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SUMMARY

A method for the numerical simulation of diffusive transport with moving boundaries is developed and tested. The variable domain is mapped onto a fixed region, which introduces a term of convective form to the transformed governing equation. The resulting convection/diffusion equation is solved by a finite-difference method. An "immersed interface" method (IIM) is introduced in order to retain second-order accuracy near discontinuities in material properties, where the solution is not smooth. The method performs well in benchmark calculations against an analytical solution. The IIM scheme is capable of treating a strong discontinuity in the gradient, and it is readily extended to two or three dimensions. The methods are illustrated through a calculation for the temperature profile in a growing continental ice sheet, in which the thermal properties are discontinuous at the rock/ice interface.

INTRODUCTION

Problems that involve diffusive transport of a conserved quantity in a domain with one or more moving boundaries arise frequently in the Earth and environmental sciences. Examples include: the expulsion of water from a growing column of sediment undergoing consolidation (*e.g.*, Gibson[1]); chemical transport in oceanic sediments undergoing accumulation or erosion at the sea floor (*e.g.*, Lerman and Lietzke[2]); solidification of magma in an intrusive body (*e.g.*, Delaney and Pollard[3]); and heat conduction in a growing ice sheet (*e.g.*, Heine and McTigue[4]). The last of these examples motivated us to consider the moving-boundary problem in some detail, and provides a context for the developments presented here.

A challenge for numerical analysis of such problems is that the boundaries of the computational domain are not fixed in space. In some cases, the location of at least one interface is itself part of the solution. One of a number of artifices must be introduced in order to handle the evolution of the domain. For example, one may choose to remesh at each time step in order to retain the same treatment of interfacial conditions throughout a calculation. This approach, of course, introduces considerable complexity, and is computationally intensive. A second numerical challenge arises because material properties are often discontinuous at an interface, so that the solution sought is not smooth. Conventional difference methods introduce error by smoothing out the profile in the neighborhood of an interface.

In this paper, we discuss methods that address each of these difficulties. The moving boundary is handled, in the one-dimensional problems considered, by a "front-fixing" scheme, in which a change of coordinates fixes the spatial domain, but changes the character of the governing equation. For either this approach or for "front-tracking" schemes (*e.g.*, Li[5]) that are more easily generalized to two- or three-dimensional domains, error due to the discontinuous gradient at the interface is reduced through an *immersed interface method* (IIM).[5,6,7] Benchmark calculations that illustrate the effectiveness of the IIM are presented. Finally, an application is discussed in which the transient, thermal profile in a growing, continental ice sheet is simulated.

DIFFUSIVE TRANSPORT

For present purposes, we consider one-dimensional, diffusive transport of a conserved quantity, although the methods discussed can be applied to a much broader class of problems. Heat conduction provides a generic context for the discussion. It is convenient for the development to state separately the conservation equation:

$$\frac{\partial T}{\partial t} = -\frac{1}{\rho c}\frac{\partial q}{\partial z} \quad , \tag{1}$$

and the constitutive model (Fourier's Law) for the heat flux, q:

$$q = -k\frac{\partial T}{\partial z} \quad , \tag{2}$$

where T is the temperature. The heat capacity, ρc, and thermal conductivity, k, may, in general, vary spatially due to material heterogeneity or due to temperature dependence.

The usual types of boundary conditions, such as specified temperatures and/or fluxes, are stipulated on the ends of the domain. In general, the end boundaries may be moving. In addition, there may be internal, material interfaces at which the properties are discontinuous and jump conditions on the temperature and flux are specified. Such an interface may also move. We consider here, for the sake of illustration, a single moving boundary located at $z = \alpha(t)$, a point that can be on one end of the domain or on an internal interface.

FRONT FIXING

The front-fixing scheme employs a coordinate transformation to map the evolving domain onto a fixed region. Conventional, fixed-grid methods are then directly applicable. Front fixing is a well-established method; a more detailed discussion and historical perspective are provided by Crank.[8]

Consider a domain with a stationary boundary at $z = 0$ and a moving boundary at $z = \alpha(t)$. Define a new coordinate

$$\zeta \equiv \frac{z}{\alpha} \quad , \tag{3}$$

such that the domain in ζ remains fixed between $\zeta = 0$ and $\zeta = 1$. Equations (1) and (2), written in terms of ζ rather than z, become:

$$\frac{\partial T}{\partial t} = \frac{\zeta}{\alpha}\frac{d\alpha}{dt}\frac{\partial T}{\partial \zeta} - \frac{1}{\rho c}\frac{1}{\alpha}\frac{\partial q}{\partial \zeta} \quad , \tag{4}$$

$$q = -\frac{k}{\alpha}\frac{\partial T}{\partial \zeta} \quad . \tag{5}$$

One consequence of the transformation is the appearance of a term in the conservation equation (4) of convective form, with "effective" velocity proportional to $-d\alpha/dt$, and varying linearly across the domain. In addition, the heat capacity and conductivity are rescaled by the time-dependent length, α. Thus, the thermal diffusivity, $\kappa \equiv k/\rho c$ in the untransformed problem, is replaced by an "effective" diffusivity, proportional to κ/α^2 in the transformed domain. For a growing region ($d\alpha/dt > 0$), then, the effective diffusivity decreases with time, and the characteristic time scale for the transport correspondingly increases.

NUMERICAL TREATMENT NEAR THE INTERFACE

When we use a difference scheme to discretize equations (1)–(2) or (4)–(5), we lose accuracy near an interface between contrasting materials. This is because the physical properties (here the thermal conductivity and heat capacity) are, in general, discontinuous. We require that the temperature be continuous, *i.e.*,

$$[T] = T(x^+, t) - T(x^-, t) \equiv 0 \quad . \tag{6}$$

We discuss the treatment of the interface in terms of a generic coordinate x in order to emphasize that the method can be applied either to the original problem posed (eqs. 1–2) or to the transformed problem (eqs. 4–5). In the former case, we can make the identity $x \equiv z$; in the latter, $x \equiv \zeta$. If we choose the formulation in the original frame, we also need to implement a numerical scheme to track the motion of the interface through a fixed grid (*e.g.*, Li[5]) or to remesh at each time step. In the examples presented here, we choose the front-fixing scheme.

