. \tag{2.21}$$

Here A is supposed to be an interval $[0, T]$ of time, and ρ may be a vector of parameters such as (ρ, θ) or (ρ, θ, μ), in which case $\rho - \beta t$ is replaced by $(\rho - \beta t, \theta - \beta t)$ or $(\rho - \beta t, \theta - \beta t, \mu - \beta t)$. Due to the large number of function spaces and norms, we do not have a separate notation for every norm. Instead, we will always state the function space under consideration, and then the norm will be the one for that space.

Table 1: Table of Function Spaces.
(E=Euler, E_1= first order Euler, P=Prandtl, S=Stokes, NS=Navier-Stokes)

Space	L^2	sup	$O(\partial_x)$	$O(\partial_y)$ or $O(\partial_Y)$	$O(\partial_t)$	equations
$H^{\prime l,\rho}$	x		l			E,P,S,NS
$H^{l,\rho,\theta}$	x, y		$j \leq l$	$l - j$		E
$H^{l,\rho,\theta}_{\beta,T}$	x, y	t	$j \leq l$	$k \leq l - j$	$l - k - j$	E
$K^{l,\rho,\theta,\mu}$	x	Y	$l - k$	$k \leq 2$		P
$K^{\prime l,\rho}_{\beta,T}$	x	t	$l - j$		$j \leq 1$	P,S
$K^{l,\rho,\theta,\mu}_{\beta,T}$	x	Y, t	$l - k$ $l - 1$	$k \leq 2$ 0	0 1	P,S
$L^{l,\rho,\theta}$	x, Y		l $l - 2$	0 $0 < k \leq 2$		S, NS
$L^{l,\rho,\theta}_{\beta,T}$	x, Y	t	$l - 2j$ $l - 2$	0 $0 < k \leq 2$	$j \leq 1$ 0	S,NS
$N^{l,\rho,\theta}$	x, y		l $l - 2$	0 $0 < k \leq 2$		E_1
$N^{l,\rho,\theta}_{\beta,T}$	x, y	t	$l - 2j$ $l - 2$	0 $0 < k \leq 2$	$j \leq 1$ 0	E_1

The following are function spaces that will be used in the sequel. We begin with the space of functions depending only on the transversal variable (the x-variable). All the functions used below will belong, for a fixed time t and for a fixed value of the normal coordinate y, to this space. A summary of these function spaces is made in Table 1.

Definition 2.1 *$H'^{l,\rho}$ is the set of all complex functions $f(x)$ such that*

- *f is analytic in $D(\rho)$*
- *$\partial_x^\alpha f \in L^2(\Gamma(\Im x))$ for $\Im x \in (-\rho, \rho)$, $\alpha \leq l$; i.e. if $\Im x$ is inside $(-\rho, \rho)$, then $\partial_x^\alpha f(\Re x + i\Im x)$ is a square integrable function of $\Re x$*
- *$|f|_{l,\rho} = \sum_{\alpha \leq l} \sup_{\Im x \in (-\rho,\rho)} \|\partial_x^\alpha f(\cdot + i\Im x)\|_{L^2(\Gamma(\Im x))} < \infty$.*

We now introduce the dependence on the transversal variable:

Definition 2.2 *$H^{l,\rho,\theta}$ is the set of all functions $f(x, y)$ such that*

- *f is analytic inside $D(\rho) \times \Sigma(\theta, a)$*
- *$\partial_y^{\alpha_1} \partial_x^{\alpha_2} f(x, y) \in L^2 \left(\Gamma(\theta', a); H'^{0,\rho}\right)$ with $|\theta'| \leq \theta$, $\alpha_1 + \alpha_2 \leq l$*
- *$|f|_{l,\rho,\theta} = \sum_{\alpha_1+\alpha_2 \leq l} \sup_{|\theta'| \leq \theta} \| |\partial_y^{\alpha_1} \partial_x^{\alpha_2} f(\cdot, y)|_{0,\rho} \|_{L^2(\Gamma(\theta',a))} < \infty$.*

For a fixed time t all the functions used in the proof of existence and uniqueness for Euler equations will belong to the above space. We now introduce the functions depending on time:

Definition 2.3 *The space $H^{l,\rho,\theta}_{\beta,T}$ is defined as*

$$M^{l,\rho,\theta}_{\beta,T} = \bigcap_{j=0}^{l} B^j_\beta \left([0, T], H^{l-j,\rho,\theta}\right).$$

If a function $f(x, y, t)$ belongs to this space its norm is

$$|f|_{l,\rho,\theta,\beta,T} = \sum_{j=0}^{l} \sup_{0 \leq t \leq T} |\partial_t^j f(\cdot, \cdot, t)|_{l-j,\rho-\beta t,\theta-\beta t}\,.$$

In the above spaces we shall prove the existence of a solution of the Euler equations.

We now pass to the function spaces for Prandtl equations. The main difference with respect to the Euler equations is the presence of the heat operator $(\partial_t - \partial_{YY})$, which breaks the symmetry between the normal and transversal coordinates. We can require differentiability with respect to the transversal coordinate Y only up to the second order, and with respect to time t only up to the first order. Moreover, for Prandtl equations we shall use, in the Y variable, the sup norm. This will allow us to observe the behavior of the Prandtl solution outside the boundary layer. The Prandtl solution will in fact turn out to exponentially match the boundary data of the Euler solution.

Definition 2.4 *$K^{l,\rho,\theta,\mu}$, with $\mu > 0$, is the set of all functions $f(x, Y)$ such that*

- *f is analytic inside $D(\rho) \times \Sigma(\theta)$*
- *$\partial_Y^{\alpha_1} \partial_x^{\alpha_2} f(x, Y) \in C^0 \left(\Sigma(\theta); H'^{0,\rho}\right)$ with $\alpha_1 \leq 2$ and $\alpha_1 + \alpha_2 \leq l$*
- *$|f|_{l,\rho,\theta,\mu} = \sum_{\alpha_1 \leq 2} \sum_{\alpha_2 \leq l-\alpha_2} \sup_{Y \in \Sigma(\theta)} e^{\mu \Re Y} |\partial_Y^{\alpha_1} \partial_x^{\alpha_2} f(\cdot, Y)|_{0,\rho} < \infty$.*

Definition 2.5 *The space* $K^{\prime l,\rho}_{\beta,T}$ *is defined as*

$$K^{\prime l,\rho}_{\beta,T} = \bigcap_{j=0}^{1} B^j_\beta\left([0,T], H^{\prime l-j,\rho}\right).$$

If $f(x,t)$ *belongs to this space its norm is*

$$|f|_{l,\rho,\beta,T} = \sum_{j=0}^{1} \sum_{\alpha\le l-j} \sup_{0\le t\le T} |\partial_t^j \partial_x^\alpha f(t,\cdot)|_{0,\rho-\beta t}\,.$$

Definition 2.6 *The space* $K^{l,\rho,\theta,\mu}_{\beta,T}$ *is the set of all functions* $f(x,Y,t)$ *such that*

- $f \in C^0\left([0,T], K^{l,\rho,\theta,\mu}\right)$ *and* $\partial_t \partial_x^\alpha f \in C^0\left([0,T], K^{0,\rho,\theta,\mu}\right)$ *with* $\alpha \le l-1$
- $|f|_{l,\rho,\theta,\mu,\beta,T} = \sum_{\alpha_1\le 2}\sum_{\alpha_1+\alpha_2\le l} \sup_{0\le t\le T} |\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,Y,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t}$ $+ \sum_{\alpha\le l-1} \sup_{0\le t\le T} |\partial_t \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} < \infty.$

We now pass to the spaces for the correction terms in the Navier-Stokes solution. We shall use the spaces below only in [12].

Definition 2.7 $N^{l,\rho,\theta}$ *is the set of all functions* $f(x,y)$ *such that*

- f *is analytic inside* $D(\rho)\times\Sigma(\theta,a)$
- $\partial_y^{\alpha_1}\partial_x^{\alpha_2} f(x,y) \in L^2\left(\Gamma(\theta',a); H^{\prime 0,\rho}\right)$ *with* $|\theta'|\le\theta$, $\alpha_1+\alpha_2\le l$ *and* $\alpha_2 \le l-2$ *when* $\alpha_1 > 0$.
- $|f|_{l,\rho,\theta} = \sum_{\alpha\le l} \sup_{|\theta'|\le\theta} \| |\partial_x^\alpha f(\cdot,y)|_{0,\rho}\|_{L^2(\Gamma(\theta',a))}$ $+ \sum_{0<\alpha_1\le 2}\sum_{\alpha_2\le l-2} \sup_{|\theta'|\le\theta} \| |\partial_y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,y)|_{0,\rho}\|_{L^2(\Gamma(\theta',a))} < \infty.$

Definition 2.8 *The space* $N^{l,\rho,\theta}_{\beta,T}$ *is defined as the set of all functions* $f(x,y,t)$ *such that:*

- $f \in C^0([0,T], N^{l,\rho,\theta})$ *and* $\partial_t\partial_x^j f \in C^0([0,T], N^{0,\rho,\theta})$ *with* $j \le l-2$
- $|f|_{l,\rho,\theta,\beta,T} = \sum_{0\le j\le 1}\sum_{\alpha\le l-2j} \sup_{0\le t\le T} |\partial_t^j\partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t}$ $+ \sum_{0<\alpha_1\le 2}\sum_{\alpha_2\le l-2} \sup_{0\le t\le T} |\partial_y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} < \infty$

in which the norms on the right are in $N^{l,\rho,\theta}$.

In the above two spaces we shall prove the existence of a solution for the first order correction of the Euler flow (see Section 5 of [12]).

Definition 2.9 $L^{l,\rho,\theta}$ *is the set of all functions* $f(x,Y)$ *such that*

- f *is analytic inside* $D(\rho)\times\Sigma(\theta,a/\varepsilon)$
- $\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(x,Y) \in L^2\left(\Gamma(\theta',a/\varepsilon); H^{\prime 0,\rho}\right)$ *with* $|\theta'|\le\theta$, $\alpha_1\le 2$, $\alpha_1+\alpha_2\le l$ *and* $\alpha_2\le l-2$ *when* $\alpha_1>0$
- $|f|_{l,\rho,\theta} = \sum_{\alpha\le l} \sup_{|\theta'|\le\theta} \| \, |\partial_x^\alpha f(\cdot,Y)|_{0,\rho}\|_{L^2(\Gamma(\theta',a/\varepsilon))}$ $+ \sum_{0<\alpha_1\le 2}\sum_{0\le\alpha_2\le l-2} \sup_{|\theta'|\le\theta} \| \, |\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,Y)|_{0,\rho}\|_{L^2(\Gamma(\theta',a/\varepsilon))} < \infty.$

Definition 2.10 *The space* $L^{l,\rho,\theta}_{\beta,T}$ *is the set of functions* $f(x,Y,t)$ *such that*

- $f \in C^0([0,T], L^{l,\rho,\theta})$ *and* $\partial_t \partial_x^\alpha f \in C^0([0,T], L^{0,\rho,\theta})$ *with* $\alpha \le l-2$
- $|f|_{l,\rho,\theta,\beta,T} = \sum_{0\le j\le 1} \sum_{\alpha \le l-2j} \sup_{0\le t\le T} |\partial_t^j \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t}$
$+ \sum_{0<\alpha_1\le 2} \sum_{\alpha_2 \le l-2} \sup_{0\le t\le T} |\partial_Y^{\alpha_1} \partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} < \infty$

The above two spaces are the spaces where we shall prove existence and uniqueness for the overall Navier-Stokes correction (see Section 7 of Part II [12]).

