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Abstract

The valuation of mortgage-backed securities is a challenging problem for the quasi-Monte Carlo method, because the nominal dimension is 360 for a security involving 30-year mortgages. Paskov showed, however, that quasi-Monte Carlo is much more effective than standard Monte Carlo for this problem. We perform further analysis and computation for this problem and find that a Brownian bridge discretization of the interest rate fluctuations provides even further improvements in the quasi-Monte Carlo method. Furthermore, in this representation we find that the effective dimension of the problem is fairly small.

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1 Introduction

Quasi-random sequences have the potential to yield great improvements in the accuracy of Monte Carlo methods for many applications. Previous studies [1, 2] on test problems have shown the limitations of quasi-Monte Carlo methods for problems of high dimension or those in which the integrand is not smooth. The improved convergence of quasi-Monte Carlo can sometimes be recovered, however, by careful adaptation of standard Monte Carlo techniques to lower the effective dimension or to smooth out the problem [3].

One of the most challenging problems of mathematical finance is the valuation of securities, such as mortgage-backed securities, whose value depends on the whole trajectory of interest rates or other variables. Because each time step must be represented by a separate dimension in application of quasi-random sequences, the dimension for such problems is quite large. As an example, for a security involving 30-year mortgages with monthly payments and interest rate fluctuations the nominal dimension is 360. Nevertheless, Paskov and Traub [4, 5] showed that quasi-Monte Carlo evaluation of a particular mortgage-backed security is much more accurate than standard Monte Carlo (using pseudo-random sequences).

In this paper we re-examine the mortgage-backed security problem using quasi-Monte Carlo. Using a different representation of the problem, we obtain significant further improvements in the accuracy of the quasi-Monte Carlo method. In this representation, we show that the effective dimension of the problem is rather small (around 30). Although the problem considered here differs from that of Paskov in some details–e.g. the parameters in his computation are unknown to us, since they are proprietary, and we have used mortgages of length 256 months for convenience–we believe that these results may explain why the quasi-Monte Carlo method was so effective in his study.

The motivation for our method is the representation of this problem as the discretization of a Feynman-Kac integral, in which payments are made continuously rather than monthly and interest rate fluctuations are a function of Brownian motion rather than a random walk. For quasi-Monte Carlo evaluation of Feynman-Kac integrals, Moskowitz and Caflisch [3] showed that a discretization based on the Brownian bridge distribution is far more accurate than that using a standard discretization. The Brownian bridge discretization involves successive subdivisions of the time interval of the problem and puts most of the variance of the problem at the first few dimensions. While this has no effect on a calculation using pseudo-random points, it greatly improves the use of quasi-random points since it reduces the effective dimension of the problem.

These results show that the mortgage-backed securities value depends strongly on only a few of the random variables in the Brownian bridge representation of the interest rate fluctuations. We also perform a calculation of the sensitivity to the variance of the random variables in the problem for both the Brownian bridge and the standard discretization. This provides a quantitative measure of the dependence, which may be useful in trying further improvements of the method.

2 Mortgage-Backed Securities

Consider a security backed by mortgages of length M months with fixed interest rate i_0, which is the current interest rate at the beginning of the mortgage. The present value of the security is then

$$\begin{aligned} PV &= E(v) \\ &= E(\sum_{k=1}^{M} u_k m_k) \end{aligned} \tag{2.1}$$

in which E is the expectation over the random variables involved in the interest rate fluctuations. The variables in the problem are the following:

$$\begin{aligned} u_k &= \text{discount factor for month } k \\ m_k &= \text{cash flow for month } k \\ i_k &= \text{interest rate for month } k \\ w_k &= \text{fraction of remaining mortgages prepaying in month } k \end{aligned}$$

$$
\begin{aligned}
r_k &= \text{fraction of remaining mortgages at month } k \\
c_k &= (\text{remaining annuity at month } k)/c \\
c &= \text{monthly payment} \\
\xi_k &= \text{an } N(0,\sigma) \text{ random variable.}
\end{aligned}
$$

This notations follows that of Paskov, except that our c_k is denoted a_{M-k+1} by Paskov.