Assuming there is no heat source on the interface, we also have a natural jump condition on the flux:

$$[q] = [-k\frac{\partial T}{\partial x}] = -k^+\frac{\partial T}{\partial x}^+ + k^-\frac{\partial T}{\partial x}^- \equiv 0 \tag{7}$$

across the interface, where $+$ and $-$ stand for the limiting value from the right hand side and the left hand side of the interface $x = \alpha(t)$, respectively. It is evident that, if $k^+ \neq k^-$, then $T_x^+ \neq T_x^-$ at the interface, *i.e.*, the temperature profile is not smooth. Direct difference discretization will produce large error near the interface when $[k]$ is large or when the net flux [q] is not zero.

A new approach, the *immersed interface method* (IIM),[6, 7] has been developed to solve general PDEs with discontinuous coefficients and/or singular sources with second-order accuracy at all grid points, including those which are close to or on the interface. The main idea is to incorporate the known jumps in the solution or its derivatives into the finite difference scheme, obtaining a modified scheme whose solution is second-order accurate throughout the domain, even for quite arbitrary interface conditions. This approach has also been applied to three-dimensional elliptic equations,[9] parabolic equations,[5, 10] hyperbolic wave equations with discontinuous coefficients,[11, 12] and incompressible Stokes flow problems with moving interfaces.[13, 14] Below we briefly explain this method for our model problem. Readers are referred to Li[5, 7] for a more general treatment of moving-interface problems.

For simplicity in the present development, we assume that the interface is fixed at $x = \alpha$, an interior point in the solution domain, and that the governing equation is linear. Under these conditions, equations (1)–(2) or (3)–(4) can be written

$$\frac{\partial T}{\partial t} = f_1(t)\frac{\partial T}{\partial x} + f_2(t)\frac{\partial^2 T}{\partial x^2} \quad . \tag{8}$$

Equation (8) is only valid in the interior of the solution domain. Across the interface, there are jump conditions, which typically take two different forms:

1. The jump condition is known:

$$[T] = 0 \quad , \tag{9}$$

$$[q] = [-k\frac{\partial T}{\partial x}] = S(t) \quad , \tag{10}$$

   where $S(t)$ is the strength of the interfacial source, due, for example, to latent-heat release.

2. The temperature is known on the interface:

$$[T] = 0 \quad , \tag{11}$$

$$T(\alpha, t) = T_0(t) \quad , \tag{12}$$

   where $T_0(t)$ is a given function of time.

In our example problem for a growing ice sheet, we have the first kind of relation at the rock/ice interface (with $S(t) = 0$), and the second kind on the surface of the ice sheet, where, in general, the mean atmospheric temperature decreases as the surface elevation increases. We cast our discussion of the IIM in terms of the first type of interface relation.[4]

Assume we use a uniform grid with the spatial step size h, such that $x_i = x_0 + i\,h$, $i = 0, 1, \cdots$, and the interface is between x_j and x_{j+1}, $x_j \leq \alpha < x_{j+1}$. At a regular grid point x_i, $i \neq j, j+1$, the explicit difference scheme can be written as

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = f_1(t^n)\,\frac{T_{i+1}^n - T_{i-1}^n}{2h} + f_2(t^n)\,\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{h^2} \quad , \tag{13}$$

where Δt is the time step size. At the irregular grid point x_j, the difference scheme can be written as

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = \gamma_{j1}^n T_{j-1}^n + \gamma_{j2}^n T_j^n + \gamma_{j3}^n T_{j+1}^n + C_j^n \quad . \tag{14}$$

We need to determine the coefficients γ_{j1}^n, γ_{j2}^n, γ_{j3}^n, and the correction term C_j^n so that

$$\gamma_{j1}^n T(x_{j-1},t^n) + \gamma_{j2}^n T(x_j,t^n) + \gamma_{j3}^n T(x_{j+1},t^n) + C_j^n \approx \left(f_1(t^n)T_x + f_2(t^n)T_{xx}\right)_{(\alpha,t^n)} \quad . \tag{15}$$

The idea is simple; we expand $T(x_{j-1},t^n)$ and $T(x_j,t^n)$ in Taylor series about α from the left, and $T(x_{j+1},t^n)$ from the right of α. Then the left hand side of (15) becomes

$$\gamma_{j1}^n\left(T^- + T_x^-(x_{j-1}-\alpha) + T_{xx}^-\frac{(x_{j-1}-\alpha)^2}{2}\right) + \gamma_{j2}^n\left(T^- + T_x^-(x_j-\alpha) + T_{xx}^-\frac{(x_j-\alpha)^2}{2}\right) + \gamma_{j3}^n\left(T^+ + T_x^+(x_{j+1}-\alpha) + T_{xx}^+\frac{(x_{j+1}-\alpha)^2}{2}\right) + C_j^n + \cdots \quad , \tag{16}$$

where all quantities are calculated at (α, t^n). From the interface relations (9) and (10), we have

$$T^+ = T^- \quad , \tag{17}$$

$$T_x^+ = \frac{k^-T_x^- - S}{k^+} \quad . \tag{18}$$

Also from the differential equation (8), we have

$$f_1^+T_x^+ + f_2^+T_{xx}^+ = f_1^-T_x^- + f_2^-T_{xx}^- \quad . \tag{19}$$

Here we have used the assumption that $T(x,t)$ is continuous and the interface α is fixed. From (17)–(19) we obtain

$$T_{xx}^+ = \frac{f_2^-}{f_2^+}T_{xx}^- + \frac{f_1^-}{f_2^+}T_x^- + \frac{f_1^+}{f_2^+}\,\frac{S - k^-T_x^-}{k^+} \quad . \tag{20}$$

