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A Level Set Approach for the Motion of Soap Bubbles with Curvature Dependent Velocity or Acceleration

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requirements for the degree Doctor of Philosophy

in Mathematics

by

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ABSTRACT OF THE DISSERTATION

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Professor Stanley Osher, Chair

In this thesis, we develop the appropriate equations for bubble motions depending on curvature using the level set methodology. Our method consists of an appropriate finite difference scheme for solving our model equations, and a level set approach for capturing the complicated motion between the bubbles. Results indicate that this method can handle topology changes and complicated interfacial shapes, and that it can numerically simulate many of the physical features of bubble motions.

CHAPTER 1

Introduction

In this paper, we will derive the proper equations and numerical algorithms for a soap bubble.

It is generally accepted that many researchers in the field of CFD have difficulties in dealing with singularities of fluid interfaces . To resolve these difficulties, we will use a level set approach to compute the motion of soap bubbles. We intend to capture the interface using a level set approach instead of explicitly tracking it.

Front tracking methods usually require that we add or subtract points dynamically. In [2], an interesting front tracking method which does not explicitly reposition the points of interface was devised. The investigators cited good results, but it seems hard to implement in three space dimensions. Moreover, topological changes cause difficulties, as with all tracking methods.

In [4], a level set formulation for moving interfaces with curvature dependent velocity was introduced. The level set function is typically a smooth function, denoted as ϕ. The level set formulation eliminates the problem of repositioning the points to a moving grid and is capable of capturing geometric properties of highly complicated boundaries including topological changes without explicitly tracking the interfaces. An application of the level set formulation was used in ([1],[7]) for incompressible fluid flows. They found that it was best, at least close to the front, to keep ϕ as the signed distance from the front to prevent the development of steep or flat gradients in ϕ. This can be done by solving a simple initial value problem for ϕ which leaves the front location unchanged for fixed time.

We will use the level set approach to solve the problem for motion of soap bubbles in 2D and 3D. This includes area preserving motion by velocity or by acceleration.

CHAPTER 2

Equations of Motion

In our study, we shall consider area preserving motion for soap bubbles with curvature dependent velocity or acceleration.

2.1 Motion with curvature dependent velocity

Consider the equation

$$\boldsymbol{u} = -\sigma\kappa\boldsymbol{n} - [p]\boldsymbol{n} \tag{2.1}$$

where $\boldsymbol{u}$ is the velocity, σ is the surface tension, κ is the mean curvature, $\boldsymbol{n}$ is the normal vector at the front and $[p]$ is the jump of pressure across the interface. We need to find $[p]$ to preserve the area. The change of area is given by

$$\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{n}ds = \int_\Gamma(-\sigma\kappa - [p])ds \tag{2.2}$$

where Γ is the front of a bubble. From this, $[p]$ can be rewritten as

$$[p] = \frac{\sigma\int_\Gamma -\kappa\ ds}{\int_\Gamma ds} = -\sigma\overline{\kappa} \tag{2.3}$$

where $\overline{\kappa}$ is the average curvature at the front. Now, our governing equation becomes

$$\boldsymbol{u} = -\sigma(\kappa - \overline{\kappa})\boldsymbol{n} \tag{2.4}$$

This equation is valid both 2-D and 3-D. It is simple to model this motion numerically.

2.2 Motion with curvature dependent acceleration

In this section, we will derive the proper equations of motions for a soap bubble in 3-dimensions(which gives the 2-dimensional results as a special case). We derive the basic equation of bubble motion by applying Newton's law $dP/dt = F$ where $P = particle\ momentum, F = force\ on\ particle$ to a small element of the bubble surface. Care is required when doing this, because F must include all sources of momentum to the element —in particular, the source due to mass transfer. This can cause confusion, because most applications of Newton's law do not include mass transfer to the element under consideration, while some of our bubble models do(due to migration of mass on the surface). Let $d\boldsymbol{s}$ be a small piece of the bubble surface, with mass dm, velocity $\boldsymbol{u}$, momentum $P = dm\boldsymbol{u}$. Then Newton's law is

$$\frac{d(dm\boldsymbol{u})}{dt} = F \tag{2.5}$$

and we must specify F. If $d\boldsymbol{s}$ does not exchange any mass with other parts of the bubble, (e.g., a rubber sheet, where surface mass is not free to move within the surface) then the only momentum sources are the standard forces of pressure, surface tension, gravity, etc. However, if $d\boldsymbol{s}$ does exchange mass with the rest of the bubble surface, then the momentum brought in or carried away from $d\boldsymbol{s}$ by the mass transport process must also be included in the momentum balance:

$$\frac{d(dm\boldsymbol{u})}{dt} = F + F_{MT} \tag{2.6}$$

where F_{MT} is the momentum source due to the mass transport, and F contains the standard forces. In particular, if the element mass dm is not constant in time, then we must consider this additional term, since there must be mass transfer. However, even if dm were constant, we could hypothesize a mass transfer process that resulted in momentum transfer as well. Thus F_{MT} depends on the details of the mass transfer processes that actually occur. F_{MT} could also be used to include the effects of evaporation or condensation on bubble momentum—for example, if the bubble is absorbing raindrops, F_{MT} would include the momentum gained from each raindrop. We simply have to specify this term. The form of F_{MT} is an additional physical axiom, independent of the rest of the problem. The only restriction on the form of the term is that it must conserve total momentum of the bubble, i.e. summing F_{MT} over all surface elements should give 0 (the bubble does not transfer momentum to itself). This assumes that all mass transfer is within the bubble and not the surrounding environment. Note that assuming $F_{MT} = 0$ is consistent with this, but does not correspond to the desired physics. Given these cautions, we proceed to derive the equations.

Newton's law $dP/dt = F$ for an element of the surface is

$$\begin{aligned} \frac{d(momentum\ of\ element)}{dt} &= force\ on\ the\ element \\ &= (force\ per\ area)(area\ of\ element) \end{aligned} \tag{2.7}$$

If the element has area $|dA|$, mass per unit μ, and velocity $\boldsymbol{u}$(note, $\boldsymbol{u}$ need not be normal to the surface, e.g. a rotating spherical bubble, where $\boldsymbol{u}$ is wholly tangential) then

$$\text{(2.8)} \qquad \textit{momentum of element} = (\mu|dA|)\boldsymbol{u}$$

$$\text{(2.9)} \qquad \textit{force per area} = -[p]\boldsymbol{n} - \sigma k\boldsymbol{n} + \boldsymbol{f}$$

where $[p]$ is the pressure jump across the surface, which is chosen to preserve the volume, σ is surface tension, k curvature, $\boldsymbol{n}$ the outward unit normal and $\boldsymbol{f}$ is any additional source of momentum(per unit area), which includes the possible mass transport effects described above. (We do not assume that $\boldsymbol{f}$ is normal to the surface). Thus Newton's law becomes

$$\text{(2.10)} \qquad \frac{d(\mu|dA|\boldsymbol{u})}{dt} = (-[p]\boldsymbol{n} - \sigma k\boldsymbol{n} + \boldsymbol{f})|dA|$$

Isolating $\frac{d\boldsymbol{u}}{dt}$ and dividing out $|dA|$ yields

$$\text{(2.11)} \qquad \begin{aligned} \mu\frac{d\boldsymbol{u}}{dt} &= -[p]\boldsymbol{n} - \sigma k\boldsymbol{n} + \boldsymbol{f} - \boldsymbol{u}\frac{1}{|dA|}\frac{d\mu|dA|}{dt} \\ &= -[p]\boldsymbol{n} - \sigma k\boldsymbol{n} + \boldsymbol{f} + \boldsymbol{F} \end{aligned}$$

where $\boldsymbol{F}$ simply stands for the terms indicated. $\boldsymbol{F}$ can be broken into two parts by expanding out the derivative,

$$\text{(2.12)} \qquad \boldsymbol{F} = -\mu\boldsymbol{u}(\frac{1}{|dA|}\frac{d(|dA|)}{dt} + \frac{1}{\mu}\frac{d\mu}{dt})$$

In order to close this system, we need an expression for

$$\text{(2.13)} \qquad (a)\frac{1}{|dA|}\frac{d(|dA|)}{dt},\ (b)[p],\ (c)\mu \textit{ and } \frac{d\mu}{dt},\ (d)\boldsymbol{f}$$

The first depends only the velocity field, the second will be derived by the volume conservation constraint, the third requires additional specification. In order to represent a physical bubble, the total mass (surface integral of μ) should be constant, which is an added constraint on the possible form of (c).

2.3 Calculation of $\frac{d|dA|}{dt}$

Let dA denote the vector area of a surface element, with scalar area $|dA|$. We can derive several equivalent formulas for the fractional rate of change of $|dA|$.

$$\frac{1}{|dA|}\frac{d|dA|}{dt} = \boldsymbol{u}\cdot\boldsymbol{n}\kappa \tag{2.14}$$

$$\frac{1}{|dA|}\frac{d|dA|}{dt} = \boldsymbol{u}\cdot\boldsymbol{n}\nabla\cdot\boldsymbol{n} \tag{2.15}$$

$$\frac{1}{|dA|}\frac{d|dA|}{dt} = \nabla_s\cdot\boldsymbol{u} \tag{2.16}$$

$$\frac{1}{|dA|}\frac{d|dA|}{dt} = \nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} \tag{2.17}$$

where ∇_s is the surface divergence operator, which can be written generally as $\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n}$ when $\boldsymbol{u}$ is extended to be defined off the surface. Note that formula (2.14) and (2.15) are equivalent by the standard relationship $k = \nabla\cdot\boldsymbol{n}$ and formula (2.14) and (2.15) are only valid when the velocity is normal to the surface. Also, in the special case of normal velocity $\boldsymbol{u} = u_n\boldsymbol{n}$ formula (2.17) reduces to (2.15). We will prove (2.16) and (2.17). Let dA be the vector area element. Let

$\boldsymbol{ds}_1$ and $\boldsymbol{ds}_2$ by two-orthogonal-tangent vectors whose cross product is dA,

$$dA = \boldsymbol{ds}_1 \times \boldsymbol{ds}_2 = |\boldsymbol{ds}_1||\boldsymbol{ds}_2|\boldsymbol{n} \tag{2.18}$$

Then we have

$$\frac{d(dA)}{dt} = \frac{d(\boldsymbol{ds}_1)}{dt} \times \boldsymbol{ds}_2 + \boldsymbol{ds}_1 \times \frac{d(\boldsymbol{ds}_2)}{dt} \tag{2.19}$$

For any orthogonal tangent vector $\boldsymbol{ds}$, we have

$$\begin{aligned} \frac{d(\boldsymbol{ds})}{dt} &= \textit{change in } \boldsymbol{u} \textit{ over segment } \boldsymbol{ds} \\ &= \boldsymbol{Du} \cdot \boldsymbol{ds} \end{aligned} \tag{2.20}$$

where $\boldsymbol{Du}$ is the full derivative of $\boldsymbol{u}$. Thus we obtain

$$\frac{d(dA)}{dt} = (\boldsymbol{Du} \cdot \boldsymbol{ds}_1) \times \boldsymbol{ds}_2 + \boldsymbol{ds}_1 \times (\boldsymbol{Du} \cdot \boldsymbol{ds}_2) \tag{2.21}$$

Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be unit vectors in the ds_1, ds_2 direction, $\boldsymbol{ds}_1 = |\boldsymbol{ds}_1|\boldsymbol{v}_1$, $\boldsymbol{ds}_2 = |\boldsymbol{ds}_2|\boldsymbol{v}_2$, $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \boldsymbol{n}$. This yields

$$\frac{d(dA)}{dt} = |\boldsymbol{ds}_1||\boldsymbol{ds}_2|((\boldsymbol{Du} \cdot \boldsymbol{v}_1) \times \boldsymbol{v}_2 + \boldsymbol{v}_1 \times (\boldsymbol{Du} \cdot \boldsymbol{v}_2)) \tag{2.22}$$

Now, if we write out vector $\boldsymbol{Du} \cdot \boldsymbol{v}_1, \boldsymbol{Du} \cdot \boldsymbol{v}_2$ in terms of its $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{n}$ component using

$$\boldsymbol{w} = (\boldsymbol{w} \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{w} \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 + (\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{n} \tag{2.23}$$

for any vector $\boldsymbol{w}$, and carry out the cross product explicitly, we find

$$\frac{d(dA)}{dt} = |dA|(\boldsymbol{v}_1 \cdot \boldsymbol{Du} \cdot \boldsymbol{v}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{Du} \cdot \boldsymbol{v}_2)\boldsymbol{n} + (\textit{stuff in } \boldsymbol{v}_1, \boldsymbol{v}_2 \textit{ plane}) \tag{2.24}$$

So the normal component is

$$\begin{aligned} \boldsymbol{n} \cdot \frac{d(dA)}{dt} &= |dA|(\boldsymbol{v}_1 \cdot D\boldsymbol{u} \cdot \boldsymbol{v}_1 + \boldsymbol{v}_2 \cdot D\boldsymbol{u} \cdot \boldsymbol{v}_2) \\ &= |dA| \nabla_s \cdot \boldsymbol{u} \end{aligned} \tag{2.25}$$

where ∇_s is the surface divergence operator. If $\boldsymbol{u}$ is a 3-D vector, note that

$$\begin{aligned} \nabla_s \cdot \boldsymbol{u} &= (\boldsymbol{v}_1 \cdot D\boldsymbol{u} \cdot \boldsymbol{v}_1 + \boldsymbol{v}_2 \cdot D\boldsymbol{u} \cdot \boldsymbol{v}_2) \\ &= (\boldsymbol{v}_1 \cdot D\boldsymbol{u} \cdot \boldsymbol{v}_1 + \boldsymbol{u}_2 \cdot D\boldsymbol{u} \cdot \boldsymbol{v}_2 + \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n}) - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n} \\ &= \nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n} \end{aligned} \tag{2.26}$$

where ∇ is the usual 3D divergence. Thus we get an expression only involving the normal of the surface, as desired:

$$\boldsymbol{n} \cdot \frac{d(dA)}{dt} = |dA|(\nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n}) \tag{2.27}$$

From this, we can immediately derive $\frac{1}{|dA|}\frac{d|dA|}{dt}$

$$\begin{aligned} \frac{1}{|dA|}\frac{d|dA|}{dt} &= \frac{1}{2}\frac{1}{|dA|^2}\frac{d(|dA|^2)}{dt} \\ &= \frac{1}{2}\frac{1}{|dA|^2}\frac{d}{dt}dA \cdot dA \\ &= \frac{1}{|dA|^2}dA \cdot \frac{d}{dt}dA \\ &= \frac{1}{|dA|^2}|dA|\boldsymbol{n} \cdot \frac{d}{dt}dA \end{aligned} \tag{2.28}$$

So that using our above result yields

$$\frac{1}{|dA|}\frac{d|dA|}{dt} = \nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n} \tag{2.29}$$

2.4 Calculation of [p]

We derive $[p]$ from the constraint that volume remains constant. We will give this derivation using the level set representation. The bubble volume can be written as

$$Volume : V = \int_\Omega H(\phi) d\Omega \tag{2.30}$$

where $H(\phi)$ is the Heaviside function,

$$H(\phi) = \begin{cases} 1 & \text{if } \phi \geq 0 \\ 0 & \text{if } \phi < 0 \end{cases} \tag{2.31}$$

We will also use $\delta(\phi)$ where the Dirac delta function

$$\delta(x) = \frac{d}{dx} H(x) \tag{2.32}$$

in the sense of distribution. Then

$$\frac{dV}{dt} = \int_\Omega H(\phi)_t d\Omega \tag{2.33}$$

The level set evolution equation is

$$\phi_t + \boldsymbol{u} \cdot \nabla \phi = 0 \tag{2.34}$$

Therefore

$$\begin{aligned} [H(\phi)]_t &= H'(\phi)\phi_t \\ &= H'(\phi)(-\boldsymbol{u} \cdot \nabla \phi) \\ &= H'(\phi)|\nabla \phi| \boldsymbol{u} \cdot \boldsymbol{n} \end{aligned} \tag{2.35}$$

This yields

$$\frac{dV}{dt} = \int_\Omega H'(\phi)|\nabla\phi|\boldsymbol{u}\cdot\boldsymbol{n}d\Omega \tag{2.36}$$

To preserve volume, if we have $\frac{dV}{dt}|_{t=0} = 0$, then we must have $\frac{d}{dt}(\frac{dV}{dt}) = 0$.

$$\begin{aligned}\frac{d}{dt}(\frac{dV}{dt}) &= \frac{d}{dt}\int_\Omega \boldsymbol{u}\cdot\boldsymbol{n}\delta(\phi)|\nabla\phi|d\Omega \\ &= -\underbrace{\int_\Omega \delta'(\phi)\phi_t\boldsymbol{u}\cdot\nabla\phi d\Omega}_{I} - \underbrace{\int_\Omega \delta(\phi)\boldsymbol{u}_t\cdot\nabla\phi d\Omega}_{II} \\ &\quad -\underbrace{\int_\Omega \delta(\phi)\boldsymbol{u}\cdot\nabla\phi_t d\Omega}_{III}\end{aligned} \tag{2.37}$$

$$\begin{aligned}(II) &= [p]\int_\Omega \frac{1}{\mu}\delta(\phi)|\nabla\phi|d\Omega + \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n}\delta(\phi)|\nabla\phi|d\Omega \\ &\quad + \int_\Omega \frac{1}{\mu}(\sigma\kappa - \boldsymbol{f}\cdot\boldsymbol{n} - \boldsymbol{F}\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega\end{aligned} \tag{2.38}$$

$$\begin{aligned}(III) &= -\int_\Omega \delta'(\phi)\phi_t\boldsymbol{u}\cdot\nabla\phi d\Omega \\ &\quad -\int_\Omega (\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega\end{aligned} \tag{2.39}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}(\frac{dV}{dt}) &= -[p]\int_\Omega \frac{1}{\mu}\delta(\phi)|\nabla\phi|d\Omega - \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n}\delta(\phi)|\nabla\phi|d\Omega \\ &\quad + \int_\Omega (\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega \\ &\quad - \int_\Omega \frac{1}{\mu}(\sigma\kappa - \boldsymbol{f}\cdot\boldsymbol{n} - \boldsymbol{F}\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega\end{aligned} \tag{2.40}$$

From our volume preserving constraint,

$$\begin{aligned}[p]\int_\Omega \frac{1}{\mu}\delta(\phi)|\nabla\phi|d\Omega &= \int_\Omega ((\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega \\ &\quad - \int_\Omega \frac{1}{\mu}(\sigma\kappa - \boldsymbol{f}\cdot\boldsymbol{n} - \boldsymbol{F}\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega\end{aligned}$$

From this, we can get $[p]$.