The large number of function spaces is needed because of the various types of data and equations that are considered here:

- The H function spaces, which are $L^2(x,y)$, are natural function spaces for the Euler equations.
- The K function spaces, which are $L^\infty(L^2(x), Y)$ with decay in Y, are natural for the Prandtl equations.
- For Navier-Stokes, the L spaces, which are $L^2(x,Y)$ with a restricted number of derivatives in Y, are used to allow combination of the Euler and Prandtl results.
- For the first order Euler terms, the N spaces, which are $L^2(x,y)$ with a restricted number of derivatives in y, are used.

We shall also use a large number of operators in this analysis. For convenience and clarity, we list the operators used in this paper as well as in the second paper, in the following table, with reference to the location of their definitions.

Table 2: Table of Operators.
(E=Euler, E_1 first order Euler, P=Prandtl, S=Stokes, NS=Navier-Stokes, I=this paper, II=Ref [12])

Operator	Description	Definition	Use
P^∞	projection	I (4.4)	E , E_1
P	half-space projection	I (4.24), (4.25)	E , E_1
P_t	integrated (in time) half-space projection	I (4.35)	E , E_1
$E_0(t)$	heat op with IC, diff in Y	I (5.4)	P
E_1	heat op with BC, diff in Y	I (5.8)	P
E_2	heat op with force, diff in Y	I (5.11)	P
N'	$i\xi'/\lvert\xi'\rvert$	I (4.10)	E,S,NS
$\tilde{E}_1$	heat op with BC, diff in x, Y	II (3.21)	S, NS
$\tilde{E}_2$	heat op with force, diff in x, Y	II (7.4)	NS
$\overline{P}^\infty$	rescaled projection	II (7.12)(7.13)	NS
$\mathcal{S}$	Stokes op	II (3.36)	S, NS
$\mathcal{N}_0$	projected heat op	II (7.15)	NS
$\mathcal{N}^*$	Navier-Stokes operator	II (7.20)	NS

In the following Sections we shall often estimate products of functions belonging to the above spaces; if $l \geq 4$ such products can be estimated using the following Sobolev inequalites:

Proposition 2.1 *Let $f, g \in H^{l,\rho,\theta}_{\beta,T}$ and $l \geq 4$. Then $f \cdot g \in H^{l,\rho,\theta}_{\beta,T}$, and*

$$|f \cdot g|_{l,\rho,\theta,\beta,T} \leq c|f|_{l,\rho,\theta,\beta,T}|g|_{l,\rho,\theta,\beta,T} \,. \tag{2.22}$$

Proposition 2.2 *Let $f, g \in L^{l,\rho,\theta}_{\beta,T}$ and $l \geq 4$. Then $f \cdot g \in L^{l,\rho,\theta}_{\beta,T}$, and*

$$|f \cdot g|_{l,\rho,\theta,\beta,T} \leq c|f|_{l,\rho,\theta,\beta,T}|g|_{l,\rho,\theta,\beta,T} \,. \tag{2.23}$$

A similar statement holds if we are using the sup norm in the Y variable:

Proposition 2.3 *Let $f \in K^{l,\rho,\theta,0}_{\beta,T}$, $g \in K^{l,\rho,\theta,\mu}_{\beta,T}$ and $l \geq 3$ ($l \geq 4$ in 3D). Then $f \cdot g \in K^{l,\rho,\theta,\mu}_{\beta,T}$, and*

$$|f \cdot g|_{l,\rho,\theta,\beta,\mu,T} \leq c|f|_{l,\rho,\theta,\beta,0,T}|g|_{l,\rho,\theta,\beta,\mu,T} \,. \tag{2.24}$$

3 Cauchy-Kowalewski Theorem

By a Banach scale $\{X_\rho :\ 0 < \rho \leq \rho_0\}$ with norms $|\ |_\rho$ we mean a collection of Banach spaces such that $X_{\rho'} \subset X_{\rho''}$ and $|\ |_{\rho''} \leq |\ |_{\rho'}$ when $\rho'' \leq \rho' \leq \rho_0$.

Let $\tau > 0$, $0 < \rho \leq \rho_0$ and $R > 0$.

Definition 3.1 *$X_{\rho,\tau}$ is the set of all functions $u(t)$ from $[0, \tau]$ to X_ρ endowed with the norm*

$$|u|_{\rho,\tau} = \sup_{0 \leq t \leq \tau} |u(t)|_\rho \,. \tag{3.1}$$

Definition 3.2 *$X_{\rho,\tau}(R)$ is the set of all functions $u(t)$ from $[0, \tau]$ to X_ρ such that*

$$|u|_{\rho,\tau} \leq R \,. \tag{3.2}$$

Definition 3.3 *$Y_{\rho,\beta,\tau}$ is the set of all functions $u(t)$ from $[0, \tau]$ to X_ρ endowed with the norm*

$$|u|_{\rho,\beta,\tau} = \sup_{0 \leq t \leq \tau} |u(t)|_{\rho-\beta t} \,. \tag{3.3}$$

Definition 3.4 *$Y_{\rho,\beta,\tau}(R)$ is the set of all functions $u(t)$ from $[0, \tau]$ to X_ρ such that*

$$|u|_{\rho,\beta,\tau} \leq R \,. \tag{3.4}$$

For t in $[0, T]$, consider the equation

$$u + F(t, u) = 0 \,. \tag{3.5}$$

The basic existence theorem for this system is the following Abstract Cauchy-Kowalewski (ACK) Theorem, which is a slight modification of the version proved by Safonov [11].

Theorem 3.1 *Suppose that $\exists R > 0$, $T > 0$, $\rho_0 > 0$, and $\beta_0 > 0$ such that if $0 < t \leq T$, the following hold:*

1. *$\forall 0 < \rho' < \rho \leq \rho_0 - \beta_0 T$ and $\forall u \in X_{\rho,T}(R)$ the function $F(t, u) : [0, \tau] \longrightarrow X_{\rho'}$ is continuous.*
2. *$\forall 0 < \rho \leq \rho_0 - \beta_0 T$ the function $F(t, 0) : [0, T] \longrightarrow X_{\rho,T}(R)$ is continuous in $[0, T]$ and*

$$|F(t, 0)|_{\rho_0 - \beta_0 t} \leq R_0 < R \,. \tag{3.6}$$

3. $\forall 0 < \rho' < \rho(s) \le \rho_0 - \beta_0 s$ *and* $\forall\, u^1$ *and* $u^2 \in Y_{\rho_0,\beta_0,T}(R)$,

$$|F(t,u^1) - F(t,u^2)|_{\rho'} \le C \int_0^t \frac{|u^1 - u^2|_{\rho(s)}}{\rho(s) - \rho'} ds\,. \tag{3.7}$$

Then $\exists \beta > \beta_0$ *and* $T_1 > 0$ *such that Eq. (3.5) has a unique solution in* Y_{ρ_0,β,T_1}.

In the applications below, this theorem will be applied with ρ replaced by a vector of parameters (ρ,θ) or (ρ,θ,μ), and the fraction $(\rho(s)-\rho')^{-1}$ is replaced by $(\rho(s)-\rho')^{-1} + (\theta(s)-\theta')^{-1}$ or $(\rho(s)-\rho')^{-1} + (\theta(s)-\theta')^{-1} + (\mu(s)-\mu')^{-1}$. This does not change the proof of the theorem.

In the analysis below, the Euler Prandtl and Navier-Stokes equations will be solved in a time integrated form. That is, the system $w_t = A(w)$ will replaced by $u = A(\int u dt)$ in which $u = w_t$. In this form the natural estimate on the difference of F is the right hand side of (3.7) plus an additional term like

$$\left(\int_0^t |u^1 - u^2|_{\rho'} ds\right)\left(\int_0^t (\rho(s)-\rho')^{-1} ds\right)\,. \tag{3.8}$$

This additional term can bounded by the right hand side of (3.7) as follows: First replace $|u(t)|_\rho|$ by $|u|_{\rho,t}|$ everywhere, so that the norms are increasing in t. Also restrict to $\rho(s)$ which is decreasing in s, so that $(\rho(s)-\rho')^{-1}$ is also increasing. Then use the following simple Lemma.

Lemma 3.1 *Suppose that* $a(s)$ *and* $b(s)$ *are positive increasing functions. Then*

$$\left(\int_0^t a(s) ds\right)\left(\int_0^t b(s) ds\right) \le t\left(\int_0^t a(s)b(s) ds\right)\,. \tag{3.9}$$

4 Existence and uniqueness for the Euler equations

In this section we shall prove the following theorem:

Theorem 4.1 *Suppose that* $\boldsymbol{u}_0^E \in H^{l,\rho,\theta}$, $l \ge 4$, *with* $\nabla \cdot \boldsymbol{u}_0^E = 0$ *and* $\gamma_n \boldsymbol{u}_0^E = 0$. *Then the Euler equations Eq. (2.1)-Eq. (2.4) in either 2D or 3D admit a unique solution* $\boldsymbol{u}^E$ *in* $H^{l,\rho_0,\theta_0}_{\beta_0,T}$ *for some* $0 < \rho_0 < \rho$, $0 < \theta_0 < \theta$, $0 < \beta_0$, $0 < T$. *This solution satisfies the following bound in* $H^{l,\rho_0,\theta_0}_{\beta_0,T}$:

$$|\boldsymbol{u}^E|_{l,\rho_0,\theta_0,\beta_0,T} < c|\boldsymbol{u}_0^E|_{l,\rho,\theta}. \tag{4.1}$$

The proof of this Theorem will be based on the ACK Theorem in the function spaces $X_\rho = H^{l,\rho,\theta}$ and $Y_{\rho,\beta,T} = H^{l,\rho,\theta}_{\beta,T}$. The key idea in recasting the Euler equations into a form suitable for an iterative procedure is to introduce a new variable $\boldsymbol{u}^\star$, essentially a projected velocity (see Eq. (4.36) below), so that the boundary, initial and incompressibility conditions are automatically satisfied. The core of this Section is devoted to introduction of the half space projection operator, and to the estimate of this operator (Subsections 4.1 and 4.2). After that we shall introduce an estimate on the convective part of the Euler equation, Eq. (4.34), as a consequence of the Cauchy estimate for the derivative of an analytic function. In Subsection 4.4 we shall solve the Euler equation, and recast it in the form given by Eq. (4.37). Using the estimates on the projection operator, and the Cauchy estimate, it will then be straightforward to verify all the hypotheses of the ACK Theorem.

