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ANALYSIS OF GENERALIZED SCHWARZ ALTERNATING PROCEDURE FOR DOMAIN DECOMPOSITION

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Abstract.
A generalized Schwarz alternating method(SAM) is proposed, which passes a convex combination of Dirichlet data u and Neumann data $\frac{\partial u}{\partial n}$, i.e. $Au + \frac{\partial u}{\partial n}$, at the artificial boundary, where A is some 'positive' operator for well-posedness. A concrete convergence analysis is done on two model elliptic problems in two subdomain case by Fourier analysis and pseudo-differential operator theory. We will clearly see how the operator A, overlapping size and the relative size of the subdomains will affect the convergence speed. No overlap case is also considered and comparison is made to the classical SAM. These methods can then be easily extended to the general coercive elliptic partial differential equations in general two subdomain cases by the equivalence of elliptic operators. Also some interpretations in other domain decomposition context is given. Numerically, in order to have good convergence rate and easy implementation, first and second order approximations of the Dirichlet to Neumann operator by local operators are constructed using asymptotic expansions. The optimal convergence rate can be explicitly determined by the partial differential equations.

Key Words.
domain decomposition, Schwarz alternating method(SAM), Dirichlet to Neumann operator, pseudo-differential operator, spectral equivalent, enhanced system, partial differential equation(PDE)

AMS subject classifications: 65N55, 65N12, 65P05

1. Introduction. In domain decomposition method, the full problem is decomposed(decoupled) into subproblems by decomposing the whole physical domain into subdomains where the subproblems in each subdomain can be more easily and efficiently solved. The crucial step is how to compose(couple) the solution of the subproblems together so that we can get the solution for the whole problem. Some compatibility conditions have to be imposed for the Dirichlet data and Neumann data between each subdomain. Schwarz alternating method for elliptic PDE is one of the most important theoretical basis for domain decomposition method in both parallel computations and computations in complicated domains.

The classical SAM method is an iterative method which solves the subproblem in each subdomain alternatively and compose(couple) the subproblems together by overlapping each subdomain where function value (Dirichlet data) is exchanged at the artificial boundaries. The classical SAM method is convergent for elliptic PDEs and its convergence rate is related to the size of overlap of the subdomains. This is mainly due to two reasons: first, maximum principle and second variational interpretation (iterated projections into subspaces of a Hilbert space). (see [14],[10],[11]). The main drawback of the classical SAM method is that its convergence speed is quite sensitive to the overlapping size.

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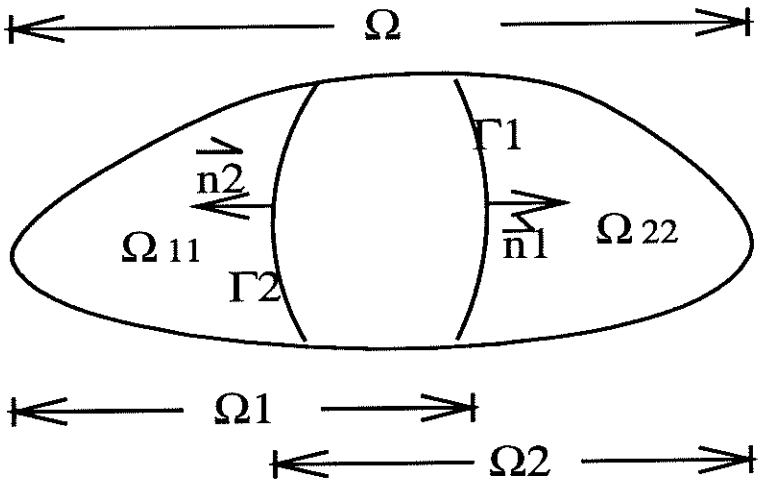
Recently many new variants of classical SAM have been proposed for domain decomposition method with or without overlap. (see [5],[1],[6],[13]) Our paper is motivated by the new modified SAM method, proposed by P.L.Lions ([12]), which exchanges a convex combination of Neumann and Dirichlet data at the artificial boundary. It has been proved to be convergent in very general setting even with no overlap. The proof is based on technical energy estimates and Sobolev inequalities. Yet no estimations on the convergence rate has been done. Here we will treat all these in a more general frame work and analyze its convergence property. We will see its improved convergence property is due to the 'positive' property of the Dirichlet to Neumann operator.

A very interesting application for the generalized SAM, which is under further investigation, is to interface problems where the solutions to the elliptic PDE and their derivatives may be discontinuous across some interfaces due to the discontinuities of the coefficients or/and singular source along the interfaces. This kind of interface problems occur very often in real applications, e.g. multiphase fluid problems, immersed boundary problems. A domain decomposition along the interfaces is very natural since now we can solve well behaved problems in each subdomain using any good numerical scheme and only couple the subproblems together by the prescribed interface condition. In this situation the classical overlapping SAM can not work well due to the discontinuities. Even the traditional non-overlapping method in domain decomposition will not work since the Steklov-Poincaré operator or the Schur complement at the interface is not well defined.

For simplicity we first consider the case where the whole domain Ω is decomposed into two subdomains Ω_1 and Ω_2 (i.e. $\Omega = \Omega_1 \cup \Omega_2$) with the two artificial boundaries Γ_1, Γ_2 intersecting $\partial\Omega$. (see figure 1)



The elliptic PDE with Dirichlet boundary data is:

$$\begin{aligned} P(D)u &= f \quad in\ \Omega \\ u &= g \quad on\ \partial\Omega \end{aligned} \tag{1}$$

where $P(D)$ is some elliptic differential operator. We generalize the classical SAM method to the following two formulations:

(i) additive(parallel) iteration.
Set u_i^0 initially in Ω_i, $i = 1, 2$ and construct u_i^n iteratively,

$$
\begin{aligned}
P(D)u_1^n &= f && \text{in } \Omega_1 \\
u_1^n &= g && \text{on } \partial\Omega_1 \backslash \Gamma_1 \\
\Lambda_1 u_1^n + \lambda_1 \frac{\partial u_1^n}{\partial n_1} &= \Lambda_1 u_2^{n-1} + \lambda_1 \frac{\partial u_2^{n-1}}{\partial n_1} && \text{on } \Gamma_1
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
P(D)u_2^n &= f && \text{in } \Omega_2 \\
u_2^n &= g && \text{on } \partial\Omega_2 \backslash \Gamma_2 \\
\Lambda_2 u_2^n + \lambda_2 \frac{\partial u_2^n}{\partial n_2} &= \Lambda_2 u_1^{n-1} + \lambda_2 \frac{\partial u_1^{n-1}}{\partial n_2} && \text{on } \Gamma_2 \\
& && n = 1, 2, \ldots,
\end{aligned} \tag{3}
$$

(ii)multiplicative(successive) iteration.
Set u^0 initially in Ω and construct u_1^{2k+1} in Ω_1 and u_2^{2k+2} in Ω_2 iteratively,

$$
\begin{aligned}
P(D)u_1^{2k+1} &= f && \text{in } \Omega_1 \\
u_1^{2k+1} &= g && \text{on } \partial\Omega_1 \backslash \Gamma_1 \\
\Lambda_1 u_1^{2k+1} + \lambda_1 \frac{\partial u_1^{2k+1}}{\partial n_1} &= \Lambda_1 u_2^{2k} + \lambda_1 \frac{\partial u_2^{2k}}{\partial n_1} && \text{on } \Gamma_1
\end{aligned} \tag{4}
$$

and

$$
\begin{aligned}
P(D)u_2^{2k+2} &= f && \text{in } \Omega_2 \\
u_2^{2k+2} &= g && \text{on } \partial\Omega_2 \backslash \Gamma_2 \\
\Lambda_2 u_2^{2k+2} + \lambda_2 \frac{\partial u_2^{2k+2}}{\partial n_2} &= \Lambda_2 u_1^{2k+1} + \lambda_2 \frac{\partial u_1^{2k+1}}{\partial n_2} && \text{on } \Gamma_2 \\
& && k = 0, 1, 2, \ldots,
\end{aligned} \tag{5}
$$

u^n in Ω can be composed in many ways from u_1^n and u_2^n such that $u^n \in H^1(\Omega)$ and, $u^n = u_1^n$, in $\Omega_1 \setminus (\Omega_1 \cap \Omega_2)$, $u^n = u_2^n$ in $\Omega_2 \setminus (\Omega_1 \cap \Omega_2)$. Λ_1, Λ_2 are some operators on u. λ_1, λ_2 are two constants. It is easily seen that if $\Lambda_1 = \Lambda_2 = I$ and $\lambda_1 = \lambda_2 = 0$, then it is the classical SAM. If $\Lambda_1 = \Lambda_2 = constants > 0$ and $\lambda_1 = \lambda_2 = 1$ then it is the modified SAM proposed by P.L. Lions (see [12]). Also we can let Λ_i, λ_i's change with each iteration. In particular we can pass Dirichlet data and Neumann data alternatively and we get the Dirichlet to Neumann type or the relaxization procedure in domain decomposition context. ([13]) Many variants of the classical SAM method can be regarded as a particular choice of Λ_i and λ_i. In general we can think of nonzero λ_1, λ_2 as absorbed in Λ_1, Λ_2. We will take $\lambda_1 = \lambda_2 = 1$ for the rest of the discussions.