(2.41)

$$[p] = \frac{\int_\Omega((\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n} - \frac{1}{\mu}(\sigma\kappa - \boldsymbol{f}\cdot\boldsymbol{n} - \boldsymbol{F}\cdot\boldsymbol{n}))\delta(\phi)|\nabla\phi|d\Omega}{\int_\Omega \frac{1}{\mu}\delta(\phi)|\nabla\phi|d\Omega}$$

2.5 Specification of μ, $\frac{d\mu}{dt}$ and $\boldsymbol{f}$

To completely specify the equations of motion, μ, $\frac{d\mu}{dt}$ (convective derivative of μ) and $\boldsymbol{f}$(additional momentum sources, including mass transfer effects) are required. No matter how these are specified, the motion will conserve volume and momentum, since the equations were derived from those principles. However, a physical bubble motion should also conserve the mass of bubble ($M = \int_{surface}\mu|dA|$), unless we are modeling a process like evaporation or condensation, which can alter the mass. There are four cases for μ and $\boldsymbol{f}$. Each of these cases also has simple restriction to 2-D.

Case 1. μ = constant on the surface and in time(e.g. $\mu = 1$,say)

This does not conserve bubble mass, since mass will vary in proportion to the area, which varies in time. Thus, it does not correspond to a physical bubble motion. Since it is artificial, we may as well also take $\boldsymbol{f} = 0$, for simplicity. This choice results in the simplest set of equation.

$$(2.42)\ [p] = \frac{\int_\Omega((\boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n} - \sigma\kappa)\delta(\phi)|\nabla\phi|d\Omega}{\int_\Omega\delta(\phi)|\nabla\phi|d\Omega}$$

$$= -\sigma\overline{\kappa} + \frac{\int_\Omega((\boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\delta(\phi)|\nabla\phi|d\Omega}{\int_\Omega\delta(\phi)|\nabla\phi|d\Omega}$$

where $\overline{\kappa}$ is the average curvature over the surface area.

Thus, the final equation of motion becomes

$$
\begin{aligned}
(2.43) \qquad \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= -\sigma(\kappa-\overline{\kappa})\boldsymbol{n} - \boldsymbol{u}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n}) \\
&+\frac{1}{A}\boldsymbol{n}\int_\Omega(\boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n})\delta(\boldsymbol{\phi})|\nabla\boldsymbol{\phi}|d\Omega \\
&-\frac{1}{A}\boldsymbol{n}\int_\Omega(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n}\delta(\boldsymbol{\phi})|\nabla\boldsymbol{\phi}|d\Omega
\end{aligned}
$$

Case 2. $\mu =$ constant over the surface, but varies in time to conserve mass($\mu = \frac{M}{A(t)}$ and $M = 1$, say)

This is the simplest motion that conserves mass, and corresponds to a real bubble with a fluid surface. So the mass is mobile enough to stay uniformly distributed. In this case there is mass transfer to a surface element during the motion. Since $\frac{d}{dt}(\mu|dA|)$ is not zero, and we need to postulate a form of the additional source of momentum $\boldsymbol{f}$. $\boldsymbol{f} = 0$ will not do, because that does not correspond to the intuitive physics. We imagine that when mass transfers into/out of the element being considered, it takes its momentum with it. This suggests that the force $\boldsymbol{F}_{MT}$ on the element $d\boldsymbol{s}$, as described above, is proportional to the local rate of change of element mass $\frac{d}{dt}(\mu|dA|)$. Also, since we are considering here an idealized mass transfer that is so rapid it keeps the mass density uniform on the surface, it in some sense samples and averages things over the whole surface, so that the mass undergoing transfer carries with it the average velocity of the entire bubble. This intuition leads one to postulate that the additional momentum transfer to the surface element ds

due to the mass transfer is

$$\boldsymbol{F}_{MT} = \overline{\boldsymbol{u}}\frac{d}{dt}(\mu|dA|) \tag{2.44}$$

where $\overline{\boldsymbol{u}}$ is the average vector velocity of the bubble,

$$\overline{\boldsymbol{u}} = \frac{1}{A}\int \boldsymbol{u}|dA| \tag{2.45}$$

and, since $\boldsymbol{f} = \frac{\boldsymbol{F}_{MT}}{|dA|}$,

$$\begin{aligned} \boldsymbol{f} &= \overline{\boldsymbol{u}}\frac{1}{|dA|}\frac{d}{dt}(\mu|dA|) & (2.46)\\ &= \mu\overline{\boldsymbol{u}}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\mu}\frac{d\mu}{dt}) & (2.47)\end{aligned}$$

In this case, we can again make the same simplifications as in Case 1 to get pressure jump $[p]$,

$$\begin{aligned} [p] &= -\sigma\overline{\kappa} + \frac{1}{A^2}\int_\Omega((\boldsymbol{u}\cdot\boldsymbol{n})(\nabla\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega \\ &\quad + \frac{1}{A^2}\int_\Omega(\overline{\boldsymbol{u}}-\boldsymbol{u})\cdot\boldsymbol{n}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\mu}\frac{d\mu}{dt})\delta(\phi)|\nabla\phi|d\Omega \end{aligned} \tag{2.48}$$

and the equation of motion is

$$\begin{aligned} \mu(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) &= -\sigma(\kappa - \overline{\kappa})\boldsymbol{n} + \mu(\overline{\boldsymbol{u}}-\boldsymbol{u})(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\mu}\frac{d\mu}{dt}) \\ &\quad - \frac{\boldsymbol{n}}{A^2}\int_\Omega(\overline{\boldsymbol{u}}-\boldsymbol{u})\cdot\boldsymbol{n}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\mu}\frac{d\mu}{dt})\delta(\phi)|\nabla\phi|d\Omega \\ &\quad - \frac{\boldsymbol{n}}{A^2}\int_\Omega((\boldsymbol{u}\cdot\boldsymbol{n})(\nabla\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega \end{aligned} \tag{2.49}$$

Case 3. $\mu|dA|$ is constant during the motion. (i.e. $\frac{d}{dt}(\mu|dA|) = 0$)

This also conserves mass; it corresponds to an elastic membrane, where the mass is not free to migrate. In this case, we envision no mass transfer effect, thus $\boldsymbol{f} = 0$. We get, from $\frac{d}{dt}(\mu|dA|) = 0$, that

$$\begin{aligned} \frac{1}{\mu}\frac{d\mu}{dt} &= -\frac{1}{|dA|}\frac{d|dA|}{dt} \\ &= -(\nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n}) \end{aligned} \tag{2.50}$$

Therefore

$$[p] = \frac{\int_\Omega((\nabla \cdot \boldsymbol{u})(\boldsymbol{u} \cdot \boldsymbol{n}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{n} - \frac{1}{\mu}\sigma\kappa)\delta(\phi)|\nabla\phi| d\Omega}{\int \frac{1}{\mu}\delta(\phi)|\nabla\phi| d\Omega} \tag{2.51}$$

Case 4. μ flows with some mass preserving velocity on the surface.

This velocity would generally require additional equations to specify it. The local momentum source due to mass transfer would be $\boldsymbol{f} = \nabla_s(\mu \boldsymbol{u}_\mu)$, the surface divergence of momentum flux associated with the μ velocity $\boldsymbol{u}_\mu$.

CHAPTER 3

Summary of Bubble Equations

We summarize our equations of bubble motion for various cases. All equations are valid in both 2-D and 3-D.

3.1 Notational convention

We review the notation used below:

dA : the vector area of surface element in the direction normal to the surface

$|dA|$: scalar area of the surface element

$\boldsymbol{n}$: the unit outward normal to the surface

κ : mean curvature

σ : surface tension

$\frac{dQ}{dt} = Q_t + \boldsymbol{u} \cdot \nabla Q$ for any quantity Q

$Q_{ave} = \frac{1}{A} \int_{surface} Q|dA|$ where A is the total area ($A = \int |dA|$)

3.2 2-D versions of equation

The 2-D versions of all equations are obtained by replacing the area of bubble surface by the length of the bubble curve in all relevant places:

$dA \longrightarrow dL$

$|dA| \longrightarrow |dL|$

$A \longrightarrow L$

Note that $dL = \boldsymbol{n}|dL|$ is an outward normal.

3.3 Initial data

In order to have volume preserving motions, the initial velocity of the bubble must also be volume preserving. (i.e. $\frac{dV}{dt} = 0$ $at\ time = 0$). From equation (2.33), we see this means that the initial velocity $\boldsymbol{u}_0$ must satisfy

$$0 = \int_{surface} \boldsymbol{u}_0 \cdot dA \tag{3.1}$$

If this condition is violated, there will be very large acceleration produced when the time evolution starts. The easiest way to insure (3.1) is to either take $\boldsymbol{u}_0 = 0$ which corresponds to a bubble at rest initially, or $\boldsymbol{u}_0 \cdot \boldsymbol{n} = 0$ which corresponds to a bubble "rotating" initially (i.e., velocity tangent to surface everywhere).

3.4 General equation of bubble motion

The general equation for bubble motion is

$$\mu \frac{d\boldsymbol{u}}{dt} = -[p]\boldsymbol{n} - \sigma\kappa\boldsymbol{n} + \boldsymbol{f} - \mu\boldsymbol{u}\left(\frac{1}{|dA|}\frac{d|dA|}{dt} + \frac{1}{\mu}\frac{d\mu}{dt}\right) \tag{3.2}$$

where

$$\frac{1}{|dA|} d\frac{|dA|}{dt} = \nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n} \tag{3.3}$$

and

(3.4)

$$[p] = \frac{\int_\Omega((\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n} - \frac{1}{\mu}(\sigma\kappa - \boldsymbol{f}\cdot\boldsymbol{n} - \boldsymbol{F}\cdot\boldsymbol{n}))\delta(\phi)|\nabla\phi|d\Omega}{\int_\Omega \frac{1}{\mu}\delta(\phi)|\nabla\phi|d\Omega}$$

In addition, we specify conservation of mass

$$Mass\ :\ M = \int \mu|dA| = constant\ in\ time \tag{3.5}$$

and no self-induced force

$$0 = \int \boldsymbol{f}|dA| \tag{3.6}$$

3.5 Volume preserving acceleration motion

Consider

$$\frac{d\boldsymbol{u}}{dt} = -[p]\boldsymbol{n} - \sigma\kappa\boldsymbol{n} \tag{3.7}$$

This preserves the bubble volume if

$$[p] = -\sigma\overline{\kappa} + \frac{1}{A}\int_\Omega((\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega \tag{3.8}$$

This model is volume preserving, but it does not conserve momentum or mass for the bubble, so it is not a physical bubble motion. However it is useful to compare it with volume preserving velocity case.

3.6 Volume and momentum preserving model

This is the model described in section 2.5, Case 1, with $\mu = 1$

$$\frac{d\boldsymbol{u}}{dt} = -[p]\boldsymbol{n} - \sigma\kappa\boldsymbol{n} - \boldsymbol{u}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n}) \tag{3.9}$$

where

$$[p] = -\sigma\overline{\kappa} + \frac{\int_\Omega ((\boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{n}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega}{\int_\Omega \delta(\phi)|\nabla\phi|d\Omega} \tag{3.10}$$

This model does not conserve bubble mass, but it does conserve volume and momentum.

3.7 Fluid bubble

This is the model described in section 2.5, Case 2, with bubble mass $M = 1$.

$$\mu\frac{d\boldsymbol{u}}{dt} = -[p]\boldsymbol{n} - \sigma\kappa\boldsymbol{n} + \boldsymbol{f} - \mu\boldsymbol{u}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\mu}\frac{d\mu}{dt}) \tag{3.11}$$

$$\mu(t) = \frac{1}{A(t)} \tag{3.12}$$

$$\boldsymbol{f} = \mu\overline{\boldsymbol{u}}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\mu}\frac{d\mu}{dt}) \tag{3.13}$$

$$\begin{aligned} [p] &= -\sigma\overline{\kappa} + \frac{1}{A^2}\int_\Omega (\overline{\boldsymbol{u}} - \boldsymbol{u})\cdot\boldsymbol{n}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\mu}\frac{d\mu}{dt})\delta(\phi)|\nabla\phi|d\Omega \\ &+\frac{1}{A^2}\int_\Omega ((\boldsymbol{u}\cdot\boldsymbol{n})(\nabla\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\boldsymbol{n})\delta(\phi)|\nabla\phi|d\Omega \end{aligned} \tag{3.14}$$

This preserves bubble volume, mass and momentum and so is physically realistic bubble motion. It corresponds to a bubble with an ideal fluid surface on which mass is very mobile and maintains a uniform mass distribution.

3.8 Elastic membrane bubble

This is the model described in section 2.5, Case 3.

$$\mu\frac{d\boldsymbol{u}}{dt} = -[p]\boldsymbol{n} - \sigma\kappa\boldsymbol{n} \tag{3.15}$$

$$\mu_{t=0} = a\ positive\ valued\ function\ on\ surface \tag{3.16}$$

$$\frac{1}{\mu}\frac{d\mu}{dt} = -(\nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n}) \tag{3.17}$$

$$[p] = \frac{\int_\Omega ((\nabla \cdot \boldsymbol{u})(\boldsymbol{u} \cdot \boldsymbol{n}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{n} - \frac{1}{\mu}\sigma\kappa)\delta(\boldsymbol{\phi})|\nabla\boldsymbol{\phi}|d\Omega}{\int \frac{1}{\mu}\delta(\boldsymbol{\phi})|\nabla\boldsymbol{\phi}|d\Omega} \tag{3.18}$$

This preserves bubble volume, mass and momentum, so it is a physically realistic bubble motion. It corresponds to a bubble modeled as an elastic membrane—i.e. no migration of mass within the surface. Various interesting effects can be created by starting with non-uniform mass distributions.

CHAPTER 4

Numerical Methods

4.1 Description of the level set approach

We construct a level set function ϕ such that bubble interface is the zero level set of ϕ. We also initialize ϕ to be the signed distance from the interface such that ϕ is positive inside the bubble and negative outside the bubble. This is easy to do using the re-distance algorithm of [7]. We represent the interface by

$$surface\ of\ bubble = \{x|\phi(x) = 0\}$$

$$\phi(\boldsymbol{x}) > 0 \ \textit{inside bubble}$$

$$\phi(\boldsymbol{x}) < 0 \ \textit{outside bubble} \tag{4.1}$$

The idea of the level set method is to move ϕ with the correct speed $\boldsymbol{u}$ at the front using the following differential equation:

$$\phi_t + \boldsymbol{u} \cdot \nabla\phi = 0 \tag{4.2}$$

Next, we reinitialize, using the algorithm[7] keeping ϕ to be signed distance, at least near the front. Additionally, we save computational time performing these calculations only near the front. There are several localization algorithms available; we use the relatively simple algorithm developed in [8].

4.2 Notations on using level set function

We can rewrite the variables by using the level set function $\boldsymbol{\phi}$.

$$\boldsymbol{n} = -\frac{\nabla \boldsymbol{\phi}}{|\nabla \boldsymbol{\phi}|} \tag{4.3}$$

$$\kappa = -\nabla \cdot \frac{\nabla \boldsymbol{\phi}}{|\nabla \boldsymbol{\phi}|} \tag{4.4}$$

$$\overline{\kappa} = \frac{\iint \kappa \delta(\phi)|\nabla \phi|\, dx\, dy}{\int \int \delta(\phi)|\nabla \phi| dx dy} \tag{4.5}$$

$$H_\alpha(\boldsymbol{\phi}) = \begin{cases} 1 & \text{if } \phi > \alpha \\ 0 & \text{if } \phi < \alpha \\ \frac{1}{2}(1 + \frac{\phi}{\alpha} + \frac{1}{\pi} sin(\frac{\pi x}{\alpha})) & \text{otherwise} \end{cases} \tag{4.6}$$

$$\delta_\alpha(\boldsymbol{\phi}) = \begin{cases} \frac{1}{2\alpha}(1 + cos(\frac{\pi \phi}{\alpha})) & \text{if } |\phi| < \alpha \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

$$\int_{surface} Q|dA| = \int_\Omega Q\delta(\boldsymbol{\phi})|\nabla \boldsymbol{\phi}| d\Omega \tag{4.8}$$

$$\int_{volume} QdV = \int_\Omega QH_\alpha(\alpha) dV \tag{4.9}$$

where α is the prescribed "thickness" of the interface (usually 3 grid points in our calculations) and Q is any quantity.

4.3 Outline of the numerical method

We can now summarize our algorithm.

Step 1. Initialize $\boldsymbol{\phi}(\boldsymbol{x},t)$ such that $\boldsymbol{\phi}$ is a signed distance function to the front.

Step 2. Solve the governing equation and get the velocity $\boldsymbol{u}$ and update the level function $\boldsymbol{\phi}$.

The method for updating the level function $\boldsymbol{\phi}$ was proposed by S. Osher and J. Sethian[4]. Consider the following differential equation.

$$\boldsymbol{\phi}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{\phi} = 0 \tag{4.10}$$

For $\boldsymbol{u}$ a given function of space and time, we obtain higher order accuracy by using ENO type schemes both in time and in space[3,4]. For the time discretization, we use Heun's method[3]. Let

$$\frac{\partial}{\partial t}\boldsymbol{\phi} = -L[\boldsymbol{\phi}] \tag{4.11}$$

Then we apply Heun's method.

$$\overline{\boldsymbol{\phi}}^{m+1} = \boldsymbol{\phi}^m - \Delta t L[\boldsymbol{\phi}^m] \tag{4.12}$$

$$\boldsymbol{\phi}^{m+1} = \frac{1}{2}\boldsymbol{\phi}^m + \frac{1}{2}\overline{\boldsymbol{\phi}}^{m+1} - \frac{\Delta t}{2}L[\overline{\boldsymbol{\phi}}^{m+1}]$$

For the space discretization, first consider the motion induced by curvature dependent velocity.

We can rewrite equation(4.10) as

$$\boldsymbol{\phi}_t - (\kappa - \overline{\kappa})\boldsymbol{n} \cdot \nabla \boldsymbol{\phi} = 0 \tag{4.13}$$

Using the relation(4.3) and (4.4), equation(4.13) becomes

$$\phi_t + \kappa|\nabla\phi| = \overline{\kappa}|\nabla\phi| \tag{4.14}$$

When we apply the numerical scheme to a Hamilton-Jacobi equation, care is required. For the term corresponding to $\overline{\kappa}|\nabla\phi|$, we use ENO scheme(we will explain it at step 3). All the derivatives of the term $\kappa|\nabla\phi|$ are approximated by central differences. The term $\kappa|\nabla\phi|$ depends on the multiplication and/or division by $(\phi_x^2 + \phi_y^2)^{\frac{1}{2}}$, which might be close to zero near some points. This implies that we should be consistent in our approximation to κ and $|\nabla\phi|$ in the term $\kappa|\nabla\phi|$ to avoid unnecessary errors.