Substituting (17), (18), and (20) into (16) and arranging terms we have

$$\left(\gamma_{j1}^n + \gamma_{j2}^n + \gamma_{j3}^n\right)T^- + \left\{\gamma_{j1}^n(x_{j-1}-\alpha) + \gamma_{j2}^n(x_j-\alpha) + \gamma_{j3}^n\left(\frac{k^-}{k^+}(x_{j+1}-\alpha) + \left(\frac{f_1^-}{f_2^+} - \frac{k^-}{k^+}\frac{f_1^+}{f_2^+}\right)\frac{(x_{j+1}-\alpha)^2}{2}\right)\right\}T_x^- + \left(\gamma_{j1}^n\frac{(x_{j-1}-\alpha)^2}{2} + \gamma_{j2}^n\frac{(x_j-\alpha)^2}{2} + \gamma_{j3}^n\frac{f_2^-}{f_2^+}\frac{(x_{j+1}-\alpha)^2}{2}\right)T_{xx}^- - \gamma_{j3}^n S\left(\frac{(x_{j+1}-\alpha)}{k^+} - \frac{f_1^+}{f_2^+}\frac{(x_{j+1}-\alpha)^2}{2}\right) + C_j^n + \cdots \quad . \tag{21}$$

Comparing (21) with the right-hand side of (15), we obtain three equations for γ_{j1}^n, γ_{j2}^n, and γ_{j3}^n as follows:

$$\gamma_{j1}^n + \gamma_{j2}^n + \gamma_{j3}^n = 0 \quad , \tag{22}$$

$$\gamma_{j1}^n(x_{j-1} - \alpha) + \gamma_{j2}^n(x_j - \alpha)$$
$$+ \gamma_{j3}^n \left\{ \frac{k^-}{k^+}(x_{j+1} - \alpha) + \left(\frac{f_1^-}{f_2^+} - \frac{k^-}{k^+}\frac{f_1^+}{f_2^+} \right) \frac{(x_{i+1} - \alpha)^2}{2} \right\} = f_1(t^n) \quad , \tag{23}$$

$$\gamma_{j1}^n \frac{(x_{j-1} - \alpha)^2}{2} + \gamma_{j2}^n \frac{(x_j - \alpha)^2}{2} + \gamma_{j3}^n \frac{f_2^-}{f_2^+} \frac{(x_{j+1} - \alpha)^2}{2} = f_2(t^n) \quad , \tag{24}$$

and the correction term is determined as

$$C_j^n = \gamma_{j3}^n S \left(\frac{(x_{j+1} - \alpha)}{k^+} - \frac{f_1^+}{f_2^+} \frac{(x_{j+1} - \alpha)^2}{2} \right) \quad , \tag{25}$$

which is known after γ_{j3}^n is obtained from (22)–(24). If $S(t) = 0$, as in our model problem, then $C_j^n = 0$. Similarly, we can derive the modified difference scheme at the regular grid point x_{j+1}.

IMPLEMENTATION: METHOD OF LINES

The development of the IIM was outlined in the previous section in the context of the linear PDE given by (8), and in difference form by (13) and (14). In a more general treatment allowing for continuously varying, temperature-dependent properties, we discretize (4) and (5) at regular grid points in the form:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\xi_i}{\alpha} \frac{d\alpha}{dt} \frac{T_{i+1}^n - T_{i-1}^n}{2h} - \frac{1}{(\rho c)_i} \frac{1}{\alpha} \frac{q_{i+\frac{1}{2}}^n - q_{i-\frac{1}{2}}^n}{h} \quad , \tag{26}$$

where α and $d\alpha/dt$ are evaluated at t^n, and the fluxes are evaluated at midpoint nodes corresponding to $\xi_i \pm \frac{1}{2}h$:

$$q_{i+\frac{1}{2}} = -\frac{k_{i+\frac{1}{2}}}{\alpha} \frac{T_{i+1} - T_i}{h} \quad , \qquad q_{i-\frac{1}{2}} = -\frac{k_{i-\frac{1}{2}}}{\alpha} \frac{T_i - T_{i-1}}{h} \quad , \tag{27}$$

with all quantities in (27) evaluated at time n. For irregular grid points, adjacent to interfaces across which the gradient is discontinuous, we implement the IIM, modified to account for the nonlinearity in the two contrasting materials.

The temperatures are computed for grid points, while, from (27), we must compute the temperature-dependent conductivity at the midpoints. We invoke the following approximation:[15]

$$k_{i+\frac{1}{2}} = \frac{2k_i k_{i+1}}{k_i + k_{i+1}} \quad , \qquad k_{i-\frac{1}{2}} = \frac{2k_{i-1} k_i}{k_{i-1} + k_i} \quad , \tag{28}$$

where, for example, the conductivity k_i is computed from the temperature T_i^n. The weightings given in (28) are constructed by matching the heat fluxes at the midpoints, which are approximated by backward and forward differences over the adjacent half-intervals. In the special case of piecewise-constant properties, equations (26)–(28) reduce to the form of (13).

We call the SLATEC backward-difference solver DEBDF[16] to perform the time integration. This is a variable-order, variable-step routine that utilizes a Newton-type algorithm to solve the system of coupled, nonlinear, algebraic equations for the nodal temperatures given by (26)–(28). This general scheme is a particular form of the *method of lines* (see, *e.g.*, Schiesser[17]).

BENCHMARK PROBLEM

In order to verify that we have implemented the solution scheme correctly, to test its performance, and to demonstrate its merits, we construct a relatively difficult numerical problem having an analytical solution. We consider two materials, separated by a moving interface. The material properties are assumed to be piecewise constant, so that equations (1) and (2) are linear. However, we consider an arbitrary contrast in the conductivity, k, across the material interface. This implies that, while the temperature and heat flux are continuous at the interface, the temperature gradient may be strongly discontinuous (differing by the ratio of the conductivities, *cf.* (7)), providing a rigorous test for the numerical method.

