UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

An Adiabatic Law for Self-Focusing of Optical Beams

Gadi Fibich

August 1996

CAM Report 96-27

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

An adiabatic law for self-focusing of optical beams

Gadi Fibich
Department of Mathematics, University of California Los-Angeles, Los-Angeles CA 90095-1555

An adiabatic approach is used to derive a new law for self-focusing in the nonlinear Schrödinger equation which is valid from the early stages of self-focusing until the blowup point. The adiabatic law leads to an analytical formula for the location of the blowup point and can be used to estimate the effects of various small perturbations on self-focusing. The results of the analysis are confirmed by numerical simulations.

The study of blowup of solutions of the nonlinear Schrödinger equation (NLS)

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi = 0 \ , \quad \psi(0,r) = \psi_0(r) \qquad (1)$$

has been ongoing for more than thirty years, ever since Kelley used eq. (1) to predict the possibility of catastrophic self-focusing of optical beams [1]. Here, $\psi(z,r)$ is the electric field envelope of a laser beam propagating in media with a Kerr nonlinearity, z is the distance in the direction of the propagation, $r = \sqrt{x^2+y^2}$ is the radial coordinate and $\Delta_\perp = \partial^2/\partial r^2 + (1/r)(\partial/\partial r)$ is the Laplacian in the transverse 2D plane. The initial approach in self-focusing analysis was to assume that the solution maintains a Gaussian profile. This approach was successful in predicting the critical power for self-focusing (but only up to a constant!), finding the critical dimension for blowup etc [2]. However, in critical transverse dimension $D = 2$ the Gaussian approximation fails to capture the delicate balance between the focusing nonlinearity and radial dispersion which increase in magnitude while almost completely canceling each other. Indeed, resolving the local structure of ψ near the blowup point Z_c had long defied research efforts until Fraiman and independently, Landman, LeMesurier, Papanicolaou, Sulem and Sulem showed that as the beam approaches Z_c it follows the loglog law [3]:

$$L(z) \sim \sqrt{\frac{2\pi(Z_c - z)}{\ln\ln 1/(Z_c - z)}} \qquad (2)$$

where L corresponds to the beam width and is also inversely proportional to the amplitude $|\psi|$. Although NLS singularity was resolved mathematically, it turned out that even when the solution has focused by a factor of 10^{10} the loglog law is still not valid. However, the validity of NLS as a model for beam propagation breaks down much earlier when the field intensity reaches the material breakdown threshold. Even at sub-threshold intensities, some terms that have been neglected during the derivation of NLS from Maxwell's equations (nonparaxial terms [4,5], time dispersion [6,7] etc.) may become important. These terms may be small in magnitude yet have a large effect on self-focusing and even lead to its arrest. Therefore, there is still a need for a description of NLS self-focusing which is valid in the domain of physical interest and that can be extended to the analysis of small perturbations. In this letter we derive a new adiabatic law that satisfies both requirements.

Previous studies [3,8] have shown that as the beam propagates forward, it splits into an inner part ψ_s which self-focuses towards the center axis and an outer part ψ_{nf} which diffracts and diverges. Until the beam gets close to the blowup point, self-focusing is a non-adiabatic process in which ψ_s transfers most of its excess power above critical to ψ_{nf} while focusing and approaching the quasi self-similar form:

$$\psi_s(r,z) = \frac{V(\xi,\zeta)}{L(z)} \exp(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}) \ , \qquad (3)$$

where $L(z)$, the function to be determined, is used to rescale ψ_s and the independent variables:

$$\xi = \frac{r}{L} \ , \quad \frac{d\zeta}{dz} = \frac{1}{L^2} \ .$$

From the corresponding equation for V it follows that $V \sim R + O(\beta)$, where R is the positive solution of

$$\Delta_\perp R - R + R^3 = 0 \ , \quad R'(0) = 0 \ , \quad R(\infty) = 0 \qquad (4)$$

and β is the adiabaticity parameter

$$\beta = -L^3 L_{zz} \ . \qquad (5)$$

During self-focusing $\beta \searrow 0$. Near the blowup point the rate of self-focusing accelerates and the following hold [8,9]: i) $0 < \beta \ll 1$. ii) β is proportional to the excess power of ψ_s above critical:

$$\beta \sim \frac{N_s - N_c}{M} \ , \quad N_s = \int |\psi_s|^2 \, r dr \ , \qquad (6)$$