For the curvature dependent acceleration, we can rewrite our governing equation as

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}(\nabla\cdot\boldsymbol{u} - \boldsymbol{n}\cdot D\boldsymbol{u}\cdot\boldsymbol{n}) = F(\kappa, ...) \tag{4.15}$$

Since this system is only weakly well-posed(see appendix), care is required in the numerical scheme. We use a 2nd order Lax-Friedrichs scheme. To apply Lax-Friedrichs scheme, we rewrite the equation(4.15) in matrix form.

$$\boldsymbol{u}_t + [A(\boldsymbol{u})\boldsymbol{u}_x] + [B(\boldsymbol{u})\boldsymbol{u}_y] = F(\kappa, ...) \tag{4.16}$$

The 1st order Lax-Friedrichs scheme is defined as

$$\begin{aligned} (4.17)[A(\boldsymbol{u})\boldsymbol{u}_x]_j &= \frac{1}{\Delta x}[A(\boldsymbol{u}_j)\Delta_-(\frac{\boldsymbol{u}_{j+1}+\boldsymbol{u}_j}{2}) - c|\boldsymbol{u}^*|\Delta_-(\frac{\boldsymbol{u}_{j+1}-\boldsymbol{u}_j}{2})] \\ |\boldsymbol{u}^*| &= max_{\{i|j-k\le i\le j+k\}}|\boldsymbol{u}_i|, \quad c \ge 1 \end{aligned}$$

We take k=2 and c=2 in our simulations.

The 2nd order TVD type Lax-Friedrichs scheme is defined by using

$$
\begin{aligned}
(4.18) \qquad \boldsymbol{u}_{j+1} &\longrightarrow \boldsymbol{u}_{j+1} - \frac{\boldsymbol{s}_{j+1}}{2} \\
\boldsymbol{u}_j &\longrightarrow \boldsymbol{u}_j + \frac{\boldsymbol{s}_j}{2} \\
\boldsymbol{s}_j &= minmod[\Delta_+ \boldsymbol{u}_j, \Delta_- \boldsymbol{u}_j] \\
minmod(x,y) &= \begin{cases} sign(x)min(|x|,|y|) & if\ x \cdot y > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

After we get the velocities from the above system, we update the level set by solving equation(4.10) using the 2nd order ENO scheme for the space discretization. For the x -direction in 2D(or 3D),

(4.19)

$$
\boldsymbol{u} \cdot \nabla \boldsymbol{\phi} = \begin{cases} \boldsymbol{u}_{i,j}(D_-^x \boldsymbol{\phi}_{i,j} + \frac{h}{2} minmod(D_-^x D_+^x \boldsymbol{\phi}_{i,j}, D_-^x D_-^x \boldsymbol{\phi}_{i,j}) & \text{if } \boldsymbol{u}_{i,j} > 0 \\ \boldsymbol{u}_{i,j}(D_+^x \boldsymbol{\phi}_{i,j} - \frac{h}{2} minmod(D_+^x D_+^x \boldsymbol{\phi}_{i,j}, D_+^x D_-^x \boldsymbol{\phi}_{i,j}) & \text{otherwise} \end{cases}
$$

$$
(4.20) \qquad minmod(x,y) = \begin{cases} sign(x)min(|x|,|y|) & if\ x \cdot y > 0 \\ 0 & \text{otherwise} \end{cases}
$$

and D_-^x, D_+^x are just backward and forward divided differences. Similarly, we can apply this to the y-direction and the z-direction for 3D problems also.

Step 3. Construct a new distance function $\boldsymbol{\phi}$.

Consider the following equation.

$$
(4.21) \qquad d_t = sign(d^0)(1 - |\nabla d|) \ \ with\ d^0 = \boldsymbol{\phi}^n
$$

where $sign(d^0)$ gives the sign of d^0. Given initial data, we solve the above equation until the solution reaches a steady state near the front. To eliminate the stiffness of sign function, we approximate $sign(\phi)$ by

$$sign(\phi) = \frac{\phi}{\sqrt{\phi^2 + \epsilon^2}} \tag{4.22}$$

where ϵ is very small number(i.e. $\epsilon = h$). Following [3,7] we use the approximation.

$$|\nabla\phi| = \begin{cases} \sqrt{max((a^+)^2,(b^-)^2) + max((c^+)^2,(d^-)^2)} & if\ d^0 > 0 \\ \sqrt{max((a^-)^2,(b^+)^2) + max((c^-)^2,(d^+)^2)} & otherwise \end{cases} \tag{4.23}$$

$$\begin{aligned} a &= D_-^x\phi + \frac{h}{2} minmod(D_-^x D_+^x \phi_{i,j}, D_-^x D_-^x \phi_{i,j}) \\ b &= D_+^x\phi - \frac{h}{2} minmod(D_+^x D_+^x \phi_{i,j}, D_+^x D_-^x \phi_{i,j}) \end{aligned} \tag{4.24}$$

$$\begin{aligned} a^+ &= max(a,0) \\ a^- &= min(a,0) \\ b^+ &= max(b,0) \\ b^- &= min(b,0) \end{aligned} \tag{4.25}$$

We can define c, d using the same approximation in the y-direction.

For the criterion for steady state, we use

$$Q = \frac{\Sigma_{|d^n_{i,j}|<\alpha} |d^{n+1}_{i,j} - d^n_{i,j}|}{M} < (\Delta t)h^2 \tag{4.26}$$

where M = number of grid points where $|d^n_{i,j}| < \alpha (= ch)$, for some constant c(we use $c = 1.5$). This is a very local criterion and convergence is very rapid.

After solving this equation we let

$$\phi^{n}_{new} = steady\ state\ solution\ of\ (4.21). \tag{4.27}$$

Step 4. We have now advanced one time step. Go to step 2 and repeat.

CHAPTER 5

Results

Our experiments simulate the motions of soap bubbles. In the case of curvature dependent velocity or acceleration, we simulate the merging and breaking of the interface in 2D and 3D. Also we compare the global algorithm and local(fast) algorithm[8]. Results indicate that the fast algorithm is almost four or five times faster than the global algorithm for our smallest mesh size without any loss of area.(see Table2)

5.1 Results using curvature dependent velocity

We simulate several different shapes in the case of curvature dependent velocity. In Figure 1-5, we show the continuous evolution of non-intersecting curves collapsing smoothly to a circle.

In Figure 6-8, we show the merging of two bubbles in 2D. After merging they converge to a steady state which is, of course, a circle.

In Figure 9-10, we use a symmetric and non-symmetric dumbbell as initial data. In 2D, a dumbbell under our motion collapses to a circle. However, in 3D, a dumbbell breaks into two part and each part changes smoothly to a sphere. Both 2D and 3D motions preserve the area and the volume respectively(see Table 1).

5.2 Results using curvature dependent acceleration

We simulate the motion of bubble which was was previously described in section 3.5.-3.8. In Figure 11-26, the results show that motion has oscillations as expected. Depending on the laws of motion, each bubbles evolve slightly different ways. All the cases preserve area well.(see Table 1)

Mesh size	32 x 32	64 x 64	128x128	Mesh size	32 x 32	64 x 64	128x128
Figure 1	0.9892	0.9942	0.9969	Figure 2	0.9674	0.9850	0.9910
Figure 3	0.9949	0.9979	0.9985	Figure 4	0.9974	0.9825	0.9951
Figure 5	0.9570	0.9829	0.9947	Figure 6	0.9420	0.9627	0.9820
Figure 7	0.9112	0.9410	0.9624	Figure 8	0.9512	0.9710	0.9827
Figure 9	0.9110	0.9427		Figure 10	0.8990	0.9320	
Figure 11	0.9915	0.9962	0.9980	Figure 12	0.9031	0.9579	0.9823
Figure 13	0.9856	1.0120	1.0107	Figure 14	0.8976	0.9205	0.9582
Figure 15	0.9915	0.9962	0.9987	Figure 16	0.9017	0.9572	0.9812
Figure 17	0.9810	1.0130	1.0070	Figure 18	0.9951	0.9979	0.9985
Figure 19	0.9675	0.9821	0.9961	Figure 20	0.8562	0.9216	0.9754
Figure 21	0.9875	1.0190	1.0135	Figure 22	0.8731	0.9321	0.9734
Figure 23	0.9542	0.9715	0.9910	Figure 24	0.9351	0.9745	0.9856
Figure 25	0.9410	0.9952	1.0021	Figure 26	0.8561	0.9145	0.9656

Table 1: Ratio of area for final area to initial area

	Mesh size	Global	Local	Global/Local
Figure 1	32 x 32	2.80×10^{-2}	1.43×10^{-2}	1.96
	64 x 64	1.08×10^{-1}	4.18×10^{-2}	2.58
	128 x 128	6.95×10^{-1}	1.36×10^{-1}	5.11
Figure 2	32 x 32	2.89×10^{-2}	1.42×10^{-2}	2.04
	64 x 64	1.02×10^{-1}	4.07×10^{-2}	2.51
	128 x 128	6.02×10^{-1}	1.13×10^{-1}	5.33
Figure 3	32 x 32	2.83×10^{-2}	1.78×10^{-2}	1.59
	64 x 64	1.16×10^{-1}	5.17×10^{-2}	2.24
	128 x 128	5.62×10^{-1}	1.46×10^{-1}	3.85
Figure 4	32 x 32	3.06×10^{-2}	2.52×10^{-2}	1.21
	64 x 64	1.17×10^{-1}	7.07×10^{-2}	1.66
	128 x 128	6.88×10^{-1}	1.82×10^{-1}	3.78
Figure 5	32 x 32	2.98×10^{-2}	3.21×10^{-2}	0.93
	64 x 64	1.03×10^{-1}	9.01×10^{-2}	1.14
	128 x 128	5.54×10^{-1}	2.38×10^{-1}	2.33

Table 2: Time(second) per Iteration for Global algorithm and Local(fast) algorithm

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APPENDIX A

Ill-Posedness of System

Here, we analyze the slight ill-posedness of a system for the case of a curvature dependent acceleration.

Consider the following system.

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{u}(\nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n}) = 0 \tag{A.1}$$

Let

$$\boldsymbol{u} = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}, \quad \tilde{\boldsymbol{u}} \ll 1 \tag{A.2}$$

Substitute (A.2) to (A.1),

$$\begin{aligned} \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= (\boldsymbol{u}_0 + \tilde{\boldsymbol{u}}) \cdot \nabla(\boldsymbol{u}_0 + \tilde{\boldsymbol{u}}) \\ &= \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 + \boldsymbol{u}_0 \cdot \nabla \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}_0 + O(\boldsymbol{u}^2) \end{aligned} \tag{A.3}$$

$$\begin{aligned} \boldsymbol{u}(\nabla \cdot \boldsymbol{u}) &= (\boldsymbol{u}_0 + \tilde{\boldsymbol{u}})(\nabla \cdot (\boldsymbol{u}_0 + \tilde{\boldsymbol{u}})) \\ &= \boldsymbol{u}_0(\nabla \cdot \boldsymbol{u}_0) + \boldsymbol{u}_0(\nabla \cdot \tilde{\boldsymbol{u}}) + \tilde{\boldsymbol{u}}(\nabla \cdot \boldsymbol{u}_0) + O(\tilde{\boldsymbol{u}^2}) \end{aligned} \tag{A.4}$$

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n}) &= (\boldsymbol{u}_0 + \tilde{\boldsymbol{u}})(\boldsymbol{n} \cdot D\boldsymbol{u}_0 \cdot \boldsymbol{n} + \boldsymbol{n} \cdot D\tilde{\boldsymbol{u}} \cdot \boldsymbol{n}) \\ &= \boldsymbol{u}_0(\boldsymbol{n} \cdot D\boldsymbol{u}_0 \cdot \boldsymbol{n}) + \boldsymbol{u}_0(\boldsymbol{n} \cdot D\tilde{\boldsymbol{u}} \cdot \boldsymbol{n}) + \tilde{\boldsymbol{u}}(\boldsymbol{n} \cdot D\boldsymbol{u}_0 \cdot \boldsymbol{n}) + O(\tilde{\boldsymbol{u}^2}) \end{aligned} \tag{A.5}$$

From above equations, we can get

$$\tilde{\boldsymbol{u}}_t + (\boldsymbol{u}_0 \cdot \nabla)\tilde{\boldsymbol{u}} + \boldsymbol{u}_0(\nabla \cdot \tilde{\boldsymbol{u}} - \boldsymbol{n} \cdot D\tilde{\boldsymbol{u}} \cdot \boldsymbol{n}) = 0 \tag{A.6}$$

Use Fourier transform,

$$\boldsymbol{u}_t + \imath(\boldsymbol{u}_0 \cdot \boldsymbol{k})I\boldsymbol{u} + \imath(\boldsymbol{k} \cdot \boldsymbol{u} - (\boldsymbol{n} \cdot \boldsymbol{u})(\boldsymbol{n} \cdot \boldsymbol{k})) \;=\; 0 \tag{A.7}$$

$$\begin{aligned} where\ \boldsymbol{k} &= (k_1, k_2)\ is\ the\ Fourier\ frequency \\ \boldsymbol{u}_0 &= (u_0, v_0) \\ \boldsymbol{u} &= (u, v) \end{aligned}$$

Now, our system becomes

$$\boldsymbol{u}_t = -\imath A\boldsymbol{u} \tag{A.8}$$

$$A = \begin{pmatrix} \boldsymbol{u}_0 \cdot \boldsymbol{k} + \boldsymbol{u}_0(k_1 - (\boldsymbol{n} \cdot \boldsymbol{k})n_1) & u_0(k_2 - (\boldsymbol{n} \cdot \boldsymbol{k})n_2) \\ v_0(k_1 - (\boldsymbol{n} \cdot \boldsymbol{k})n_1 & \boldsymbol{u}_0 \cdot \boldsymbol{k} + v_0(k_2 - (\boldsymbol{n} \cdot \boldsymbol{k})n_2) \end{pmatrix}$$

So, we can get eigenvalues,

$$\begin{aligned} \lambda_1 &= \boldsymbol{u}_0 \cdot \boldsymbol{k} \\ \lambda_2 &= 2(\boldsymbol{u}_0 \cdot \boldsymbol{k}) - (\boldsymbol{u} \cdot \boldsymbol{k})(\boldsymbol{n} \cdot \boldsymbol{k}) \end{aligned} \tag{A.9}$$

if $\boldsymbol{u}_0 = (\boldsymbol{u}_0 \cdot \boldsymbol{n})(\boldsymbol{n} \cdot \boldsymbol{k})$ (*i.e.* $\boldsymbol{u}_0 = \boldsymbol{n}$), then we have $\lambda = \boldsymbol{u}_0 \cdot \boldsymbol{k}$ as a real double eigenvalue, for which the matrix A has a Jordan block. Therefore, our system is slightly ill-posed. The numerical method does however converge. We are investigating this phenomenon which is probably due to the curvature regularization.

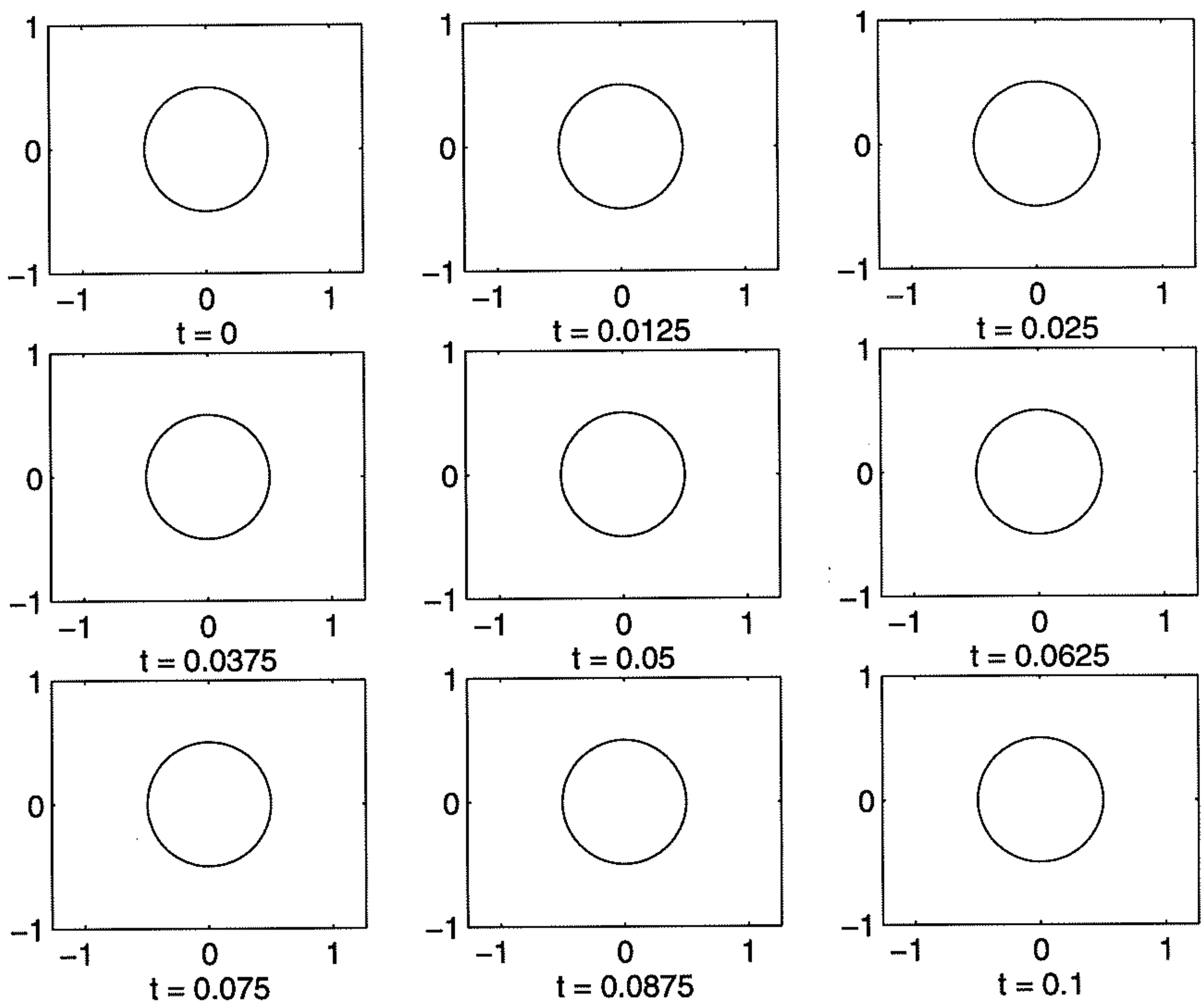

Figure 1: Area preserving curvature dependent velocity: circle is an equilibrium solution