We consider a finite domain, $0 \leq z \leq L$, in the interior of which is a moving interface located at $z = \alpha(t)$. Denote the domain $0 \leq x < \alpha$ the "minus" $(-)$ region, and $\alpha < z \leq L$ the "plus" $(+)$ region, with material properties for each region correspondingly designated, *e.g.*, k^- and k^+. The interface is assumed to be located at

$$z = \alpha = 2\sqrt{\kappa^- t} \quad . \tag{29}$$

A solution satisfying (1) and (2), as well as satisfying continuous temperature and heat flux at the interface $z = \alpha$, is:

$$T = \begin{cases} 1 - \dfrac{\mathrm{erf}(\eta)}{\mathrm{erf}(1)}, & 0 \le z \le \alpha \\ C\left[1 - \dfrac{\mathrm{erf}(R\eta)}{\mathrm{erf}(R)}\right], & \alpha \le z \le L \quad , \end{cases} \tag{30}$$

where $\eta \equiv x/2\sqrt{\kappa^- t}$,

$$C \equiv \beta R \exp(R^2 - 1)\frac{\mathrm{erf}(R)}{\mathrm{erf}(1)} \quad , \tag{31}$$

$\beta \equiv (\rho c)^-/(\rho c)^+$, and $R^2 \equiv \kappa^-/\kappa^+$. This solution satisfies the Dirichlet conditions at $z = 0$ and $z = L$:

$$T(0, t) = 1 \quad , \tag{32}$$

$$T(L, t) = C\left[1 - \frac{\mathrm{erf}(L/2\sqrt{\kappa^+ t})}{\mathrm{erf}(R)}\right] \quad . \tag{33}$$

In this example, we solve the problem with the front-fixing scheme. With reference to equation (8), $x \equiv \zeta$, $f_1 \equiv (\zeta/\alpha)d\alpha/dt$, and $f_2 \equiv \kappa/\alpha^2$, the rescaled thermal diffusivity, is piecewise constant at any particular time.

We compare our method with a conventional approach in discretizing the diffusion term κT_{zz}, the *smoothing method.* Here, we solve the untransformed problem, so that in equation (8), $x \equiv z$, $f_1 \equiv 0$, and $f_2 \equiv \kappa$, where, again, the diffusivity is piecewise constant. The smoothing method is defined as follows (Figure 1):

$$\tilde{k} = \begin{cases} k^- & \text{if } z < \alpha - \epsilon \\ \dfrac{k^- + k^+}{2} + \dfrac{k^+ - k^-}{2}\sin\dfrac{(z-\alpha)\pi}{2\epsilon} & \text{if } |z - \alpha| \le \epsilon \\ k^+ & \text{if } z > \alpha + \epsilon \quad . \end{cases} \tag{34}$$

The smoothing method is only first-order accurate and is difficult to extend to two dimensions, an exception being the level-set formulation (*e.g.* Sussman, *et al.*[17]). If we let ϵ approach zero, then we have

$$\tilde{k} = \begin{cases} k^- & \text{if } z < \alpha \\ \dfrac{k^- + k^+}{2} & \text{if } z = \alpha \\ k^+ & \text{if } z > \alpha \quad . \end{cases} \tag{35}$$

This approach is crude, but simple and widely used. One advantage of this scheme is that it can be used in two dimensional problems. It usually gives less accurate results than the smoothing method. A more subtle discretization is obtained by harmonic averaging; see Li[7] for a detailed discussion.

Table 1: Comparison of the IIM with the smoothing method. N is the number of grid points.

N	$\parallel E_N \parallel_\infty$, IIM	$\parallel E_N \parallel_\infty$, Smoothing
80	5.10×10^{-5}	7.32×10^{-3}
160	2.00×10^{-5}	9.18×10^{-4}
320	2.98×10^{-6}	6.77×10^{-4}

Table 1 shows a comparison of the error in the computed solution in the infinity norm defined as

$$\parallel E_N \parallel_\infty = \max_i \mid T(z_i, t_{out}) - T_i^{n^*} \mid, \tag{36}$$

where n^* is the final time step corresponding to the time t_{out}. The parameters are

$$t_0 = 0.5, \quad t_{out} = 0.6, \quad \beta = 1, \quad R^2 = 0.05$$

where the initial condition, at $t = t_0$, is computed from the exact solution. We take $(\rho c)^- = (\rho c)^+ = 1$, and $k^- = 1$, $k^+ = 20$ in order to consider the twenty-fold contrast in the diffusivity indicated at the interface ($R^2 = 0.05$). We see that our method, *the immersed interface method* (IIM), gives a much better result. Figure 2(a) shows the solution near the interface for a calculation with 160 grid points, while Figure 2(b) shows the error for the IIM and the smoothing method for $N = 320$.

EXAMPLE: THERMAL EVOLUTION IN A GROWING ICE SHEET

We use the model described to simulate the temperature profile at the center of an ice sheet throughout the course of a glaciation. The base of an ice sheet is either frozen to its bed ("cold-based"), or at the pressure melting point ("warm-based"). According to theory, an ice sheet becomes warm-based when the geothermal heat flux and internal heating by viscous dissipation provide enough heat to melt the base of the ice sheet, while the overlying ice mass insulates the base against the very cold air at the surface (Paterson[19]). Determination of the basal conditions of ice sheets is of paramount importance for the reconstruction of their topography and flow characteristics. Slip at the bed of a warm-based ice sheet results in far greater erosion than that produced by ice frozen to the substrate. We consider the center of the ice sheet, where horizontal flow vanishes and the heat transfer may be approximated by a one-dimensional model.

