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Valuation of Mortgage Backed Securities Using the Quasi-Monte Carlo Method

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Abstract

The quasi-Monte Carlo method for financial valuation and other integration problems has error bounds of size $O((\log N)^k N^{-1})$, which suggests significantly better performance than the error size $O(N^{-1/2})$ for standard Monte Carlo. We present a brief introduction to quasi-random sequences, which are deterministic sequences that are more uniform than random sequences. Through computations of the discrepancy (a measurement of uniformity) and two-dimensional projections of the sequences, we show that the improved uniformity is decreased as the dimension increases. For the valuation of mortgage-backed securities, which is nominally of dimension 360, we apply the Brownian bridge representation to reduce the effective dimension. This allows for the effective application of quasi-Monte Carlo, in particular in conjunction with antithetic variance reduction. The results show an error reduction by a factor of 100 or more. A Taylor expansion of the integrand is used to interpret the success of the various methods applied to the problem.

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1 Introduction

Monte Carlo is often the only effective numerical method for the accurate valuation of securities whose value depends on the whole trajectory of interest rates or other variables. The standard Monte Carlo method using pseudo-random sequences can be quite slow, however, because its convergence rate is only $O(N^{-1/2})$ for N sample paths. Quasi-Monte Carlo methods, using deterministic sequences that are more uniform than random, can be much faster with errors approaching size $O(N^{-1})$ in optimal cases.

This dramatic improvement in convergence rate has the potential for significant gains both in computational time and in range of application of Monte Carlo methods for finance problems. This improvement has been noted by a number of earlier studies [1, 8, 10], which were all motivated by the results of Paskov [9] on mortgage backed securities.

The effectiveness of quasi-Monte Carlo methods does have some important limitations. First, quasi-Monte Carlo methods are valid for integration problems, but may not be directly applicable to simulations, due to the correlations between the points of a quasi-random sequence. This problem can be overcome in many cases by writing the desired result of a simulation as an integral, but the resulting integral is often of very high dimension (e.g. dimension 360 for a mortgage of length 360 years).

This leads to a second limitation: The improved accuracy of quasi-Monte Carlo methods is generally lost for problems of high dimension or problems in which the integrand is not smooth. This loss of effectiveness has been documented for a series of test problems in [3, 4, 5]. Several researchers in computational finance have recently reported great success with quasi-Monte Carlo computation of problems of very high dimension. One purpose of our presentation is to show that, at least for some of these results, the problems are actually of moderate dimension when cast in the proper form.

A third limitation of quasi-Monte Carlo methods is that there is no theoretical basis for empirical estimates of their accuracy, as provided by the central limit theorem for standard Monte Carlo. In practice this is not a major difficulty, since confidence in the computational results can come through repeated trials.

While quasi-Monte Carlo may be very successful on relatively simple problems, obtaining the same success on more complicated applications can be difficult due to these limitations. In this study we show how to overcome some of these limitations for the the mortgage backed security problem by a reformulation of the Monte Carlo representation so that the effective dimension is of moderate size. One of our main conclusions is that the range of application of quasi-Monte Carlo methods can be significantly extended by modification of the standard Monte Carlo techniques.

The outline of this paper is the following: Section 2 gives a brief introduction to quasi-random sequences and their properties, including the Koksma-Hlawka inequality which is the basic estimate on integration error for quasi-Monte Carlo. The dependence on dimension for the properties of quasi-random sequences is described in Section 3, and the character of two-dimensional projections of quasi-random sequences is discussed in Section 4. The mortgage-backed security problem is formulated in Section 5. Our main technical tool for formulating the problem with reduced effective dimension is the Brownian bridge representation of a random walk, which is described in Section 6. Computational results for the mortgage-backed security problem are presented in Section 7 along with an interpretation of these results in terms of a Taylor expansion of the integrand. Conclusions are discussed in Section 8.

We are grateful to Spassimir Paskov and Joseph Traub for a number of discussions and for providing us with their quasi-random number generator FINDER. We are grateful to Art Owen for the observation that the first mortgage-backed security problem considered here is nearly linear.

2 Quasi-Random Sequences, Discrepancy and Integration Error

The origin of the improved accuracy of quasi-Monte Carlo methods is the improved uniformity of quasi-random sequences. Figure 1 shows two plots, each of 4096 points in two dimensions. The top is a pseudo-random sequence and the bottom is a quasi-random (Sobol') sequence. In the pseudo-random sequence there is clumping of points, which limits their uniformity. The cause of this clumping is that, since points in a pseudo-random sequence are (nearly) independent, they have a certain chance of landing very near to each other. The correlation of points in a quasi-random sequence, on the other hand, prevents them from clumping together.

The uniformity of a sequence of points in the s-dimensional unit cube is measured in terms of its *discrepancy.* This is defined by considering the number of points in rectangular subsets of the cube. For a set $J \subseteq I^s$ and a sequence of N points $\{x_n\}$ in I^s, define

$$R_N(J) = \frac{1}{N}\sum_{n=1}^{N} \mathcal{X}_J(x_n) - m(J).$$

Here $\mathcal{X}_J$ is the characteristic function of the set J, and $m(J)$ is its volume. Various kinds of discrepancy can be defined. If E is the set of all subrectangles of I^s, then the L_∞ and L_2 norms are defined as:

$$D_N = \sup_{J \in E} \mid R_N(J) \mid \tag{2.1}$$

$$T_N = [\int_{(x,y)\in I^{2s}, x_i<y_i} (R_N(J(x,y)))^2 \, dx \, dy]^{\frac{1}{2}} \tag{2.2}$$

Here $J(x,y)$ indicates the rectangle with opposite corners at (x,y). If E^* is the set of subrectangles with one corner at $\mathbf{0}$, then the star discrepancies are defined as:

$$D_N^* = \sup_{J \in E^*} \mid R_N(J) \mid \tag{2.3}$$

$$T_N^* = [\int_{I^s} (R_N(J(x)))^2 \, dx]^{\frac{1}{2}} \tag{2.4}$$

Here $J(x)$ is the rectangle with a corner at $\mathbf{0}$ and a corner at x.