Several of these variables are easily defined:

$$
\begin{aligned}
u_k &= \prod_{j=0}^{k-1}(1+i_k)^{-1} \\
m_k &= cr_k((1-w_k)+w_k c_k) \\
r_k &= \prod_{j=1}^{k-1}(1-w_j) \\
c_k &= \sum_{j=0}^{M-k}(1+i_0)^{-j}.
\end{aligned}
$$

Following Paskov, we use a model for the interest rate fluctuations and the prepayment rate given by

$$
\begin{aligned}
i_k &= K_0 e^{\xi_k} i_{k-1} \\
&= K_0^k e^{\xi_1+\ldots+\xi_k} i_0 \\
w_k &= K_1 + K_2 \arctan\left(K_3 i_k + K_4\right)
\end{aligned}
\tag{2.2}
$$

in which K_1, K_2, K_3, K_4 are constants of the model. The constant $K_0 = e^{-\sigma^2/2}$ is chosen to normalize the log-normal distribution, i.e. so that $E(i_k) = i_0$. The initial interest rate i_0 is an additional constant that must be specified.

In this study we do not divide the cash flow of the security among a group of tranches, as in [4], but only consider the total cash flow. Nevertheless, the results should be indicative of a more general computation involving a number of tranches.

The expectation PV can be written as in integral over R^M with Gaussian weights

$$
g(\xi) = (2\pi\sigma^2)^{-1/2} e^{-\xi^2/2\sigma^2}. \tag{2.3}
$$

This is transformed into an unweighted integral by a mapping $\xi = G(x)$ with $G'(x) = g(\xi)$, which takes a uniformly distributed variable x to an $N(0,\sigma)$

variable ξ. The formula for PV is

$$\begin{aligned} PV &= \int_{R^M} v(\xi_1, \ldots, \xi_M) g(\xi_1) \ldots g(\xi_1) d\xi_1 \ldots d\xi_M \\ &= \int_{[0,1]^M} v(G(\xi_1), \ldots, G(\xi_M)) dx_1 \ldots dx_M. \end{aligned} \tag{2.4}$$

Note that in quasi-Monte Carlo evaluation of an expectation involving a stochastic process with M time steps, the resulting integral is M dimensional.

In the numerical study below, we have chosen i_0 so that the yearly interest rate is 8%. In addition we have chosen the constants $K_1, K_2, K_3, K_4, \sigma$ according to the following guidelines:
(i) The function w_k is a positive, decreasing function of i_k so that we take $K_1 > 0$, $K_2 < 0$, $K_3 > 0$ and $K_4 > 0$.
(ii) The natural period for prepayments and fluctuations of the interest rate is approximately one year. For the period Δt (=1/12 in this example), the constants scale as follows:

$$(i_k, w_k, K_1, K_2, K_3, K_4) \approx (\Delta t, \Delta t, \Delta t, \Delta t, \Delta t^{-1}, 1). \tag{2.5}$$

This is discussed further in the next section.
(iii) The average length of a mortgage before prepayment is approximately 6 years. Following these guidelines, we have used the values

$$(i_0, K_1, K_2, K_3, K_4, \sigma^2) = (.007, .01, -.005, 10, .5, .0004). \tag{2.6}$$

The value of σ is exactly that used by Paskov [4]. For simplification in writing our numerical program, we have taken the length of the mortgage to be 256 months (a power of 2). This is not essential, but simplifies the Brownian bridge representation.

3 Continuum Limit

Since the monthly mortgage payments and interest rate fluctuations are small compared to those over the life of a mortgage, it is useful to consider the continuum limit of this problem. We emphasize that this approximation is used only for understanding the problem and as motivation for the Brownian bridge discretization. It is not used in our computation of the present value of the security.