The model accounts for conduction in both the ice and the underlying rock. In previous calculations,[4] we have considered temperature-dependent thermal conductivity and heat capacity for the ice. However, this introduces only a weak nonlinearity that has minimal influence on the final results. Here, we restrict our sample calculation to the linear problem for piecewise constant properties. For the bedrock, we take $k^- = 2.5$ W/m/K and $(\rho c)^- = 2.3 \times 10^6$ J/m^3/K; for the ice we take $k^+ = 2.3$ W/m/K and $(\rho c)^+ = 1.8 \times 10^6$ J/m^3/K. At the start of a model run, subsurface temperatures are in equilibrium with the geothermal heat flux and a given ground surface temperature. The geothermal heat flux is applied at a depth of 4000 m, which was found, in the present context, to be effectively an infinite boundary. Ice is deposited on the surface at the surface temperature. Ice sheet growth is exponential, $\alpha/\alpha_\infty = 1 - exp(-t/t_r)$, allowing for fast growth at its inception, decreasing with time. The characteristic rise time, t_r, is 6000 years (*i.e.*, at this time 63% of the final thickness has been reached), and the final ice sheet thickness, α_∞, is 3000 m. The surface temperature is held constant at -20 °C. The geothermal heat flux warms the bedrock and ice, until basal melting occurs at about -2 °C, the melting point at a pressure corresponding to 3000 m of ice.

Figure 3 displays the temperature profile within the ice/rock body at various times. Note that at 0 years, no ice sheet exists, while the rock domain displays a linear, geothermal profile. After 6,592 years, the ice sheet has reached 2000 m thickness and is still growing. The basal melting point is reached after 64,000 years (Figure 4). For a more detailed description of the ice sheet model, including the effects of temperature-dependent thermal properties, vertical advection due to accumulation and divergent, horizontal flow, and adiabatic cooling of the surface, see Heine and McTigue.[4] Although this simple model neglects several processes that affect the thermal regime in an ice sheet, the qualitative result is significant. Because the thermal diffusion time, α_∞^2/κ, and the time scale for a typical glacial cycle are both of the order of 10^5 years, the transient evolution of the temperature profile must be considered. In the past, some models have assumed a steady-state thermal balance. For reasonable surface temperature and sufficiently thick ice, this inevitably leads to the conclusion that the ice sheet is 'warm-based," *i.e.*, it reaches the melting point at the rock-ice interface. However, the transient model discussed here suggests that the time required for the base of the ice sheet to approach its steady-state temperature is so long that the center of the ice sheet remains "cold-based," at least through a large portion of a glacial cycle. This conclusion is consistent with some geomorphological evidence that indicates relatively little glacial erosion under past continental ice sheets.

In the context of this particular application of the numerical methods discussed here, the handling of the moving boundary is particularly important. The time scale for the growth of the ice sheet, of order 10^4 years, approaches that for thermal diffusion across the layer. Therefore, the thermal profile is evolving at the same time that the domain is changing its scale, and an accurate method that accounts for the moving boundary is essential. The IIM, a technique that maintains accuracy even in the case of a strongly discontinuous gradient at an interface, is not as critical to the problem considered here, in that the conductivity contrast at the rock/ice interface is small: $k^-/k^+ = 1.09$. Thus, conventional methods that result in some smoothing of the temperature profile near the interface may be satisfactory in this context. However, the IIM is of increasing utility as the contrast in material properties increases, as demonstrated in the benchmark problem treated in the preceding section.

CONCLUSION

A front-fixing scheme and an immersed interface method (IIM) have allowed us to solve a class of problems involving moving boundaries and discontinuous material properties with second-order accuracy throughout the domain. The front-fixing approach is quite straightforward in one dimension, but is considerably more difficult to implement for multidimensional domains. In contrast, the IIM can be embedded in any number of schemes for treating the moving interface, and therefore is readily adapted to two- or three-dimensional calculations. We also note that the IIM has been applied successfully not only to parabolic equations, as in the present case, but also to elliptic[9] and hyperbolic[11, 12] equations. The accuracy maintained by the IIM in the neighborhood of a strong discontinuity in conductivity (or, equivalently, in the temperature gradient) is demonstrated in a benchmark comparison to an analytical solution. Finally, the methods prove to be quite successful in simulating the evolution of the thermal profile through an accumulating continental ice sheet. In this application, the temperature history at the material discontinuity is the central issue, and accuracy near the interface is of particularly importance.

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FIGURE CAPTIONS

Figure 1. Coefficient smoothed over a finite thickness 2ϵ.

Figure 2(a). Local solution plot. The result obtained with the IIM is so close to the exact solution that they appear identical on this plot. $N = 160$.

Figure 2(b). Error plot for the IIM and the smoothing method. $N = 320$.

Figure 3. Temperature profiles through the ice sheet at times corresponding to 1000 m increments of growth. Final profile is for 64,000 years, when the interface reaches the melting point (−2 °C). The thermal properties are discontinuous at the rock/ice interface, at elevation = 0.

Figure 4. Temperature rise at the rock/ice interface. The melting point, −2 °C for an overburden of 3000 m of ice, is reached after 64,000 years.