Remark: Although the practical implementation of additive scheme and multiplicative scheme can be quite different, the convergent analysis in our paper is about the same for both schemes. We will mainly do analysis on the additive scheme for the

easier bookkeeping. The corresponding results for multiplicative scheme, if different, will be mentioned without proof.

In our generalized SAM, at the artificial boundary a Robin (Fourier) type of boundary condition is imposed. In order to have well-posedness for each subdomain problem, we need some requirement on the operators Λ_i, $i = 1, 2$

Definition: A bounded linear operator $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is said to be positive for the second order elliptic differential operator $P(D)$ if for

$$\forall u \; P(D)u = 0 \qquad then \qquad \int_{\partial\Omega} u\Lambda u \geq 0$$

It is said to be strictly positive if $\int_{\partial\Omega} u\Lambda u > 0$, if $u \neq 0$.

If we denote $\Lambda(x, \eta)$ to be the pseudo-differential operator of Λ, then if Λ is (strictly) positive,

$$\Lambda(x, \eta) \geq (>)0, \qquad \forall x \in \partial\Omega, \; \forall \eta$$

It is not hard to see that the restriction operator $\Lambda u = u|_{\partial\Omega}$ is a trivial strictly positive operator. If $P(D) = -\triangle$ then the Dirichlet to Neumann operator $\Lambda u = \frac{\partial u}{\partial n}|_{\partial\Omega}$ is a positive operator.

It is very easy to see the following lemma is true for the well-posedness of the two subproblems in Ω_1, Ω_2

LEMMA 1.1. *If $\Lambda_i : H^{1/2}(\partial\Omega_i) \longrightarrow H^{-1/2}(\partial\Omega_i)$, $i = 1, 2$ are positive operator, then the two boundary value problems (2),(3),(4),(5) are well-posed.*

So we can actually imagine that the right choice of the positive operator should be some interpolation between the restriction operator and the Dirichlet to Neumann operator. The perfect choice of $\Lambda_1(\Lambda_2)$ is the Dirichlet to Neumann operator in $\Omega_2(\Omega_1)$ which gives the convergence in two steps.

LEMMA 1.2. *The second-order elliptic PDE $P(D)u = f$ in Ω can be solved by domain decomposition without overlap in finite steps if $\Lambda_1(\Lambda_2)$ is the Dirichlet to Neumann operator at the artificial boundary for the corresponding homogeneous PDE in $\Omega_2(\Omega_1)$. (see figure 1)*

Proof: We will show that in two steps we can get the exact solution u in Ω. Let $e_i^n = u - u^n$, $i = 1, 2$ be the error function. Then e_i^n will satisfy the following homogeneous equation and boundary condition:

$$\begin{cases} P(D)e_1^1 = 0 & in \; \Omega_1 \\ e_1^1 = 0 & on \; \partial\Omega_1 \backslash \Gamma_1 \\ \dfrac{\partial e_1^1}{\partial n_1} + \Lambda_1 e_1^1 = \dfrac{\partial e_2^0}{\partial n_1} + \Lambda_1 e_2^0 & on \; \Gamma_1, \; where \; n_1 \; is \; the \; outer \; normal \; for \; \Gamma_1 \end{cases}$$

similarly we have the following problem in Ω_2,

$$\begin{cases} P(D)e_2^1 = 0 & in \; \Omega_2 \\ e_2^1 = 0 & on \; \partial\Omega_2 \backslash \Gamma_2 \\ \dfrac{\partial e_2^1}{\partial n_2} + \Lambda_2 e_2^1 = \dfrac{\partial e_1^0}{\partial n_2} + \Lambda_2 e_1^0 & on \; \Gamma_2, \; where \; n_2 \; is \; the \; outer \; normal \; for \; \Gamma_2 \end{cases}$$

Since Λ_1 (Λ_2) is the Dirichlet to Neumann operator for Γ_1,(Γ_2) in Ω_2,(Ω_1), so we must have,

$$\frac{\partial e_2^0}{\partial n_1} + \Lambda_1 e_2^0 = -\frac{\partial e_2^0}{\partial n_2} + \Lambda_1 e_2^0 = 0$$
$$(\frac{\partial e_1^0}{\partial n_2} + \Lambda_2 e_1^0 = -\frac{\partial e_1^0}{\partial n_1} + \Lambda_2 e_1^0 = 0)$$

then we get $e_1^1 = 0$ *in* Ω_1 ($e_2^1 = 0$ *in* Ω_2). i.e. we get the exact solution in two steps.

Remark: In general the Dirichlet to Neumann operator is a global operator and to find its action on u is equivalent to solve a Dirichlet problem. But in 1-D for ordinary differential equations with constant coefficients it can be easily found(see [18],[2]). For more general elliptic PDE in some simple geometry it is possible to incorporate this into the solving procedure by some appropriate finite element formulation(see [9]). Also we will try to use some local operators to approximate it in section 5.

Since we are concerned with the convergence, we only need to look at error function $e^n = u - u^n$, $e_i^n = u - u_i^n$, $i = 1, 2$ which satisfies the corresponding homogeneous equation and homogeneous boundary conditions. So from now on without loss of generality we can assume that $f = 0$, $g = 0$ in (1) and see how u_i^n converges to 0 in Ω_i, $i = 1, 2$

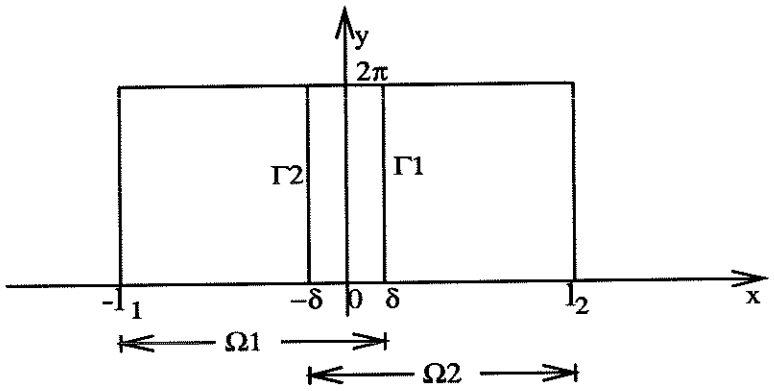
2. Convergence Analysis of the Generalized SAM for two model problems. In this section we will use Fourier analysis on the Laplace equation in two special geometries in two subdomain case. The choice of these two model problems can let us see very intuitively why the generalized SAM should work and how the convergence rate depends on Λ_i, the overlapping size and the size of the subdomains, since for these two model problems the explicit formula of the convergence rate can be derived. Using the spectral equivalence of the elliptic differential operators we will see that the analysis for these two simple model problems can be generalized to more general elliptic PDEs and geometries in two subdomain cases or even in some general multi-domain cases.