4.1 The projection operator

To prove the above theorem we need to define several operators. We start with the Fourier transform of the functions $f(x)$ and $g(x,y)$:

$$\hat{f}(\xi') = \frac{1}{(2\pi)^{1/2}} \int dx f(x) e^{-ix\xi'} \ , \ \hat{g}(\xi', \xi_n) = \frac{1}{2\pi} \int dx dy g(x,y) e^{-ix\xi' - iy\xi_n} \tag{4.2}$$

where the above integrals are on the whole real line and real plane respectively. Later we shall restrict y to be nonnegative. In the rest of this paper we shall adopt the convention of using ξ' and ξ_n as the dual of x and y respectively. In 3D, x and ξ' are vectors, and $x\xi'$ is replaced by $x \cdot \xi'$.

Suppose that $\sigma(T)(\xi')$ is a function of ξ' such that

$$\widehat{Tf}(\xi') = \sigma(T)(\xi')\hat{f}(\xi') \tag{4.3}$$

where T is an operator acting on function of one variable. Then $\sigma(T)$ is called the symbol of the pseudodifferential operator T. If T acts on functions of two (or more) variables, the definition is analogous.

We can now define the free space projection operator P^∞ acting on vectorial functions $\boldsymbol{u}$ (x,y), as the operator whose symbol is (here and in the rest of the paper we shall often omit the distinction between the operator and its symbol):

$$P^\infty = \frac{1}{\xi'^2 + \xi_n^2} \begin{pmatrix} \xi_n^2 & -\xi'\xi_n \\ -\xi'\xi_n & \xi'^2 \end{pmatrix} . \tag{4.4}$$

It is easy to see that the action of the above operator consists in projecting vectorial functions onto their divergence-free part, i.e.

$$\nabla \cdot P^\infty \boldsymbol{u} = 0 . \tag{4.5}$$

The operator P^∞ can be thought as

$$P^\infty = 1 - \nabla \Delta^{-1} \nabla \cdot \tag{4.6}$$

We define the operators N, D as the operators whose symbols are:

$$D = e^{-|\xi'|y} \ , \ N = -\frac{1}{|\xi'|} e^{-|\xi'|y} . \tag{4.7}$$

The above operators solve the Laplace equation in the half plane with Dirichlet and Neumann boundary condition respectively. In fact the problems

$$\Delta u(x,y) = 0 \ , u(x, y=0) = f(x) \tag{4.8}$$

and

$$\Delta u(x,y) = 0 \ , \partial_y u(x, y=0) = f(x) \tag{4.9}$$

admit the solutions $u = Df$ and $u = Nf$ respectively. Another operator which is useful to introduce is

$$N' = \frac{i\xi'}{|\xi'|} . \tag{4.10}$$

Now we express the components of the projection operator on the free space in a different form; i.e.

$$P^\infty{}_n = (-\xi'\xi_n, \xi'^2) \frac{1}{\xi'^2 + \xi_n^2} = \frac{1}{2} \left(-N' \left[\frac{|\xi'|}{|\xi'| + i\xi_n} - \frac{|\xi'|}{|\xi'| - i\xi_n} \right], \left[\frac{|\xi'|}{|\xi'| + i\xi_n} + \frac{|\xi'|}{|\xi'| - i\xi_n} \right] \right) . \tag{4.11}$$

This will be useful in proving existence and uniqueness for the Euler equations,as well as for the Navier-Stokes error equation [12]. We now give an expression for the projection operator involving the Fourier transform only in the x variable, which will simplify the subsequent estimates.

Lemma 4.1 *Let $f(\xi', y)$ be a function admitting the Fourier transform in y. Then*

$$\frac{|\xi'|}{|\xi'| + i\xi_n} f(\xi', y) = |\xi'| \int_{-\infty}^{y} e^{-|\xi'|(y-y')} f(\xi', y') dy' \tag{4.12}$$

and

$$\frac{|\xi'|}{|\xi'| - i\xi_n} f(\xi', y) = |\xi'| \int_{y}^{\infty} e^{|\xi'|(y-y')} f(\xi', y') dy' . \tag{4.13}$$

We prove the second of these equalities. Define the function

$$k_2(\xi', y) = \begin{cases} |\xi'| e^{|\xi'| y} & y \leq 0 \\ 0 & y > 0 \end{cases} \tag{4.14}$$

so that the integral operator in Eq. (4.13) can be written as the convolution of k_2 and f; i.e.

$$|\xi'| \int_{y}^{\infty} e^{|\xi'|(y-y')} f(\xi', y') dy' = k_2(\xi', y) * f(\xi', y) . \tag{4.15}$$

To prove Eq. (4.13) it is enough to apply the Fourier transform with respect to the y variable to Eq. (4.15), and notice that

$$\hat{k}_2 = \frac{1}{(2\pi)^{1/2}} \frac{|\xi'|}{|\xi'| - i\xi_n} . \tag{4.16}$$

The proof of Eq. (4.12) is similar, and is done using the function

$$k_1(\xi', y) = \begin{cases} |\xi'| e^{-|\xi'| y} & y \geq 0 \\ 0 & y < 0. \end{cases} \tag{4.17}$$

As a result of Lemma 4.1, the normal and transversal components ($P^{\infty}{}_n$ and $P^{\infty\prime}$) of the projection operator can be written as

$$P^{\infty}{}_n \boldsymbol{u} = \frac{1}{2} \left[|\xi'| \int_{-\infty}^{y} dy' e^{-|\xi'|(y-y')} (-N'u + v) + |\xi'| \int_{y}^{\infty} dy' e^{|\xi'|(y-y')} (N'u + v) \right] \tag{4.18}$$

$$P^{\infty\prime} \boldsymbol{u} = u + \frac{1}{2} \left[-|\xi'| \int_{-\infty}^{y} dy' e^{-|\xi'|(y-y')} (u + N'v) - |\xi'| \int_{y}^{\infty} dy' e^{|\xi'|(y-y')} (u - N'v) \right] \tag{4.19}$$

in which $\boldsymbol{u} = (u, v)$.

We now define the projection operator P on the half plane $y \geq 0$, with vanishing normal component at the boundary, i.e. for $\gamma_n \boldsymbol{u}\ (y = 0) = 0$, as

$$P = P^{\infty} - \nabla N \gamma_n P^{\infty} . \tag{4.20}$$

It is easy to see that the following properties hold for all $\boldsymbol{u}$:

$$\nabla \cdot P\boldsymbol{u} = 0 , \tag{4.21}$$

$$\gamma_n P\boldsymbol{u} = 0 \tag{4.22}$$

$$P^2 = P . \tag{4.23}$$

Explicit formulas for the half-space projection P are given by

$$\begin{aligned} P'\boldsymbol{u} = u - \frac{1}{2}|\xi'| \Big[& \int_{0}^{y} dy' e^{-|\xi'|(y-y')} (u + N'v) + \int_{0}^{y} dy' e^{-|\xi'|(y+y')} (u - N'v) \\ & + \left(1 + e^{-2|\xi'| y}\right) \int_{y}^{\infty} dy' e^{|\xi'|(y-y')} (u - N'v) \Big] \end{aligned} \tag{4.24}$$

$$P_n u = \frac{1}{2}|\xi'| \left[\int_0^y dy' e^{-|\xi'|(y-y')}(-N'u+v) - \int_0^y dy' e^{-|\xi'|(y+y')}(N'u+v) \right.$$
$$\left. + \left(1 - e^{-2|\xi'|y}\right) \int_y^\infty dy' e^{|\xi'|(y-y')}(N'u+v) \right] . \tag{4.25}$$

If y is complex, these must be understood as contour integrals. The derivation above, based on the free space projection operator, is simple, but one can also directly check that the formulas (4.24) and (4.25) satisfy the conditions (4.21), (4.22) and (4.23). In the following subsection we shall introduce some bounds on the norm of the projection operator which we shall use in the rest of this section.

The formulas Eq. (4.24) and Eq. (4.25) can be extended to 3D by the following modifications: Replace u, ξ' and $N' = i\xi'/|\xi'|$ by vectors. In Eq. (4.24), replace u in the integrals by $N'N' \cdot u$. In Eq. (4.25) replace $N'u$ by $N' \cdot u$.

4.2 Estimates on the projection operators

Let $f \in H'^{l,\rho}$ and consider the norm

$$|f|_{l,\rho} = \sum_{k \le l} \left\{ \int d\xi' \left| e^{\rho|\xi'|} |\xi'|^k \hat{f}(\xi') \right|^2 \right\}^{1/2} . \tag{4.26}$$

It is easy to see that the above norm is equivalent to the one we have previously introduced in $H'^{l,\rho}$. In the rest of this paper we shall use both of them according to convenience, sometime switching from one to the other during the same estimate. Occasionally we shall omit the distinction between the function and its Fourier transform.

Using the norm we just introduced, Jensen's inequality, and the expressions (4.24), (4.25) for P, one can easily prove the estimates in the next two lemmas.

Lemma 4.2 *Let* $u \in H^{l,\rho,\theta}$. *Then* $Pu \in H^{l,\rho,\theta}$ *and*

$$|Pu|_{l,\rho,\theta} \le c|u|_{l,\rho,\theta}. \tag{4.27}$$

We now define the function χ as

$$\chi(y) = \min(1, |y|) . \tag{4.28}$$

What we want to show is that the normal component of P goes to zero linearly fast near the origin. A precise statement is the following:

Lemma 4.3 *Let* $u \in H^{l,\rho,\theta}$ *with* $l > 0$. *Then*

$$\sup_{y \in \Sigma(\theta)} \frac{|P_n u|_{0,\rho}}{\chi(y)} \le c|u|_{l,\rho,\theta} . \tag{4.29}$$

The significance of Lemma 4.3 will be clear only when we introduce the Cauchy estimate for normal derivatives. We shall use it in this paper to estimate the convective part of the Euler equation, and it will be crucial in [12] to handle the large (i.e. $O(\nu^{-1/2})$) generation of vorticity in the boundary layer. The proof of (4.29) can be easily achieved by distinguishing the two cases: $|y| \ge 1$ and $|y| < 1$.

4.3 The Cauchy estimates

To deal with the convective part of the Euler equations we introduce the Cauchy estimate of the derivative of an analytic function:

Lemma 4.4 *Let* $f \in H'^{l,\rho''}$. *If* $\rho' < \rho''$ *then*

$$|\partial_x f|_{l,\rho'} \le \frac{|f|_{l,\rho''}}{\rho'' - \rho'} . \tag{4.30}$$

If the derivative is with respect to the y variable, because of the angular shape of the region of analyticity, we must multiply by $|y|$ for y near 0.