where $N_c = \int R^2 \, r dr \cong 1.86$ is the critical power for self-focusing and $M = (1/4)\int R^2 r^3 dr \cong 0.55$. iii) The Hamiltonian of ψ_s is given by:

$$H_s := \int |\nabla \psi_s|^2 \, r dr - \frac{1}{2}\int |\psi_s|^4 \, r dr \sim M\left(L_z^2 - \frac{\beta}{L^2}\right) , \quad (7)$$

where $\nabla = \partial/\partial r$. iv) Power losses of ψ_s (to ψ_{nf}) become exponentially small compared with its focusing rate ($\partial N_s/\partial\zeta \sim -\exp(-\pi/\sqrt{\beta})$), indicating that near the blowup point self-focusing is essentially an adiabatic process.

Our new approach is to use the dual interpretations for β (5,6) and the multiple scales method. If we ignore the slow-scale power loss ($\beta_z \sim -\exp(-\pi/\sqrt{\beta})/L^2$), *adiabatic self-focusing* follows the fast-scale equation:

$$-L^3 L_{zz} = \beta \ , \quad \beta \equiv \beta_0 := \beta(0) \ . \qquad (8)$$

If we multiply (8) by $2L_z L^{-3}$, integrate and use (7), we observe that in addition to N_s, H_s is also constant over the fast-scale:

$$L_z^2 = \frac{\beta}{L^2} + \frac{H_s}{M} \ , \quad H_s \equiv H_s(0) \ .$$

Multiplying by L^2 and integrating one more time leads to the new adiabatic law:

$$L = \sqrt{L_0^2 - 2\sqrt{\beta + \frac{H_s L_0^2}{M}} z + \frac{H_s}{M} z^2} \ , \quad L_0 = L(0) \ . \quad (9)$$

By setting $L = 0$ in (9) we can get an equation for the blowup point Z_c:

$$Z_c = \begin{cases} \dfrac{L_0^2}{\sqrt{\beta} + \sqrt{\beta + H_s L_0^2/M}} & L_z(0) \le 0 \\[2ex] \dfrac{L_0^2}{\sqrt{\beta} - \sqrt{\beta + H_s L_0^2/M}} & L_z(0) > 0 \ , \quad H_s < 0 \\[2ex] \text{no blowup} & L_z(0) > 0 \ , \quad H_s > 0 \ . \end{cases} \qquad (10)$$

In the case of a collimated beam (ψ_0 real) $L_z(0) = 0$, $H_s(0) \sim -M\beta_0/L_0^2$ (7) and the 'pure' adiabatic law is:

$$L = L_0 \sqrt{1 - \frac{z^2}{Z_c^2}} \ , \quad Z_c = \frac{L_0^2}{\sqrt{\beta_0}} \ . \qquad (11)$$

If we add a lens with focal length F at $z = 0$, the initial condition becomes $\tilde{\psi}_0 = \psi_0 exp(-ir^2/4F)$. Therefore, $\tilde{L}_0 = L_0$, $\tilde{\beta}_0 \sim \beta_0$ and $\tilde{H}_s(0) \sim H_s(0) + ML_0^2/F^2$ (6,7), where the tildes denote the corresponding parameters for $\tilde{\psi}_0$. Thus, the new blowup point is (10):

$$\tilde{Z}_c = \frac{L_0^2}{\sqrt{\beta_0} + L_0^2/F} \ .$$

Note that $1/\tilde{Z}_c = 1/Z_c + 1/F$, in agreement with Talanov's lens transformation property for NLS [10].

The adiabatic law (9) can be rewritten in the form

$$L = \sqrt{2\sqrt{\beta}(Z_c - z) + \frac{H_s}{M}(Z_c - z)^2} \ . \qquad (12)$$

As z approaches the singularity point, the quadratic term becomes negligible and (12) reduces to Malkin's law [8]:

$$L = \sqrt{2\sqrt{\beta}(Z_c - z)} \ . \qquad (13)$$

Thus, (12) and (13) agree asymptotically but (12) becomes valid earlier, since in addition to beam power it also incorporates the initial focusing angle. Likewise, the loglog law can be derived as the asymptotic limit of (13) [8]. Therefore, the three laws are not in disagreement; only their domain of validity differ.

Note that all adiabatic relations (9–13) are only $O(\beta)$ accurate due to the approximations (6,7) used in their derivation. In order to maintain this accuracy in (12) or (13) the slow scale (non-adiabatic) changes in β and H_s have to be included.