For a fixed time period T and a variable time unit of Δt, with $M = T/\Delta t$, the interest rate and prepayment rate should be proportional to the time period Δt and should vary slowly; i.e. we assume that

$$\begin{aligned} i_k &= \Delta t \ \iota(k\Delta t) \\ w_k &= \Delta t \omega(k\Delta t). \end{aligned} \tag{3.1}$$

In addition, the constants of the problem should scale as

$$\begin{aligned} \sigma &= \sqrt{\Delta t}\Sigma \\ c &= \Delta t C \\ K_1 &= \Delta t \kappa_1 \\ K_2 &= \Delta t \kappa_2 \\ K_3 &= \Delta t^{-1} \kappa_3 \\ K_4 &= \kappa_4. \end{aligned} \tag{3.2}$$

Under these scaling assumptions we may make a number of approximations. First, the discount factor is approximated by

$$\begin{aligned} u_k &= \prod_{j=0}^{k-1} (1 + i_k)^{-1} \\ &\approx \prod_{j=0}^{k-1} e^{-\Delta t \ \iota(k\Delta t)} \\ &\approx e^{-\int_0^{k\Delta t} \iota(t) dt} \\ &\equiv U(k\Delta t). \end{aligned} \tag{3.3}$$

Similarly the prepayment factor, cash flow and remaining annuity are approximated by

$$\begin{aligned} r_k &= \prod_{j=1}^{k-1} (1 - w_k) \\ &\approx e^{-\int_0^{k\Delta t} \omega(t) dt} \end{aligned}$$

$$
\begin{aligned}
&\equiv R(k\Delta t) && (3.4)\\
m_k &= \Delta t C r_k(1 - w_k + w_k b_k) &&\\
&\approx \Delta t C R(k\Delta t)(1 + \omega(k\Delta t)\alpha(k\Delta t)) &&\\
&\equiv \Delta t M(k\Delta t) && (3.5)\\
c_k &= \sum_{j=0}^{M-k} (1 + i_0)^{-j} &&\\
&\approx (\Delta t)^{-1} \int_0^{T-k\Delta t} e^{-t\iota_0} dt &&\\
&= (\iota_0 \Delta t)^{-1}(1 - e^{-(T-k\Delta t)\iota_0}) &&\\
&= (\Delta t)^{-1}\alpha(k\Delta t) && (3.6)
\end{aligned}
$$

Next the random walk in the interest rate fluctuation model is approximated by Brownian motion $b(t)$ as

$$\xi_1 + \ldots + \xi_k \approx \Sigma b(k\Delta t). \tag{3.7}$$

The model for interest rate fluctuations and prepayments are then

$$
\begin{aligned}
\iota(t) &= \kappa_0^t e^{\sigma b(t)} \iota_0 && (3.8)\\
\omega(t) &= \kappa_1 + \kappa_2 \arctan(\kappa_3 \iota(t) + \kappa_4). && (3.9)
\end{aligned}
$$

Finally the present value v is approximately

$$
\begin{aligned}
v &= \sum_{k=1}^{M} u_k m_k &&\\
&\approx \sum_{k=1}^{M} \Delta t U(k\Delta t) M(k\Delta t) &&\\
&\approx \int_0^T U(t) M(t) dt. && (3.10)
\end{aligned}
$$

The resulting continuum approximation for the present value is then PV_c given by

$$PV_c = E\left(\int_0^T U(t) M(t) dt\right) \tag{3.11}$$

in which

$$\begin{aligned} U(t) &= e^{-\int_0^t \iota(t')dt'} \\ M(t) &= CR(t)(1+\omega(t)\alpha(t)) \\ R(t) &= e^{-\int_0^t \omega(t')dt'} \\ \alpha(t) &= \iota_0^{-1}(1-e^{-(T-t)\iota_0}). \end{aligned} \tag{3.12}$$

4 Brownian Bridge and an Alternative Discretization

Since Brownian motion is a Markov process, it is most natural to generate its value $b(t+\Delta t)$ as a random jump from a past value $b(t)$ as

$$b(t+\Delta t) = b(t) + \sqrt{\Delta t}\ \nu \tag{4.1}$$

in which ν is an $N(0,1)$ random variable. On the other hand, the value $b(t+\Delta t)$ can also be generated from knowledge of both a past value $b(t)$ and a future value $b(t+2\Delta t)$, according to the *Brownian bridge* formula

$$b(t+\Delta t) = \frac{1}{2}(b(t) + b(t+2\Delta t)) + \frac{1}{\sqrt{2}}\sqrt{\Delta t}\nu. \tag{4.2}$$

Note that variance of the random part of the Brownian bridge formula (4.2) is half of that in (4.1) due to the factor $1/\sqrt{2}$. These formulas apply equally well to a random walk with Gaussian increments that is evaluated at times t that are a multiple of Δt.