Model Problem1:

$$\begin{cases} -\triangle u = 0 & in\ \Omega \\ u = 0 & on \partial\Omega \end{cases}$$

$\Omega = \{(x, y) | -l_1 \le x \le l_2, 0 \le y \le 2\pi\}$
$\Omega_1 = \{(x, y) | -l_1 \le x \le \delta, 0 \le y \le 2\pi\}$, $\Omega_2 = \{(x, y) | -\delta \le x \le l_2, 0 \le y \le 2\pi\}$
$0 \le \delta < min(l_1, l_2)$
$\Gamma_1 = \{(x, y) | x = \delta,\ 0 \le y \le 2\pi\}$, $\Gamma_2 = \{(x, y) | x = -\delta,\ 0 \le y \le 2\pi\}$,

Claim1:
For model problem1 the generalized additive Schwarz alternating method, (2) - (3). ($\lambda_1 = \lambda_2 = 1$) is convergent if $\Lambda_1 = \Lambda_2 = \Lambda$ is strictly positive. If there is overlap,



then there is a constant $0 < \rho < 1$ such that

$$\| u - u_i^n \|_{H^1(\Omega_i)} \leq \rho \| u - u_i^{n-2} \|_{H^1(\Omega_i)} \qquad i = 1, 2$$

where ρ depends on Λ, the size of $\Omega_i(l_i)$ and the overlapping size δ. If there is no overlap ($\delta = 0$) then

$$\lim_{n\to\infty} \| u - u_i^n \|_{H^1(\Omega_i)} \to 0 \qquad i = 1, 2$$
$$\lim_{n\to\infty} \| u - u_i^n \|_{H^{1/2}(\Gamma_i)} \to 0 \qquad i = 1, 2$$

Proof: After Fourier transforming in y in Ω_1 Ω_2, we have,

$$-\hat{u_i}^n_{xx}(x,\eta) + \eta^2 \hat{u_i}(x,\eta) = 0, \ i = 1, 2 \tag{6}$$

where $\hat{u_i}$ is the Fourier transform of u_i, η is the Fourier mode for y. Multiply both sides by $\overline{\hat{u_i}}$ and integrate by part we have the following relation,

$$\begin{aligned} &\int_{-l_1}^{\delta} | \hat{u_1}^n_x |^2 + \eta^2 | \hat{u_1}^n |^2 \, dx = Re\overline{\hat{u_1}^n} \hat{u_1}^n_x |_{x=\delta} \\ &\int_{-\delta}^{l_2} | \hat{u_2}^n_x |^2 + \eta^2 | \hat{u_2}^n |^2 \, dx = -Re\overline{\hat{u_2}^n} \hat{u_2}^n_x |_{x=-\delta} \end{aligned} \tag{7}$$

These two relations actually are the Fourier transform of the following variational identity

$$\int_\Omega | \nabla u |^2 = \int_{\partial\Omega} u \frac{\partial u}{\partial n}$$

i.e. the H^1 seminorm of solution is bounded by some boundary integral and the Dirichlet to Neumann operator is positive.

From (6) and the homogeneous boundary condition on $\partial\Omega$ we have,

$$\begin{aligned} \hat{u_1}^n(x,\eta) &= A_1^n(\eta)(e^{\lambda x} - e^{-2\lambda l_1} e^{-\lambda x}) \ x \in [-l_1, \delta] \\ \hat{u_2}^n(x,\eta) &= A_2^n(\eta)(e^{-\lambda x} - e^{-2\lambda l_2} e^{\lambda x}) \ x \in [-\delta, l_2] \end{aligned}$$

where $\lambda(\eta) = |\eta|, \eta = \pm 1, \pm 2, \ldots$, which is the Fourier transform of Dirichlet to Neumann operator for our PDE if $l_i = \infty$, $i = 1, 2$. According to equation (7), we

have

$$\begin{aligned}(8)\qquad &\int_{-l_1}^{\delta} \mid \hat{u_1}_x^n \mid^2 +\eta^2 \mid \hat{u_1}^n \mid^2 dx = \lambda \mid A_1^n(\eta) \mid^2 (e^{2\lambda\delta} - e^{-4\lambda l_1} e^{-2\lambda\delta}) \\ &\int_{-\delta}^{l_2} \mid \hat{u_2}_x^n \mid^2 +\eta^2 \mid \hat{u_2}^n \mid^2 dx = \lambda \mid A_2^n(\eta) \mid^2 (e^{2\lambda\delta} - e^{-4\lambda l_2} e^{-2\lambda\delta})\end{aligned}$$

Denote $\Lambda(\eta)$ to be the Fourier transform (pseudo-differential operator) of the operator Λ. At Γ_1,

$$\hat{u_1}_x^n + \Lambda(\eta)\hat{u_1}^n \mid_{x=\delta} = A_1^n(\eta)[(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_1} e^{-\lambda(\eta)\delta}]$$

$$\hat{u_2}_x^n + \Lambda(\eta)\hat{u_2}^n \mid_{x=\delta} = -A_2^n(\eta)[(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_2} e^{\lambda(\eta)\delta}]$$

So if we impose the generalized boundary condition at Γ_1 we have

$$(9)\qquad \frac{A_1^n(\eta)}{A_2^{n-1}(\eta)} = -\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_2} e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_1} e^{-\lambda(\eta)\delta}}$$

From the boundary condition at Γ_2, similarly we have,

$$(10)\qquad \frac{A_2^n(\eta)}{A_1^{n-1}(\eta)} = -\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_1} e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_2} e^{-\lambda(\eta)\delta}}$$

Now we have the convergence rate for A_i^n, $i = 1, 2$ is,

$$r_G(\eta, \delta, l_1, l_2, \Lambda) \stackrel{def}{=} \left|\frac{A_1^n(\eta)}{A_1^{n-2}(\eta)}\right| = \left|\frac{A_2^n(\eta)}{A_2^{n-2}(\eta)}\right|$$

$$(11)\qquad = \sqrt{\left|\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_2} e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_2} e^{-\lambda(\eta)\delta}}\right|}$$

$$\times \sqrt{\left|\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_1} e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_1} e^{-\lambda(\eta)\delta}}\right|}$$

Now we observe the following two facts for the two fractions in (11):

(i) two terms in the numerator and in the denominator have the same product.
(ii) The term with the largest absolute value and the term with the smallest absolute value are in the denominator. Since $\delta < min(l_1, l_2)$, and $\Lambda(\eta)$ is strictly positive.

From the simple algebraic fact we know the norm of each fraction is less than 1.So

$$(12)\qquad r_G(\eta, \delta, l_1, l_2, \Lambda) < 1 \quad \forall \eta \text{ and } \delta \geq 0$$

Even if $\delta = 0$, i.e. there is no overlap, $r_G(\eta, \delta, l_1, l_2, \Lambda) < 1$. And it is a decreasing function of δ, because the larger the δ the bigger the difference between the two terms in the denominator. If there is overlap, i.e. $\delta > 0$, then

$$\lim_{|\eta| \to \infty} r_G(\eta, \delta, l_1, l_2, \Lambda) = \left|\frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)}\right| e^{-2\lambda(\eta)\delta} \to 0$$

since $\Lambda(\eta)$ is a pseudo-differential operator from $H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$, So

$$\sup_{\eta} r_G(\eta, \delta, l_1, l_2, \Lambda) = c < 1$$

then we can simply take $\rho = c$, since from equation (8), we have

$$\| \nabla u_i^n \|^2_{L^2(\Omega_i)} = \int_{\Gamma_i} \frac{\partial u_i^n}{\partial n_i} u_i^n = \sum_{\eta} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (e^{2\lambda(\eta)\delta} - e^{-4\lambda(\eta)l_i} e^{-2\lambda(\eta)\delta})$$

and by Poincaré inequality we have the estimate of the convergence rate in H^1 norm.