Lemma 4.5 *Let $f \in H^{l,\rho,\theta''}$. If $\theta' < \theta'' < \pi/4$ then*

$$|\chi(y)\partial_y f|_{l,\rho,\theta'} \leq \frac{|f|_{l,\rho,\theta''}}{\theta'' - \theta'}. \tag{4.31}$$

With the above two Cauchy estimates and using the Sobolev inequality (see e.g. Proposition 2.1), it is easy to prove that

Lemma 4.6 *Let f and $g \in H^{l,\rho'',\theta''}$ with $l \geq 4$, and let $\rho' < \rho''$. Then*

$$|g\partial_x f|_{l,\rho',\theta''} \leq c|g|_{l,\rho',\theta''} \frac{|f|_{l,\rho'',\theta''}}{\rho'' - \rho'}. \tag{4.32}$$

Lemma 4.7 *Let f and $g \in H^{l,\rho'',\theta''}$, with $l \geq 4$ and with $g(y=0) = 0$, and let $\theta' < \theta''$. Then*

$$|g\partial_y f|_{l,\rho'',\theta'} \leq c|g|_{l,\rho'',\theta'} \frac{|f|_{l,\rho'',\theta''}}{\theta'' - \theta'}. \tag{4.33}$$

We can finally estimate the convective part of the Euler equations.

Lemma 4.8 *Suppose that $\boldsymbol{u}^{\ 1}$ and $\boldsymbol{u}^{\ 2}$ are in $H^{l,\rho,\theta}_{\beta,T}$ $l \geq 4$, and that $\gamma_n \boldsymbol{u}^{\ 1} = \gamma_n \boldsymbol{u}^{\ 2} = 0$. Moreover let ρ' and ρ'' satisfy*

$$\rho - \beta t \geq \rho'' > \rho'$$
$$\theta - \beta t \geq \theta'' > \theta'$$

for $0 \leq t \leq T$. Then

$$|\boldsymbol{u}^{\ 1} \cdot \nabla \boldsymbol{u}^{\ 1} - \boldsymbol{u}^{\ 2} \cdot \nabla \boldsymbol{u}^{\ 2}|_{l,\rho',\theta'} \leq c \left[\frac{|\boldsymbol{u}^{\ 1} - \boldsymbol{u}^{\ 2}|_{l,\rho'',\theta'}}{\rho'' - \rho'} + \frac{|\boldsymbol{u}^{\ 1} - \boldsymbol{u}^{\ 2}|_{l,\rho',\theta''}}{\theta'' - \theta'} \right] \tag{4.34}$$

where the constant c depends only on $|\boldsymbol{u}^{\ 1}|_{l,\rho,\theta,\beta,T}$ and $|\boldsymbol{u}^{\ 2}|_{l,\rho,\theta,\beta,T}$.

4.4 Pressure-free Euler equations

The usual problem with the Euler equations is the presence of the pressure gradient in the conservation of momentum equations and the corresponding coupling of these evolution type equations to the incompressibility equation. There are two ways to circumvent these problems: the projection method, which is employed here, and the vorticity formulation.

First we define the operator P_t, whose action on a vector function $\boldsymbol{u}\ (x,y,t)$ is given by

$$P_t \boldsymbol{u}\ (x,y,t) = P \int_0^t ds \boldsymbol{u}\ (x,y,s)\ , \tag{4.35}$$

and pose

$$\boldsymbol{u}^{\ E}(x,y,t) = \boldsymbol{u}^{\ E}_{\ 0}(x,y) + P_t \boldsymbol{u}^{\ \star}\ . \tag{4.36}$$

It is clear that once $\boldsymbol{u}^{\ E}$ is expressed in the above form, the initial and boundary conditions for the Euler equations and the incompressibility condition are automatically satisfied. If we put (4.35) into the conservation of momentum equation we get

$$\boldsymbol{u}^{\ \star} + \boldsymbol{H}\ (\boldsymbol{u}^{\ \star}, t) = 0 \tag{4.37}$$

where

$$\boldsymbol{H}\ (\boldsymbol{u}^{\ \star}, t) = (\boldsymbol{u}^{\ E}_{\ 0} + P_t \boldsymbol{u}^{\ \star}) \cdot \nabla(\boldsymbol{u}^{\ E}_{\ 0} + P_t \boldsymbol{u}^{\ \star})\ . \tag{4.38}$$

Existence and uniqueness of the solution $\boldsymbol{u}^{\ \star}$ of Eq. (4.37), which implies existence and uniqueness for the Euler equations, is stated in the following theorem:

Theorem 4.2 *Suppose $u_0^E \in H^{l,\rho,\theta}$ $l \geq 4$, with $\nabla \cdot u_0^E = 0$ and $\gamma_n u_0^E = 0$. Then Eq. (4.37) admits a unique solution $u^\star$ in $H^{l,\rho_0,\theta_0}_{\beta_0,T}$ for some $0 < \rho_0 < \rho$, $0 < \theta_0 < \theta$, $\beta_0 > 0$, $T > 0$.*

Theorem 4.1 follows directly from Theorem 4.2, using the following Proposition, which is a consequence of Lemma 4.2:

Proposition 4.1 *Let $u^\star \in H^{l,\rho_0,\theta_0}_{\beta_0,T}$. Then $P_t u^\star \in H^{l,\rho_0,\theta_0}_{\beta_0,T}$ and*

$$|P_t u^\star|_{l,\rho_0,\theta_0,\beta_0,T} \leq c|u^\star|_{l,\rho_0,\theta_0,\beta_0,T}\,. \tag{4.39}$$

We have also the following bound on P_t :

Proposition 4.2 *Let $u^\star \in H^{l,\rho_0,\theta_0}_{\beta_0,T}$ and let $\rho' < \rho_0 - \beta_0 T$ and $\theta' < \theta_0 - \beta_0 T$. Then $P_t u^\star \in H^{l,\rho',\theta'}$ for each $0 < t < T$ and*

$$|P_t u^\star|_{l,\rho',\theta'} \leq c\int_0^t ds|u^\star(\cdot,\cdot,s)|_{l,\rho',\theta'} \leq c|u^\star|_{l,\rho_0,\theta_0,\beta_0,T}\,. \tag{4.40}$$

In the rest of this section we shall be concerned with proving Theorem 4.2. To do this we shall verify that the operator $\boldsymbol{H}$ satisfies all the hypotheses of the ACK Theorem in the function spaces $X_{\rho,\theta} = H^{l,\rho,\theta}$ (at each fixed t) and $Y_{\rho,\theta,\beta,T} = H^{l,\rho,\theta}_{\beta,T}$ (as a function of t), and with ρ replaced by the vector (ρ,θ).

4.5 The forcing term

It is obvious that $\boldsymbol{H}$ satisfies the first condition of the ACK Theorem in the norms $H^{l,\rho,\theta}$. In this subsection we shall prove that there exists a constant R_0 such that

$$|\boldsymbol{H}\,(t,0)|_{l,\rho_0-\beta t,\theta_0-\beta t} \leq R_0 \tag{4.41}$$

in $H^{l,\rho,\theta}$ for $0 \leq t \leq T$, which verifies the second assumption of the Theorem. The constant R_0 will of course depend on $|u_0^E|_{l,\rho,\theta}$ and on the difference between ρ and ρ_0, θ and θ_0. From Eq. (4.38), we see that

$$\boldsymbol{H}\,(t,0) = u_0^E \cdot \nabla u_0^E \tag{4.42}$$

and Lemmas 4.6 and 4.7 imply

$$|u_0^E \cdot \nabla u_0^E|_{l,\rho_0-\beta t,\theta_0-\beta t} \leq c|u_0^E|^2_{l,\rho,\theta} \tag{4.43}$$

which gives the desired bound (4.41). We now pass to the Cauchy estimate.

4.6 The Cauchy estimate

In this subsection we shall be concerned with proving that the operator $\boldsymbol{H}$ satisfies the last hypothesis of the ACK Theorem. We have to show that, if $\rho' < \rho(s) \leq \rho_0 - \beta s$, $\theta' < \theta(s) \leq \theta_0 - \beta s$, and if u^1 and u^2 are in $H^{l,\rho_0,\theta_0}_{\beta,T}$ $l \geq 4$, with

$$|u^{\star 1}|_{l,\rho_0,\theta_0,\beta,T} \leq R\,,\ |u^{\star 2}|_{l,\rho_0,\theta_0,\beta,T} \leq R\,, \tag{4.44}$$

then in $H^{l,\rho,\theta}$.

$$|\boldsymbol{H}\,(t,u^{\star 1}) - \boldsymbol{H}\,(t,u^{\star 2})|_{l,\rho',\theta'} \leq C\int_0^t ds\left\{\frac{|u^{\star 1}-u^{\star 2}|_{l,\rho(s),\theta'}}{\rho(s)-\rho'} + \frac{|u^{\star 1}-u^{\star 2}|_{l,\rho',\theta(s)}}{\theta(s)-\theta'}\right\}. \tag{4.45}$$

First estimate the nonlinear term of $\boldsymbol{H}$. Using Lemma 4.8 to estimate the convective part of the operator $\boldsymbol{H}$ and then Proposition 4.2 leads to

$$\begin{aligned}
|\,P_t u^{\star 1}\cdot\nabla P_t u^{\star 1} - P_t u^{\star 2}\cdot\nabla P_t u^{\star 2}\,|_{l,\rho',\theta'} &\leq c\int_0^t ds\left[\frac{|u^{\star 1}-u^{\star 2}|_{l,\rho(s),\theta'}}{\rho(s)-\rho'} + \frac{|u^{\star 1}-u^{\star 2}|_{l,\rho',\theta(s)}}{\theta(s)-\theta'}\right] \\
+\int_0^t |u^{\star 1}-u^{\star 2}|_{l,\rho_0,\theta_0,\beta_0,T}ds\int_0^t &\left[\frac{|u^{\star 1}|_{l,\rho(s),\theta'}+|u^{\star 2}|_{l,\rho(s),\theta'}}{\rho(s)-\rho'} + \frac{|u^{\star 1}|_{l,\rho',\theta(s)}+|u^{\star 2}|_{l,\rho',\theta(s)}}{\theta(s)-\theta'}\right]ds \\
&\leq C\int_0^t ds\left[\frac{|u^{\star 1}-u^{\star 2}|_{l,\rho(s),\theta'}}{\rho(s)-\rho'} + \frac{|u^{\star 1}-u^{\star 2}|_{l,\rho',\theta(s)}}{\theta(s)-\theta'}\right]
\end{aligned} \tag{4.46}$$

in $H^{l,\rho,\theta}$, using Lemma 3.1 and the bound (4.44) in the last step. The estimate of the linear part is similar.