The standard method of generating a random walk $y_k = \sigma b(k\Delta)$ is based on the updating formula (4.1). The initial value is $y_0 = 0$. Each subsequent value y_{k+1} is generated from the previous value y_k using formula (4.1), with independent normal variables ν_k.

Another method, which we refer to as the *alternative discretization* can be based on (4.2). Suppose we wish to determine the path $y_0, y_1, \ldots, y_M$, and for convenience assume that M is a power of 2. The initial value is $y_0 = 0$.

The next value generated is $y_N = \sigma\sqrt{N\Delta t}\ \nu_0$. Then the value at the mid point $y_{N/2}$ is determined from the Brownian bridge formula (4.2). Subsequent values are found at the successive mid-points; i.e. $y_{N/4}, y_{3N/4}, y_{N/8}, \ldots$. Using a more general version of (4.1), one can subdivide an interval into any number of pieces, so that the alternative discretization can also be performed for values of M other than powers of 2.

Although the total variance in this representation is the same as in the standard discretization, much more of the variance is contained in the first few steps of the alternative discretization due to the factor of 2 reduction in the variance in the Brownian bridge formula. This reduces the effective dimension of the random walk simulation, which increases the accuracy of quasi-Monte Carlo. Moskowitz and Caflisch [3] applied this method to the evaluation of Feynman-Kac integrals and showed the error to be substantially reduced when the number of time steps, which is equal to the dimension of the corresponding integral, is large. Since the mortgage-backed securities problem described above depends on a random walk, and can be written as a discretization of a Feynman-Kac integral, we were naturally led to apply the alternative discretization to this problem .

Another possibility is just a rearrangement of the alternative discretization. Determine the values y_k in the following order:

$$y_0,\ y_N,\ y_{N/2},\ y_{N/4},\ y_{N/8},\ \ldots,\ y_1, y_{3N/4},\ y_{3N/8},\ \ldots,\ y_3,\ \ldots y_{(N-1)/N} \quad (4.3)$$

We call this a a depth-first method, as opposed to the breadth-first method of the first alternative discretization. Although it looks promising, we have not yet tested it.

5 Numerical Method and Results

We performed computations of the present value PV using Three numerical methods, standard Monte Carlo using a pseudo-random sequence, quasi-Monte Carlo using a Sobol sequence with the standard discretization, and the same quasi-Monte Carlo with the alternative discretization. The quasi-random computations used a numerical generator for Sobol sequences with a maximum dimension of 40. The remaining 216 dimensions were taken to be pseudo-random. Inversion of the error function *erf* was performed through generating a table of values of *erf*, searching for the interval of the desired value and interpolating in between.

First we present results on the performance of the model finance problem. The present value PV, the variance var of PV, and the average lifetime l of a mortgage are

$$\begin{aligned} PV &= 119.215 \\ var &= 25. \\ l &= 76.7\ (months). \end{aligned} \tag{5.1}$$

The average cash flow and the average discounted cash flow for each month are plotted in Figure 1.

Next we present results, which are plotted in Figure 2, on the convergence of the three numerical methods as the number of sample points N is increased. For the pseudo-random computation, the rms error $var\ N^{-1/2}$ is plotted. For the quasi-random computations, we compute PV for values of N ranging from 2000 to $64,000$. An empirical error is determined by comparison of the value PV for each N to the average value over all of the computations. In addition, for each value of N the computation is performed 16 times, and the error for that value of N is averaged as a rms over those 8 runs. For each of the 8 runs and each value of N the computations were performed independently, that is the Sobol points and pseudo-random points in one computation are not used in any of the others (except in the determination of the empirical error).