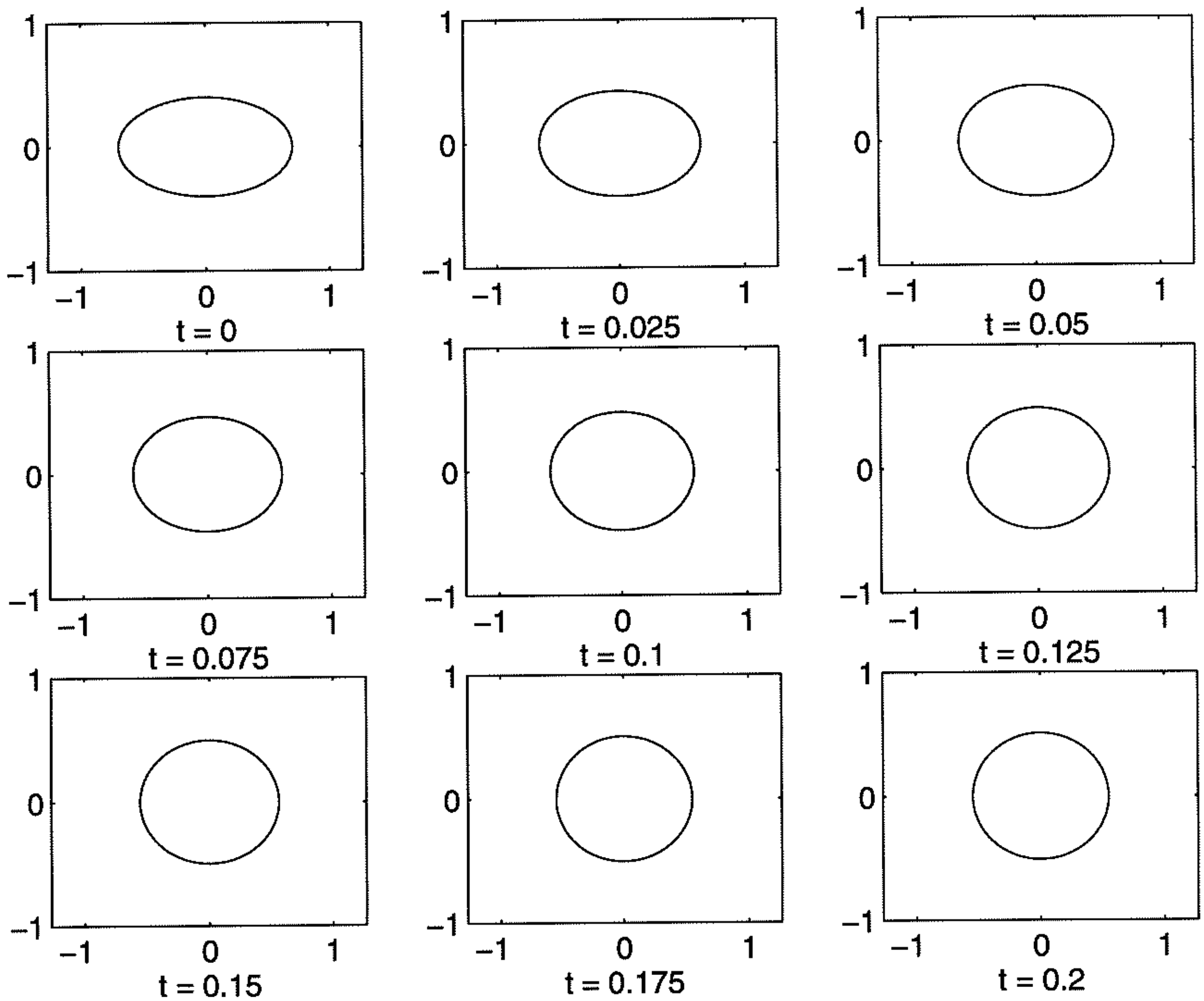

Figure 2: Area preserving curvature dependent velocity: ellipse relaxes to circle

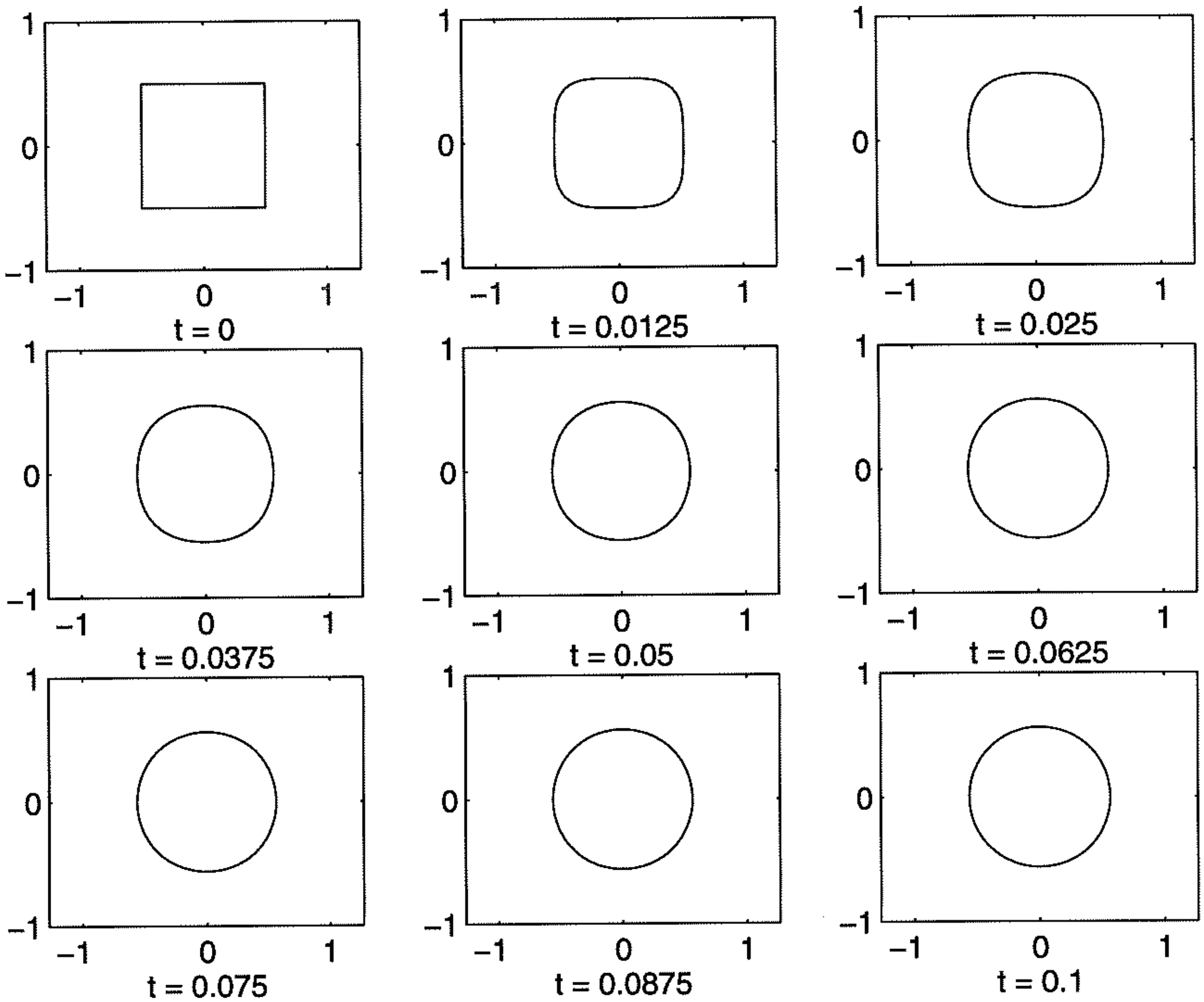

Figure 3: Area preserving curvature dependent velocity: square relaxes to circle

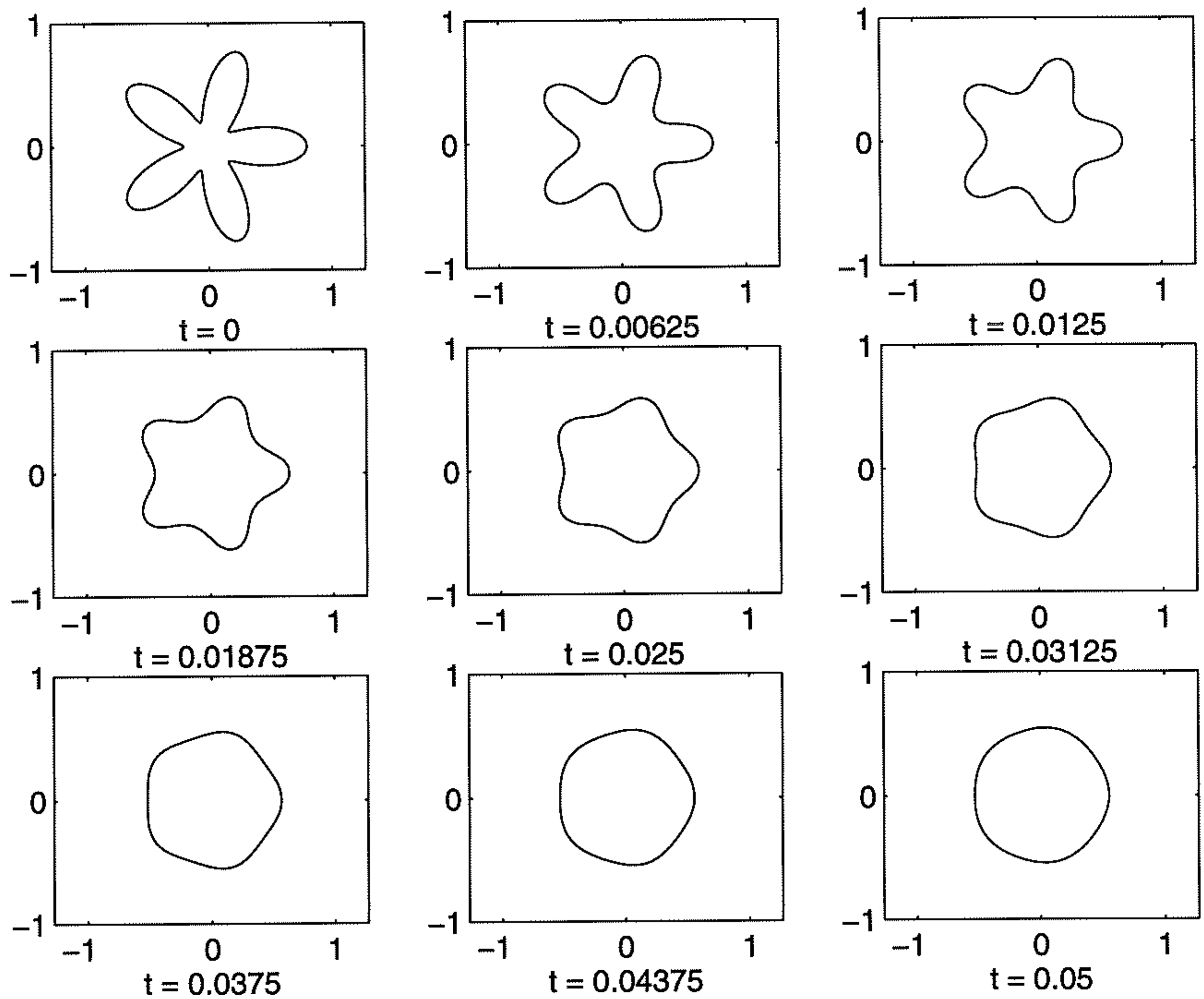

Figure 4: Area preserving curvature dependent velocity: starfish relaxes towards circle

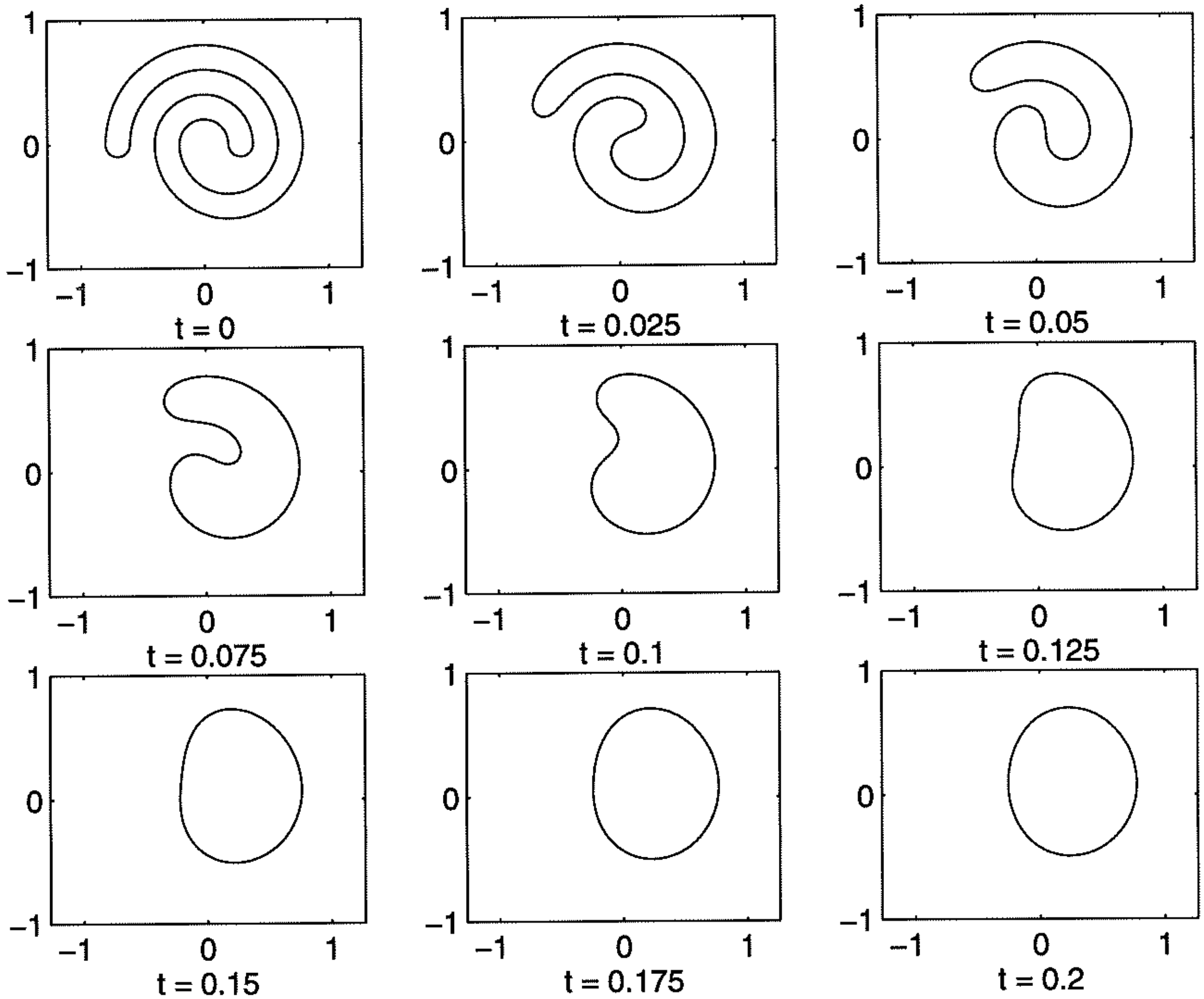

Figure 5: Area preserving curvature dependent velocity: spiral relaxes towards circle

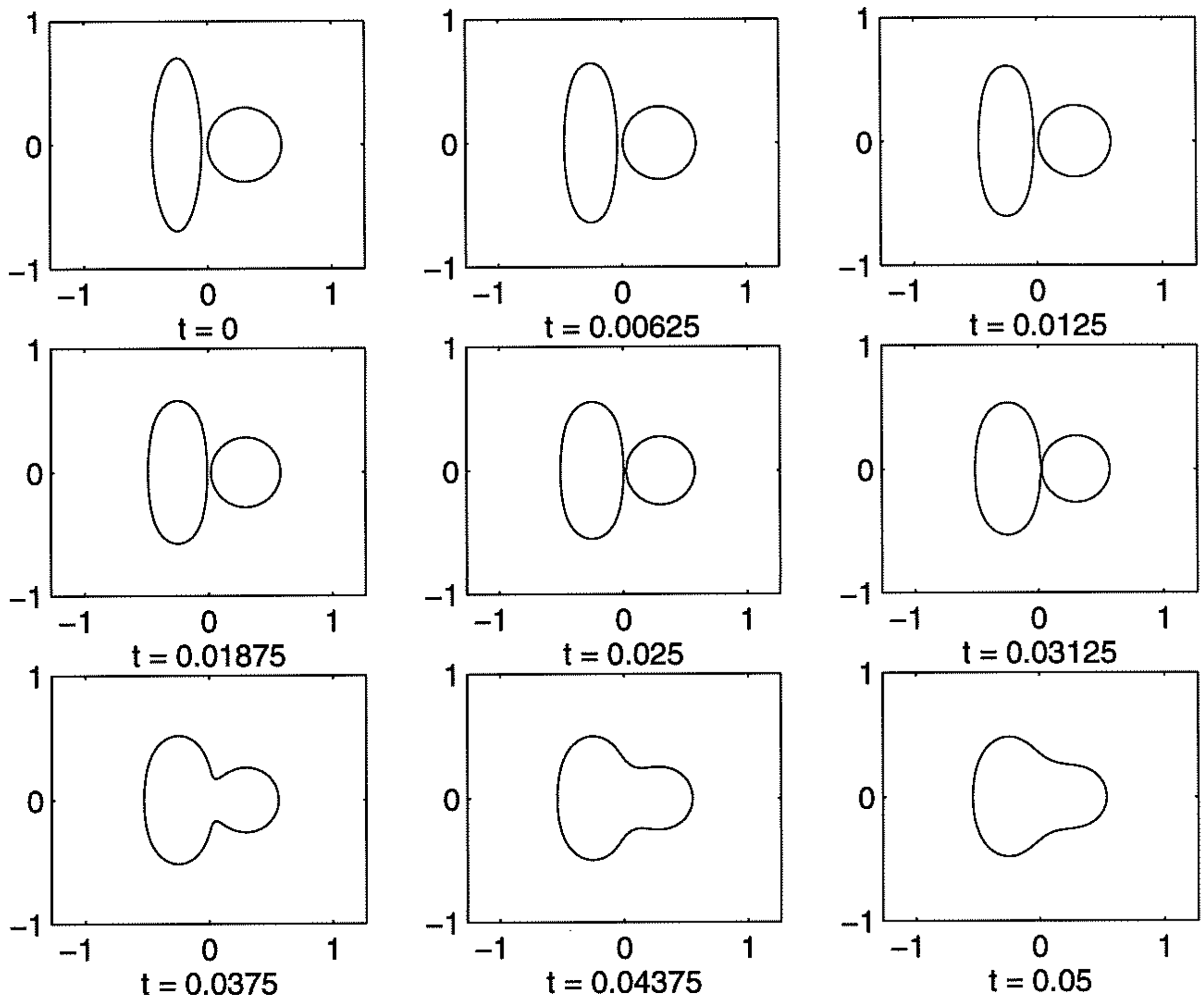

Figure 6: Area preserving curvature dependent velocity: merging case, relaxes towards circle