4.7 Conclusion of the Proof of Theorem 4.1

Since all of the hypotheses of the ACK Theorem have been verified, the proof of Theorem 4.2 has been achieved. There exist $0 < \rho_0 < \rho$, $0 < \theta_0 < \theta$, and a $\beta_0 > 0$ such that Eq. (4.37) admits a unique solution in $H^{l,\rho_0,\theta_0}_{\beta_0,T}$. This also concludes the proof of Theorem 4.1 for the Euler equations.

5 Existence and Uniqueness for Prandtl's Equations

We want to prove that the Prandtl equations (2.6)-(2.11) admit a unique solution in an appropriate function space. The main result of this section is the following Theorem:

Theorem 5.1 *Suppose that u_0^P satisfies the compatibility conditions (2.14) and (2.15), that $\boldsymbol{u}\,{}_0^E \in H^{l+1,\rho,\theta}$, and that $u_0^P - \gamma u_0^E \in K^{l+1,\rho_0,\theta_0,\mu_0}$ $l \geq 3$ ($l \geq 4$ in 3D). Then there exists a unique solution u^P of the Prandtl equations (2.6)-(2.11). This solution can be written as:*

$$u^P(x,Y,t) = \tilde{u}^P(x,Y,t) + \gamma u^E \tag{5.1}$$

where $\tilde{u}^P \in K^{l,\rho_1,\theta_1,\mu_1}_{\beta_1,T}$, with $0 < \rho_1 < \rho_0$, $0 < \theta_1 < \theta_0$, $0 < \mu_1 < \mu_0$, $\beta_1 > \beta_0 > 0$. This solution satisfies the following bound in $K^{l,\rho_1,\theta_1,\mu_1}_{\beta_1,T}$:

$$|\tilde{u}^P|_{l,\rho_1,\theta_1,\mu_1,\beta_1,T} < c\left(|u_0^P - \gamma u_0^E|_{l+1,\rho_0,\theta_0,\mu_0} + |\boldsymbol{u}\,{}_0^E|_{l+1,\rho,\theta}\right) . \tag{5.2}$$

In particular this shows that if the initial condition for Prandtl equations exponentially approaches the initial value of the Euler flow calculated at the boundary, the same property will be true for the Prandtl solution at least for short time.

The proof of this theorem will occupy the remainder of this section. As in the proof of existence and uniqueness for Euler equation, we shall recast the Prandtl equations in a form suitable for the use of the ACK Theorem (see Eq. (5.37) below). In Prandtl's equations a second order operator (the heat operator) is present. The key idea is to invert this operator, taking into account boundary and initial conditions. Therefore we shall first introduce the heat operators, and prove some bounds on them.

In Subsection 5.2 we find an operator form for Prandtl equations, Eq. (5.37). The resulting operator F consists of two terms: The first is a forcing term that accounts for BC and IC. The second is the composition of a convective operator and the inverse of the heat operator with zero BC and IC.

With the bounds on the heat and convective operators, it is then straightforward to get the desired bounds, which is performed in Subsections 5.3 and 5.4.

In the rest of this section we shall always suppose $l \geq 4$ ($l \geq 5$ in $3D$), as needed for Proposition 2.3.

5.1 Estimates on heat operators

To solve Prandtl equations we introduce the heat kernel:

$$E_0(Y,t) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-Y^2/4t\right) , \tag{5.3}$$

and the heat operators acting on functions $f(Y)$ with $\Re Y \geq 0$ and $t \geq 0$.

$$E_0(t)f = \int_0^\infty dY' \left[E_0(Y - Y', t) - E_0(Y + Y', t)\right] f(Y') , \tag{5.4}$$

The operator $E_0(t)$ is obtained by convolution with the heat kernel $E_0(Y,t)$, with respect to Y, once the function $f(Y)$ is extended in an odd manner to $\Re Y < 0$. Note that for $t \geq 0$ and $Y \geq 0$,

$$\begin{aligned} (\partial_t - \partial_{YY})\, E_0(t)f &= 0 && (5.5)\\ E_0(t)f|_{t=0} &= f(Y) && (5.6)\\ \gamma E_0(t)f &= 0\,. && (5.7)\end{aligned}$$

We need the following operator E_1, acting on functions defined on the boundary:

$$E_1 g(x,t) = \int_0^t ds\, h(Y,t-s)g(x,s) \tag{5.8}$$

where $h(Y,t)$ is defined by:

$$h(Y,t) = \frac{Y}{t}\frac{\exp(-Y^2/4t)}{(4\pi t)^{1/2}}\,. \tag{5.9}$$

The function $E_1 g$ solves the heat equations with zero initial data and with boundary value g; i.e.

$$\begin{aligned} (\partial_t - \partial_{YY})\, E_1 g &= 0\\ E_1 g|_{t=0} &= 0\\ \gamma E_1 g &= g. \end{aligned} \tag{5.10}$$

Using $E_0(t)$ we define the operator E_2 by

$$\begin{aligned} E_2 f &= \int_0^t ds E_0(t-s)f(s)\\ &= \int_0^t ds \int_0^\infty dY' \left[E_0(Y-Y',t-s) - E_0(Y+Y',t-s)\right] f(Y',s). \end{aligned} \tag{5.11}$$

The operator E_2 inverts the heat operator with zero initial data and boundary data; i.e.

$$\begin{aligned} (\partial_t - \partial_{YY})\, E_2 f &= f\\ E_2 f|_{t=0} &= 0\\ \gamma E_2 f &= 0\,. \end{aligned} \tag{5.12}$$

We now recall some basic properties of the heat operators. In the estimates below, c is a constant depending (at most) only on ρ, θ, β, μ and T. Notice that the restriction $\theta < \pi/4$ is needed here.

Lemma 5.1 *Let $f(Y)$ and $g(Y)$ be two continuous bounded functions, and let g be exponentially decaying at infinity; i.e. there exists a positive μ such that $\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}|g(Y)| < \infty$. Let $0 < \theta < \pi/4$. Then*

$$\sup_{Y\in\Sigma(\theta)} \left| \int_0^\infty dY' |E_0(Y \pm Y',t)| f(Y') \right| \leq c \sup_{Y\in\Sigma(\theta)} |f(Y)|. \tag{5.13}$$

$$\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} \left| \int_0^\infty dY' |E_0(Y \pm Y',t)| g(Y') \right| \leq c \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} |g(Y)| \tag{5.14}$$

in which the constant c depends only on θ and μ.

Lemma 5.2 *Let $f \in C^1([0,T])$, with $f(0) = 0$, $0 < \theta < \pi/4$ and $j = 0,1,2$. Then*

$$\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} |E_1 f| \leq c \sup_{t>0} |f(t)| \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} \left|\partial_Y^j E_1 f\right| \leq c\left\{\sup_{t>0}|f(t)| + \sup_{t>0}|f'(t)|\right\}\,. \tag{5.15}$$

The following bounds on analytic norms of the heat operators will be used throughout the rest of this paper:

Proposition 5.1 *Let $u \in K^{l,\rho,\theta,\mu}$ with $\gamma u = 0$. Then $E_0(t)u \in K^{l,\rho,\theta,\mu}$ for all t and*

$$\sup_{0\le t\le T} |E_0(t)u|_{l,\rho,\theta,\mu} \le c\,|u|_{l,\rho,\theta,\mu} \,. \tag{5.16}$$

The above estimate obviously implies that $E_0(t)u \in K^{l,\rho,\theta,\mu}_{\beta,T}$ for all β and T and that

$$|E_0(t)u|_{l,\rho,\theta,\mu,\beta,T} \le c|u|_{l,\rho,\theta,\mu} \,. \tag{5.17}$$

Corollary 5.1 *Let $u' = u + f$ with $u \in K^{l,\rho,\theta,\mu}$, $f \in H'^{l,\rho}$ constant with respect to Y and t; moreover $\gamma u = -f$. Then $E_0(t)u' - f \in K^{l,\rho,\theta,\mu}$ for all t and:*

$$\sup_{0\le t\le T} |E_0(t)u' - f|_{l,\rho,\theta,\mu} \le c\left(|u|_{l,\rho,\theta,\mu} + |f|_{l,\rho}\right) \,. \tag{5.18}$$

The above estimate obviously implies that $E_0(t)u' - f \in K^{l,\rho,\theta,\mu}_{\beta,T}$ for all β and T and that

$$|E_0(t)u' - f|_{l,\rho,\theta,\mu,\beta,T} \le c\left(|u|_{l,\rho,\theta,\mu} + |f|_{l,\rho}\right) \,. \tag{5.19}$$

In the next Proposition we give an estimate of $E_0(t)u$ in a different space, namely in $L^{l,\rho,\theta}$; we shall not use this estimate in this paper but in Part II [12].

Proposition 5.2 *Let $u \in L^{l,\rho,\theta}$ with $\gamma u = 0$. Then $E_0(t)u \in L^{l,\rho,\theta}$ for all t and*

$$\sup_{0\le t\le T} |E_0(t)u|_{l,\rho,\theta} \le c|u|_{l,\rho,\theta} \,. \tag{5.20}$$

The above estimate obviously implies that $E_0(t)u \in L^{l,\rho,\theta}_{\beta,T}$ for all β and T and that

$$|E_0(t)u|_{l,\rho,\theta,\beta,T} \le c|u|_{l,\rho,\theta}. \tag{5.21}$$

Proposition 5.3 *Let $\phi \in K'^{l,\rho}_{\beta,T}$ with $\phi(t=0) = 0$. Then $E_1\phi \in K^{l,\rho,\theta,\mu}_{\beta,T}$ and*

$$|E_1\phi|_{l,\rho,\theta,\mu,\beta,T} \le c|\phi|_{l,\rho,\beta,T}. \tag{5.22}$$

We have the following estimate for E_2:

Proposition 5.4 *Let $u \in K^{l,\rho,\theta,\mu}_{\beta,T}$. Then $E_2u \in K^{l,\rho,\theta,\mu}_{\beta,T}$ and*

$$|E_2u|_{l,\rho,\theta,\mu,\beta,T} \le c|u|_{l,\rho,\theta,\mu,\beta,T} \,. \tag{5.23}$$

The following estimates will also be useful:

Proposition 5.5 *Let $u \in K^{l,\rho,\theta,\mu}_{\beta,T}$ with $\gamma u = 0$. If $\rho' < \rho - \beta t$, $\theta' < \theta - \beta t$ and $\mu' < \mu - \beta t$, then*

$$|E_2u|_{l,\rho',\theta',\mu'} \le c\int_0^t ds|u(\cdot,\cdot,s)|_{l,\rho',\theta',\mu} \le c|u|_{l,\rho,\theta,\mu,\beta,T} \,. \tag{5.24}$$