If there is no overlap i.e. $\delta = 0$, and $\lim_{|\eta|\to\infty} \left| \frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)} \right| \to 1$ then

$$\lim_{|\eta|\to\infty} r_G(\eta, \delta, l_1, l_2, \Lambda_1, \Lambda_2) \to 1 \quad (e.g.\ \Lambda_i = constant > 0)$$

But now we have $u_i^n \in H^1(\Omega_i)$, so

$$\frac{\partial u_i^n}{\partial n_i} \in H^{-\frac{1}{2}}(\Gamma_i), \qquad u_i^n \in H^{\frac{1}{2}}(\Gamma_i) \qquad \Rightarrow \qquad \frac{\partial u_i^n}{\partial n_i} u_i^n \in L_1(\Gamma_i) \qquad n = 0, 1, \ldots$$

and $\quad \sum_\eta \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \infty$

Now $\forall \epsilon > 0$, we can have a positive integer M such that

$$\sum_{|\eta|\geq M} \lambda(\eta) \mid A_i^0(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \frac{\epsilon}{2}$$

Also we can find N large enough such that for $n > N$,

$$\sum_{|\eta|< M} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \frac{\epsilon}{2}$$

since (12) is still true. Then

$$\| \nabla u_i^n \|^2_{L^2(\Omega_i)} = \sum_{\eta} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i})$$

$$= \sum_{|\eta|< M} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) + \sum_{|\eta|\geq M} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i})$$

$$\leq \frac{\epsilon}{2} + \sum_{|\eta|\geq M} \lambda(\eta) \mid A_i^0(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \epsilon$$

again by Poincaré inequality this shows that the domain decomposition algorithm is convergent even with no overlap. By the trace theorem or just look at the expression

$$\| u_i^n \|^2_{H^{1/2}(\Gamma_i)} = \sum_{\eta} (1 + |\eta|)^{\frac{1}{2}} |\hat{u}_i^n(0, \eta)|^2$$

$$= \sum_{\eta} (1 + |\eta|)^{\frac{1}{2}} (1 - e^{-2\lambda(\eta)l_i})^2 \leq C \| \nabla u_i^n \|^2_{L^2(\Omega_i)}$$

We can get the convergence of $u_i^n|_{\Gamma_i} \to 0$ in $H^{1/2}(\Gamma_i)$. That completes the proof. □

We can actually do the similar analysis (or just let $\Lambda = \infty$, $\Lambda = 0$) to get the convergence rate $r_D(\eta)$ for passing Dirichlet data, (classical SAM), and $r_N(\eta)$ for passing Neumann data at the artificial boundary.

$$r_D(\eta) \stackrel{def}{=} \sqrt{\frac{e^{-\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_1}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_1}e^{-\lambda(\eta)\delta}}} \times \sqrt{\frac{e^{-\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_2}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_2}e^{-\lambda(\eta)\delta}}} \tag{13}$$

$$r_N(\eta) \stackrel{def}{=} \sqrt{\frac{e^{-\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_1}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_1}e^{-\lambda(\eta)\delta}}} \times \sqrt{\frac{e^{-\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_2}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_2}e^{-\lambda(\eta)\delta}}} \tag{14}$$

It is easy to see that, if there is overlap, i.e. if $\delta > 0$ then

$$0 < r_D(\eta) < r_N(\eta) < 1, \qquad and \qquad \delta \to 0,\ r_N(\eta) \to 1,\ r_D(\eta) \to 1,\ \forall \eta$$

So without overlap, these two algorithms alone do not have contraction factor on the error and thus do not converge. It is also not hard to see that the convergence rate $r_G(\eta)$ for the generalized SAM is a weighted form of the classical SAM $r_D(\eta)$.

If we do some asymptotic analysis for the most interesting case when $l_1 \approx l_2 \gg \delta$, then

$$\begin{aligned} r_G(\eta) &\approx \left|\frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)}\right| e^{-2\lambda(\eta)\delta} \\ r_D(\eta) &\approx e^{-2\lambda(\eta)\delta} \\ r_N(\eta) &\approx e^{-2\lambda(\eta)\delta} \end{aligned}$$

Remark: Here we see that with small overlap the classical SAM only have a good convergence property on the highly oscillatory part in the solution, which corresponds to the well-known facts that iterative methods for elliptic PDE smooth out the solution very quickly. For the generalized SAM we can also have a good control of the smooth part or any range of modes in the solution by an appropriate choice of operator Λ approximating the Dirichlet to Neumann operator.

Definition Two positive operators Λ_1, Λ_2 are said to be equivalent in some Hilbert space H if

$$\exists 0 < C_1 < C_2 \quad such\ that \quad \forall u \in H\ 0 \le C_1(\Lambda_1 u, u) \le (\Lambda_2 u, u) \le C_2(\Lambda_1 u, u)$$

where $(\cdot, \cdot)$ is the inner product in the Hilbert space.

COROLLARY 2.1. *For the generalized SAM (2) - (3), if the operators Λ_i is equivalent to the Dirichlet to Neumann operator in $H^{1/2}(\partial\Omega_i)$, then we have uniform convergence, i.e. $0 < r_G(\eta, \delta, l_1, l_2, \Lambda_1, \Lambda_2) < \rho < 1, \forall \eta, \delta$.*

Another interesting case is when the Dirichlet boundary condition at the physical boundary is changed to the Neumann boundary condition. Actually for generalized

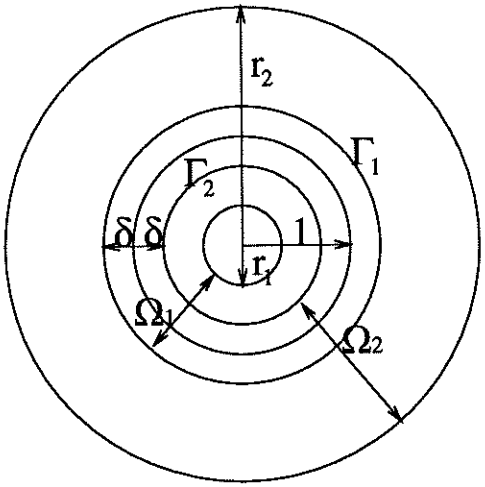
SAM we will get exactly the same result.

COROLLARY 2.2. *For the generalized SAM (2) - (3), if the Dirichlet data at $\partial\Omega$ is changed to Neumann data, then Claim1 still holds.*

Remark: All these analysis can be easily extended to higher dimension case or the case where the whole domain can be split into two strips instead of two rectangles.

Model Problem2:

$$\begin{cases} -\triangle u = 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$



$\Omega = \{(x,y)|r_1 \le \sqrt{x^2+y^2} \le r_2\}$
$\Omega_1 = \{(x,y)|r_1 \le \sqrt{x^2+y^2} \le 1+\delta\}$, $\Omega_2 = \{(x,y)|1-\delta \le \sqrt{x^2+y^2} \le r_2\}$
$0 \le r_1 \le 1-\delta \le 1 \le 1+\delta \le r_2 < \infty$
$\Gamma_1 = \{(x,y)|\sqrt{x^2+y^2} = 1+\delta\}$, $\Gamma_2 = \{(x,y)|\sqrt{x^2+y^2} = 1-\delta\}$

Claim2:
For model problem2 Claim1 holds for the generalized additive Schwarz alternating method, (2) - (3).

Remark: If $0 < r_1 < r_2 < \infty$ then it is very easy to see that model problem2 can be reduced to model problem1 by an analytical transformation. But in the case $r_1 = 0$, i.e. $\Omega_1 \cap \partial\Omega = 0$ which is a very typical case in two or multi-subdomain situation, we can not use the standard Poincaré inequality as in the proof of claim1 to bound the H^1 norm by the H^1 seminorm. We are going to use the following lemma which is a direct extension of the lemma2 in [12] and proof can be found there.

LEMMA 2.3. *Let Ω be a bounded, open, smooth domain in R^N and γ_0 be an open(relative to $\partial\Omega$) subset of $\partial\Omega$. If $\Lambda : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is a bounded positive operator, then there is a positive constant C such that*

$$\| u \|_{L_2(\Omega)} \leq C \left(\| \nabla u \|_{L_2(\Omega)} + \| \frac{\partial u}{\partial n} + \Lambda u \|_{H^{-\frac{1}{2}}(\gamma_0)} \right) \tag{15}$$

for all $u \in H^1(\Omega)$ satisfying: $\Delta u = 0$ in Ω weakly, where n denotes the outward unit normal to $\partial\Omega$.