Some improvements on these definitions have been made by Hickernell [2] who included the discrepancy over the sides of the unit cube as well.

The importance of discrepancy as an error bound for Monte Carlo integration can be seen from the Koksma-Hlawka inequality for integration error. For the integral of a function f on the s-dimensional unit cube, the Monte Carlo estimate of the integral is

$$I_N(f) = \frac{1}{N}\sum_{n=1}^{N} f(x_n) \tag{2.5}$$

and the Monte Carlo integration error is

$$\varepsilon(f) = \left| \int_{I^s} f(x)\, dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right|. \tag{2.6}$$

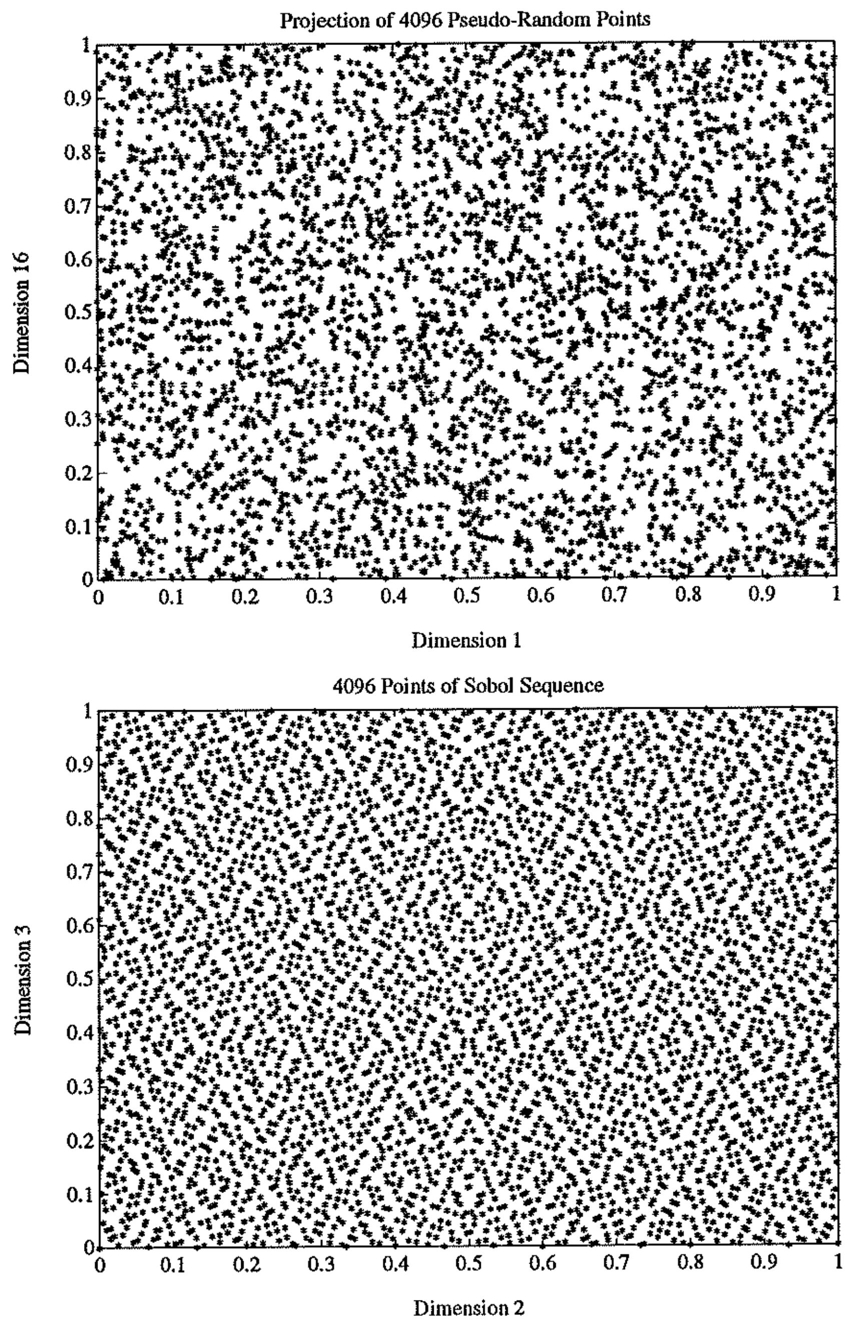

Figure 1: 2-D Projection of a Pseudo-Random Sequence (top) and a Sobol' Sequence (bottom)

The Koksma-Hlawka inequality says that

$$\varepsilon(f) \leq V(f)\, D_N^* \tag{2.7}$$

where D_N^* is the discrepancy of the sequence $\{x_n\}$ and $V(f)$ is the variation of f.

In one dimension, the variation is $V(f) = \int_0^1 |df|$. The definition in higher dimension is more complicated. Define for all $k \leq s$ and all sets of k integers $1 \leq i_1 < i_2 < \ldots < i_k \leq s$ the quantity

$$V^{(k)}(f; i_1, \ldots, i_k) = \int_{I^k} \left| \frac{\partial^k f}{\partial t_{i_1} \cdots \partial t_{i_k}} \right|_{t_j=1, j\neq i_1,\ldots,i_k} dt_{i_1} \cdots dt_{i_k} \, .$$

The variation of f (in the sense of Hardy and Krause) is defined as

$$V(f) = \sum_{k=1}^{s} \sum_{1\leq i_1<i_2<\ldots<i_k\leq s} V^{(k)}(f; i_1, \ldots, i_k) \, .$$

The Koksma-Hlawka inequality (2.7) should be compared with the formula for root-mean-square error of Monte Carlo integration using a random sequence. If the sequence x_n is uniformly distributed on I^s, then

$$E[\varepsilon(f)^2]^{1/2} = \sigma(f) N^{-1/2} \tag{2.8}$$

in which E is the usual expectation and $\sigma(f)$ is the square root of the variance of f given by

$$\sigma(f) = \left(\int_{I^s} (f(x) - \bar{f})^2 dx \right)^{1/2} \tag{2.9}$$

with $\bar{f} = \int_{I^s} f dx$.