5.2 The final form of Prandtl's equations

It is useful to introduce the new variable $\tilde{u}^P$:

$$\tilde{u}^P = u^P - \gamma u^E . \tag{5.25}$$

It is more natural to write Prandtl equations in terms of this new variable: First because the matching condition with the outer Euler flow, Eq. (2.10), will be simply a consequence of the fact that $\tilde{u}^P$ is exponentially decaying in Y, i.e. of the fact that $\tilde{u}^P \in K^{l,\rho_1,\theta_1,\mu_1}_{\beta_1,T}$. Second the gradient of the pressure will not show up in the equation. Equation (2.6) in terms of $\tilde{u}^P$ becomes:

$$(\partial_t - \partial_{YY})\,\tilde{u}^P + \tilde{u}^P \partial_x \gamma u^E + \gamma u^E \partial_x \tilde{u}^P + \tilde{u}^P \partial_x \tilde{u}^P - \left[\int_0^Y \partial_x \tilde{u}^P dY' + Y \partial_x \gamma u^E\right] \partial_Y \tilde{u}^P = 0 \,, \tag{5.26}$$

where we have used

$$v^P = -\int_0^Y \partial_x u^P dY' = -\left[\int_0^Y \partial_x \tilde{u}^P dY' + Y \partial_x \gamma u^E\right] \tag{5.27}$$

and the Euler equation at the boundary

$$\gamma\left[\partial_t u^E + u^E \partial_x u^E + \partial_x p^E\right] = 0 \,. \tag{5.28}$$

The initial condition for Eq. (5.26) is

$$\tilde{u}^P\,(x, Y, t=0) = u_0^P(x,Y) - \gamma u_0^E = \tilde{u}_0^P \tag{5.29}$$

while the boundary condition is

$$\gamma \tilde{u}^P = -\gamma u^E \,. \tag{5.30}$$

Equation (5.26) for $\tilde{u}^P$ with (5.29) as initial condition and (5.30) as boundary condition, and with v^P given by (5.27), is equivalent to (2.6)-(2.11) for $\boldsymbol{u}^{\,P}$. To prove existence and uniqueness for (5.26)-(5.30), we shall use the ACK Theorem with the norms $K^{l,\rho,\theta,\mu}$ and $K^{l,\rho,\theta,\mu}_{\beta,T}$. To put Eq. (5.26) in a suitable form for the application of the ACK Theorem, we have to invert the heat operator in Eq. (5.26), taking into account the IC and BC. We define U to be

$$U = -\gamma u_0^E - E_1\left(\gamma u^E - \gamma u_0^E\right) + E_0(t)\left(\tilde{u}_0^P + \gamma u_0^E\right) \,. \tag{5.31}$$

It is easy to see that U solves the heat equation with (5.29) as IC and (5.30) as BC; i.e.

$$\begin{aligned} (\partial_t - \partial_{YY})\,U &= 0 && (5.32)\\ U(t=0) &= \tilde{u}_0^P && (5.33)\\ \gamma U &= -\gamma u^E. && (5.34) \end{aligned}$$

Define the operators $K(\tilde{u}^P, t)$, which is (minus) the convective part of Eq. (5.26), and F as

$$\begin{aligned} K(\tilde{u}^P, t) &= -\left\{\tilde{u}^P \partial_x \gamma u^E + \gamma u^E \partial_x \tilde{u}^P + \tilde{u}^P \partial_x \tilde{u}^P - \left[\int_0^Y \partial_x \tilde{u}^P dY' + Y \partial_x \gamma u^E\right] \partial_Y \tilde{u}^P\right\} && (5.35)\\ F(\tilde{u}^P, t) &= E_2 K(\tilde{u}^P, t) + U \,. && (5.36) \end{aligned}$$

The following equation is then equivalent to Eqs. (5.26)-(5.30):

$$\tilde{u}^P = F(\tilde{u}^P, t) \,. \tag{5.37}$$

The rest of this section is devoted to proving that the operator $F(\tilde{u}^P, t)$ satisfies all the hypotheses of the ACK Theorem with $X = K^{l,\rho,\theta,\mu}$ and $Y = K^{l,\rho,\theta,\mu}_{\beta,T}$.

5.3 The forcing term

It is obvious that the operator F satisfies the first condition of the ACK Theorem. In this subsection we shall prove that the operator F satisfies the second condition of the ACK Theorem. Namely we prove that

$$|F(t,0)|_{l,\rho_1-\beta_0 t,\theta_1-\beta_0 t,\mu_1-\beta_0 t} \le R_0 \tag{5.38}$$

in $K^{l,\rho_1-\beta_0 t,\theta_1-\beta_0 t,\mu_1-\beta_0 t}$ for $0 \le t \le T$, where R_0 is a constant.

Since

$$F(t,0) = U\ , \tag{5.39}$$

Corollary 5.1 and Proposition 5.3 show that

Proposition 5.6 *Given that $\tilde{u}_0^P \in K^{l+1,\rho_1,\theta_1,\mu_1}$ with $\gamma\tilde{u}_0^P = -\gamma u_0^E$ and $u_0^E \in H^{l+1,\rho_1,\theta_1}$, then $\gamma u^E \in K^{\prime l,\rho_1}_{\beta_0,T}$ and $U \in K^{l,\rho_1,\theta_1,\mu_1}_{\beta_0,T}$ satisfying*

$$|U|_{l,\rho_1,\theta_1,\mu_1,\beta_0,T} \le c\left(|u_0^E|_{l+1,\rho_1,\theta_1} + |\tilde{u}_0^P|_{l+1,\rho_1,\theta_1,\mu_1}\right)\ . \tag{5.40}$$

Notice how, to get that $\gamma u^E \in K^{\prime l,\rho_1}_{\beta_0,T}$, a Sobolev estimate in the y variable has been used. With this proposition one sees that the forcing term is estimated in terms of the initial conditions for Prandtl equations and of the outer Euler flow. This concludes the proof of the estimate (5.38)

5.4 The Cauchy estimate

In this and in the next subsections we shall prove that the operator F as given by Eq. (5.36) satisfies the last hypothesis of the ACK Theorem. Namely we want to show that if $\rho' < \rho(s) \le \rho_1 - \beta_0 s$, $\theta' < \theta(s) \le \theta_1 - \beta_0 s$, $\mu' < \mu(s) \le \mu_1 - \beta_0 s$, and if $u^{(1)}$ and $u^{(2)}$ are in $K^{l,\rho_1,\theta_1,\mu_1}_{\beta_0,T}$ with

$$|u^{(1)}|_{l,\rho_1,\theta_1,\mu_1,\beta_0,T} < R \quad \text{and} \quad |u^{(2)}|_{l,\rho_1,\theta_1,\mu_1,\beta_0,T} < R \tag{5.41}$$

then

$$\begin{aligned} & |F(t,u^{(1)}) - F(t,u^{(2)})|_{l,\rho',\theta',\mu'} \\ \le\ & C\int_0^t ds \left\{ \frac{|u^{(1)}-u^{(2)}|_{l,\rho(s),\theta',\mu'}}{\rho(s)-\rho'} + \frac{|u^{(1)}-u^{(2)}|_{l,\rho',\theta(s),\mu'}}{\theta(s)-\theta'} + \frac{|u^{(1)}-u^{(2)}|_{l,\rho',\theta',\mu(s)}}{\mu(s)-\mu'} \right\}\ . \end{aligned} \tag{5.42}$$

In this subsection we shall be concerned with the operator K. The operator K involves three different kinds of terms:

1. The nonlinear term involving the x-derivative $\tilde{u}^P\partial_x\tilde{u}^P$;
2. the nonlinear term involving the Y-derivative, $\int_0^Y dY'\partial_x\tilde{u}^P \cdot \partial_Y\tilde{u}^P$;
3. the linear terms.

Before going into the details we anticipate that term (1) will be estimated using the Cauchy estimate in the x-variable, term (2) will be estimated using the Cauchy estimate in the Y-variable, the linear growth in Y of the coefficient of term (3) will be estimated using the exponential decay in Y of the solution. Here and in the rest of this section

$$\begin{aligned} 0 < \rho' < \rho'' \le \rho_1 - \beta_0 t \\ 0 < \theta' < \theta'' \le \theta_1 - \beta_0 t \\ 0 < \mu' < \mu'' \le \mu_1 - \beta_0 t\ . \end{aligned}$$

We now state some Lemmas used to estimate the convective operator.

Lemma 5.3 *Suppose that* $u^{(1)}$ *and* $u^{(2)}$ *are in* $K^{l,\rho_1,\theta_1,\mu_1}_{\beta_0,T}$. *Then*

$$|u^{(1)}\partial_x u^{(1)} - u^{(2)}\partial_x u^{(2)}|_{l,\rho',\theta',\mu'} \leq c\frac{|u^{(1)} - u^{(2)}|_{l,\rho'',\theta',\mu'}}{\rho'' - \rho'} \tag{5.43}$$

in the $K^{l,\rho,\theta,\mu}$ *norm, where the constant c depends only on* $|u^{(1)}|_{l,\rho_1,\theta_1,\mu_1,\beta_0,T}$ *and* $|u^{(2)}|_{l,\rho_1,\theta_1,\mu_1,\beta_0,T}$.

In fact

$$\begin{aligned} & |u^{(1)}\partial_x u^{(1)} - u^{(2)}\partial_x u^{(2)}|_{l,\rho',\theta',\mu'} \\ \leq\ & |u^{(1)}\partial_x\left(u^{(1)} - u^{(2)}\right)|_{l,\rho',\theta',\mu'} + |u^{(2)}\partial_x\left(u^{(1)} - u^{(2)}\right)|_{l,\rho',\theta',\mu'} \\ & +|\partial_x\left[u^{(2)}\left(u^{(1)} - u^{(2)}\right)\right]|_{l,\rho',\theta',\mu'} \\ \leq\ & c\frac{|u^{(1)} - u^{(2)}|_{l,\rho'',\theta',\mu'}}{\rho'' - \rho'}. \end{aligned} \tag{5.44}$$

We now pass to estimation of terms involving the Y-derivative. First we give a version of the Cauchy estimate Lemma 4.5 for analytic functions exponentially decaying in the Y-variable.

Lemma 5.4 *Let* $f \in H^{l,\rho',\theta'',\mu''}$. *Then*

$$|\chi(Y)\partial_Y f|_{l,\rho',\theta',\mu'} \leq \frac{|f|_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'} + \mu'|f|_{l,\rho',\theta',\mu'} \tag{5.45}$$

$$|Y\partial_Y f|_{l,\rho',\theta',\mu'} \leq \frac{|f|_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'} + \mu'\frac{|f|_{l,\rho',\theta',\mu''}}{\mu'' - \mu'} + |f|_{l,\rho',\theta',\mu'}\,. \tag{5.46}$$

To estimate the nonlinear term involving the Y-derivative we have to use the fact that the normal component of the velocity, as expressed by the integral from 0 to Y, goes to zero linearly fast. We now state a lemma similar to Lemma 5.3.