While eq. (11) was derived under the assumptions that ψ_s has approached its asymptotic form (3) and $\beta \ll 1$, we can try to extrapolate it to predict Z_c for general initial conditions. The value of β is 'determined' from (6) with $N_s \sim \int |\psi_0|^2$ and that of L_0 by 'matching' $\psi_0(r) \sim R_{L_0} := L_0^{-1} R(r/L_0)$. For example, if we impose $\int |\nabla\psi_0|^2 = \int \nabla |R_{L_0}|^2$, then $L_0 = \sqrt{N_c / \int |\nabla\psi_0|^2}$ and

$$Z_c \sim \sqrt{\frac{MN_c}{p-1}} \Big/ \int |\nabla\psi_0|^2 \, r dr \ , \ p = \frac{\int |\psi_0|^2 \, r dr}{N_c} \ . \quad (14)$$

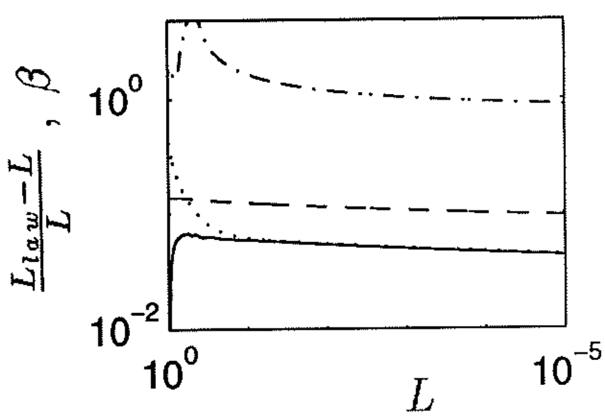

FIG. 1. The relative error in L based on the new adiabatic law (eq. 12, solid line), Malkin's law (eq. 13, dotted line) and the loglog law (eq. 2, dash-dot line) and β (eq. 6, dashed line). Initial condition is $\psi_0 = 1.02R(r)$.

In the simulations eq. (1) was solved using the method of dynamic rescaling [11] and β was evaluated using (6). In Figure 1 we plot the relative error in the prediction for L based on the new adiabatic law (12), Malkin's adiabatic law (13) and the loglog law (2). The initial condition is $\psi_0 = 1.02R(r)$, whose power is 4% above critical and whose profile is close to the asymptotic one (3). The two adiabatic laws become $O(\beta)$ accurate and agree asymptotically, with (12) being accurate from the beginning and (13) after focusing by a factor of 10. After focusing by 100,000 β has only decreased by 30% and the loglog law is still not valid. Note that if we add a focusing lens term ($\psi_0 = 1.02\exp(-ir^2/4F)R(r)$) only the new adiabatic law would maintain the same accuracy. In Figure 2 we compare 'pure' adiabatic self-focusing (11) and 'almost' adiabatic self-focusing ((12) with the slowly varying $\beta(z)$ and $H_s(z)$ and Z_c from the numerics) for the same initial condition. While both (11) and (12) are in reasonable agreement with the numerical solution in the prefocal region, only (12) maintains $O(\beta)$ accuracy near the focal point (Figure 1). In Figure 3 the adiabatic predictions for Z_c (14) are compared with simulations using the initial conditions: A: $\psi_0 = cR(r)$, B: $\psi_0 = c\exp(-r^2)$ and C: $\psi_0 = c\exp(-r^4)$, where c is varied so that $1 < p \leq 2$. Naturally, the best agreement is in Figure 3A where ψ_0 is closest to the asymptotic profile. However, even in Figure 3B–C the agreement is quite good, considering that we neglected non-adiabatic changes, that ψ_0 is not close to (3) and that the excess power above critical is not small. As Figure 3B indicates, the accuracy of the formula of Dawes and Marburger [12]

$$Z_c = 0.367[(p^{1/2} - 0.852)^2 - 0.0219]^{-1/2} \qquad (15)$$

and of (14) are of comparable magnitude. However, (15) is only valid for the special case of Gaussian initial conditions and was derived by curve fitting values of Z_c obtained from simulations.