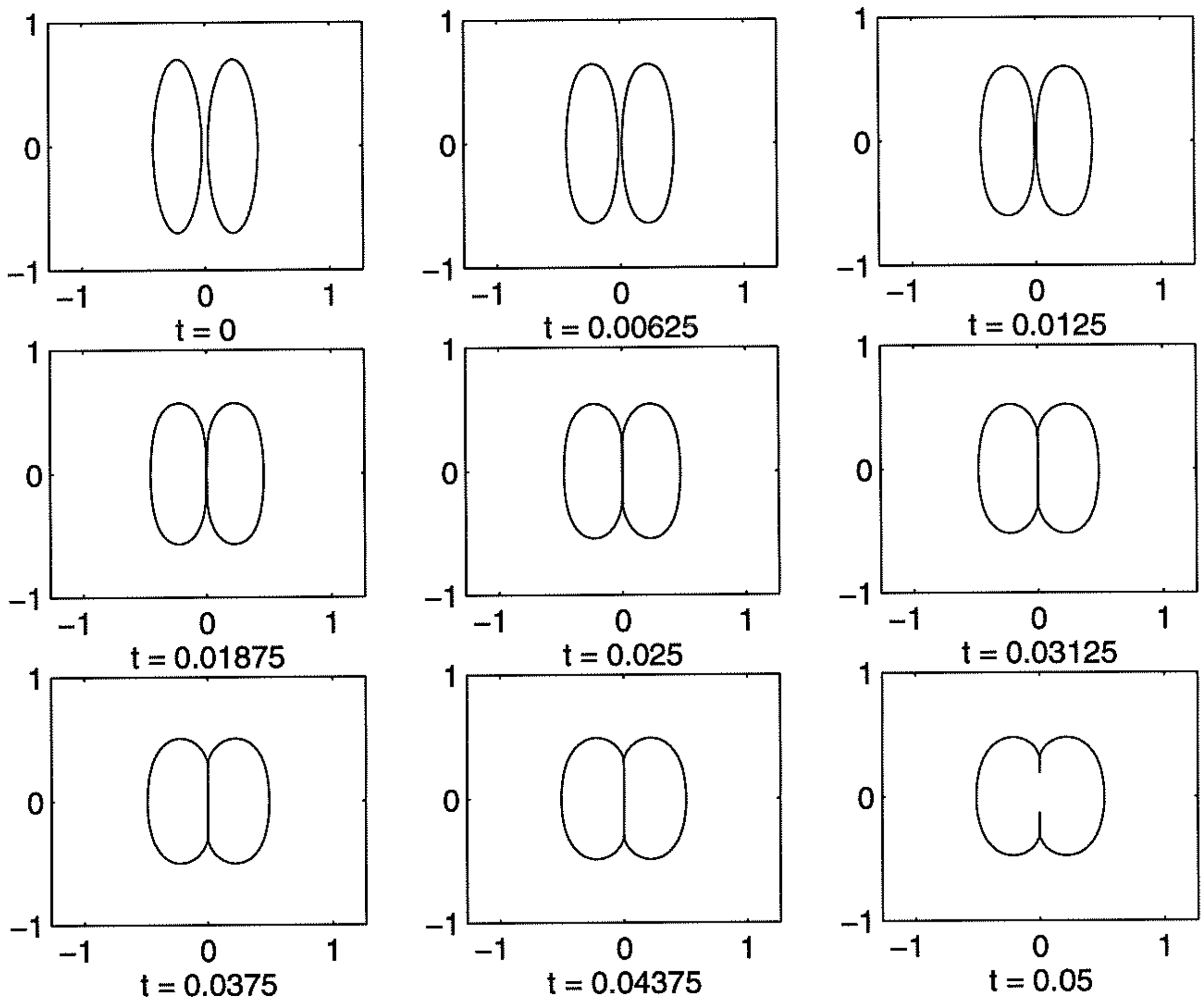

Figure 7: Area preserving curvature dependent velocity: merging case, relaxes towards circle

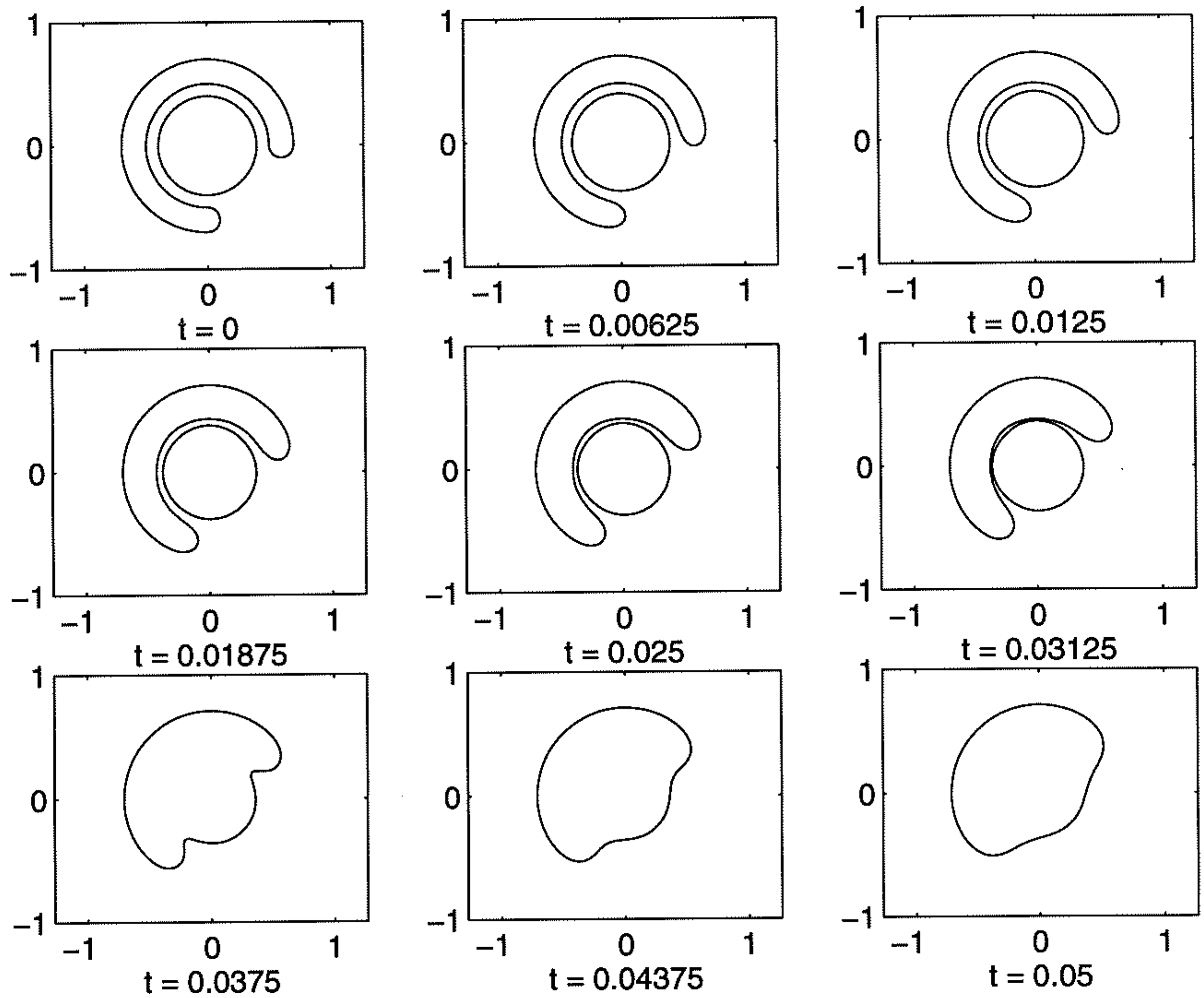

Figure 8: Area preserving curvature dependent velocity: merging case, relaxes towards circle

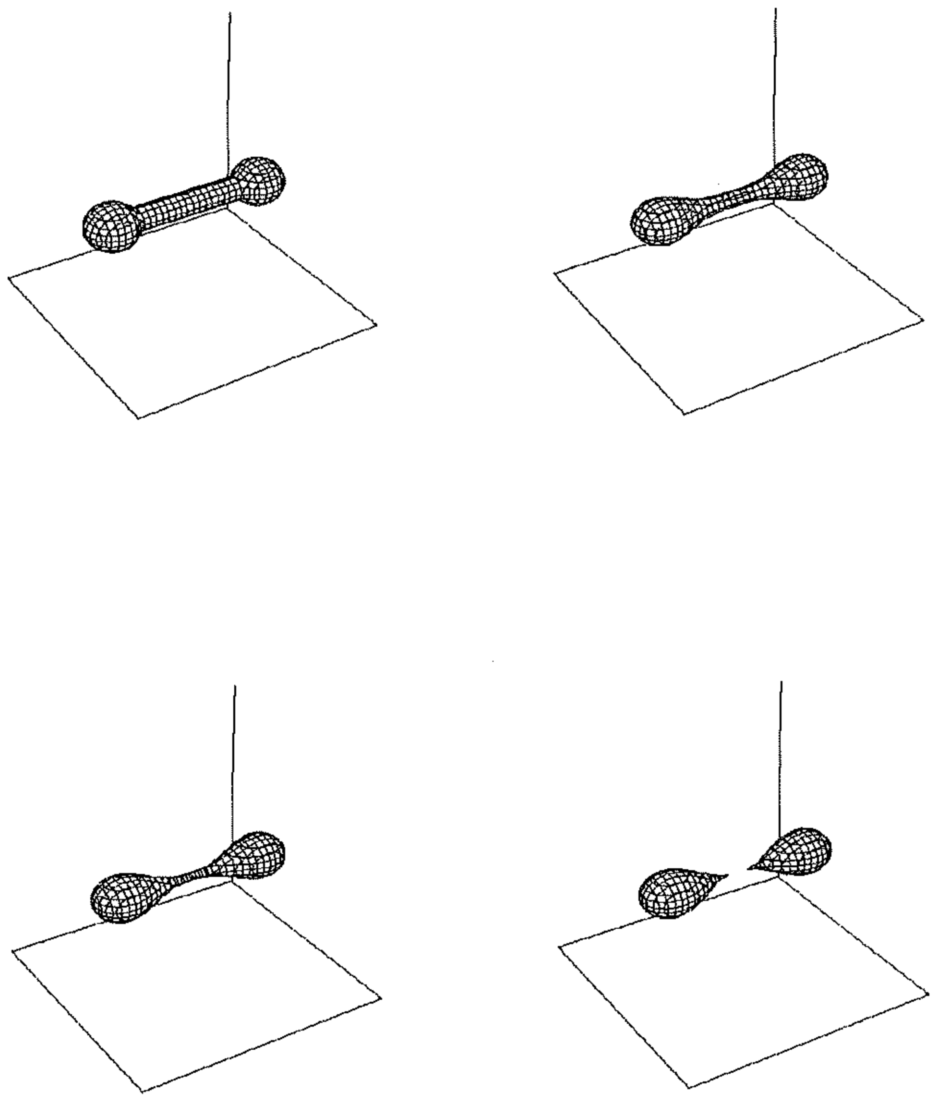

Figure 9: Three dimensional volume preserving curvature dependent velocity: symmetric pinchoff, relaxes towards two spheres

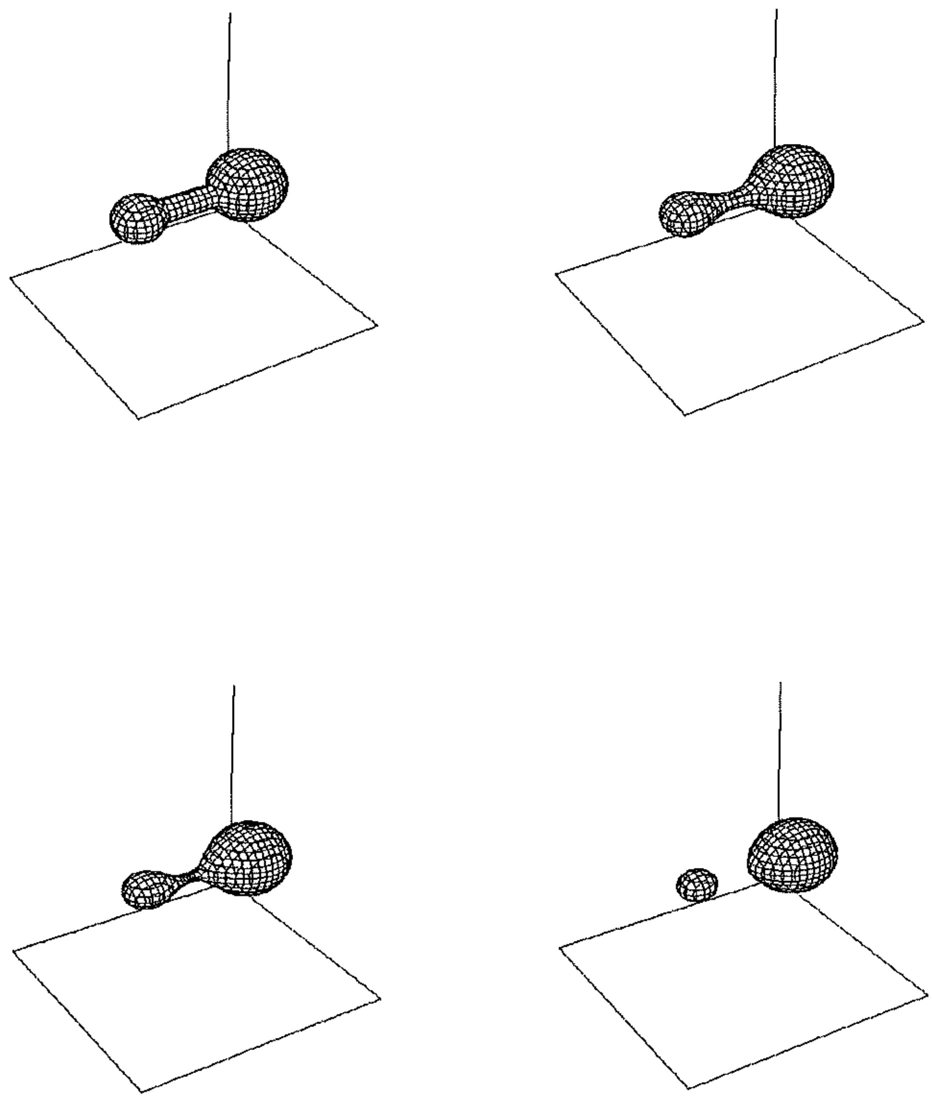

Figure 10: Three dimensional volume preserving curvature dependent velocity: non-symmetric pinchoff, relaxes towards two spheres

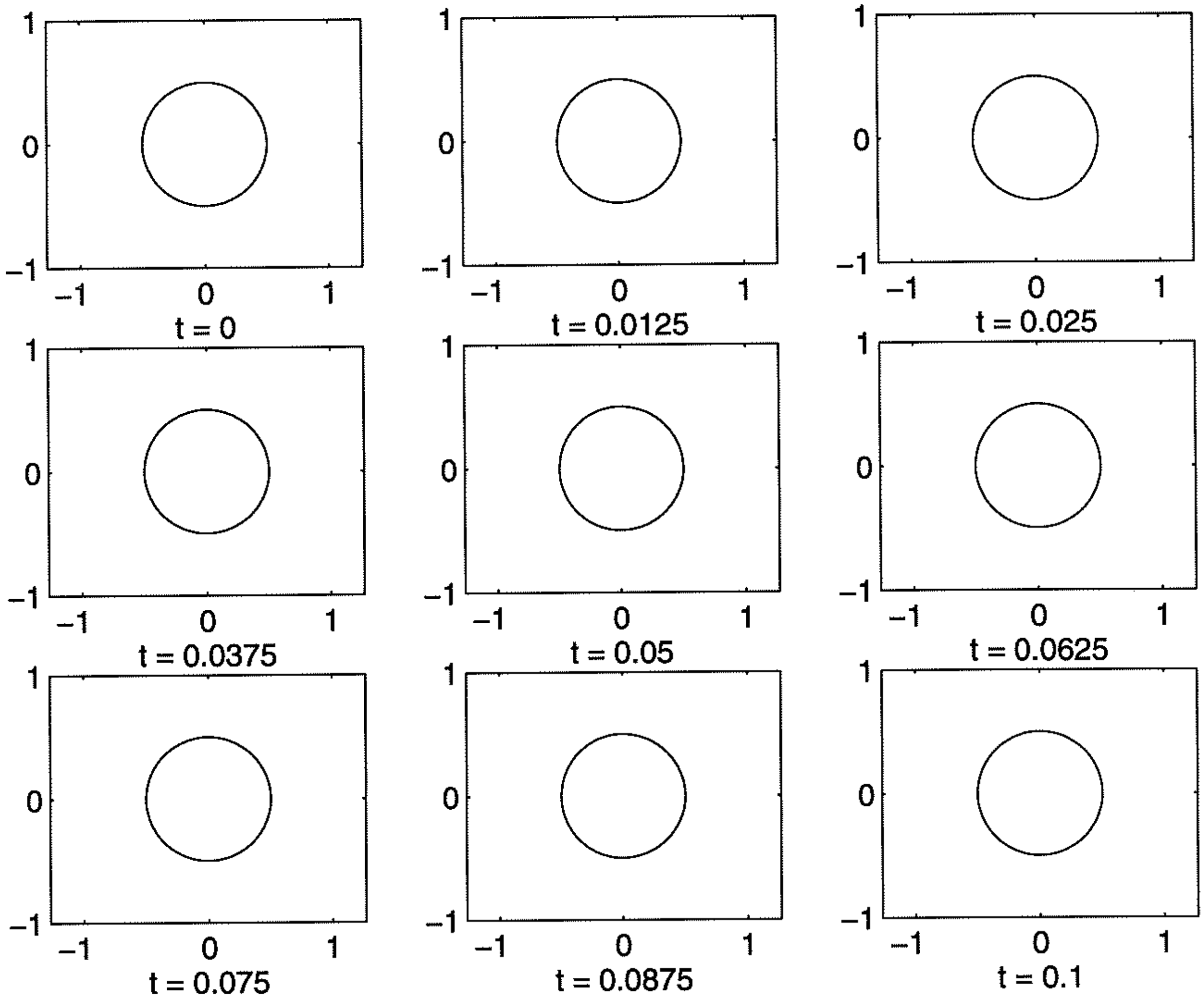

Figure 11: Curvature dependent acceleration(volume preserving): circle remains circle