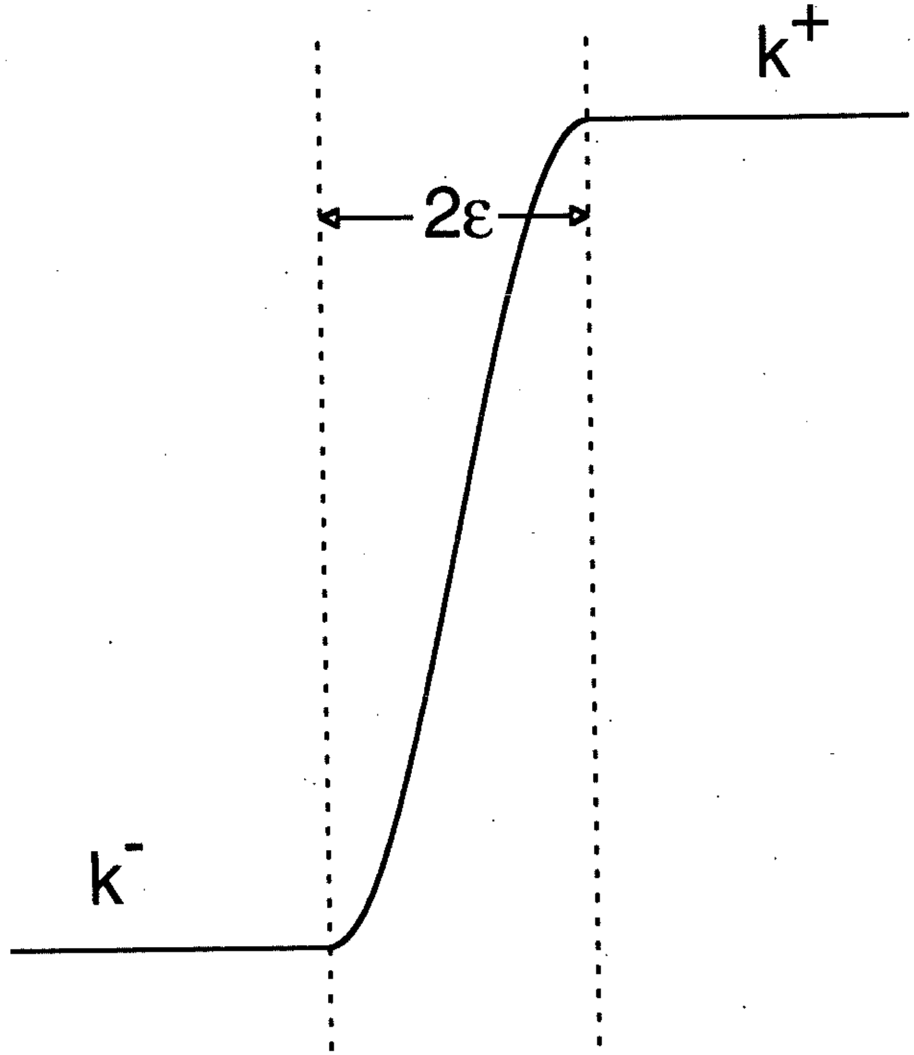

Fig. 1

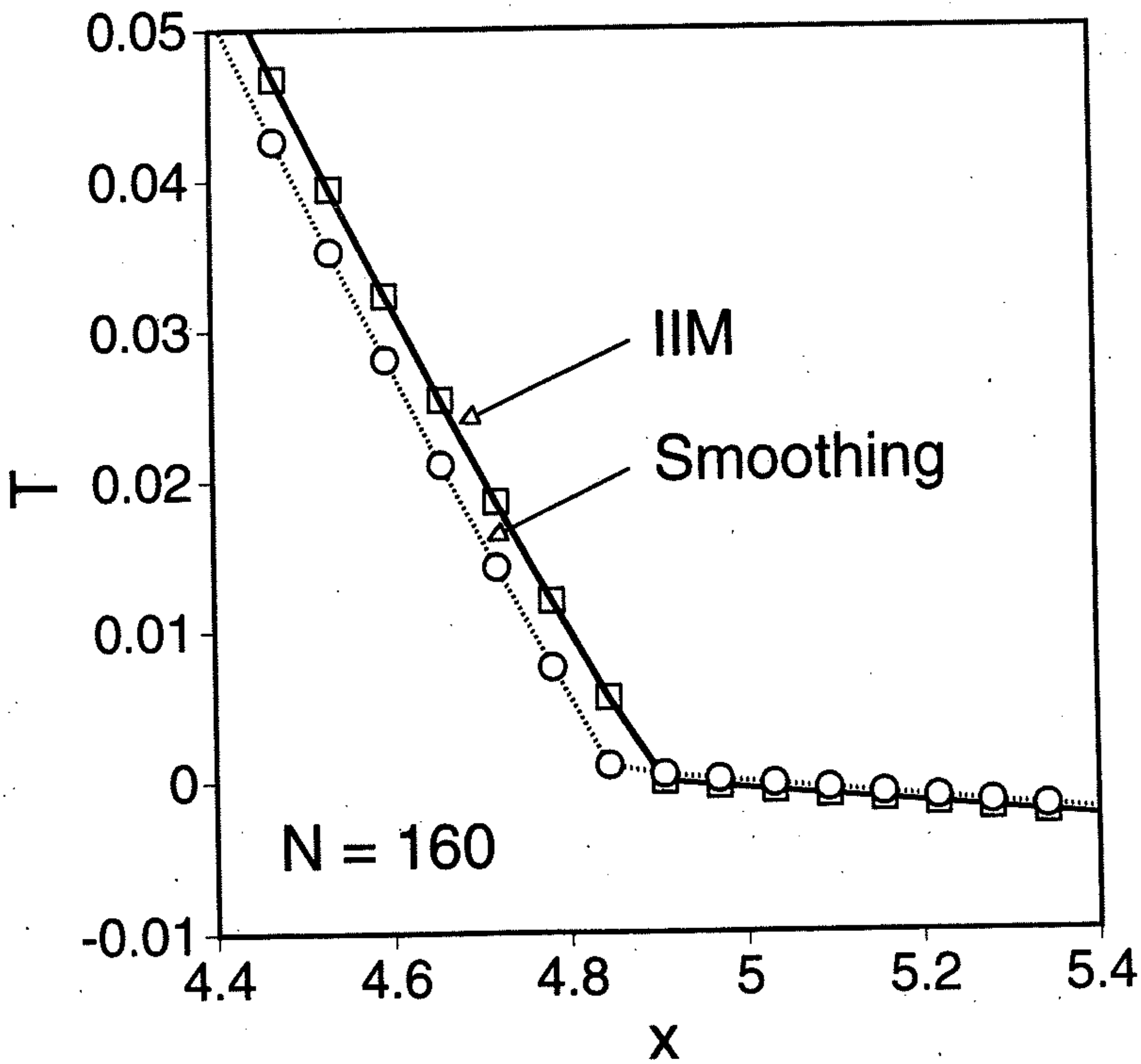

Fig. 2(a)