Proof of Claim2:
In polar coordinates $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$ So if we do Fourier transform in θ, we can write u_i^n in the following form.

$$u_i^n = c_i^n + \sum_{m=1}^{\infty} (a_{i,m}^n r^m + b_{i,m}^n r^{-m}) e^{im\theta} \qquad in\ \Omega_i, \qquad i = 1, 2$$

where $a_{i,m}, b_{i,m}, c_i^n$ are determined by the boundary condition. By the homogeneous Dirichlet boundary condition at $\partial\Omega$, we have

$$b_{1,m}^n = -a_{1,m}^n r_1^{2m}, \qquad a_{2,m}^n = -b_{2,m}^n r_2^{-2m} \qquad m = 1, 2, \ldots$$

By divergence theorem

$$\begin{aligned} \int_{\Omega_1} |\nabla u_1^n|^2 &= \int_{\Gamma_1} \frac{\partial u_1^n}{\partial n} u_1^n = \sum_{m=1}^{\infty} |a_{1,m}^n|^2 \frac{m}{1+\delta}[(1+\delta)^{2m} - r_1^{4m}(1+\delta)^{-2m}] \\ \int_{\Omega_2} |\nabla u_2^n|^2 &= \int_{\Gamma_2} \frac{\partial u_2^n}{\partial n} u_2^n = \sum_{m=1}^{\infty} |b_{2,m}^n|^2 \frac{m}{1-\delta}[(1-\delta)^{-2m} - r_2^{-4m}(1-\delta)^{2m}] \end{aligned} \tag{16}$$

Denote $\Lambda(m)$ to be the Fourier transform of the positive operator Λ in θ. From the generalized boundary condition at the artificial interface $\Gamma_1(r = 1+\delta)$ we have

$$\left| \frac{a_{1,m}^n}{b_{2,m}^{n-1}} \right| = \left| \frac{[\frac{m}{1+\delta} + \Lambda(m)] r_2^{-2m}(1+\delta)^m + [\frac{m}{1+\delta} - \Lambda(m)](1+\delta)^{-m}}{[\frac{m}{1+\delta} + \Lambda(m)](1+\delta)^m + [\frac{m}{1+\delta} - \Lambda(m)] r_1^{2m}(1+\delta)^{-m}} \right| \tag{17}$$

Similarly at $\Gamma_2(r = 1-\delta)$ we have

$$\left| \frac{b_{2,m}^n}{a_{1,m}^{n-1}} \right| = \left| \frac{[\frac{m}{1-\delta} + \Lambda(m)] r_1^{2m}(1-\delta)^{-m} + [\frac{m}{1-\delta} - \Lambda(m)](1-\delta)^m}{[\frac{m}{1-\delta} + \Lambda(m)](1-\delta)^{-m} + [\frac{m}{1-\delta} - \Lambda(m)] r_2^{-2m}(1-\delta)^m} \right| \tag{18}$$

and $c_1^n = c_2^n$. By the homogeneous boundary condition $c_i^n = 0$ for $n \geq 2$.

We can define the convergence rate for the generalized SAM method for our

problem in each mode to be,

$$r_G(m,\delta,r_1,r_2,\Lambda) \stackrel{def}{=} \left|\frac{a^n_{1,m}(\eta)}{a^{n-2}_{1,m}(\eta)}\right| = \left|\frac{b^n_{2,m}(\eta)}{b^{n-2}_{2,m}(\eta)}\right|$$

$$(19) = \sqrt{\left|\frac{\frac{m}{1+\delta}[r_2^{-2m}(1+\delta)^m + (1+\delta)^{-m}] - \Lambda(m)[(1+\delta)^{-m} - r_2^{-2m}(1+\delta)^m]}{\frac{m}{1-\delta}[(1-\delta)^{-m} + r_2^{-2m}(1-\delta)^m] + \Lambda(m)[1-\delta)^{-m} - r_2^{-2m}(1-\delta)^m]}\right|}$$

$$\times\sqrt{\left|\frac{\frac{m}{1-\delta}[(r_1^{2m}(1-\delta)^{-m} + (1-\delta)^m] - \Lambda(m)[(1-\delta)^m - r_1^{2m}(1-\delta)^{-m}]}{\frac{m}{1+\delta}[(1+\delta)^m + r_1^{2m}(1+\delta)^{-m}] + \Lambda(m)[(1+\delta)^m - r_1^{2m}(1+\delta)^{-m}]}\right|}$$

$$< \sqrt{max\left(\frac{1-\delta}{1+\delta}\left[\frac{r_2^{-2m}(1+\delta)^m + (1+\delta)^{-m}}{(1-\delta)^{-m} + r_2^{-2m}(1-\delta)^m}\right], \frac{(1+\delta)^{-m} - r_2^{-2m}(1+\delta)^m}{(1-\delta)^{-m} - r_2^{-2m}(1-\delta)^m}\right)}$$

(20)

$$\times\sqrt{max\left(\frac{1+\delta}{1-\delta}\left[\frac{(r_1^{2m}(1-\delta)^{-m} + (1-\delta)^m}{(1+\delta)^m + r_1^{2m}(1+\delta)^{-m}}\right], \frac{(1-\delta)^m - r_1^{2m}(1-\delta)^{-m}}{(1+\delta)^m - r_1^{2m}(1+\delta)^{-m}}\right)}$$

If we look at each possible combination of (20), by the same reason as in the proof of Claim1 we can easily see that

$$r_G(m,\delta,r_1,r_2,\Lambda) < 1, \qquad \forall m \geq 1,\ \delta \geq 0$$

except for the combination of

$$\sqrt{\left[\frac{(1+\delta)^{-m} - r_2^{-2m}(1+\delta)^m}{(1-\delta)^{-m} - r_2^{-2m}(1-\delta)^m}\right]\frac{1+\delta}{1-\delta}\left[\frac{(r_1^{2m}(1-\delta)^{-m} + (1-\delta)^m}{(1+\delta)^m + r_1^{2m}(1+\delta)^{-m}}\right]}$$

$$= \sqrt{\left[\frac{(1+\delta)^{-m+1} - r_2^{-2m}(1+\delta)^{m+1}}{(1-\delta)^{-m+1} - r_2^{-2m}(1-\delta)^{m+1}}\right]\left[\frac{(r_1^{2m}(1-\delta)^{-m} + (1-\delta)^m}{(1+\delta)^m + r_1^{2m}(1+\delta)^{-m}}\right]}$$

By the homogeneous Dirichlet boundary condition, we must have $m \geq 1$. So

$$(1-\delta)^{-m+1} \geq 1 \geq (1+\delta)^{-m+1} \geq r_2^{-2m}(1+\delta)^{m+1} \geq r_2^{-2m}(1-\delta)^{m+1}$$

So in all we have

$$(21) \qquad r_G(m,\delta,r_1,r_2,\Lambda) < 1, \qquad \forall m \geq 1,\ \delta \geq 0$$

if $\delta > 0$, then $\lim_{m\to\infty} r_G(m,\delta,r_1,r_2,\Lambda) = \sqrt{\left|\frac{\frac{m}{1-\delta} - \Lambda(m)}{\frac{m}{1-\delta} + \Lambda(m)}\right|\left|\frac{\frac{m}{1+\delta} - \Lambda(m)}{\frac{m}{1+\delta} + \Lambda(m)}\right|}\left(\frac{1-\delta}{1+\delta}\right)^m \to 0$

So

$$\sup_{0\leq m<\infty} r_G(m,\delta,r_1,r_2,\Lambda) = c < 1$$

If $0 < r_1 < r_2 < \infty$ by Poincaré inequality $\| u_i^n \|_{H^1(\Omega_i)} \leq C \| \nabla u_i^n \|_{L_2(\Omega_i)}$ then we can take c to be the convergence rate for the generalized SAM method in H^1.

If $r_1 = 0$, i.e. in Ω_1 we do not see the boundary information, we need to use lemma 2.3 in Ω_1.

$$\begin{aligned}
\| u_1 \|_{L_2(\Omega_1)} &\leq C\left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| \frac{\partial u_1}{\partial n_1} + \Lambda u_1 \|_{H^{-\frac{1}{2}}(\Gamma_1)} \right) \\
&\leq C\left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| \frac{\partial u_2}{\partial n_2} \|_{H^{-\frac{1}{2}}(\Gamma_1)} + \| \Lambda u_2 \|_{H^{-\frac{1}{2}}(\Gamma_1)} \right) \\
&\leq C\left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| u_2 \|_{H^1(\Omega_2)} \right) \\
&\leq C\left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| \nabla u_2 \|_{L_2(\Omega_2)} \right)
\end{aligned}$$

where C denote some generic constant.

If $\delta = 0$, (21) is still true. Then by the same argument of Claim1 and again the use of lemma 2.3 (if $r_1 = 0$) we can complete the proof of claim2. □

Claim3:
For model problem1 (or 2), Claim1 (or 2) is true for the generalized multiplicative Schwarz alternating method, (4) - (5).

Remark: If we look at (9), (10), (17) and (18), each expression is not necessary less than 1 when the relative size of each subdomain is arbitrary, i.e. each one sided(half) iteration may not be contractive. But a complete iteration will balance this effect and as a whole the generalized SAM is still convergent which agrees with the general energy estimate by P.L.Lions. (see [12]) In most usual application, we have comparable size of subdomains which are much larger than the overlapping size. Then each iteration will contribute about the same to the convergence.