The inequalities (2.7) and (2.9) are similar in that the bound is a product of one term depending on properties of the integrand function and a second term depending on properties of the sequence. They differ in that the Koksma-Hlawka inequality is an absolute bound, whereas (2.9) is an equality in expectation, and thus holds only probabilistically.

The infinite sequence $\{x_n\}$ is said to be *uniformly distributed* if $\lim_{N\to\infty} D_N = 0$, where D_N refers to the discrepancy of the first N terms of the sequence. The sequence is said to be *quasi-random* if

$$D_N \leq c(\log N)^k N^{-1} \tag{2.10}$$

in which the constant c and the logarithmic exponent k may depend on the dimension s. For Monte Carlo integration using quasi-random sequences, the Koksma-Hlawka inequality say that the integration error is size $O((\log N)^k N^{-1})$, which for large N is much more accurate than standard Monte Carlo using random or pseudo-random sequences.

The simplest example of a quasi-random sequence is the van der Corput sequence in one dimension which is formed as follows: To obtain the n-th term x_n, write the number n in base 2 as

$$n = a_m a_{m-1} \dots a_1 a_0 \quad (\text{base } 2). \tag{2.11}$$

Then transpose these digits around the "decimal" point to get

$$x_n = .a_0 a_2 \dots a_{m-1} a_m \quad (\text{base } 2). \tag{2.12}$$

Additional examples of quasi-random sequences have been constructed by Halton, Faure, Sobol', Niederreiter and others. For a comprehensive review see the monograph of Niederreiter [7].

Although the variation of f requires s derivatives of f, we have found in practice that only a minimal amount of smoothness of f is needed for effectiveness of quasi-Monte Carlo integration. For problems in which f is discontinuous, however, the improvements of quasi-Monte Carlo integration are diminished.

3 Dependence on Dimension

It is possible to derive an exact formula for T_N for any given sequence $\{\boldsymbol{a}_n\}$ of N terms (the notation for a sequence is changed here from $\boldsymbol{x}$ to $\boldsymbol{a}$ to help distinguish between the terms of the sequence and the points defining the rectangles). Use the Heaviside function

$$\theta(y) = \begin{cases} 1, & y > 0 \\ 0, & y \leq 0 \end{cases} ,$$

to rewrite R_N as

$$R_N(J(\boldsymbol{y}, \boldsymbol{z})) = \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{s} \theta(z_i - a_{n,i}) \cdot \theta(a_{n,i} - y_i) - \prod_{i=1}^{s} (z_i - y_i) \, .$$

Squaring this quantity and integrating over the domain described above leads to T_N^2, which can be expressed as

$$
\begin{aligned}
(T_N)^2 &= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}\prod_{i=1}^{s}[1-\max(a_{n,i},a_{m,i})]\cdot\min(a_{n,i},a_{m,i}) \\
&\quad -\frac{2^{-s+1}}{N}\sum_{n=1}^{N}\prod_{i=1}^{s}a_{n,i}(1-a_{n,i})+12^{-s}\,.
\end{aligned}
$$

Hickernell [2] found a similar formula for the discrepancy including the discrepancy on sides of the cube.

As an example, one can compute the expected value of this quantity for a random sequence. This root mean square (rms) expectation of T_N is given by

$$
\begin{aligned}
E(T_N^2) &= \int_{I^{sN}} T_N^2 \prod_{n=1}^{N}\prod_{i=1}^{s} da_{n,i} \\
&= \frac{1}{N}6^{-s}(1-2^{-s})
\end{aligned}
$$

Thus the average L_2 discrepancy of a random sequence decreases like $N^{-\frac{1}{2}}$, corresponding nicely with the random Monte Carlo bound.

Figures 2 shows plots of T_N (solid line) on a log log base 2 scale. The rms expectation of T_N for a random sequence (dashed line) is also plotted, along with the function $1/N$ (dotted line) for reference.

An important observation to be made from these plots is that for small N the discrepancy of a quasi-random sequence is almost exactly the same as that of a random sequence, with size $O(N^{-1/2})$. Only when N gets rather large does the discrepancy decay like $O(N^{-1})$, as predicted (neglecting $\log N$ terms, which cannot be accurately detected on this plots). The transition value of N appears to be grow exponentially with the dimension, but there is no theoretical explanation of this behavior. This indicates that in high dimensions, unless one uses a very large number of points, quasi-random sequences are no more uniform than random sequences. Thus the advantages of quasi-Monte Carlo are lost for moderate values of N if the dimension s is sufficiently large. On the other hand, we have found almost no problems for which quasi-Monte Carlo is worse than standard Monte Carlo. In addition,