The results show that the quasi-Monte Carlo computations using the standard discretization are significantly more accurate than those from standard Monte-Carlo, by about a factor of 3. This is roughly consistent with the results of Paskov [4], although he did not make exactly these comparisons, and some of his results are affected by apparent errors in the random number generators, as he pointed out. The quasi-Monte Carlo computations using the Brownian bridge, however, are much more accurate, about a factor of 10 better than quasi-Monte Carlo using the standard discretization. In addition the alternative quasi-Monte Carlo error converges at a faster rate than the standard quasi-Monte Carlo error, which converges faster than the standard Monte Carlo error.

Next we attempt to determine the effective dimension of the mortgage-

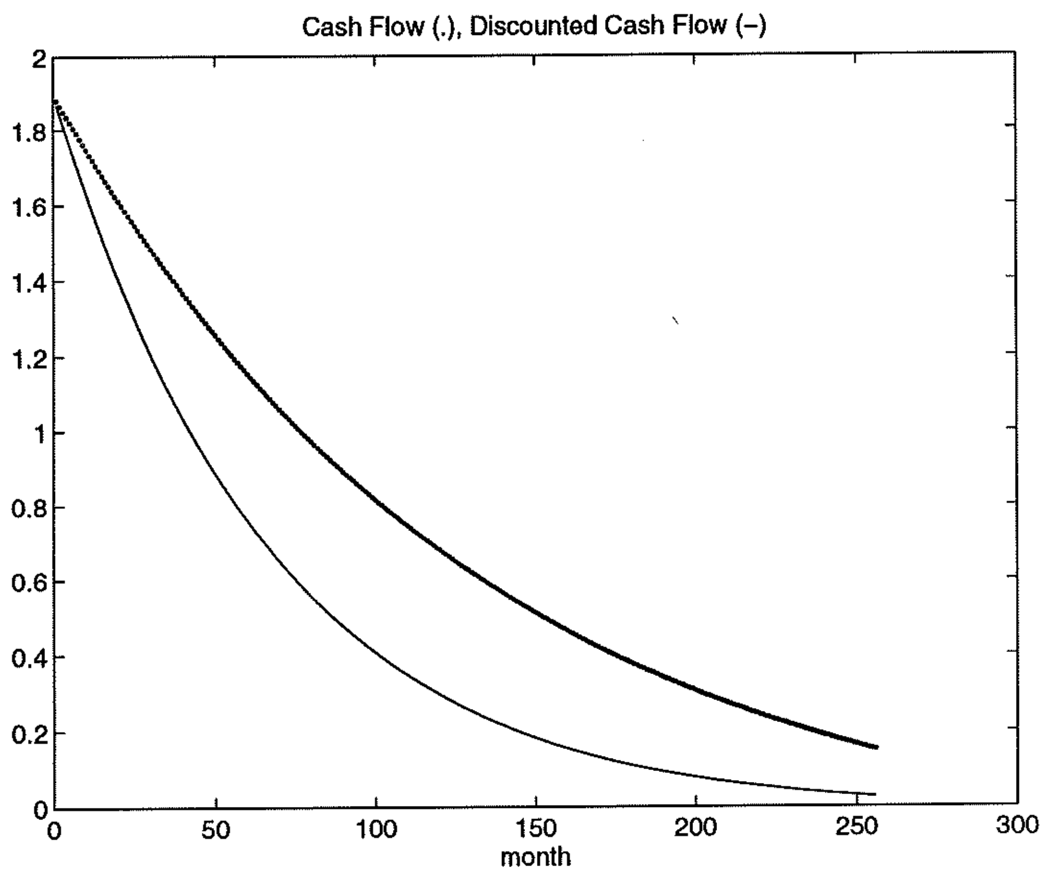

Figure 1: Average Cash Flow and Discounted Cash Flow for Each Month.