Lemma 5.5 *Suppose that* $u^{(1)}$ *and* $u^{(2)}$ *are in* $K^{l,\rho_1,\theta_1,\mu_1}_{\beta_0,T}$. *Then*

$$\left|\partial_Y u^{(1)}\int_0^Y dY'\partial_x u^{(1)} - \partial_Y u^{(2)}\int_0^Y dY'\partial_x u^{(2)}\right|_{l,\rho',\theta',\mu'} \leq c\left[\frac{|u^{(1)} - u^{(2)}|_{l,\rho'',\theta',\mu'}}{\rho'' - \rho'} + \frac{|u^{(1)} - u^{(2)}|_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'}\right] \tag{5.47}$$

in the $K^{l,\rho,\theta,\mu}$ *norm.*

The proof of Lemma 5.5 goes like the proof of Lemma 5.3. The only thing to be noticed is the fact that, because of the presence of a derivative in both the terms, one cannot use Sobolev estimate right away, but has to pay attention to the way the l derivatives distribute between them. If all the l derivatives hit the term involving the integral one has to Cauchy estimate the x-derivatives inside it. If instead all derivatives hit the term involving the Y-derivative one has to Cauchy estimate that derivative.

The estimate of the linear term whose coefficient grows linearly in Y is expressed in the following Lemma.

Lemma 5.6 *Suppose that* $u^{(1)}$ *and* $u^{(2)}$ *are in* $K^{l,\rho_1,\theta_1,\mu_1}_{\beta_0,T}$. *Then*

$$\begin{aligned} & |Y\partial_x\gamma u^E\partial_Y u^{(1)} - Y\partial_x\gamma u^E\partial_Y u^{(2)}|_{l,\rho',\theta',\mu'} \\ \leq\ & c\left[\frac{|u^{(1)} - u^{(2)}|_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'} + \frac{|u^{(1)} - u^{(2)}|_{l,\rho',\theta',\mu''}}{\mu'' - \mu'} + |u^{(1)} - u^{(2)}|_{l,\rho',\theta',\mu'}\right]. \end{aligned} \tag{5.48}$$

Using Lemmas 5.3, 5.5 and 5.6 we can conclude this subsection with the following estimate on the convective operator of Prandtl equations:

Proposition 5.7 *Suppose $u^{(1)}$ and $u^{(2)}$ are in $K^{l,\rho_1,\theta_1,\mu_1}_{\beta_0,T}$. Then*

$$\begin{aligned} & \left| K(u^{(1)},t) - K(u^{(2)},t) \right|_{l,\rho',\theta',\mu'} \\ \leq \; & c \left[\frac{|u^{(1)} - u^{(2)}|_{l,\rho'',\theta',\mu'}}{\rho'' - \rho'} + \frac{|u^{(1)} - u^{(2)}|_{l,\rho',\theta'',\mu'}}{\theta'' - \theta'} + \frac{|u^{(1)} - u^{(2)}|_{l,\rho',\theta',\mu''}}{\mu'' - \mu'} + |u^{(1)} - u^{(2)}|_{l,\rho',\theta',\mu'} \right] . \end{aligned} \tag{5.49}$$

5.5 Conclusion of the Proof of Theorem 5.1

To conclude the proof of estimate (5.42), first notice that in the iterative construction of the solution of (5.37), each term satisfies $\gamma\tilde{u}^P = -\gamma u^E$. The difference $K(u^{(1)}) - K(u^{(2)})$ need be considered only for these functions. As a result, we may assume that

$$\gamma \left[K(u^{(1)},t) - K(u^{(2)},t) \right] = 0 . \tag{5.50}$$

This will allow us to use Proposition 5.5. In fact:

$$\begin{aligned} & \left| F(u^{(1)},t) - F(u^{(2)},t) \right|_{l-1,\rho',\theta',\mu'} \\ = \; & \left| E_2 \left[K(u^{(1)},t) - K(u^{(2)},t) \right] \right|_{l-1,\rho',\theta',\mu'} \\ \leq \; & c \int_0^t ds \left| K(u^{(1)},t) - K(u^{(2)},t) \right|_{l-1,\rho',\theta',\mu'} \\ \leq \; & c \int_0^t ds \left[\frac{|u^{(1)}(\cdot,\cdot,s) - u^{(2)}(\cdot,\cdot,s)|_{l-1,\rho(s),\theta',\mu'}}{\rho(s) - \rho'} + \frac{|u^{(1)}(\cdot,\cdot,s) - u^{(2)}(\cdot,\cdot,s)|_{l-1,\rho',\theta(s),\mu'}}{\theta(s) - \theta'} \right. \\ & \left. + \frac{|u^{(1)}(\cdot,\cdot,s) - u^{(2)}(\cdot,\cdot,s)|_{l-1,\rho',\theta',\mu(s)}}{\mu(s) - \mu'} \right] . \end{aligned} \tag{5.51}$$

With the above estimate we conclude this subsection. The proof of the estimate (5.42) has been finally achieved. Therefore operator F satisfies all the hypotheses of the ACK Theorem, and Theorem 5.1 has been proved.

5.6 A final remark

The main result of this section is Theorem 5.1 stating the existence and the uniqueness of a solution u^P of equations (2.6)-(2.11), and that this solution is the sum of a function exponentially decaying outside the boundary layer and of the value at the boundary of the Euler flow.

What about normal velocity v^P? Corresponding to $\tilde{u}^P$ define $\tilde{v}^P$ by

$$\tilde{v}^P = -\int_0^Y \partial_x \tilde{u}^P dY' . \tag{5.52}$$

Using this expression, the fact that $\tilde{u}^P$ is exponentially decaying in the Y variable, and a Cauchy estimate in the x variable, it follows that $\tilde{v}^P$ differs for a constant (in Y) from a function in $K^{l-1,\rho_1',\theta_1,\mu_1}_{\beta_1,T}$ with $\rho_1' < \rho_1$. Renaming ρ_1', just to simplify the notation, we can therefore conclude that

$$\tilde{u}^P \in K^{l-1,\rho_1,\theta_1,\mu_1}_{\beta_1,T} \tag{5.53}$$

$$\bar{v}^P = \tilde{v}^P - \tilde{v}^P(y = \infty) \in K^{l-1,\rho_1,\theta_1,\mu_1}_{\beta_1,T} . \tag{5.54}$$

6 Conclusions

This concludes the proofs of existence for the Euler and Prandtl equations with analytic initial data. These results will be used in Part II [12] as the leading order terms in an asymptotic expansion for the solution of the Navier-Stokes equations with small viscosity. The solution will be found as a composite expansion, using the Prandtl solution near the boundary and the Euler solution far from the boundary.

Appendix A: The estimates for the heat operators

Proof of Lemma 5.1 We prove the estimate (5.14); the estimate (5.13) can be proved analogously. Set $\eta = (Y' \pm Y)/\sqrt{4t}$ so that

$$\begin{aligned}\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}\left|\int_0^\infty dY' \frac{e^{-(Y\pm Y')^2/4t}}{\sqrt{4\pi t}} g(Y')\right| &= c\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}\left|\int_{\pm Y/\sqrt{4t}}^\infty d\eta e^{-\eta^2} g(\mp Y + \eta\sqrt{4t})\right| \\ &\leq c\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}|g(Y)|. \end{aligned} \tag{A.1}$$

This uses the restriction that $0<\theta<\pi/4$, so that $e^{-\eta^2} \leq e^{-k\Re\eta^2}$, for some constant k and thus

$$\int_{\pm Y/\sqrt{4t}}^\infty d\eta e^{-\eta^2} \exp -(\mp Y + \eta\sqrt{4t}) \leq c. \tag{A.2}$$

Proof of Lemma 5.2 We begin with the estimate (5.15). Use the change of variable $\zeta = Y/\sqrt{4(t-s)}$ to obtain

$$\begin{aligned}\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}\left|\int_0^t ds \frac{Y}{t-s}\frac{e^{-Y^2/4(t-s)}}{\sqrt{4\pi(t-s)}} f(s)\right| &= \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}\left|2\int_{Y/\sqrt{4t}}^\infty d\zeta e^{-\zeta^2} f(t - Y^2/4\zeta^2)\right| \\ &\leq \sup_t |f(t)| \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} 2\int_{Y/\sqrt{4t}}^\infty d\zeta e^{-\zeta^2} \\ &= c\sup_t |f(t)|. \end{aligned} \tag{A.3}$$

We now pass to the estimate (5.15) with $j=1$. Since $f(0)=0$, then

$$\begin{aligned}\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}\left|\partial_Y \int_0^t ds \frac{Y}{t-s}\frac{e^{-Y^2/4(t-s)}}{\sqrt{4\pi(t-s)}} f(s)\right| &= 2\sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y}\left|\int_{Y/\sqrt{4t}}^\infty d\zeta e^{-\zeta^2}\frac{Y}{\zeta^2} f'(t - Y^2/4\zeta^2)\right| \\ &\leq \sup_t |f'(t)| \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} Y\int_{Y/\sqrt{4t}}^\infty d\zeta e^{-\zeta^2}/\zeta^2 \\ &= c\sup_t |f'(t)|. \end{aligned} \tag{A.4}$$

The estimate (5.15) with $j=2$ can be proved in a similar way, using $\partial_Y^2 E_1 f = \partial_t E_1 f$.

Proof of Propositions 5.1 Denote with ∂^j a j-th derivative in x and Y where the derivative in Y does not show up more than twice (we stress again that in our functional setting a Y derivative is required up to order two, see e.g. Definition 2.4). Then

$$\begin{aligned} |E_0(t)u|_{l,\rho,\theta,\mu} &= \sum_{j\le l} \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} \sup_{|\Im x|<\rho} \left\| \partial^j \int_0^\infty dY' \left[E_0(Y-Y',t) - E_0(Y+Y',t)\right] u(x,Y') \right\|_{L^2} \\ &\le \sum_{j\le l} \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} \int_0^\infty dY' \left[|E_0(Y-Y',t)| + |E_0(Y+Y',t)|\right] \sup_{|\Im x|<\rho} \|\partial^j u(\cdot,Y')\|_{L^2} \\ &\le c|u|_{l,\rho,\theta,\mu} \end{aligned} \tag{A.5}$$

where Lemma 5.1 has been used in the last step. For the first derivative in Y, the boundary terms at $Y=0$ vanished because $u(Y=0)=0$; for the second derivative, they vanished due to cancelation of the two E_0 factors.