3. Extension to General Equations in General Two Subdomain case. For the more general equation, we replace the simple Laplace operator by the following elliptic operator,

$$P(D)u(\boldsymbol{x}) = -\nabla \cdot (a(\boldsymbol{x}) \nabla u(\boldsymbol{x})) + \vec{b}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) + c(\boldsymbol{x}) u(\boldsymbol{x}) \tag{22}$$

where $\boldsymbol{x} \in R^N$, $a(\boldsymbol{x})$, $\vec{b}(\boldsymbol{x})$, $c(\boldsymbol{x})$ are smooth and $a(\boldsymbol{x}) > 0$, $c(\boldsymbol{x}) \geq 0$. Though we can not do Fourier analysis for (22). But as long as we have the coercivity assumption as in [12], i.e. $\exists\ c > 0$, such that $\forall u \in H_0^1(\Omega)$

$$\int_\Omega a(\boldsymbol{x}) |\nabla u(\boldsymbol{x})|^2 + \vec{b}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) u(\boldsymbol{x}) + c(\boldsymbol{x}) u^2(\boldsymbol{x}) d\boldsymbol{x} \geq c \int_\Omega |\nabla u(\boldsymbol{x})|^2 d\boldsymbol{x} \tag{23}$$

If $u(\boldsymbol{x})$ satisfies the homogeneous equation, $P(D)u(\boldsymbol{x}) = 0$, then by divergence theorem we can get

$$\int_\Omega a(\boldsymbol{x}) |\nabla u(\boldsymbol{x})|^2 + \vec{b}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) u(\boldsymbol{x}) + c(\boldsymbol{x}) u^2(\boldsymbol{x}) d\boldsymbol{x} = \int_{\partial\Omega} \frac{\partial u(\boldsymbol{x})}{\partial n} a(\boldsymbol{x}) u(\boldsymbol{x}) \tag{24}$$

By Poincaré inequality we can bound the H^1 norm of the solution by the boundary data. Now the only difference is that we have two bounded positive weight functions

$a(\boldsymbol{x})$ and $c(\boldsymbol{x})$. But we have the spectral equivalence of the general elliptic differential operator to the Laplace operator in finite domain (see [8]). Thus we also have the spectral equivalence for the corresponding Dirichlet to Neumann operator.

For the general geometry in two subdomain case, if the artificial boundaries are orthogonal to the real boundary, then by a smooth transformation we can transform them to one of the geometries in the model problems. If the artificial boundaries are not orthogonal to the real boundary or there are corners in the artificial boundaries, i.e. the Dirichlet to Neumann operator maybe singular at some points, then the convergence rate may deteriorate, which depends how singular it is, but the generalized SAM still converges as long as the Dirichlet to Neumann operator is a positive operator and $\int_{\Gamma_i} \frac{\partial u_i^n}{\partial n_i} u_i^n < \infty$ which is true by the trace theorem for Lipschitz domain. This can be seen from either the proof of the model problem or the proof of P.L.Lions by using energy estimate.(see [12])

THEOREM 3.1. *The generalized additive or multiplicative SAM (2)-(5) ($\lambda_1 = \lambda_2 = 1$) for elliptic differential operator (22) in two subdomain case is convergent if*

$$\Lambda_i = \Lambda : \ H^{\frac{1}{2}}(\partial\Omega_i) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega_i) \qquad i = 1, 2$$

is a strictly positive operator. If there is a uniform overlap, i.e. $dis(\Gamma_1, \Gamma_2) = \delta > 0$ then there is a constant $0 < \rho < 1$ such that

$$\| u - u_i^n \|_{H^1} \leq \rho^n \| u - u_i^0 \|_{H^1} \ \text{ in } \Omega_i, \ i = 1, 2$$

where ρ depends on Λ, the size of Ω_i, the overlapping size δ and the partial differential equations.

If we look at the analysis in the model problems, we can actually see that the convergent property of the generalized SAM is better than the classical SAM due to two reasons,
(i) The generalized SAM inherits the same factor from the classical SAM, that is because of the overlap, maximum principle or iterated projections takes effect.
(ii) The "positivity" of the Dirichlet to Neumann operator, which gives convergence even without overlap.

Both reasons are still true in the general case. In the most interesting case the subdomains are of a comparable size and is much larger than the overlap. Then by asymptotic analysis we can see that the the overlap will give a contraction rate of $e^{-c\delta}$ (see [11]), where δ is the size of the overlap. The constant c will depend on the differential operator and the relative size of the subdomains. Also if we expand the solution in terms of the eigenfunctions of the elliptic operator, then c will be bigger for the larger eigenvalues, i.e. the highly oscillatory parts. So the classical SAM can have a very slow convergence speed for the smooth part of the solution which is the most important part of the elliptic PDEs. But if coupled with the second factor in the generalized SAM, then we also have a good control on the smooth part of the solution, if the coefficients and boundary are smooth which do not give rise to singularity in the Dirichlet to Neumann operator. Now let us do some formal analysis on these two factors in the general case using operator formulation.

Let us denote

$$\Omega_{12} = \Omega_1 \cap \Omega 2 \quad \Omega_{11} = \Omega_1 \backslash \Omega_{12} \quad \Omega_{22} = \Omega_2 \backslash \Omega_{12}$$

We define the projection operator

$$P_i : \ H^1(\Omega_i) \rightarrow H^1(\Omega_{ii})$$

to be the restriction from Ω_i to Ω_{ii}. Denote $D_i(\tilde{D}_i)$ to be the Dirichlet to Neumann operator in $\Omega_i(\Omega_{ii})$ on Γ_i. Define R_i to be the restriction operator from $H^1(\Omega_i)$ to $H^{1/2}(\Gamma_i)$. Define R_i^* to be the right adjoint operator of R_i, i.e. $R_i R_i^* = I$ and $\forall g \in H^{1/2}(\Gamma_i)$,

$$P(D)(R_i^* g) = 0, \qquad R_i^* g = g \ at \ \Gamma_i \qquad R_i^* g = 0 \ at \ \partial\Omega_i \setminus \Gamma_i$$

Denote $\tilde{R}_i, \tilde{R}_i^*$ to be the corresponding operator in Ω_{ii}

If we look at one cycle of iteration on the error function $e_i^n = u - u_i^n$ in Ω_i for the generalized additive SAM (2), (3), formally we have

$$(25) \quad e_i^n = R_i^*(\Lambda_i + D_i)^{-1}(\Lambda_i - \tilde{D}_j)\tilde{R}_j P_j R_j^*(\Lambda_j + D_j)^{-1}(\Lambda_j - \tilde{D}_i)\tilde{R}_i P_i(e_i^{n-2})$$

$\{i, j\} = \{1, 2\}$, $i \neq j$. In two extreme cases,
(i) In classical SAM case, (25) is reduced to

$$e_i^n = R_i^* \tilde{R}_j P_j R_j^* \tilde{R}_i P_i(e_i^{n-2})$$

(ii) In no overlap case, (25) is reduced to

$$e_i^n = R_i^*(\Lambda_i + D_i)^{-1}(\Lambda_i - D_j)(\Lambda_j + D_j)^{-1}(\Lambda_j - D_i)R_i(e_i^{n-2})$$

So the convergence of the generalized SAM can be thought as the product of the two factors if the overlap size is small. The first factor of convergence is the projection operator. Using the equivalence of elliptic operator, we can see from the model problems that the contraction of the projection operator depends on the relative size of the overlap and the subdomains. We can also use the maximal principle to see this as in [11], since the error function u_i^n satisfy $P(D)u_i^n = 0$. The second factor is $(\Lambda_i + D_i)^{-1}(\Lambda_i - \tilde{D}_j)$ which is due to the convex combination of u and $\frac{\partial u}{\partial n}$ at the artificial boundary,

$$(26) \qquad \begin{aligned} u_1^n &= (\Lambda_1 + D_1)^{-1}(\Lambda_1 - \tilde{D}_2)u_2^{n-1} \qquad at \ \Gamma_1 \\ u_2^n &= (\Lambda_2 + D_2)^{-1}(\Lambda_2 - \tilde{D}_1)u_1^{n-1} \qquad at \ \Gamma_2 \end{aligned}$$

and

$$(27) \qquad \begin{aligned} (\Lambda_1 + D_1)^{-1}(\Lambda_1 - \tilde{D}_2) &= I - (\Lambda_1 + D_1)^{-1}(D_1 + \tilde{D}_2) \\ (\Lambda_2 + D_2)^{-1}(\Lambda_2 - \tilde{D}_1) &= I - (\Lambda_2 + D_2)^{-1}(D_2 + \tilde{D}_1) \end{aligned}$$

where Λ_i, D_i, $\tilde{D}_i$ are all positive operators. The contraction rate for the second factor is totally determined by the spectral behavior of $\Lambda_i^{-1} D_j$ and $\Lambda_j^{-1}\tilde{D}_i$.