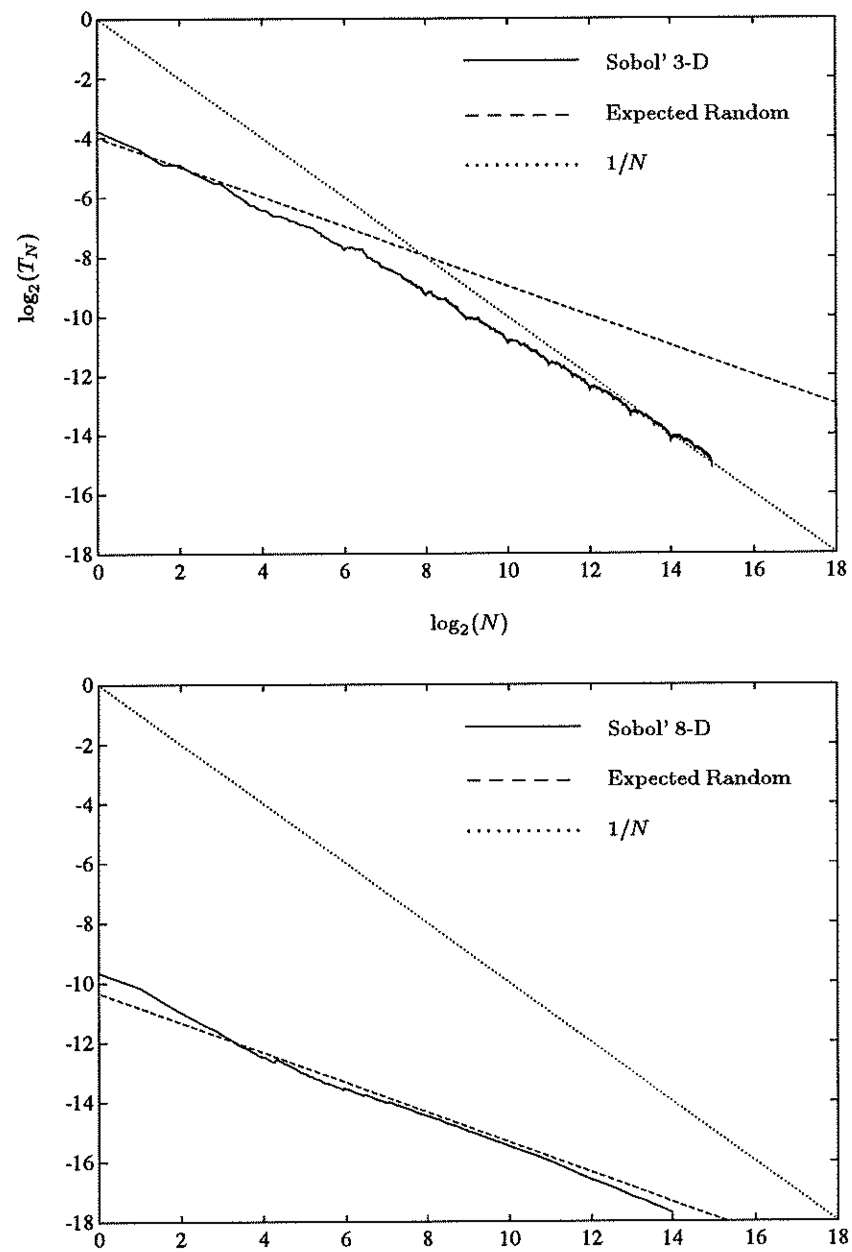

Figure 2: T_N for 3 Dimensional Sobol' Sequence (top) and 8 Dimensional Sobol' Sequence (bottom)

for problem of high dimension such as the mortgage-backed securities problem described below, it may be possible to reformulate the problem so that the effective dimension is of manageable size.

4 Orthogonal Projections

Further understanding of quasi-random sequences is gained by looking at two dimensional projections of the points in I^s . The appearance of non-uniformity in these projections is an indication of potential problems in using a quasi-random sequence for integration.

To be specific, some details of the Sobol' sequence are needed. Each dimension of this sequence is just a permutation of the the van der Corput sequence, whenever $N = 2^m$ for $m = 0, 1, 2, 3, \ldots$. These permutations are generated from irreducible polynomials over the field {0,1}. Ideally, polynomials of the lowest degree possible are used; however, as dimension increases, it is necessary to use polynomials of higher and higher degree. To generate a one dimensional sequence from a polynomial of degree d, $d-1$ odd integers $j_1, \ldots, j_{d-1}$ must be chosen with the restriction that $j_i < 2^i$. Thus there are $2^d - 1$ possible ways of picking the starting values. Sobol' has given a list of good starting values for dimension up to 16 [11], which produce sequences satisfying an additional uniformity property.

It is difficult to evaluate the uniformity of a sequence in a high dimensional space, but one indication of uniformity is the uniformity of the two-dimensional projections of the sequence, which are easily graphed. The graph at the top of Figure 1 shows a "good" pairing of dimensions using Sobol's 2^{nd} and 3^{rd} dimensions with his recommended starting values. A "bad" pairing of dimensions is presented in Figure 3, which shows two higher dimensions (following Sobol's convention for associating dimension with generating polynomial) based on the polynomials $x^7 + x^5 + x^4 + x^3 + 1$ and $x^7 + x^5 + x^4 + x^3 + x^2 + x + 1$ and the starting values are (1,3,5,11,3,3,35) and (1,1,7,5,11,59,113), respectively. Although this non-uniformity could go away if these starting values were changed, we have found that this type of non-uniformity if fairly typical of some of the two-dimensional projections of high dimensional quasi-random sequences. Moreover, it does not seem possible to tell a priori which choices will lead to bad pairing.

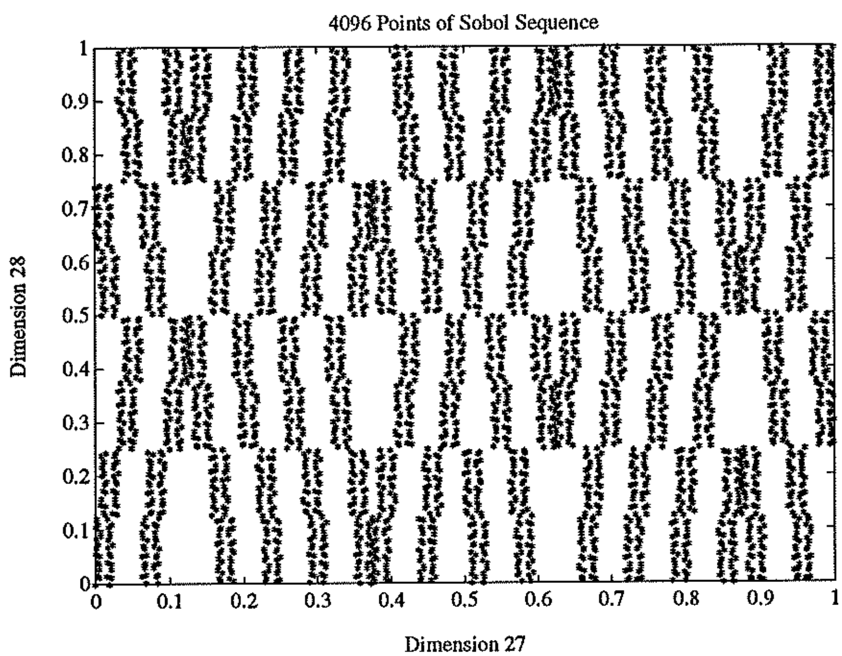

Figure 3: 2-D Projection of Sobol' Sequence

The bad behavior seen in the second plot of Figure 3 can be explained in terms of filling in holes. If 8192 (2^{13}) points are used, the plot looks almost identical to what is shown for 4096. However, the next 8192 points fall only where the gaps appear. Thus by $N = 16,384$, the projection plot is almost perfectly uniform. The problem is that the cycle for filling in holes is 2^{13}, which is too long.