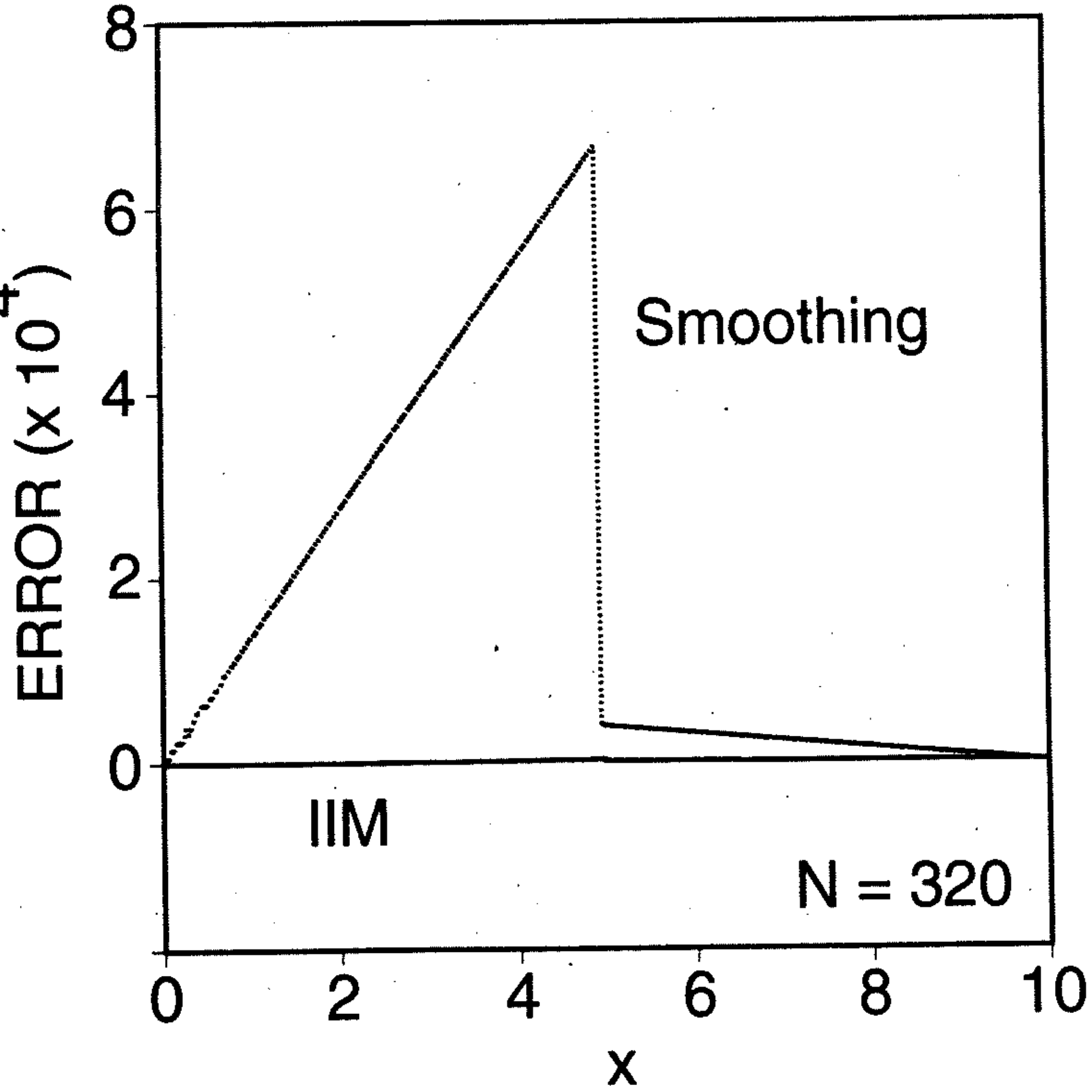

Fig. 2(b)

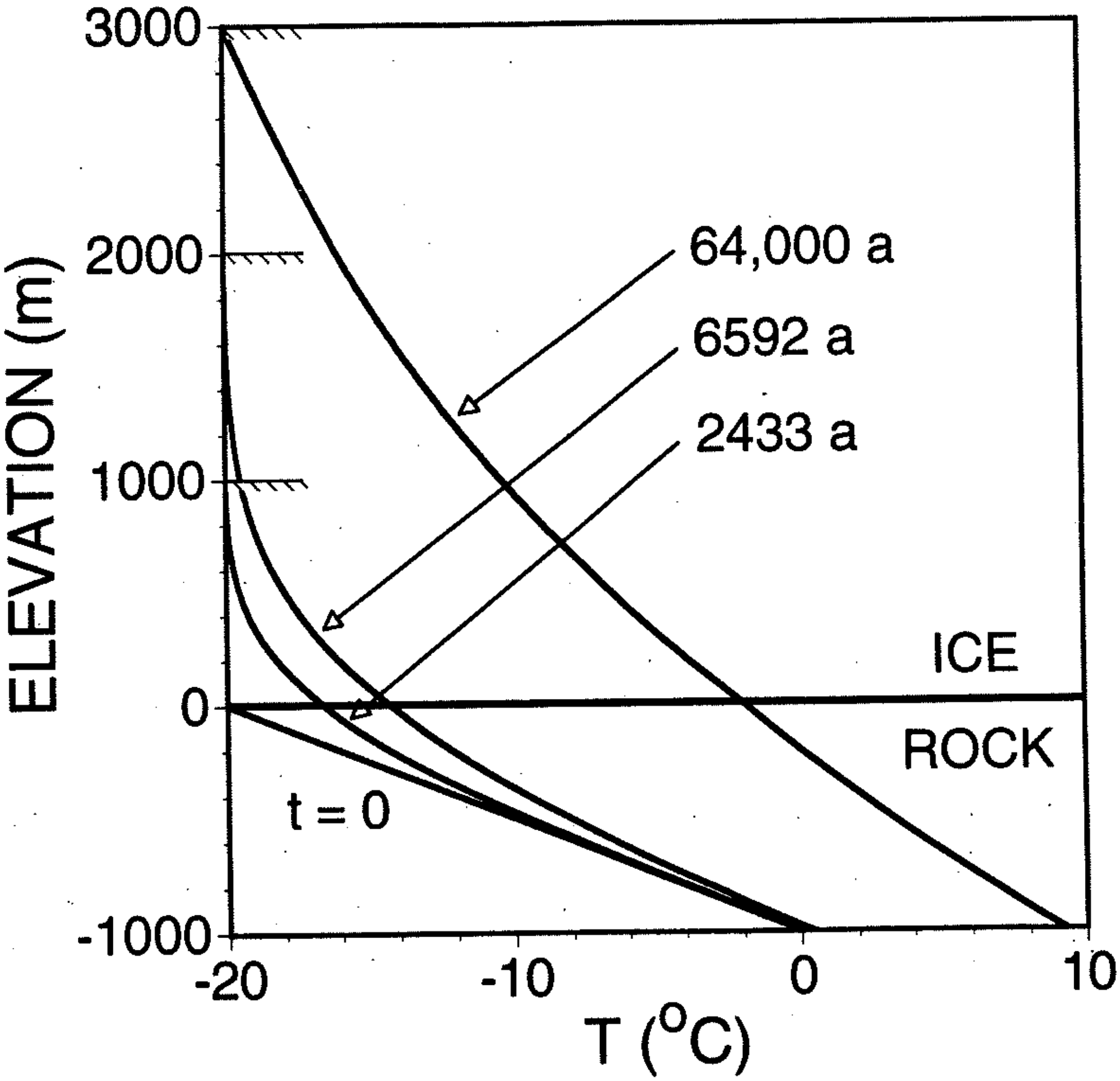
3000
2000
1000
0
-1000
ELEVATION (m)
64,000 a
6592 a
2433 a
ICE
ROCK
t = 0
-20
-10
0
10
T (°C)