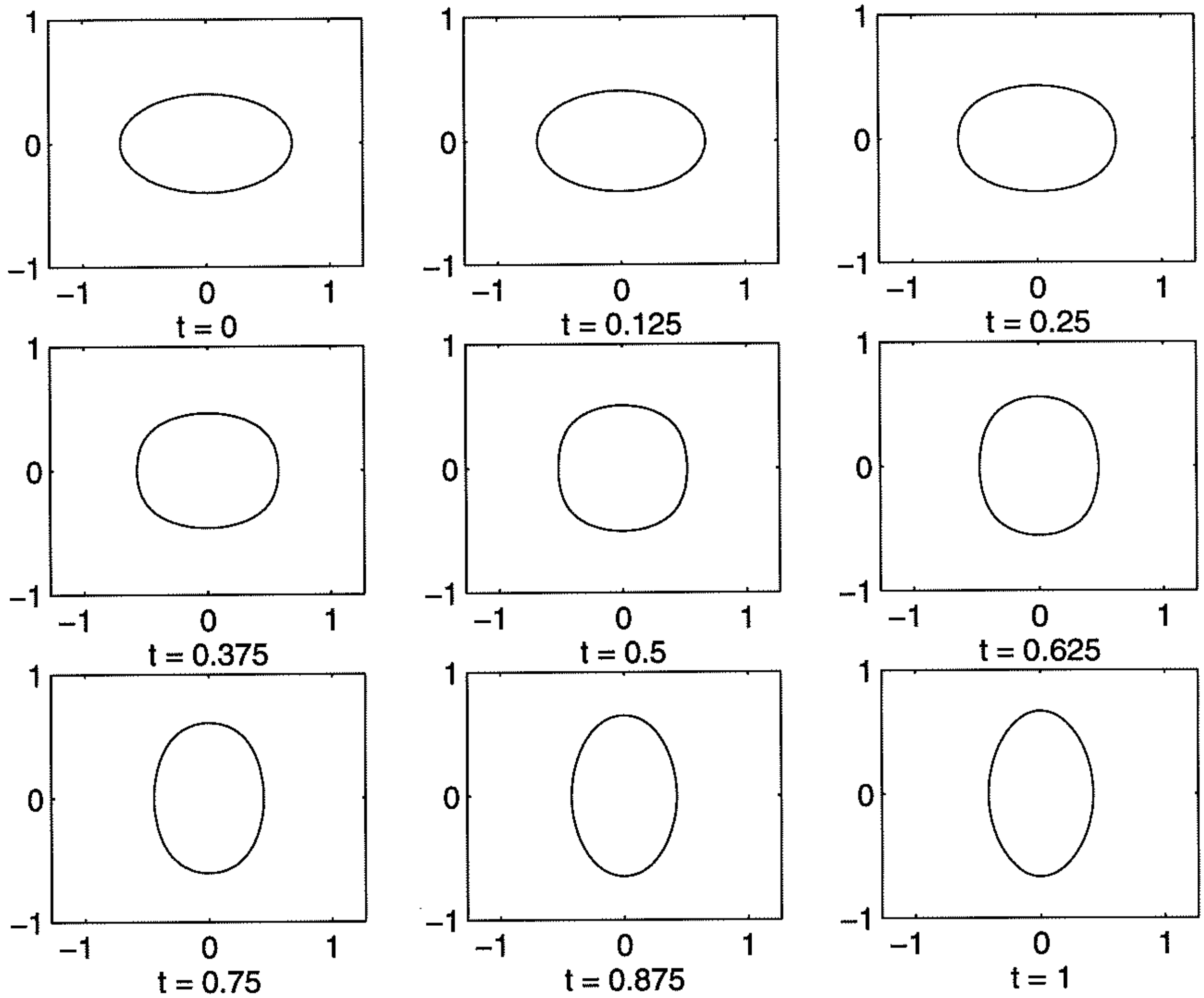

Figure 12: Curvature dependent acceleration(volume preserving): oscillating ellipse

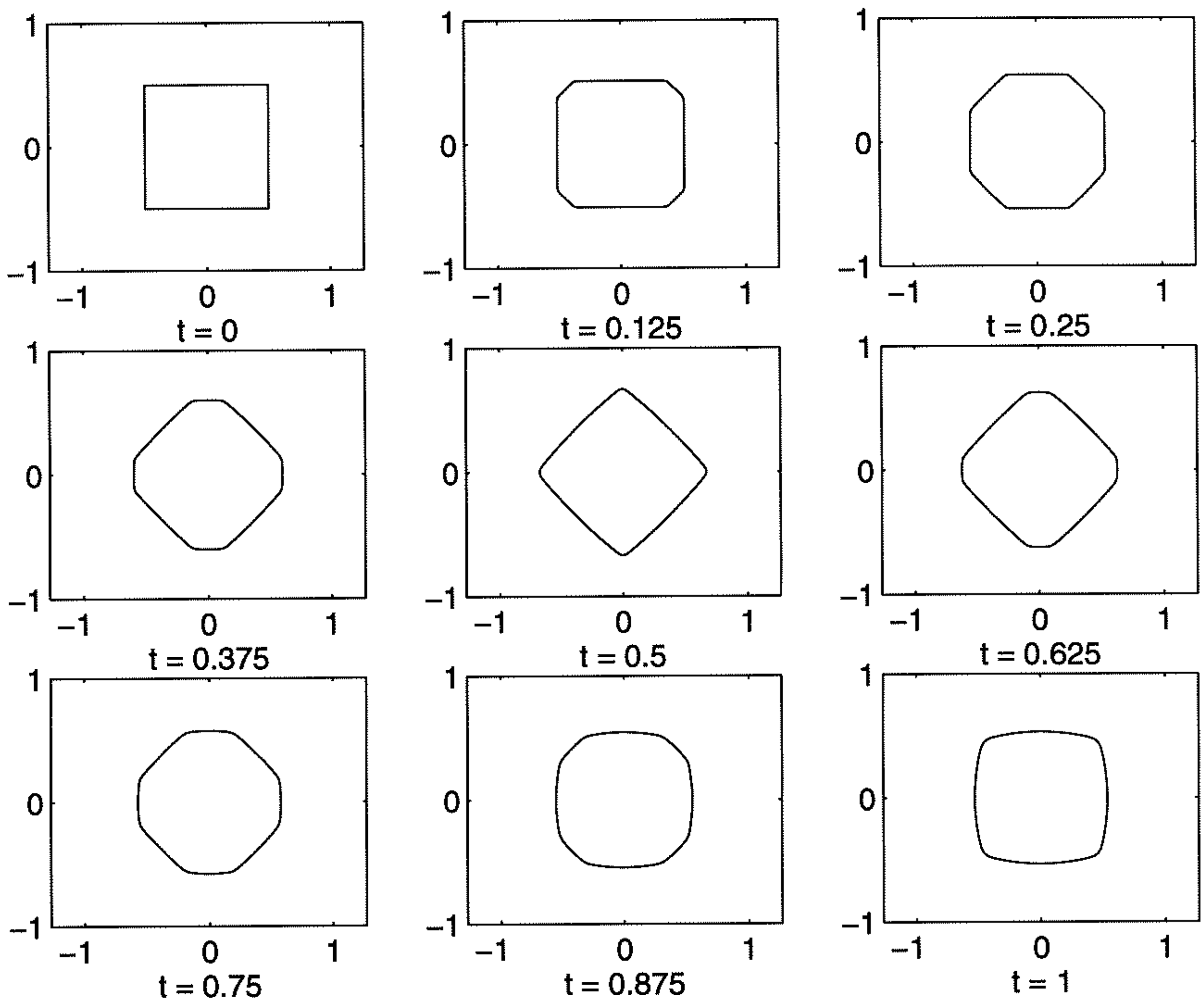

Figure 13: Curvature dependent acceleration(volume preserving): oscillating square

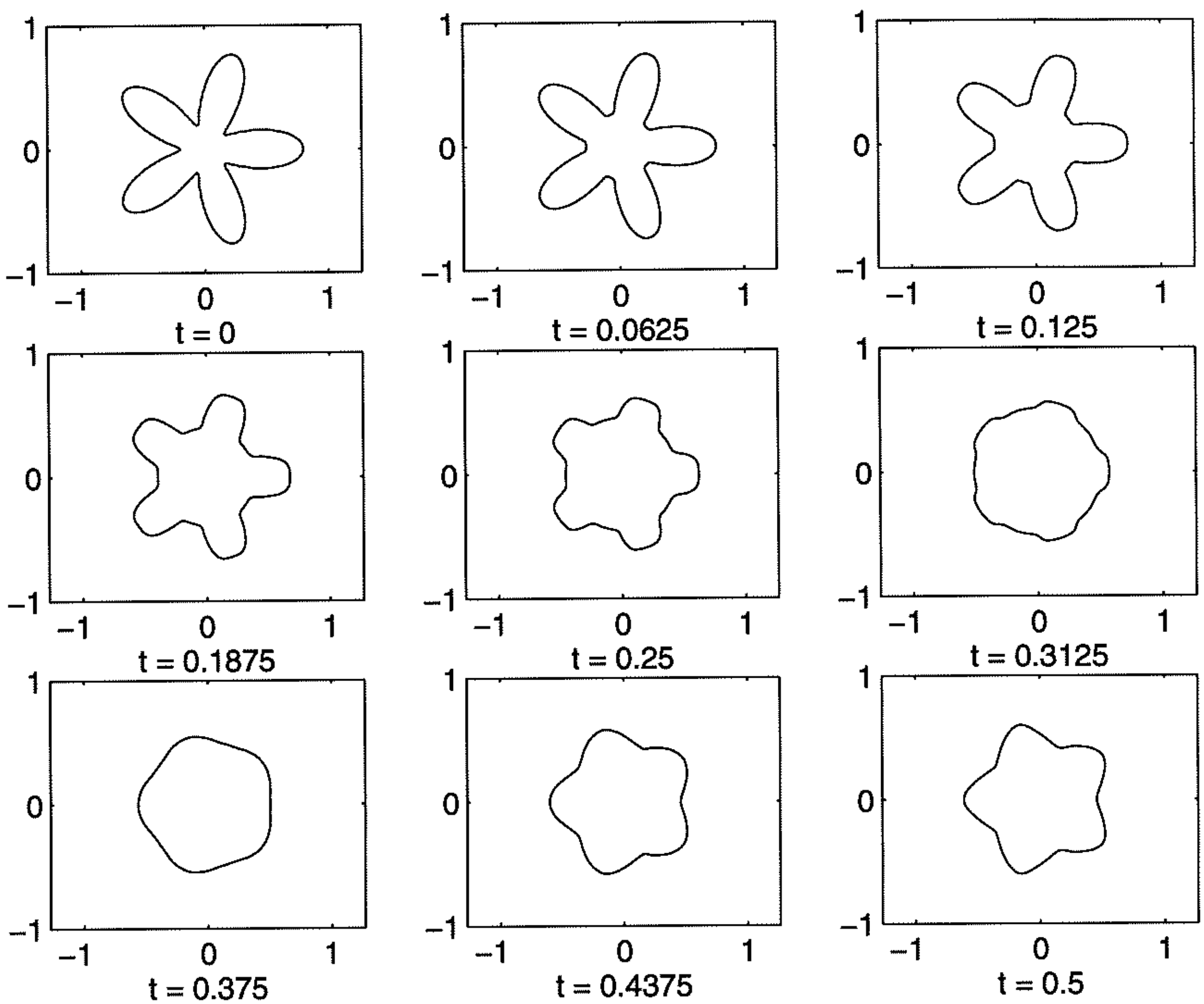

Figure 14: Curvature dependent acceleration(volume preserving): oscillating starfish

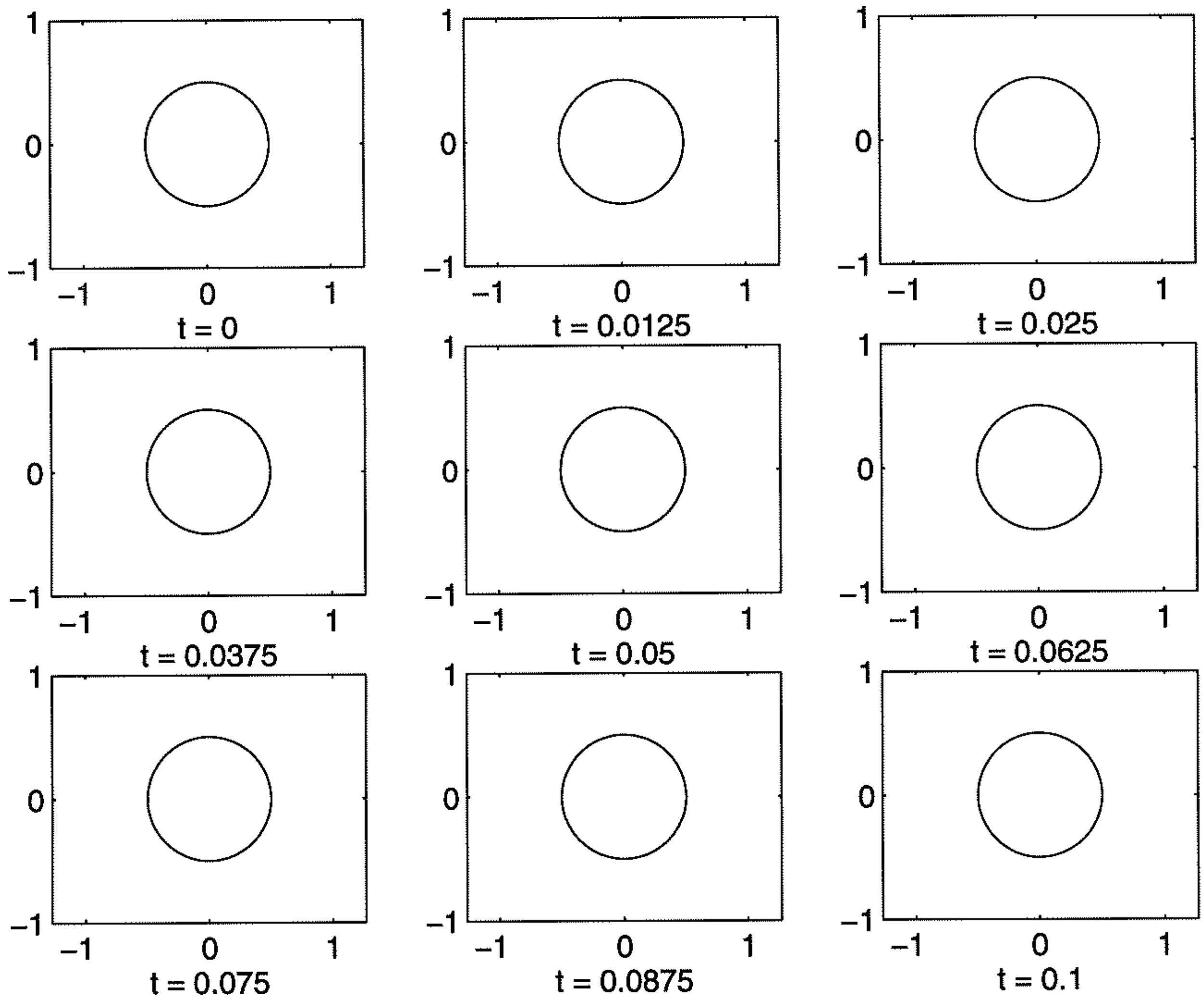

Figure 15: Curvature dependent acceleration(volume and momentum preserving): circle remains circle

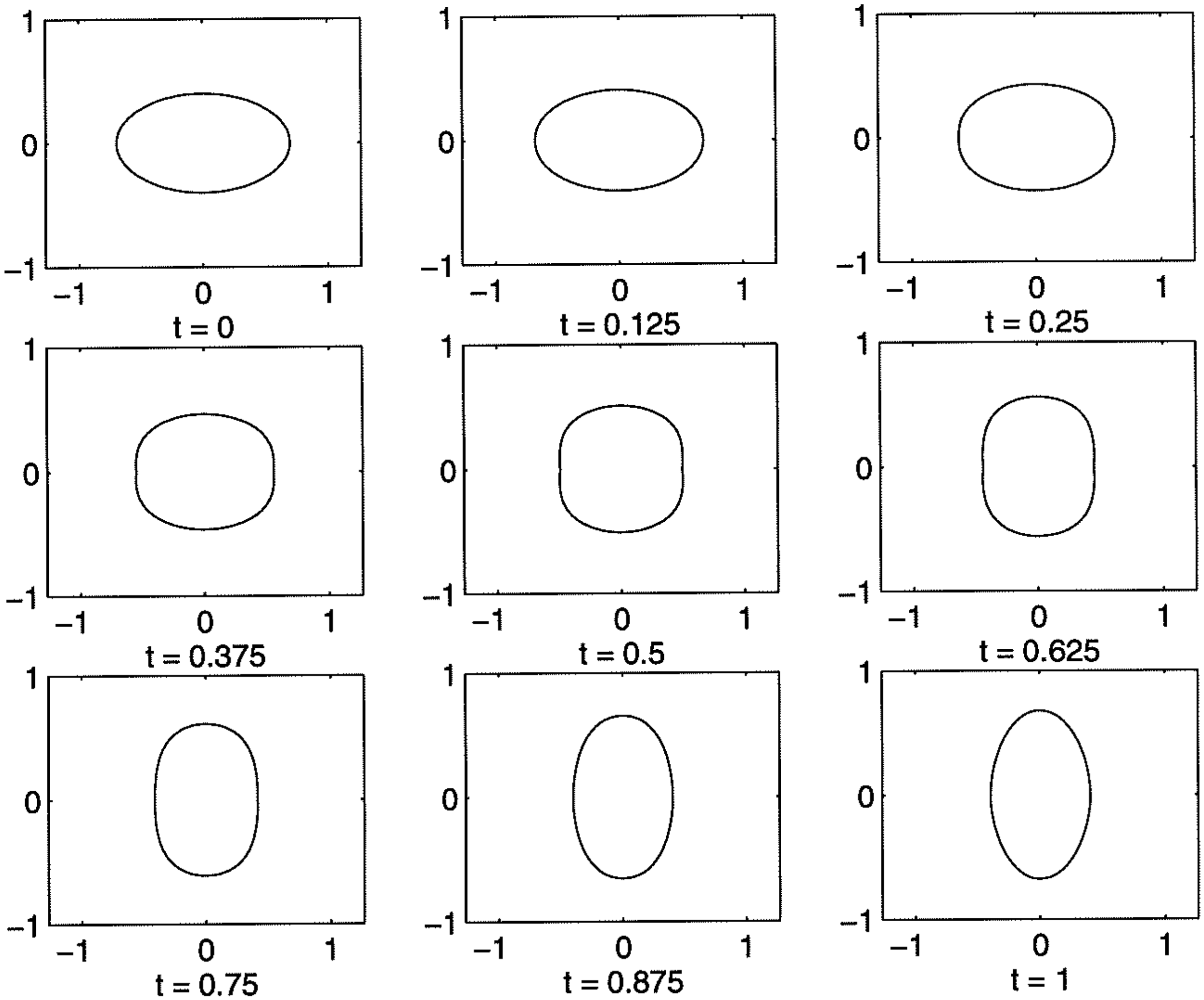

Figure 16: Curvature dependent acceleration(volume and momentum preserving): oscillating ellipse