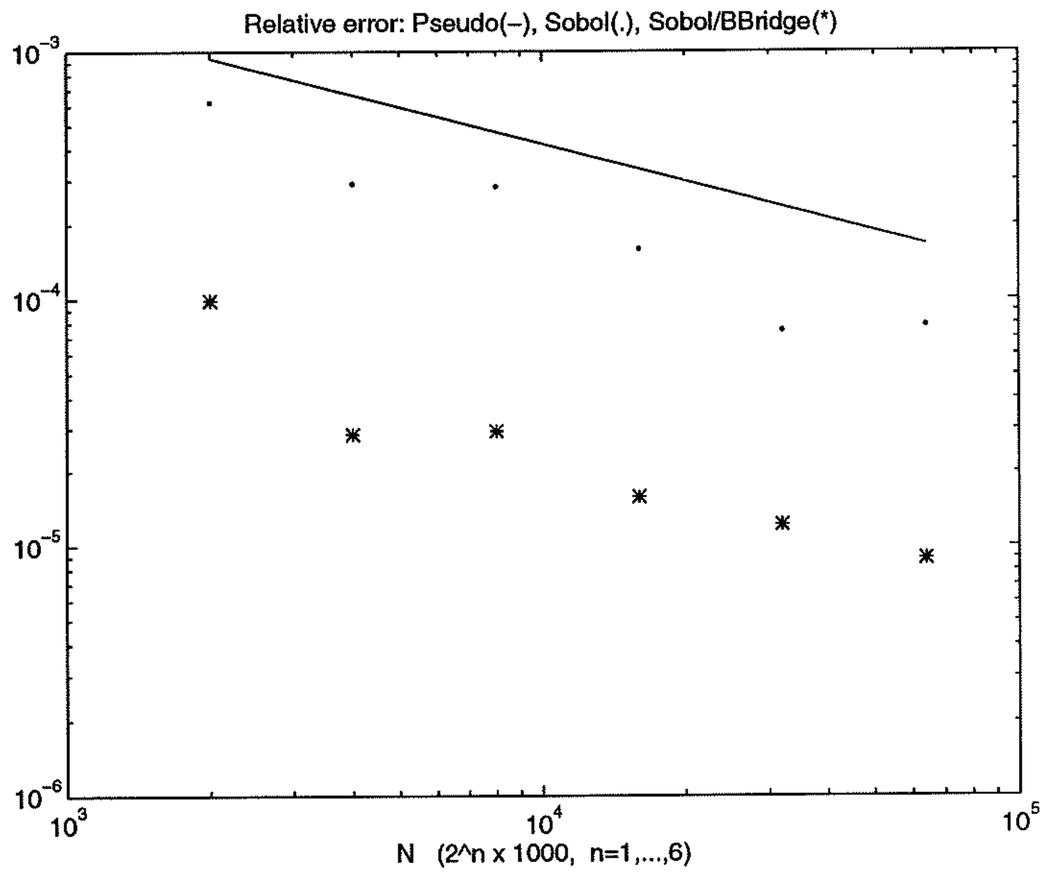

Figure 2: Relative error in the computation of PV from three methods: standard Monte Carlo, quasi-Monte Carlo, and quasi-Monte Carlo with the alternative discretization.

backed securities problem. We roughly define the effective dimension to be the minimum number of random variables that are required to compute the value PV with a specified level of accuracy, which we take to be 10^{-3} to 10^{-4}. For the quasi-Monte Carlo using the standard discretization, we use the first n variables, and take the remaining variables to be 0. For the quasi-Monte Carlo with the alternative discretization, we take the first n_1 variables in the order determined from the Brownian bridge and the first n_2 variables in the order of time. All of the others are set to 0. The result are plotted for the alternative discretization with $n_2 = 16$ and a range of values of n_1 in Figure 3. Note that for the standard discretization, the interest rate is constant over the times for which the random variable is replaced by 0; whereas for the alternative discretization, the interest rate is linearly interpolated in those regions since the random value is added to an average rate.

The results show that the number of random variables n needed to obtain good accuracy in the standard discretization is rather large, about . On the other hand for the alternative discretization, good accuracy is obtained with a small number of random variables, such as $n_1 = n_2 = 16$, which gives a total effective dimension of 32.

Finally we attempt to measure the sensitivity of the computation of PV to the variables in the problem. For each i, we double the variance of the $i - th$ variable while leaving the other variables unchanged. We take the difference between this modified value and the value without modification, averaged over 8000 samples, then take the rms of this value over 16 runs. The result is plotted as a function of i for the alternative discretization in Figure 4. These results show that the significance of the random variables drops off rapidly.