Fig. 3.

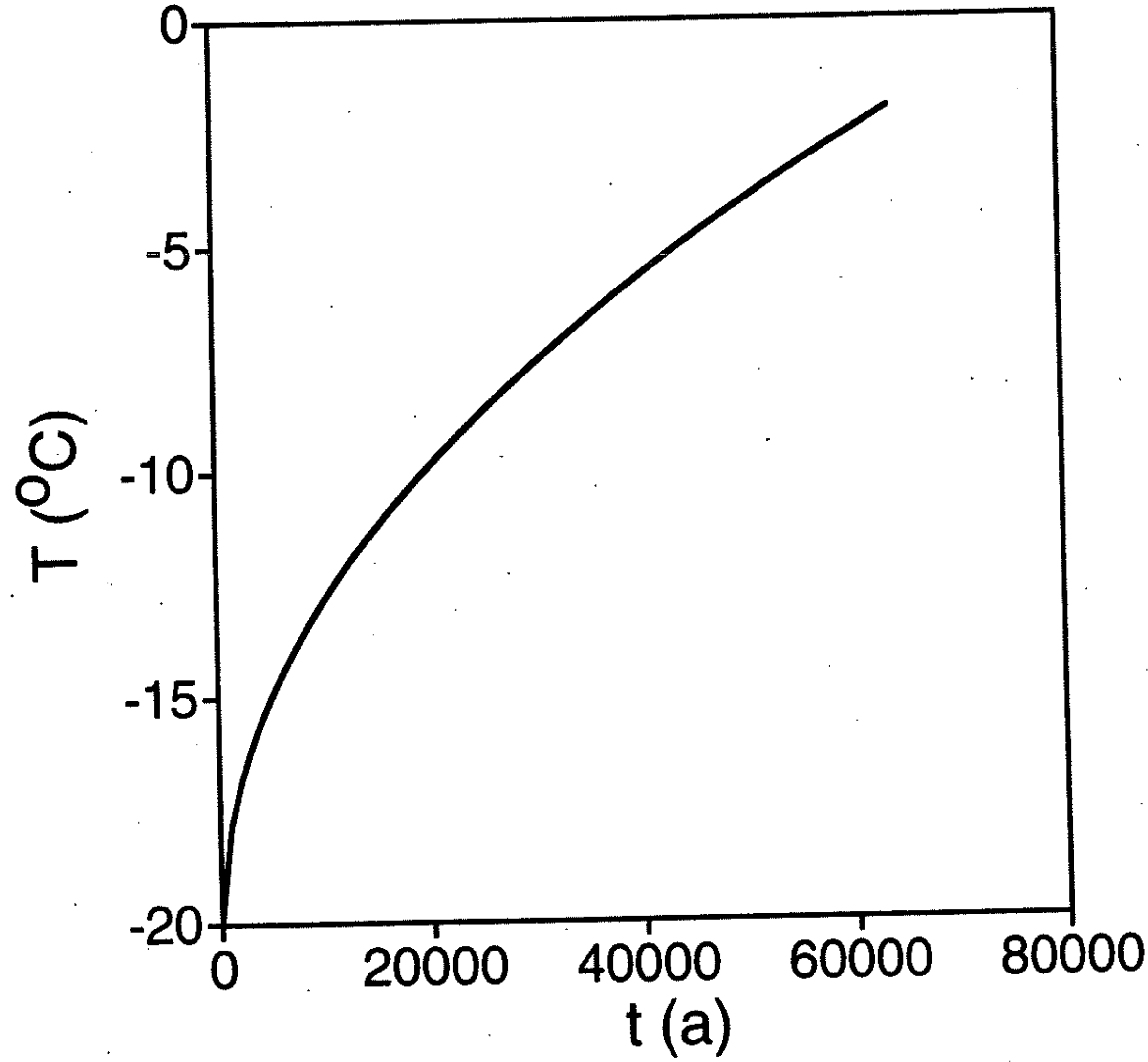

Fig. 4