It is worthwhile to note that, even if a sequence has poor two dimensional projections, it may still be fairly uniform in I^s , and there are many functions which it may integrate quite well. However, it is important to be aware of the potential problems these sequences may have, and the orthogonal projections are a good means of identifying and assessing the difficulties.

5 Mortgage-Backed Securities

Consider a security backed by mortgages of length M months with fixed interest rate i_0, which is the current interest rate at the beginning of the mortgage. The present value of the security is then

$$\begin{aligned} PV &= E(v) \\ &= E(\sum_{k=1}^{M} u_k m_k) \end{aligned} \tag{5.1}$$

in which E is the expectation over the random variables involved in the interest rate fluctuations. The variables in the problem are the following:

$$\begin{aligned} u_k &= \text{discount factor for month } k \\ m_k &= \text{cash flow for month } k \\ i_k &= \text{interest rate for month } k \\ w_k &= \text{fraction of remaining mortgages prepaying in month } k \\ r_k &= \text{fraction of remaining mortgages at month } k \\ c_k &= (\text{remaining annuity at month } k)/c \\ c &= \text{monthly payment} \\ \xi_k &= \text{an } N(0,\sigma) \text{ random variable.} \end{aligned}$$

This notations follows that of Paskov [9], except that our c_k is denoted a_{M-k+1} by Paskov.

Several of these variables are easily defined:

$$\begin{aligned} u_k &= \prod_{j=0}^{k-1}(1+i_k)^{-1} \\ m_k &= cr_k((1-w_k)+w_k c_k) \\ r_k &= \prod_{j=1}^{k-1}(1-w_j) \\ c_k &= \sum_{j=0}^{M-k}(1+i_0)^{-j}. \end{aligned}$$

Following Paskov, we use a model for the interest rate fluctuations and the prepayment rate given by

$$\begin{aligned} i_k &= K_0\, e^{\xi_k}\, i_{k-1} \\ &= K_0^k\, e^{\xi_1+\ldots+\xi_k}\, i_0 \\ w_k &= K_1 + K_2 \arctan\left(K_3 i_k + K_4\right) \end{aligned} \tag{5.2}$$

in which K_1, K_2, K_3, K_4 are constants of the model. The constant $K_0 = e^{-\sigma^2/2}$ is chosen to normalize the log-normal distribution, i.e. so that $E(i_k) = i_0$. The initial interest rate i_0 is an additional constant that must be specified.

In this study we do not divide the cash flow of the security among a group of tranches, as in [9], but only consider the total cash flow. Nevertheless, the results should be indicative of a more general computation involving a number of tranches.

The expectation PV can be written as in integral over R^M with Gaussian weights

$$g(\xi) = (2\pi\sigma^2)^{-1/2} e^{-\xi^2/2\sigma^2}. \tag{5.3}$$

This is transformed into an unweighted integral by a mapping $\xi = G(x)$ with $G'(x) = g(\xi)$, which takes a uniformly distributed variable x to an $N(0,\sigma)$ variable ξ. The formula for PV is

$$\begin{aligned} PV &= \int_{R^M} v(\xi_1,\ldots,\xi_M) g(\xi_1)\ldots g(\xi_1) d\xi_1 \ldots d\xi_M \\ &= \int_{[0,1]^M} v(G(\xi_1),\ldots,G(\xi_M)) dx_1 \ldots dx_M. \end{aligned} \tag{5.4}$$

Note that in quasi-Monte Carlo evaluation of an expectation involving a stochastic process with M time steps, the resulting integral is M dimensional.

In the numerical study below, we have used two sets of values of the parameters $i_0, K_1, K_2, K_3, K_4, \sigma$. We refer to the first set of parameters as the "Nearly Linear Example", because the integrand, when considered as a function of the Gaussian increment variables ξ_k, has a dominant linear component. For this case, the parameters are

$$(i_0, K_1, K_2, K_3, K_4, \sigma^2) = (.007, .01, -.005, 10, .5, .0004) \tag{5.5}$$

In the second example, the "Nonlinear Example", the parameters are

$$(i_0, K_1, K_2, K_3, K_4, \sigma^2) = (.007, .04, .0222, -1500.0, 7.0, .0004) \tag{5.6}$$

The interest rate corresponds to a yearly rate of 8.4%. The variance in interest rate increments σ^2 leads to yearly fluctuations of size .5%. In the Nearly Linear Example, the prepayment rate is nearly linear in the interest rate, in the range of interest; whereas for the Nonlinear Example, the prepayment rate has a step increase when the interest rate falls much below i_0. In both examples the length of the loans is taken to be 30 years ($M = 360$).

6 Brownian Bridge and an Alternative Discretization

Since Brownian motion is a Markov process, it is most natural to generate its value $b(t + \Delta t_1)$ as a random jump from a past value $b(t)$ as

$$b(t + \Delta t_1) = b(t) + \sqrt{\Delta t_1}\, \nu \tag{6.1}$$

in which ν is an $N(0, 1)$ random variable. On the other hand, the value $b(t + \Delta t_1)$ can also be generated from knowledge of both a past value $b(t)$ and a future value $b(T = t + \Delta t_1 + \Delta t_2)$, with $0 \leq \Delta t_i$, according to the *Brownian bridge* formula

$$b(t + \Delta t_1) = ab(t) + (1 - a)b(T) + c\nu \tag{6.2}$$

in which

$$a = \Delta t_1 / (\Delta t_1 + \Delta t_2)$$
$$c = \sqrt{a \Delta t_2} \tag{6.3}$$

Note that $a\Delta t_2 \leq \Delta t_1$, so that the variance of the random part of the Brownian bridge formula (6.2) is less than that that in (6.1).

The standard method of generating a random walk $y_k = \sigma b(k\Delta)$ is based on the updating formula (6.1). The initial value is $y_0 = 0$. Each subsequent value y_{k+1} is generated from the previous value y_k using formula (6.1), with independent normal variables ν_k.