Proof of Proposition 5.2 Denote by ∂^j a j-th derivative in x and Y, where the derivative in Y does not show up more than twice. Also if a Y derivative does show up, the order of the x derivative is at most $l-2$ (as required in $H^{l,\rho,\theta}$). Then

$$\begin{aligned} &|E_0(t)u|_{l,\rho,\theta} \\ &= \sum_{j\le l} \sup_{\theta'\le\theta} \left(\int_{\Gamma(\theta',a/\varepsilon)} dY \left\{ \sup_{|\Im x|<\rho} \left\| \partial^j \int_0^\infty dY' \left[E_0(Y-Y',t) - E_0(Y+Y',t)\right] u(x,Y') \right\|_{L^2(\Re x)} \right\}^2 \right)^{1/2} \\ &= \sum_{j\le l} \sup_{\theta'\le\theta} \left(\int_{\Gamma(\theta',a/\varepsilon)} dY \left\{ \sup_{|\Im x|<\rho} \left\| \partial^j \left[\int_{-Y/\sqrt{4t}}^\infty d\eta e^{-\eta^2} u(x, Y+\eta\sqrt{4t}) \right.\right.\right.\right. \\ &\qquad \left.\left.\left.\left. - \int_{Y/\sqrt{4t}}^\infty dz e^{-z^2} u(x,-Y+z\sqrt{4t}) \right] \right\|_{L^2(\Re x)} \right\}^2 \right)^{1/2} \\ &\le \sum_{j\le l} 2\int_{-\infty}^\infty d\eta e^{-\eta^2} \sup_{\theta'\le\theta} \left(\int_{\Gamma(\theta',a/\varepsilon)} dY \left\{ \sup_{|\Im x|<\rho} \|\partial^j u(\cdot,Y')\|_{L^2} \right\}^2 \right)^{1/2} \\ &\le c|u|_{l,\rho,\theta}. \end{aligned} \tag{A.6}$$

Proof of Proposition 5.3 We have

$$\begin{aligned} |E_1\phi|_{l,\rho,\theta,\mu,\beta,T} &\le c\sum_{\alpha_1\le 2}\sum_{\alpha_2\le l-\alpha_1} \sup_{0\le t\le T} \sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y} \sup_{|\Im x|\le\rho-\beta t} \|\partial_Y^{\alpha_1}\partial_x^{\alpha_2} E_1\phi\|_{L^2} \\ &\le c\sum_{\alpha_1\le 2}\sum_{\alpha_2\le l-\alpha_1} \sup_{0\le t\le T} \sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y} \left| \partial_Y^{\alpha_1} E_1 \sup_{|\Im x|<\rho-\beta t} \|\partial_x^{\alpha_2}\phi\|_{L^2} \right| \\ &\le c|\phi|_{l,\rho,\beta,T}\,. \end{aligned} \tag{A.7}$$

In passing from the first to the second line we have used the fact that $E_1 u$ solves the heat equation (so that $\partial_t E_1 u = \partial_{YY} E_1 u$); in passing from the third to the fourth line we have used Lemma 5.2 with $f(t) = \sup_{|\Im x|<\rho-\beta t} \|\partial_x^{\alpha_2}\phi\|_{L^2}$.

Proof of Proposition 5.4 To estimate $|E_2 u|_{l,\rho,\theta,\mu,\beta,T}$ we must estimate $|\partial_x^\alpha E_2 u|_{0,,\rho,\theta,\mu,\beta,T}$ with $\alpha\le l$, $|\partial_Y\partial_x^\alpha E_2 u|_{0,\rho,\theta,\mu,\beta,T}$ with $\alpha\le l-1$, $|\partial_t\partial_x^\alpha E_2 u|_{0,\rho,\theta,\mu,\beta,T}$ with $\alpha\le l-1$ and $|\partial_{YY}\partial_x^\alpha E_2 u|_{0,,\rho,\theta,\mu,\beta,T}$ with $\alpha\le l-2$. We begin with $|\partial_x^\alpha E_2 u|_{0,,\rho,\theta,\mu,\beta,T}$:

$$\begin{aligned} &|\partial_x^\alpha E_2 u|_{0,,\rho,\theta,\mu,\beta,T} \\ &= \sup_{0\le t\le T} \sup_{Y\in\Sigma(\theta-\beta T)} e^{(\mu-\beta t)\Re Y} \sup_{|\Im x|\le\rho-\beta t} \left\| \partial_x^\alpha \int_0^t ds \left[\int_{-Y/\sqrt{4(t-s)}}^\infty d\eta e^{-\eta^2} u(x, Y+\eta\sqrt{4(t-s)}, s) + \right.\right. \end{aligned}$$

$$\int_{Y/\sqrt{4(t-s)}}^{\infty} dz e^{-z^2} u(x, -Y + z\sqrt{4(t-s)}, s)\Big]\Big\|_{L^2}$$

$$\leq \sup_{0\leq t\leq T} \sup_{Y\in\Sigma(\theta-\beta T)} e^{(\mu-\beta t)\Re Y} \int_0^t ds \Big[\int_{-Y/\sqrt{4(t-s)}}^{\infty} d\eta e^{-\eta^2} \sup_{|\Im x|\leq \rho-\beta t} \|\partial_x^\alpha u(\cdot, Y + \eta\sqrt{4(t-s)}, s)\|_{L^2} +$$

$$\int_{Y/\sqrt{4(t-s)}}^{\infty} dx e^{-z^2} \sup_{|\Im x|\leq \rho-\beta t} \|\partial_x^\alpha u(\cdot, -Y + z\sqrt{4(t-s)}, s)\|_{L^2}\Big]$$

$$\leq |\partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T} \sup_{0\leq t\leq T} \sup_{Y\in\Sigma(\theta-\beta T)} \int_0^t ds \Big[\int_{-Y/\sqrt{4(t-s)}}^{\infty} d\eta e^{-\eta^2} + \int_{Y/\sqrt{4(t-s)}}^{\infty} dz e^{-z^2}\Big]$$

$$= |\partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T} \,. \qquad \text{(A.8)}$$

We now pass to $|\partial_Y \partial_x^\alpha E_2 u|_{0,\rho,\theta,\mu,\beta,T}$; the only difference from the above estimate will be the appearance of a boundary term, behaving like $\sqrt{(t-s)}$, which is nevertheless bounded for the regularizing property of the integration in time.

$$|\partial_Y \partial_x^\alpha E_2 u|_{0,\rho,\theta,\mu,\beta,T}$$

$$\leq \sup_{0\leq t\leq T} \sup_{Y\in\Sigma(\theta-\beta T)} e^{(\mu-\beta t)\Re Y} \int_0^t ds \Big[\int_{-Y/\sqrt{4(t-s)}}^{\infty} d\eta e^{-\eta^2} \sup_{|\Im x|\leq \rho-\beta t} \|\partial_Y \partial_x^\alpha u(\cdot, Y + \eta\sqrt{4(t-s)}, s)\|_{L^2}$$

$$+ \int_{Y/\sqrt{4(t-s)}}^{\infty} dx e^{-z^2} \sup_{|\Im x|\leq \rho-\beta t} \|\partial_Y \partial_x^\alpha u(\cdot, -Y + z\sqrt{4(t-s)}, s)\|_{L^2}$$

$$-2\frac{e^{-Y^2/4(t-s)}}{\sqrt{4(t-s)}} \sup_{|\Im x|\leq \rho-\beta t} \|\partial_x^\alpha u(\cdot, 0, s)\|_{L^2}\Big]$$

$$\leq |\partial_Y \partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T} + c|\partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T} \sup_{0\leq t\leq T} \sup_{Y\in\Sigma(\theta-\beta t)} \int_0^t ds \frac{e^{-Y^2/4(t-s)}}{\sqrt{4(t-s)}}$$

$$\leq |\partial_Y \partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T} + c|\partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T}. \qquad \text{(A.9)}$$

We now pass to $|\partial_t \partial_x^\alpha E_2 u|_{0,\rho,\theta,\mu,\beta,T}$:

$$|\partial_t \partial_x^\alpha E_2 u|_{0,\rho,\theta,\mu,\beta,T}$$

$$\leq \sup_{0\leq t\leq T} \sup_{Y\in\Sigma(\theta-\beta T)} e^{(\mu-\beta t)\Re Y} \left\{ \sup_{|\Im x|\leq\rho-\beta t} \|\partial_x^\alpha u(\cdot,Y,t)\|_{L^2} + \int_0^t ds \left[\frac{Y}{t-s} \frac{e^{-Y^2/4(t-s)}}{\sqrt{4(t-s)}} \sup_{|\Im x|\leq\rho-\beta t} \|\partial_x^\alpha u(\cdot,0,t)\|_{L^2} \right.\right.$$
$$+ \int_{-Y/4(t-s)}^{\infty} d\eta e^{-\eta^2} \eta \frac{1}{\sqrt{(t-s)}} \sup_{|\Im x|\leq\rho-\beta t} \|\partial_Y \partial_x^\alpha u(\cdot, Y+\eta\sqrt{4(t-s)},t)\|_{L^2}$$
$$\left.\left. - \int_{Y/4(t-s)}^{\infty} dz e^{-z^2} z \frac{1}{\sqrt{(t-s)}} \sup_{|\Im x|\leq\rho-\beta t} \|\partial_Y \partial_x^\alpha u(\cdot,-Y+z\sqrt{4(t-s)},t)\|_{L^2} \right]\right\}$$
$$\leq c|\partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T} + c|\partial_Y \partial_x^\alpha u|_{0,\rho,\theta,\mu,\beta,T}$$
$$\leq c|u|_{l,\rho,\theta,\mu,\beta,T} \,. \tag{A.10}$$

The term $|\partial_{YY}\partial_x^\alpha E_2 u|_{0,\rho,\theta,\mu,\beta,T}$ can be bounded using the estimate (A.10) and the fact that $\partial_{YY} E_2 u = \partial_t E_2 u - u$. this concludes the proof of Proposition 5.4.

Proof of Proposition 5.5 The proof of Proposition 5.5 is very similar to the proof of Proposition 5.4, the main difference being that one is not allowed to use the regularizing properties of the integration in time; no singular term appears, though, because of the requirement that $u = 0$ at the boundary. Here we present the estimate of the term $|\partial_Y \partial_x^\alpha E_2 u|_{0,\rho',\theta',\mu'}$ with $\alpha \leq l-1$:

$$|\partial_Y \partial_x^\alpha E_2 u|_{0,\rho',\theta',\mu'} \leq \sup_{Y\in\sigma(\theta')} e^{\mu'\Re Y} \int_0^t ds \left[\int_{-Y/\sqrt{4(t-s)}}^{\infty} d\eta e^{-\eta^2} \sup_{|\Im x|\leq\rho'} \|\partial_Y \partial_x^\alpha u(\cdot, Y+\eta\sqrt{4(t-s)}, s)\|_{L^2} \right.$$
$$\left. + \int_{Y/\sqrt{4(t-s)}}^{\infty} dx e^{-z^2} \sup_{|\Im x|\leq\rho'} \|\partial_Y \partial_x^\alpha u(\cdot,-Y+z\sqrt{4(t-s)},s)\|_{L^2} \right]$$
$$\leq \int_0^t ds |\partial_Y \partial_x^\alpha u(\cdot,\cdot,s)|_{0,\rho',\theta',\mu'} \,. \tag{A.11}$$

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