6 Conclusions

The results of this study confirm the conclusions of Paskov [4] that quasi-Monte Carlo performs significantly better than standard Monte Carlo on the mortgage-backed securities problem. Although our computation differs from that of Paskov in some details, it yields roughly the same order of improvement for the standard discretization which he used.

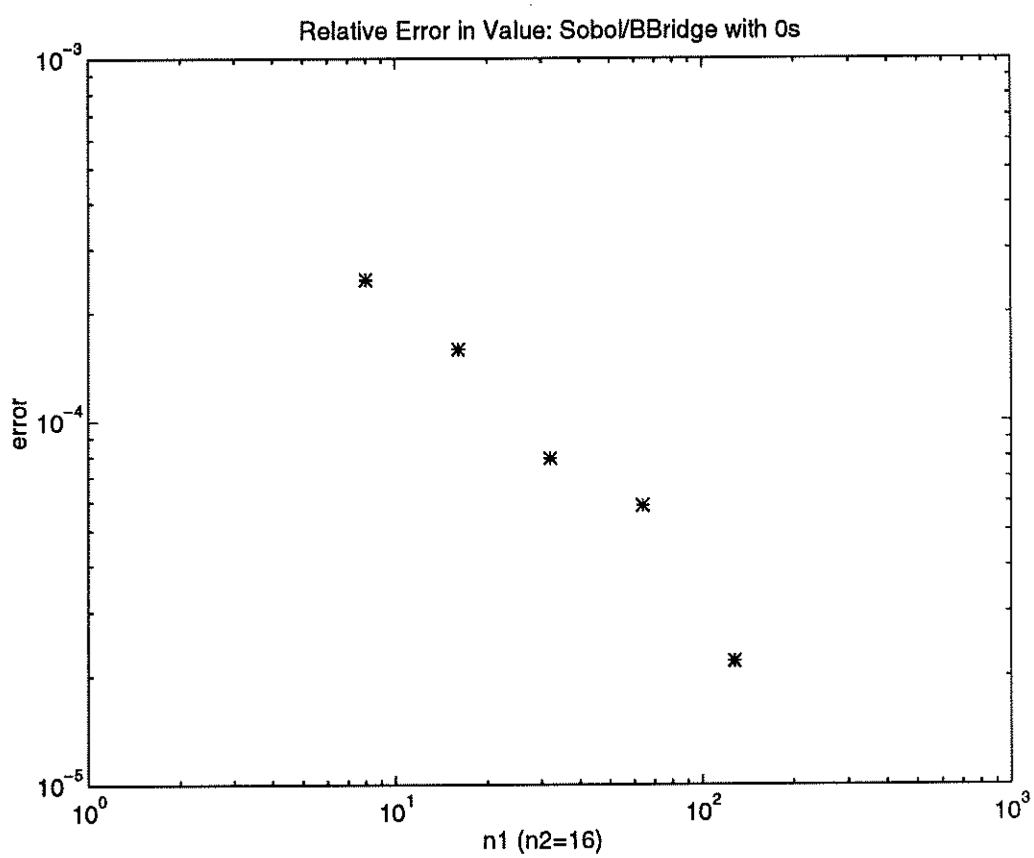

Figure 3: Relative Error for Alternative Discretization with Reduced Number of Random Variables: RMS(PV(256)-PV($n_1 = 16, n_2$))