Remark: We can also analyze this expression in more detail by eigenfunction decomposition as in the model problems. Some discussions on the eigendecompositions of the interface operators after discretization can be found in [3],[4].

4. Interpretation of the Generalized SAM in Other Domain Decomposition Context. We first interpret the generalized SAM in the linear algebra context. As formulated in [16],[15] we can regard the SAM after discretization as using block Jacobi(additive) or Gauss-Seidel(multiplicative) to solve an equivalent enhanced linear system instead of the original linear system. Suppose we need to solve linear system $Ax = f$ after discretization of the corresponding elliptic PDE. In two subdomain case, the linear system can be partitioned as

$$\begin{pmatrix} A_{11} & A_{1l} & A_{1r} & A_{12} \\ A_{l1} & A_{ll} & A_{lr} & A_{12} \\ A_{r1} & A_{rl} & A_{rr} & A_{r2} \\ A_{21} & A_{2l} & A_{2r} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_l \\ x_r \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \tag{28}$$

Let n_1, n_l, n_r, n_2 be the dimension of the subvectors respectively. Usually $n_l, n_r \ll n_1, n_2$ and the subvectors x_l, x_r contain only unknowns in the "vincity" of a subdomain interface. This linear system can be easily transformed to a enhanced system

$$\begin{pmatrix} A_{11} & A_{1l} & A_{1r} & 0 & 0 & A_{12} \\ A_{2l} & A_{ll} & A_{lr} & 0 & 0 & A_{l2} \\ 0 & C_{ll} & C_{lr} & -C_{ll} & -C_{lr} & 0 \\ 0 & -C_{rl} & -C_{rr} & C_{rl} & C_{rr} & 0 \\ A_{r1} & 0 & 0 & A_{rl} & A_{rr} & A_{r2} \\ A_{21} & 0 & 0 & A_{2l} & A_{2r} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_l \\ \tilde{x}_r \\ \tilde{x}_l \\ x_r \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_l \\ 0 \\ 0 \\ f_r \\ f_2 \end{pmatrix} \tag{29}$$

It is shown in [15] that (29) is a preconditioned enhanced system that is equivalent to (28) iff matrix $C \stackrel{def}{=} \begin{pmatrix} -C_{ll} & C_{lr} \\ C_{rl} & -C_{rr} \end{pmatrix}$ is non-singular. If we choose x_l, x_r properly and a discretization with local support then $A_{12}, A_{l2}, A_{r1}, A_{21}$ may all vanish even with a minimal overlap. So the only coupling for the enhanced system (29) is by equations

$$\begin{pmatrix} C_{ll} & C_{lr} \\ C_{rl} & C_{rr} \end{pmatrix} \begin{pmatrix} x_l \\ \tilde{x}_r \end{pmatrix} = \begin{pmatrix} C_{ll} & C_{lr} \\ C_{rl} & C_{rr} \end{pmatrix} \begin{pmatrix} \tilde{x}_l \\ x_r \end{pmatrix} \tag{30}$$

which can be viewed as an analogous version of the continuous coupling at the interface. If we take $C_{ll} = I_{nl}$, $C_{rr} = I_{nr}$, $C_{lr} = C_{rl} = 0$ then this is the classical SAM version. We can also use the 1st or 2nd order localized version of the generalized SAM in section 5. The choice of coupling matrix C can be thought as interfacial preconditioning whose purpose is to decouple the enhanced system as much as possible. If we know the discrete version of the Dirichlet to Neumann operator then we can totally decouple it into two subsystems.

Another interpretation of this result is as following: the classical proof of convergence of the domain decomposition method by iterated projection only relies on how to decompose the whole function space V into subspaces of function V_i, such that $V = \Sigma_i V_i$.(see [10],[17]). For domain decomposition the estimate of the convergence rate only depends on the overlapping size and the geometry of the subdomain. What we show here is that if we pass a "convex" combination of Dirichlet and Neumann data at the artificial boundary, using the "positivity" of the Dirichlet to Neumann

operator, we obtain a projection which gives an extra contraction in the energy(H^1 norm).

5. Numerical Implementation of the Generalized SAM. We see from the previous sections that the convergence rate of the generalized SAM depend on the geometry of the subdomains, the overlapping size of the subdomains, the PDE and on the choice of the positive operator at the artificial boundary between each subdomain. We have already shown that theoretically the best choice is the Dirichlet to Neumann operator which is a global operator in most cases and is not practical in numerical implementation. Motivated by the absorbing type boundary condition proposed by Engquist and Majda([7]), we will use some local operator to approximate the Dirichlet to Neumann operator by asymptotic expansion. We again start with a model problem

$$\begin{cases} -\triangle u + u = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{31}$$

where $\Omega, \Omega_1, \Omega_2$ and the generalized SAM are exactly the same as in model problem1 in section2. From the analysis in model problem1 we know the convergence rate of the generalized SAM in each frequency mode is

$$r_G \approx \left| \frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)} \right| e^{-2\lambda(\eta)\delta}$$

in the case $l_1 \approx l_2 \gg \delta$. Now $\lambda(\eta) = \sqrt{1+\eta^2}$. Since for elliptic differential equation the smooth part of the solution converges very slowly, we expand $\lambda(\eta)$ at 0.

$$\lambda(\eta) = \sqrt{1+\eta^2} = 1 + \frac{1}{2}\eta^2 + O(\eta^4)$$

So use the first order expansion we can get our first order numerical scheme

$$\Lambda(\eta) = 1 \ \Rightarrow\ \Lambda u = u \tag{32}$$

Use the second order expansion we get our second order numerical scheme

$$\Lambda(\eta) = 1 + \frac{1}{2}\eta^2 \ \Rightarrow\ \Lambda u = u - \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} \tag{33}$$

where τ is the tangential direction. Assume the boundary and the solution make $\frac{\partial^2 u}{\partial \tau^2}$ meaningful.

Now let us do some concrete analysis on the discretized problem. Suppose the grid size is $h \ll 1$ and has one overlapping grid (minimum overlap for numerical scheme), $\delta = h$. Then the highest possible frequency mode is $\eta_{max} = \frac{1}{h}$. Then

$$r_G \approx \max_{0 \le \eta \le \eta_{max}} \left| \frac{\sqrt{1+\eta^2} - \Lambda(\eta)}{\sqrt{1+\eta^2} + \Lambda(\eta)} \right| e^{-2h\sqrt{1+\eta^2}}$$

Differentiate this expression and find the equation for the critical value η_c is

$$\eta\Lambda(\eta) - (1+\eta^2)\Lambda'(\eta) = \eta h(1 + \eta^2 - \Lambda^2(\eta)) \tag{34}$$

For the first order approximation (32), we have $\eta_c = \sqrt{\frac{1}{h}}$ and so

$$r_G \approx \frac{\sqrt{1+h^{-1}}-1}{\sqrt{1+h^{-1}}+1} e^{-2h\sqrt{1+h^{-1}}} \approx 1 - 4h^{\frac{1}{2}}$$

For the second order approximation (33) we have $\eta_c = \sqrt{\frac{2}{h}}$ and so

$$r_G \approx \frac{\sqrt{1+h^{-1}}-(1+h^{-1})}{\sqrt{1+h^{-1}}+(1+h^{-1})} e^{-2h\sqrt{1+2h^{-1}}} \approx 1 - 8h^{\frac{1}{2}}$$

It seems that the first and second order approximation are not much different. But they are both conservative estimates for the worst cases. In real numerical computation the second order scheme converges really faster than the first order scheme if everything is smooth, since then the solutions for the elliptic problems are smooth and that is where the second order approximation to the Dirichlet to Neumann operator is better. We can also see that the corresponding classical SAM has the convergence rate

$$r_D \approx \max_{0 \le \eta \le \eta_{max}} e^{-2\lambda(\eta)\delta} \approx e^{-h} \approx 1 - h$$

Also we notice that the worst convergence occurs near $\eta = 0$, which is the most significant part of the solution. That is why the classical SAM is very sensitive to the overlapping size.