Another method, which we refer to as the *alternative discretization* can be based on (6.2). Suppose we wish to determine the path $y_0, y_1, \ldots, y_M$, and for convenience assume that M is a power of 2. The initial value is $y_0 = 0$. The next value generated is $y_N = \sigma\sqrt{N\Delta t}\ \nu_0$. Then the value at the mid point $y_{N/2}$ is determined from the Brownian bridge formula (6.2). Subsequent values are found at the successive mid-points; i.e. $y_{N/4}, y_{3N/4}, y_{N/8}, \ldots$. The procedure is easily generalized to general values of M.

Although the total variance in this representation is the same as in the standard discretization, much more of the variance is contained in the first few steps of the alternative discretization due to the reduction in variance in the Brownian bridge formula. This reduces the effective dimension of the random walk simulation, which increases the accuracy of quasi-Monte Carlo. Moskowitz and Caflisch [6] applied this method to the evaluation of Feynman-Kac integrals and showed the error to be substantially reduced when the number of time steps, which is equal to the dimension of the corresponding integral, is large. Since the mortgage-backed securities problem described above depends on a random walk, and can be written as a discretization of a Feynman-Kac integral, we were naturally led to apply the alternative discretization to this problem .

7 Numerical Results

7.1 Nearly Linear Example

The value PV for this example was calculated to be 131.787. The variance in this value is 41.84 and the variance in the antithetic computation of this value (described below) is .014. The mean length of a mortgage in this case is 100.9 months and the median length is 93 months.

We now describe the accuracy of various integration methods for this problem as a function of N, the number of paths. For each of these results, we present the root-mean-square of the error over 25 independent computations. Moreover, the computations for different values of N are all independent. The results are plotted in terms of error vs. N, both in log base 10.

First, we perform straightforward Monte Carlo evaluation, with results plotted in Figure 4. The top curve shows results from Monte Carlo using standard pseudo-random points, with the error decreasing at the expected rate of $N^{-1/2}$. The second curve is the result of using quasi-Monte Carlo (Sobol') for the first 50 dimensions (time steps), followed by pseudo random for the remaining 310 time steps. This shows almost no gain through this limited use of quasi-Monte Carlo. The third graph shows a dramatic improvement using quasi-random for all 360 dimensions of the problem. The 360 dimensional quasi-random sequence was generated using a part of the code FINDER, written by Paskov.

These results are consistent with the results of Paskov [9, 10] and Ninomiya and Tezuka [8]. They seem to contradict the observations above that the effectiveness of quasi-Monte Carlo is lost in high dimensions. This result is deceptive,however, because of the special nature of this problem in which the value is almost entirely linear in the integration variables. For such a linear problem, only the one-dimensional projections of the quasi-random sequence are significant. The high-dimensional sequences generated here have the property that all of the one-dimensional projections are equally well distributed. So they do well on linear functions of any dimensionality. We will show next that they do not necessarily do well on nonlinear functions, even quadratic functions, presumably due to the poor quality of two-dimensional projections.

The linear terms in the integrand can all be eliminated through the use of antithetic variables. This means that for every path $\{x_n\}$, we also use the path $\{-x_n\}$. The resulting computational results are plotted on Figure 5. The top curve of this figure shows the error due to standard Monte Carlo, without antithetic variables, for reference. When antithetic variables are used with standard Monte Carlo the error is reduced by more than a factor of 50, as shown in the curve labeled *MC-anti.* Straightforward use of 360 dimensional quasi-random sequences, labeled *QR-anti* does not improve this result. This shows that once the linear terms are removed, straightforward quasi-Monte Carlo is not effective, due to the high-dimension of the problem. The final result, labeled *QR-anti,BB*, shows the result of using the Brownian bridge representation with antithetic variables. This gives an additional error reduction over random antithetic variables by a factor of nearly 10 (for larger N) and increases the rate of error decay from $N^{-1/2}$ to approximately $N^{-3/4}$. This improvement is due to the decrease in effective dimension in the Brownian bridge representation.

As further evidence of the nearly linearity of this problem, we next consider the effect of the quadratic terms of the Taylor expansion of the integrand about the point $(\xi_1, \ldots, \xi_M) = (0, \ldots, 0)$. The quadratic terms can be calculated exactly by taking second derivatives of the integrand. We then use them as a control variate; i.e., we subtract the exact quadratic terms from the integrand for the Monte Carlo evaluation, and then add back the exact integral of the quadratic terms. The resulting error, for standard Monte Carlo with antithetic variables, is another factor of 5 to 10 smaller, as shown in Figure 5.

7.2 Nonlinear Example

We next consider the Nonlinear Example and display the same error curves. The value for this problem is 130.7125. The variance in this value is 18.54 and the variance in the antithetic computation of this value is 1.127. The mean length of a mortgage in this case is 76.5 months and the median length is 58 months.