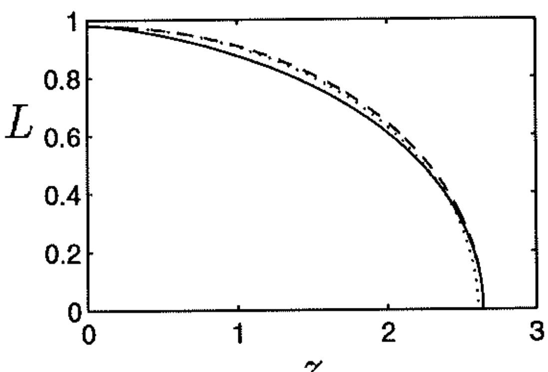

FIG. 2. Comparison of the 'pure' adiabatic law (11, dotted line) and the 'almost' adiabatic law (12, dashed line) with the numerical solution of NLS (solid line). Initial condition is $\psi_0 = 1.02R(r)$.

The adiabatic approach can be extended to analyze the effects of small perturbations on self-focusing [9]. For example, we have recently shown [5] that nonparaxial effects become important and lead to the arrest of self-focusing when $a/\lambda = O(\sqrt{N_c/M\beta_0}/4\pi)$ where a and λ are the pulse radius and wavelength, respectively. Since simulations of (1) suggest that typically in the adiabatic regime $\beta = O(0.1)$, arrest due to nonparaxiality will occur when $a \sim \lambda/2$. A similar approach can be used to analyze at which point small normal time dispersion will affect self-focusing by combining the results of this paper and [6]. Therefore, for given initial conditions it is possible to determine which of these two mechanisms will be the first to affect self-focusing.

G. Fibich would like to thank G. Papanicolaou and W. Morokoff for useful discussions. This study has been supported in part by the NSF grant DMS-9623087.

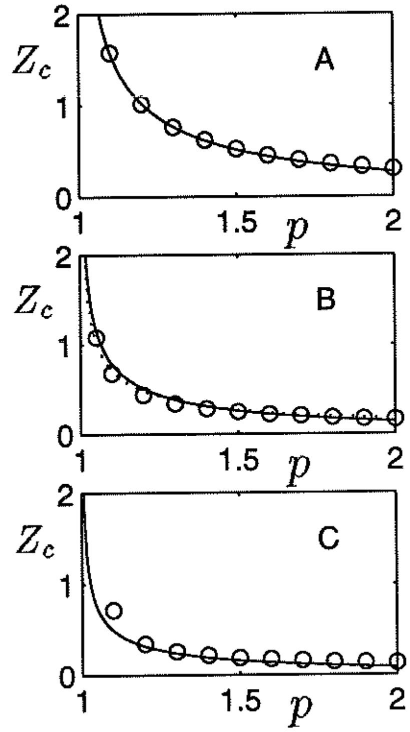

FIG. 3. The location of the blowup point Z_c as a function of beam power p according to adiabatic theory (14, solid line) and numerical simulations (circles) for A: $c\,R(r)$ B: $c\,exp(-r^2)$ (dotted line is (15)) and C: $c\,exp(-r^4)$.

[1] P. Kelley, Phys. Rev. Lett. **15**, 1005 (1965).
[2] S. Akhmanov, A. Sukhorukov, and R. Khokhlov, JETP **23**, 1025 (1966); F. Cooper, C. Lucheroni, and H. Shepard, Phys. Lett. A **70**, 184 (1992).
[3] G. Fraiman, Sov. Phys. JETP **61**, 228 (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem, Phys. Rev. A **38**, 3837 (1988); B. LeMesurier, G. Papanicolaou, C. Sulem, and P. Sulem, Physica D **32**, 210 (1988).
[4] M. Feit and J. Fleck, J. Opt. Soc. Am. B **5**, 633 (1988).
[5] G. Fibich, Phys. Rev. Lett. **76**, 4356 (1996).
[6] G. Fibich, V. Malkin, and G. Papanicolaou, Phys. Rev. A **52**, 4218 (1995).
[7] J. E. Rothenberg, Opt. Lett. **17**, 1340 (1992); G. Luther, A. Newell, and J. Moloney, Physica D **74**, 59 (1994).
[8] V. Malkin, Physica D **64**, 251 (1993).
[9] G. Fibich and G. Papanicolaou, preprint.
[10] V. Talanov, JETP Lett **11**, 199 (1970).
[11] D. McLaughlin, G. Papanicolaou, C. Sulem, and P. Sulem, Phys. Rev. A **34**, 1200 (1986).
[12] E. Dawes and J. Marburger, Phys. Rev. **179**, 862 (1969); J. Marburger, Prog. Quant. Electr. **4**, 35 (1975).