Although we have improved the convergence by generalized SAM, it is clear that the convergence rates for the first and second order schemes still depend on the overlapping size, though we still can improve it by choosing a more suitable constant. Now let us address the question of optimality of our numerical schemes by some careful asymptotic analysis. In the following presentation c always means some constant which is independent of mesh size h and they are not necessary the same.

(i) first order scheme. Let $\Lambda(\eta) = ch^\alpha$ and plug it into the equation(34) for the critical point.

$$\eta_c^2 = c^2h^{2\alpha} + ch^{\alpha-1} - 1 \qquad or \qquad \eta_c = 0 \tag{35}$$

So if
(a) $\alpha \ge 1$ then

$$r_G(0) = \frac{1-ch^\alpha}{1+ch^\alpha} e^{-2h} \approx (1-ch^\alpha)(1-2h) \approx 1 - 2h$$

(b)$0 \le \alpha < 1$ then

$$r_G(0) \approx (1-ch^\alpha)(1-2h) \approx 1 - ch^\alpha$$

and at $\eta_c^2 \approx ch^{\alpha-1}$

$$r_G(\eta_c) = \frac{\sqrt{1+ch^{\alpha-1}}-ch^\alpha}{\sqrt{1+ch^{\alpha-1}}+ch^\alpha} e^{-2h\sqrt{1+ch^{\alpha-1}}} \approx (1-ch^{\frac{1+\alpha}{2}})(1-ch^{\frac{1-\alpha}{2}}) \approx 1 - ch^{\frac{1-\alpha}{2}}$$

So the best choice is $\alpha = \frac{1}{3} \Rightarrow r_G = 1 - ch^{\frac{1}{3}}$
(c) $-1 \leq \alpha < 0$ then

$$r_G(0) \approx (1 - ch^{-\alpha})(1 - 2h) \approx 1 - ch^{-\alpha}$$

at $\eta_c^2 \approx ch^{\alpha-1}$

$$r_G(\eta_c) = \frac{\sqrt{1 + ch^{\alpha-1}} - ch^{\alpha}}{\sqrt{1 + ch^{\alpha-1}} + ch^{\alpha}} e^{-2h\sqrt{1+ch^{\alpha-1}}} \approx (1 - ch^{\frac{1+\alpha}{2}})(1 - ch^{\frac{1-\alpha}{2}}) \approx 1 - ch^{\frac{1+\alpha}{2}}$$

So the best choice is $\alpha = -\frac{1}{3} \Rightarrow r_G = 1 - ch^{\frac{1}{3}}$
(d) $\alpha < -1$ then

$$r_G(0) \approx (1 - ch^{-\alpha})(1 - 2h) \approx 1 - 2h$$

For the first order scheme the optimal convergence rate is $1 - ch^{\frac{1}{3}}$ when we take $\Lambda = O(h^{\frac{1}{3}})$ or $\Lambda = O(h^{-\frac{1}{3}})$. But we prefer to take $\Lambda = O(h^{-\frac{1}{3}})$ since the critical value in this case is at $\eta_c = O(h^{-\frac{2}{3}})$ which is at a much higher frequency than the other choice.

(ii) second order scheme. Let $\Lambda(\eta) = 1 + ch^{\alpha}\eta^2$. Again plug it into equation(34) for the critical value.

(36) $c^2h^{2\alpha+1}\eta^4 + (2ch^{\alpha+1} - h - ch^{\alpha})\eta^2 + (1 - 2ch^{\alpha}) = 0 \qquad or \qquad \eta = 0$

so

$$\eta_c^2 = \frac{(h + ch^{\alpha} - 2ch^{\alpha+1}) \pm \sqrt{(h + ch^{\alpha} - 2ch^{\alpha+1})^2 - 4c^2h^{2\alpha+1}(1 - 2ch^{\alpha})}}{2c^2h^{2\alpha+1}} \qquad or\ \eta_c = 0$$

and at $\eta = 0$, $r_G(0) \approx 0$. We now divide the situation into four cases and do some tedious asymptotic analysis again.

(a) If $\alpha \leq -1$,

$$\eta_c^2 = \frac{(h + ch^{\alpha} - 2ch^{\alpha+1}) \pm \sqrt{(h + ch^{\alpha} - 2ch^{\alpha+1})^2 - 4c^2h^{2\alpha+1}(1 - 2ch^{\alpha})}}{2c^2h^{2\alpha+1}} \approx ch^{-\frac{\alpha+1}{2}}$$

and

$$r_G \approx \left| \frac{\sqrt{1 + ch^{-\frac{\alpha+1}{2}}} - (1 + ch^{\frac{\alpha-1}{2}})}{\sqrt{1 + ch^{-\frac{\alpha+1}{2}}} + (1 + ch^{\frac{\alpha-1}{2}})} \right| e^{-2h\sqrt{1+ch^{-\frac{\alpha+1}{2}}}} \geq 1 - ch$$

(b) If $-1 < \alpha \leq 0$,

$$\eta_c^2 \approx ch^{-(\alpha+1)}$$

and $\qquad r_G \approx 1 - ch^{\frac{1-\alpha}{2}} \geq 1 - ch^{\frac{1}{2}}$
(c) If $0 < \alpha \leq 1$,

$$\eta_c^2 \approx ch^{-(\alpha+1)} \qquad or \qquad \eta_c^2 \approx ch^{-\alpha}$$

and $r_G \approx max(1 - ch^{\frac{1-\alpha}{2}}, (1 - ch^{\frac{\alpha}{2}})(1 - ch^{1-\frac{\alpha}{2}}))$
So the best choice is when $\frac{\alpha}{2} = \frac{1-\alpha}{2}$, i.e. $\alpha = \frac{1}{2}$ and $r_G \approx 1 - ch^{\frac{1}{4}}$.
(d) If $\alpha > 1$,

$$\eta_c^2 \approx ch^{-2\alpha} \qquad or \qquad \eta_c^2 \approx ch^{-1}$$

and $r_G \approx 1 - ch^{\frac{1}{2}}$

So the optimal choice for the second order scheme is

$$\Lambda u = u - ch^{\frac{1}{2}} \frac{\partial^2 u}{\partial \tau^2} \tag{37}$$

(τ is the tangential direction at the interface) for the generalized SAM and we can have a convergence rate which is $1 - ch^{\frac{1}{4}}$.

Remark1: All these analysis can be extended to more general coercive elliptic PDE by the spectral equivalence. In the variable coefficients case, we can (i) either use our first or second order approximation to the homogenized (averaged) equation (ii) or just use the local approximation at the interface. If the coefficients are highly oscillatory, then the first choice is better since the large scale behavior of the solution is more close to the homogenized equation and the highly oscillatory part can be controlled by the solver in each subdomain. If the coefficients are smooth, we suggest to use the second choice since we are using local operators to approximate the global one anyway.

Remark2: In each subdomain instead of using exact solver which is also expensive, we can again use some efficient iterative scheme. Just as in the case of classical SAM, we only need the non-exact solver is accurate enough, i.e. the error e satisfies $P(D)e = \tilde{f}$ and $\tilde{f}$ is small enough such that for e we still have the maximum principle (or iterated projection) and the positivity of the Dirichlet to Neumann operator.

Remark3: One advantage of using local operator is that we can easily incorporate it into the existing domain decomposition codes, since it does not change the structure in the interior of each subdomain. (see [15])

Remark4: We can get an accelerated version of the generalized SAM by using different Λ_i^n in each nth iteration. Then at each iteration a different range of frequency modes can have the best convergence rate, which is similar to the idea of multigrid. We can have a very fast convergence which is almost independent of mesh size. How to choose Λ_i^n for optimal convergence is under further study.

6. Conclusion. Due to the positivity of the Dirichlet to Neumann operator the generalized SAM can have improved convergence properties over the classical SAM so that the generalized SAM for domain decomposition can converge without overlap of subdomains. Since a local operator can not approximate a global operator uniformly in general, we can not expect a convergence rate which is independent of the overlapping size for the generalized SAM.

We will discuss the extension of generalized SAM to multidomain cases and the application to interface problems in some other papers.

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