Figure 7 shows the error from standard Monte Carlo (*MC*), standard

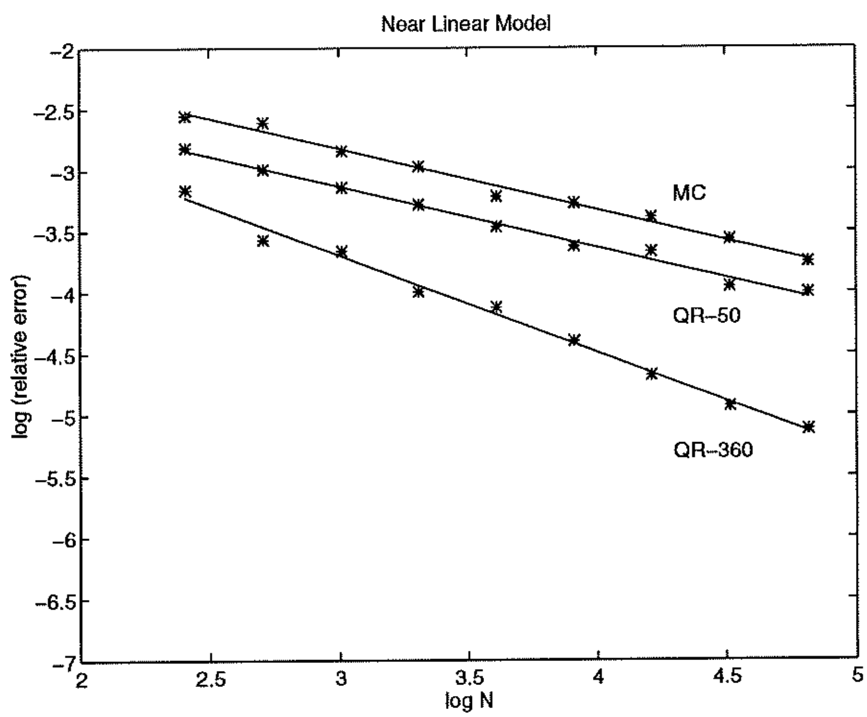

Figure 4: Error vs. N (log base 10) for the Nearly Linear Problem, using standard Monte Carlo (MC), quasi-random in the first 50 dimensions (QR-50), and quasi-random for all dimensions (QR-360).

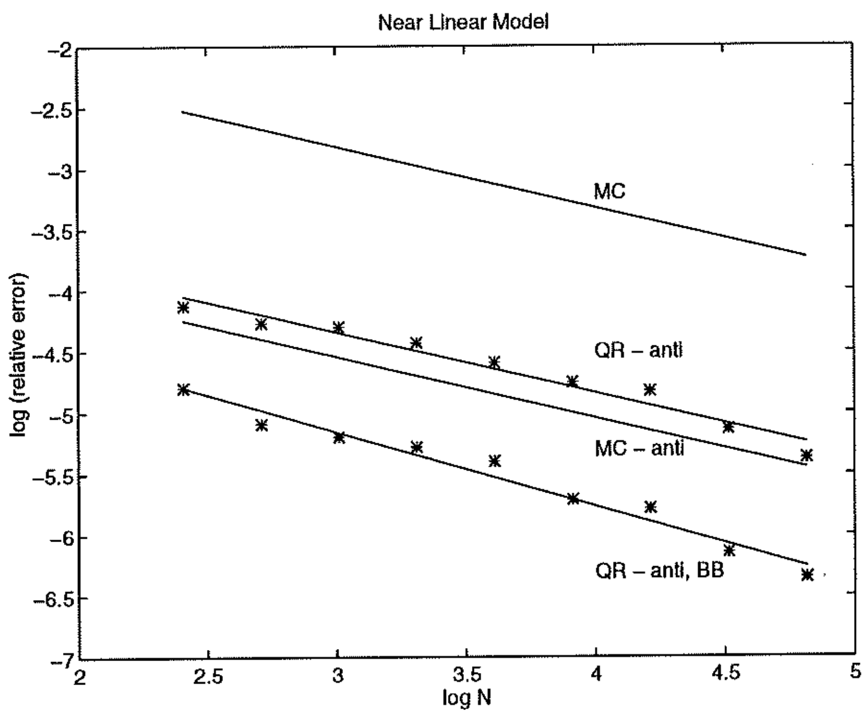

Figure 5: Error vs. N (log base 10) for the Nearly Linear Problem, using standard Monte Carlo with antithetic variables (MC-anti), quasi-random with antithetic variables (QR-anti), and quasi-random with antithetic variables and the Brownian bridge discretization (QR-anti,BB). The error for standard Monte Carlo is also plotted for reference.

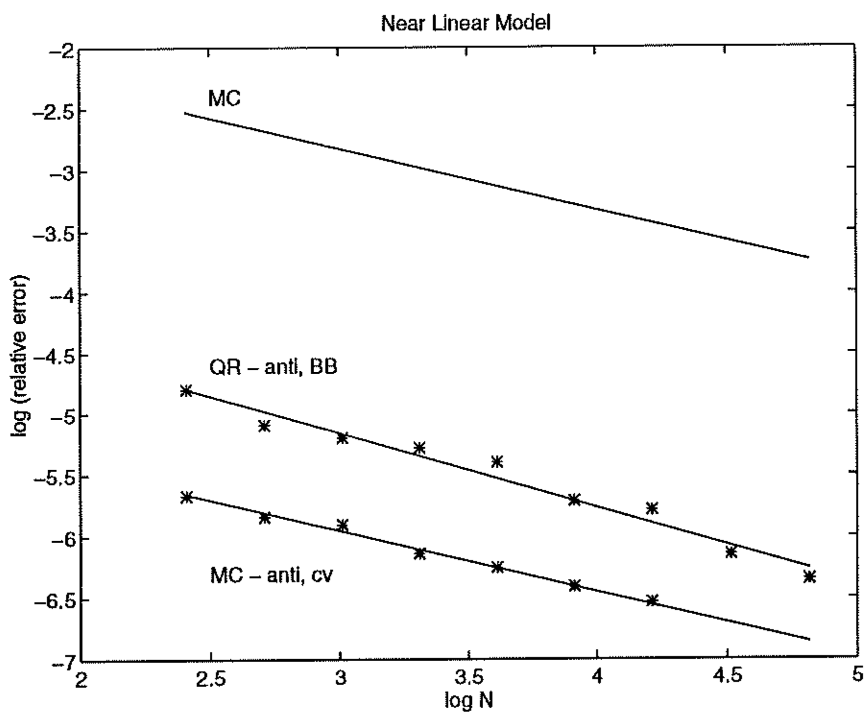

Figure 6: Error vs. N (log base 10) for the Nearly Linear Problem, using standard Monte Carlo with antithetic variables and with the exact quadratic terms as a control variate (MC-anti). The results from standard Monte Carlo and from quasi-random with antithetic variables and the Brownian bridge discretization are also plotted for reference.

Monte Carlo with antithetic variables (*MC-anti*), Sobol' in all 360 dimension (*QR*), and Sobol' with antithetic (*QR-anti*). We see that the latter three methods give roughly the same results–a factor of about 4 to 8 reduction in error compared with standard Monte Carlo.

Figure 8 shows the error from Sobol' with the Brownian bridge (*QR-BB*), and Sobol' with Brownian bridge and antithetic (*QR-anti,BB*), as well as the results for standard Monte Carlo and Sobol' for comparison. The Sobol' with Brownian bridge gives a further error reduction (over standard Sobol') by a factor of more than 3. When combined with antithetic variables, the improvement is a factor of about 30.