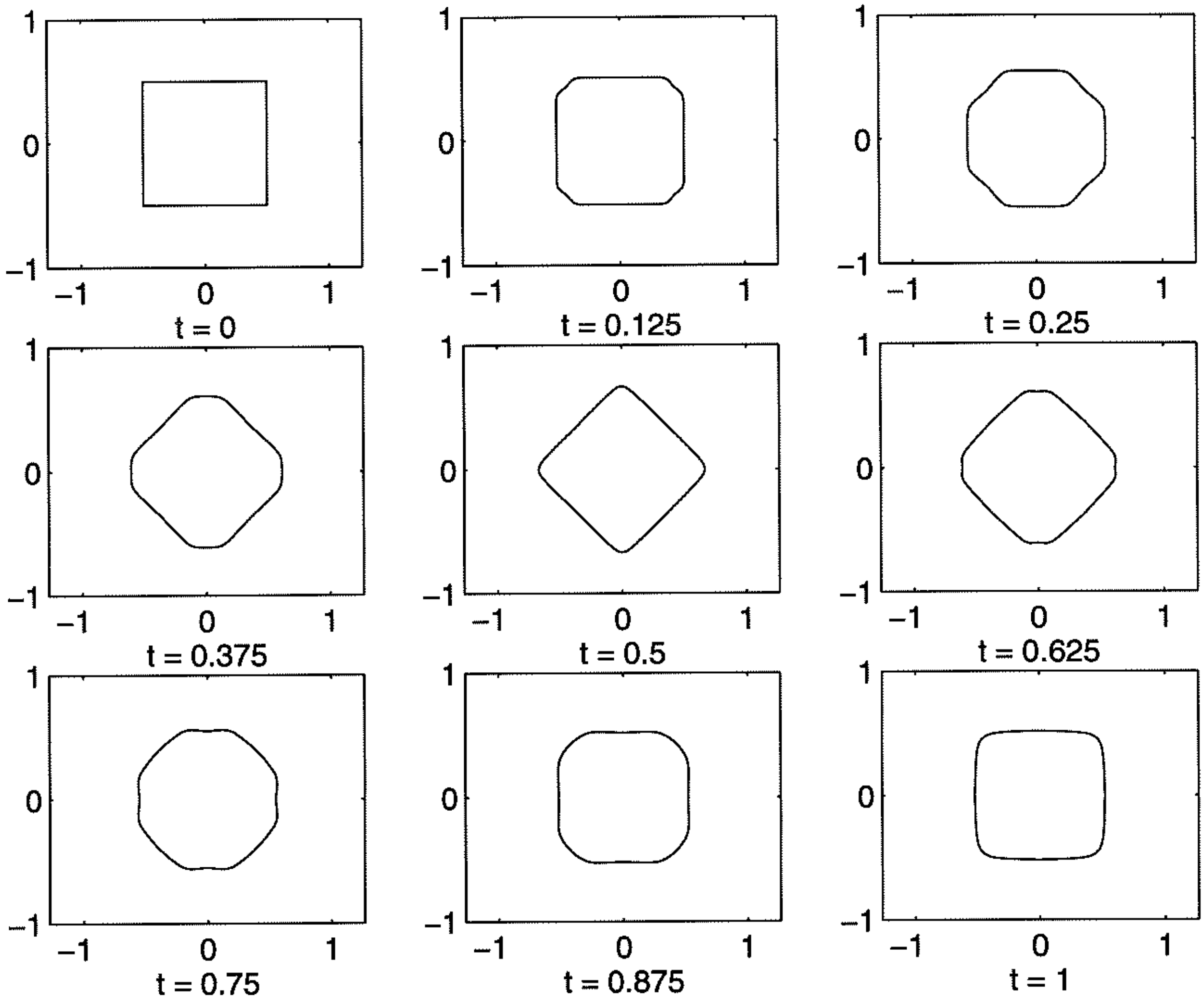

Figure 17: Curvature dependent acceleration(volume and momentum preserving): oscillating square

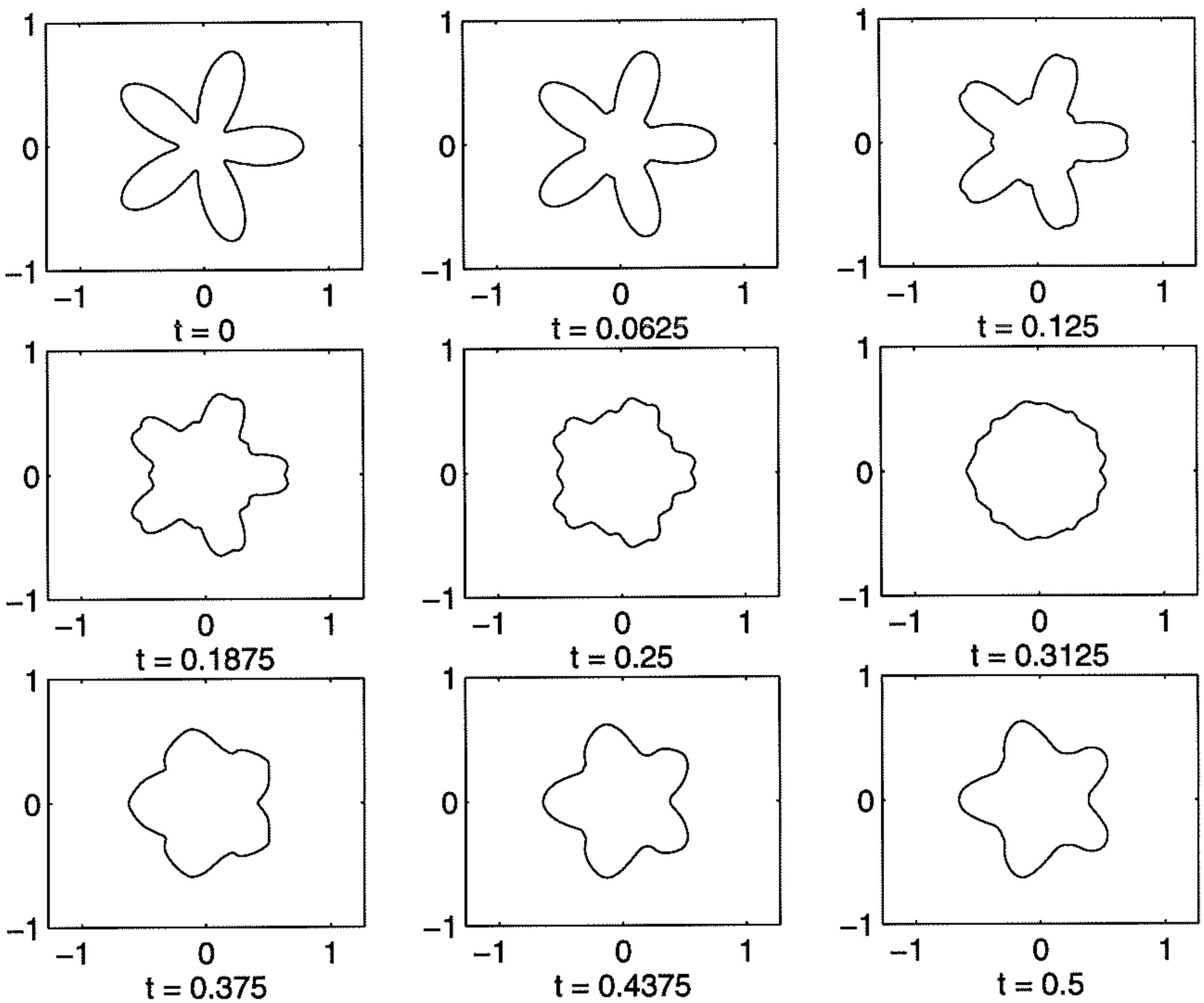

Figure 18: Curvature dependent acceleration(volume and momentum preserving): oscillating starfish