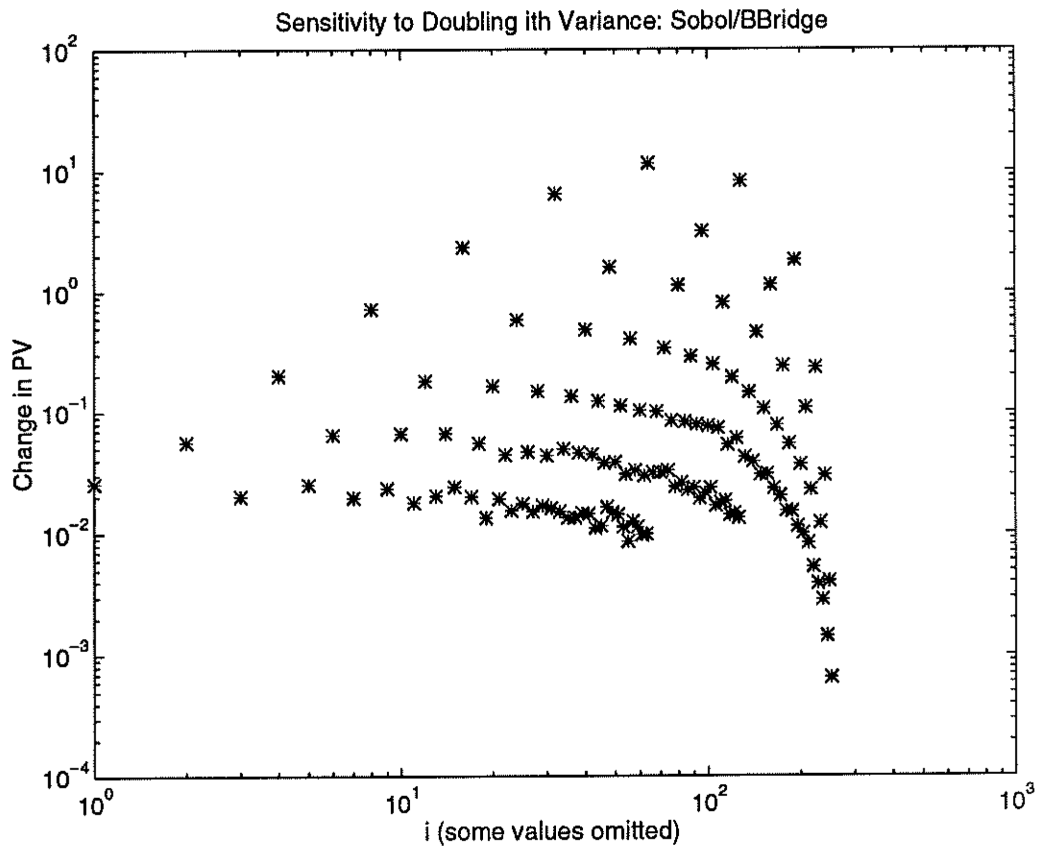

Figure 4: Sensitivity of PV Computation to Variance of ith Variable: RMS(PV(σ_i)-PV(σ_i^2))

On the other hand, we find greatly improved accuracy for the quasi-Monte Carlo method if the alternative discretization based on the Brownian bridge distribution is used. In the alternative discretization, the small monthly variations in the interest rate are efficiently represented by fluctuations over longer periods, with subsequent corrections made for the fluctuations over shorter periods. This makes the the first few variables of the problem much more important than the others. Therefore it reduces the effective dimension of the problem, which greatly improves the accuracy of the quasi-Monte Carlo method.

In this alternative method, the effective dimension of the problem with 256 months is found to be no greater than 32. Although this does not directly show that the effective dimension of the computation by the standard discretization is also small, we believe that it partly explains why quasi-Monte Carlo methods worked so well in the original computations of Paskov.

The sensitivity analysis performed in this study should serve as a guide for even further improvements in the Monte Carlo method.

References

[1] W. Morokoff and R.E. Caflisch. Quasi-random sequences and their discrepancies. *SIAM J. Sci. Stat. Computing*, 15:1251–1279, 1994.

[2] W. Morokoff and R.E. Caflisch. Quasi-Monte Carlo integration. *J. Comp. Phys.*, 122:218–230, 1995.

[3] B. Moskowitz and R.E. Caflisch. Smoothness and dimension reduction in quasi-Monte Carlo methods. *J. Math. Comp. Modeling*, 23:37–54, 1996.

[4] S.H. Paskov. New methodolgies for valuing derivatives. In S. Pliska and M. Dempster, editors, *Mathematics of Derivative Securities*. Isaac Newton Inst., Cambridge U. Press, 1996.

[5] S.H. Paskov and J.F. Traub. Faster valuation of financial derivatives. *J. Portfolio Manag.*, pages 113–120, 1995.