Our interpretation of these results is the following: Both antithetic variables and straightforward quasi-Monte Carlo effectively eliminate the error due to the linear terms, but quasi-Monte Carlo does not help with the quadratic terms. There is still significant error reduction in this problem, because the linear terms are significant, although not overwhelmingly dominant as in the previous problem.

Reduction in the error due to the quadratic and other nonlinear terms is possible through quasi-Monte Carlo using the Brownian bridge since the problem is then of lower effective dimension. When combined with antithetic variables, the method is even more effective.

8 Conclusions

Our main conclusions are the following:

- Quasi-Monte Carlo methods provide significant improvements in accuracy and computational speed for problems of moderate dimension.
- The effectiveness of quasi-Monte Carlo is lost on problems of high dimension, except in special cases, such as linear problems.
- Some problems that have a large nominal dimension, can be reformulated to have a moderate-sized effective dimension, so that the effectiveness of quasi-Monte Carlo is recovered.

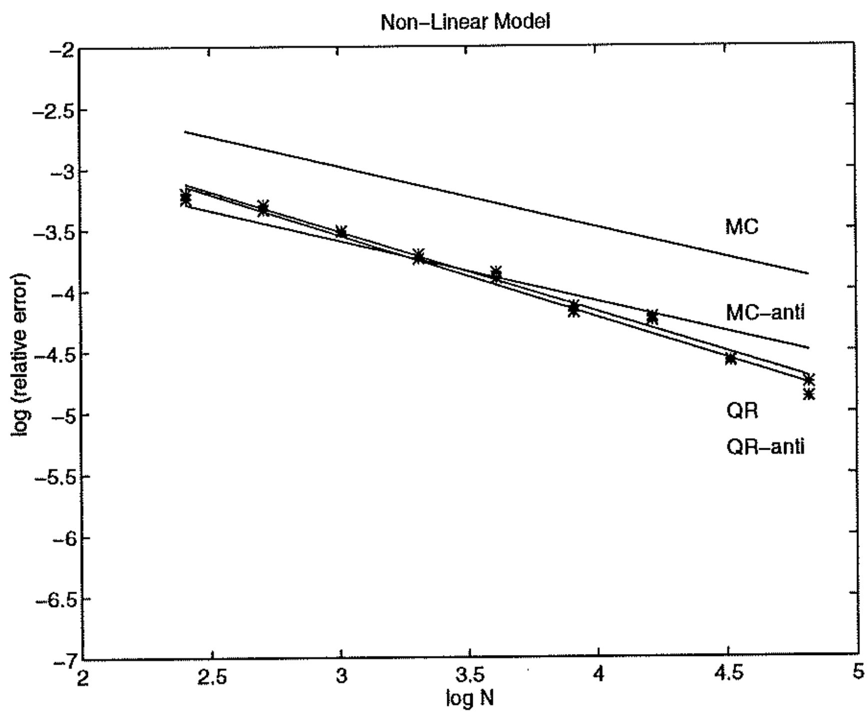

Figure 7: Error vs. N (log base 10) for the Nonlinear Problem, using standard Monte Carlo (MC), standard Monte Carlo with antithetic variables (MC-anti), quasi-random (QR), and quasi-random with antithetic variables (QR-anti).

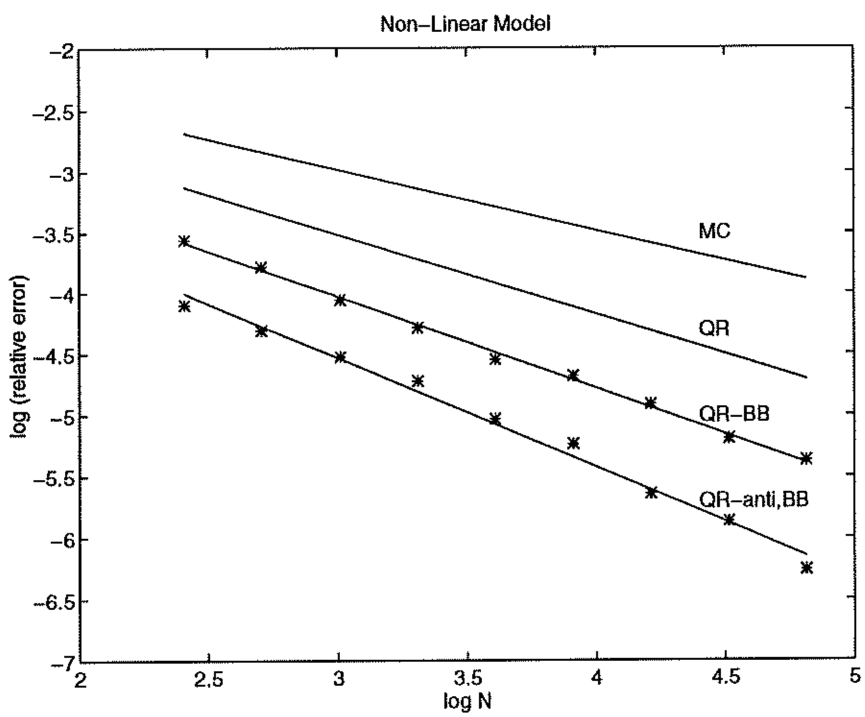

Figure 8: Error vs. N (log base 10) for the Nonlinear Problem, using quasi-random with the Brownian bridge (QR-BB), and quasi-random with the Brownian bridge and with antithetic variables (QR-anti,BB).

- The Brownian bridge representation reduces the effective dimension for problems, such as the mortgage-backed security problem described here.
- Quasi-Monte Carlo can be effectively combined with the variance reduction technique of antithetic variables.

We believe that range of applicability for quasi-Monte Carlo methods can be further increased through additional modification of standard Monte Carlo techniques for use with quasi-random sequences. For example combination of quasi-Monte Carlo with variance reduction methods, such as control variates, importance sampling and stratification, which could be combined with quasi-random sequence could lead to many improvements.

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