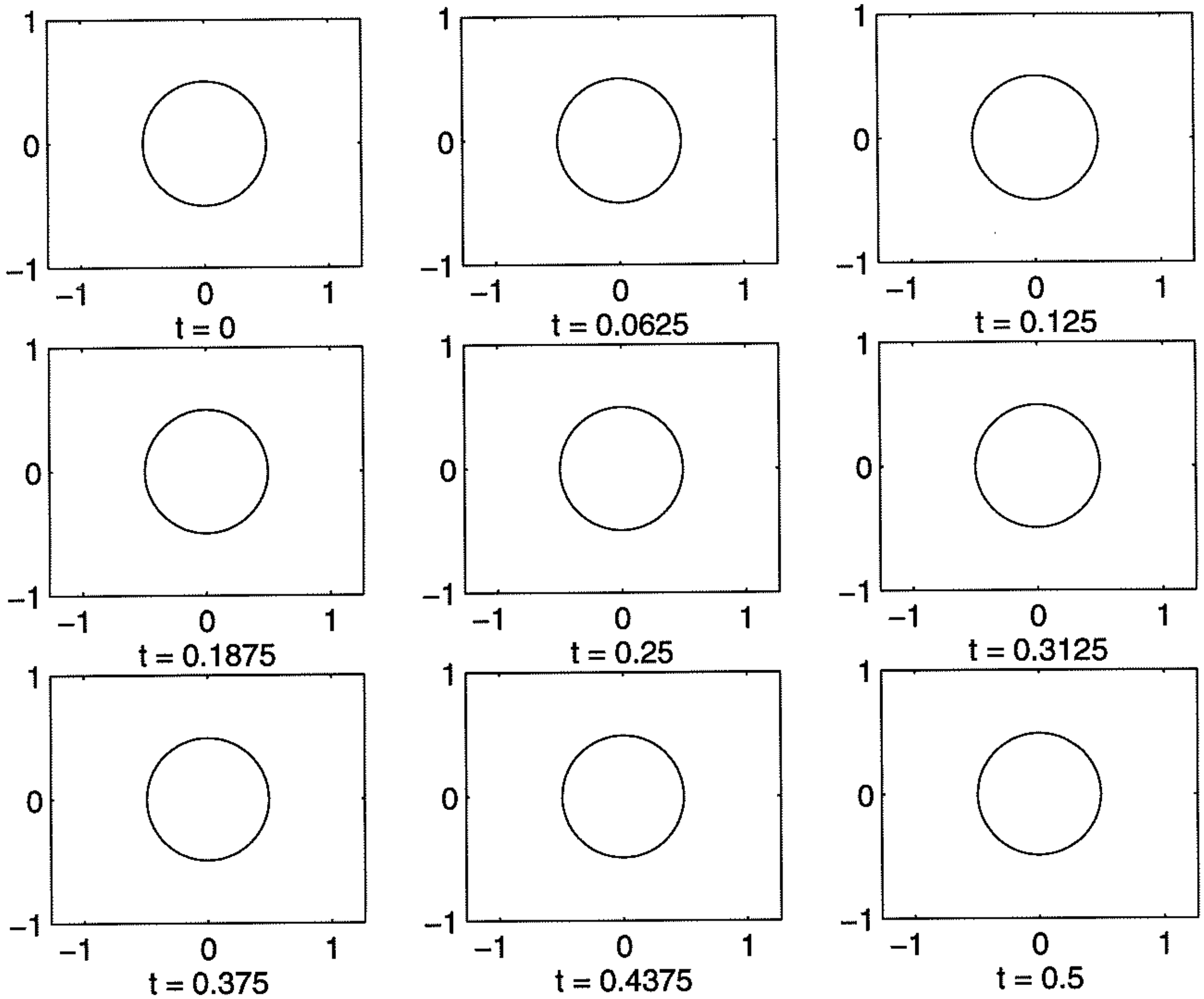

Figure 19: Curvature dependent acceleration(fluid bubble): circle remains circle

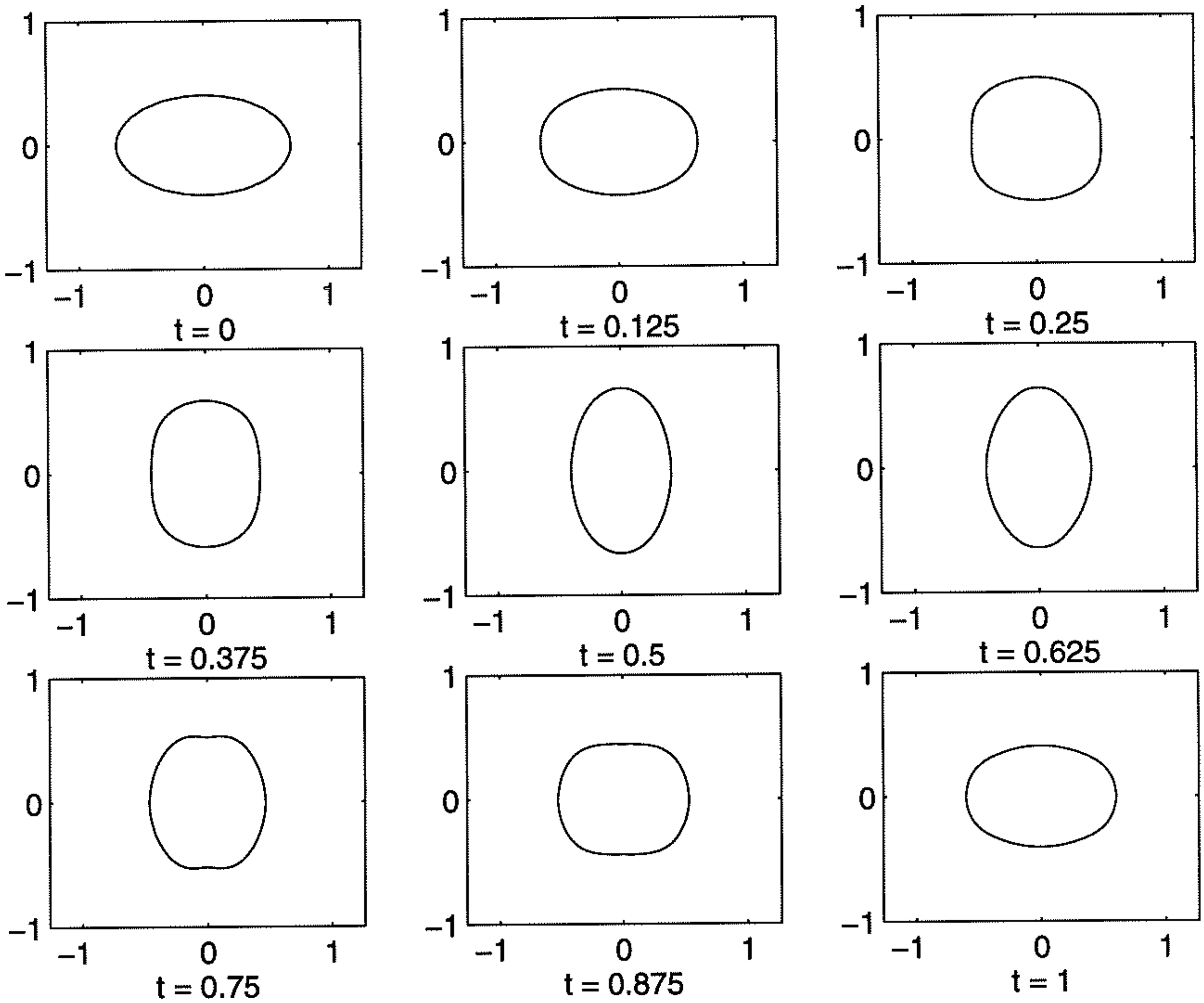

Figure 20: Curvature dependent acceleration(fluid bubble): oscillating ellipse

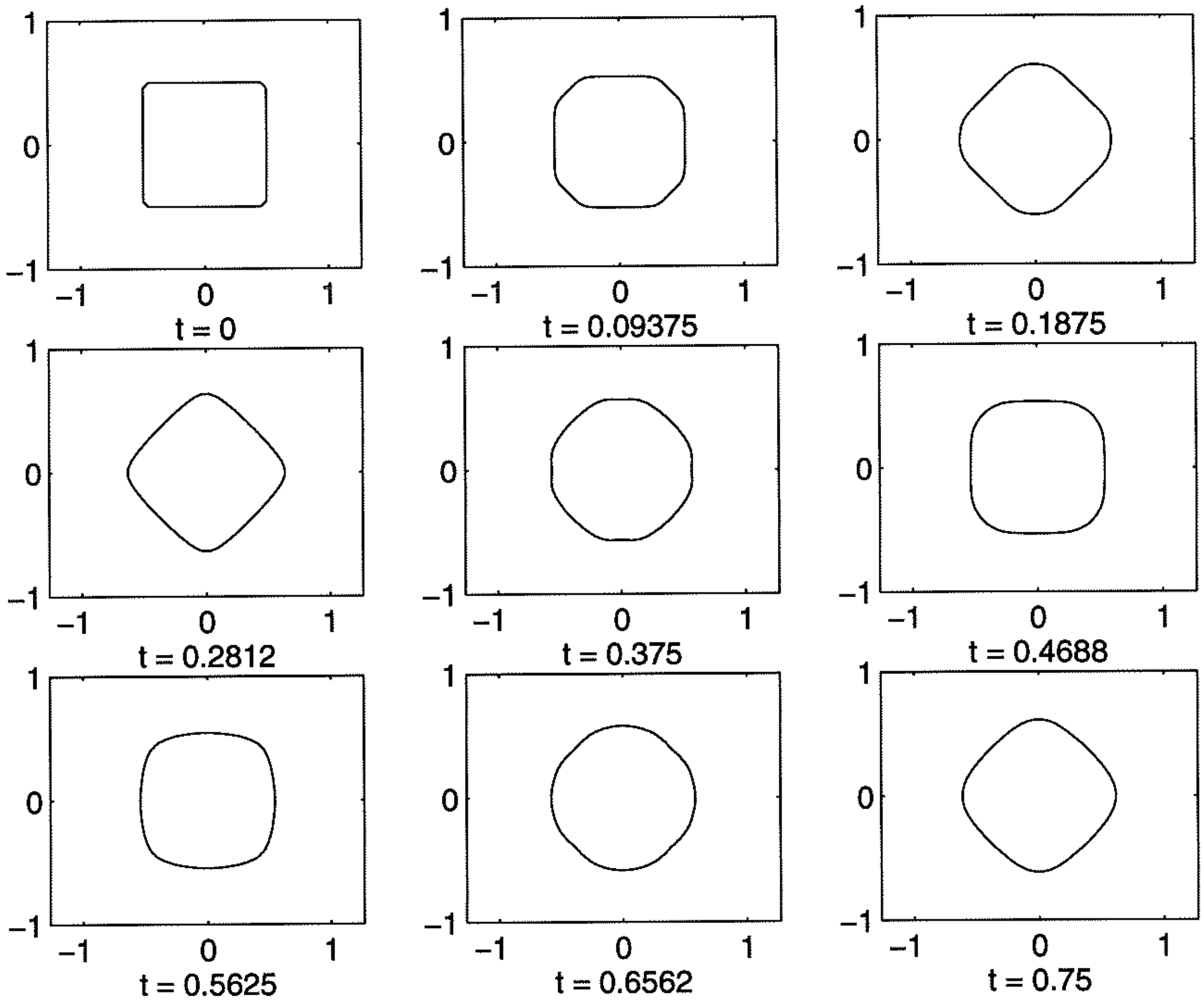

Figure 21: Curvature dependent acceleration(fluid bubble): oscillating square

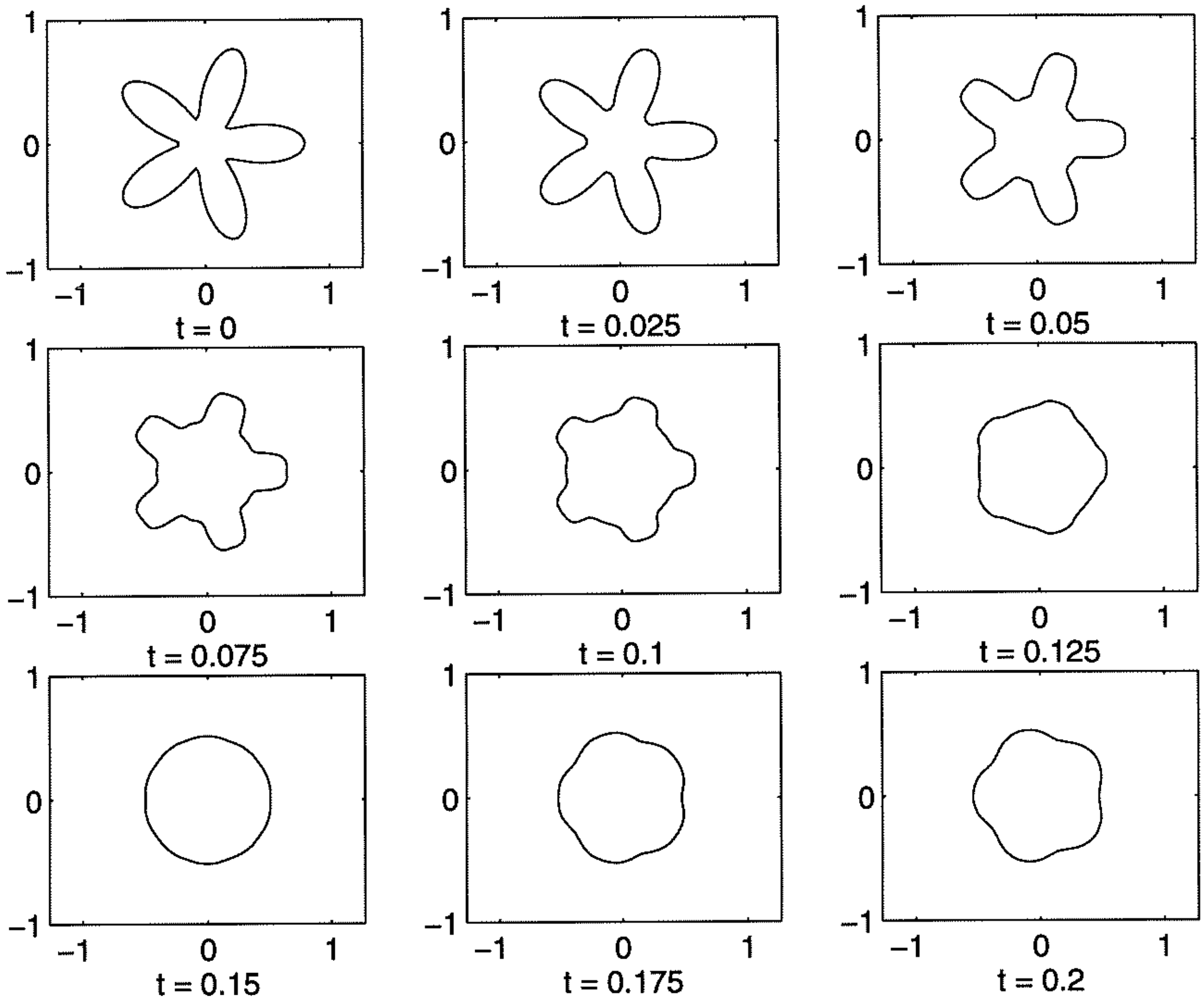

Figure 22: Curvature dependent acceleration(fluid bubble): oscillating starfish

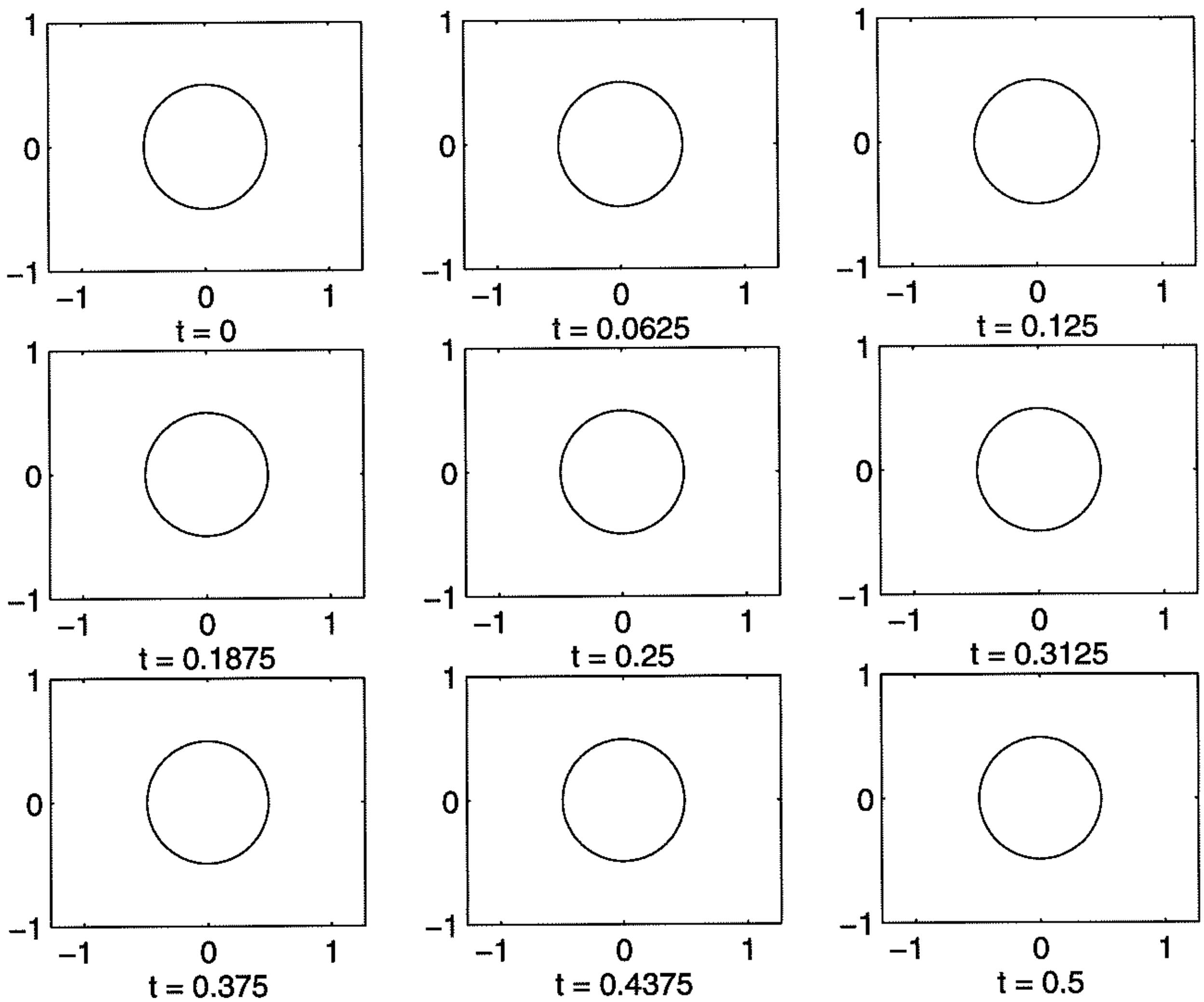

Figure 23: Curvature dependent acceleration(elastic membrane bubble): circle remains circle

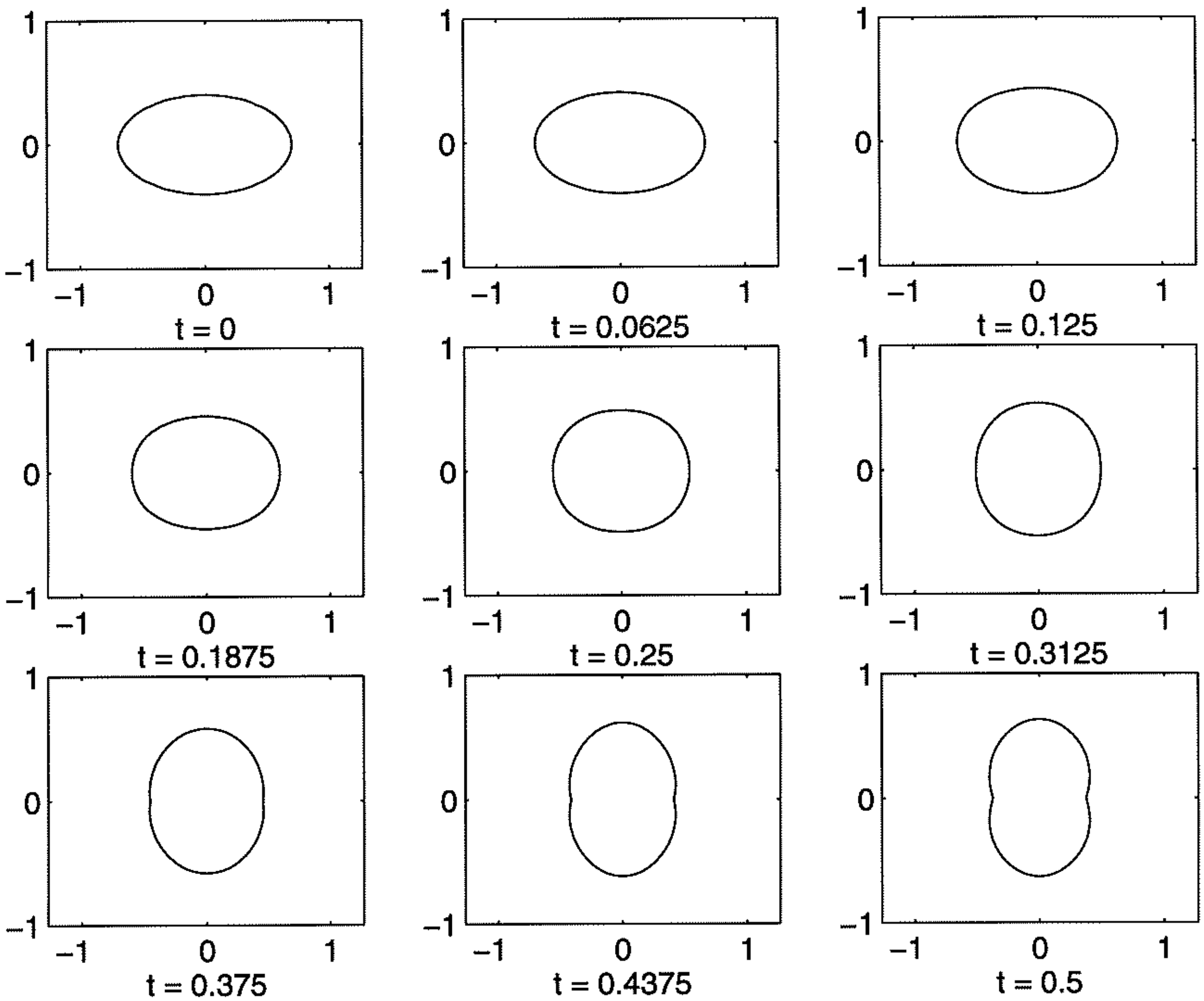

Figure 24: Curvature dependent acceleration(elastic membrane bubble): oscillating ellipse

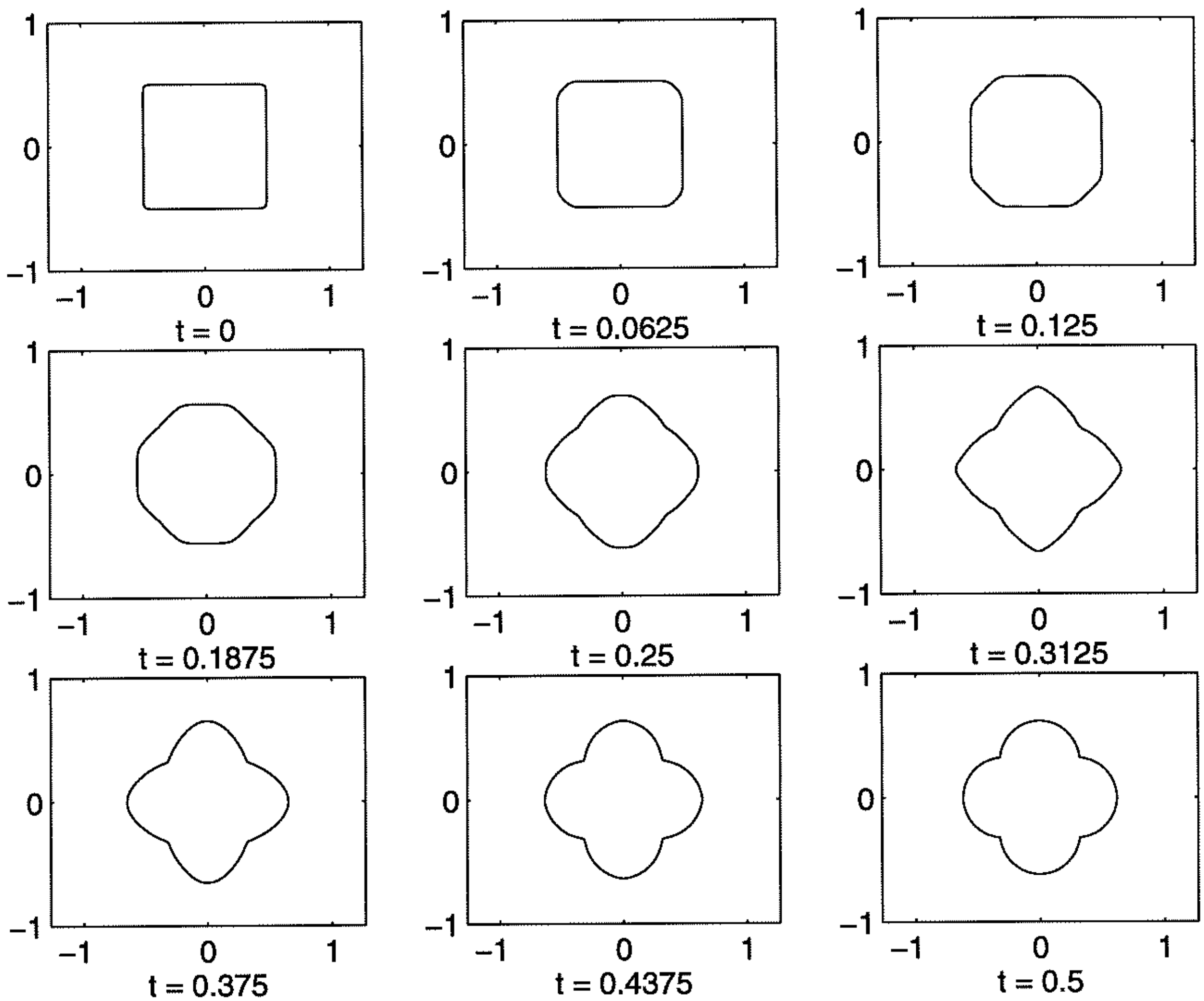

Figure 25: Curvature dependent acceleration(elastic membrane bubble): oscillating square

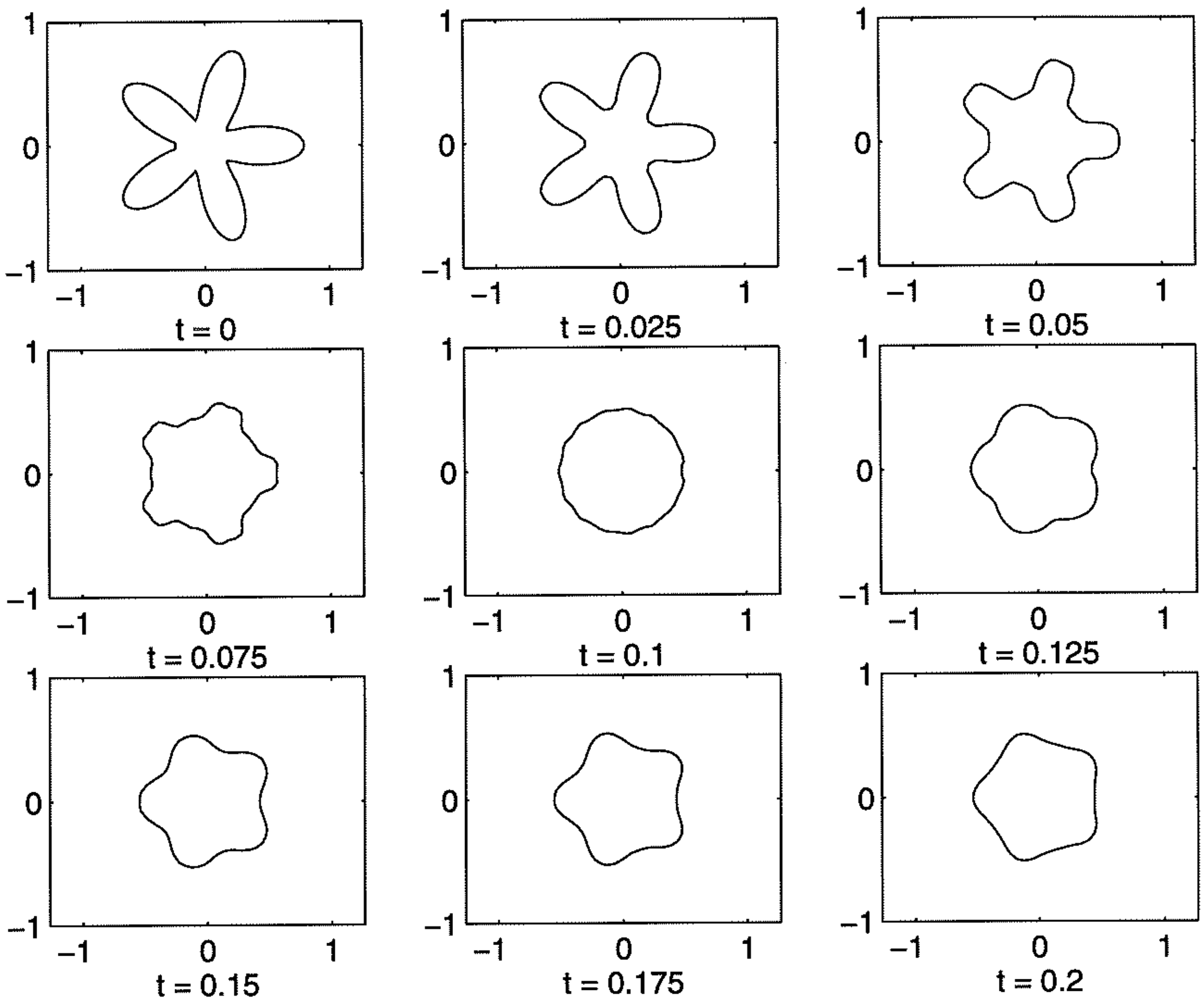

Figure 26: Curvature dependent acceleration(elastic membrane bubble): oscillating starfish