UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Subspace Correction Multilevel Methods for Elliptic Eigenvalue Problems

Tony F. Chan
Ilya Sharapov

September 1996

CAM Report 96-34

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

1

Subspace Correction Multilevel Methods for Elliptic Eigenvalue Problems

Tony F. Chan [1] and Ilya Sharapov [1]

Abstract: In this work we apply the ideas of domain decomposition and multigrid methods to PDE-based eigenvalue problems represented in two equivalent variational formulations. To find the lowest eigenpair we use a "subspace correction" framework for deriving the multiplicative algorithm for minimizing the objective function in the variational formulations. The results of the asymptotic convergence rate analysis and the global convergence proof as well as some numerical results are presented.

1.1 Introduction

Domain decomposition and multigrid methods are powerful techniques for solving elliptic linear problems. Unfortunately the straightforward implementation of the methods is limited to linear problems and relatively little work has been done for nonlinear applications. The goal of this paper is to analyze the application of the multiplicative Schwarz methods to the eigenvalue problem without linearization. An important distinction of this approach is that the subspace problem is also a generalized eigenvalue problem which allows to apply the algorithm recursively and formulate a multilevel method of optimal complexity.

Solution of eigenvalue problems by multigrid methods using linearization was discussed by Hackbusch ([Hac84]) and McCormick ([McC92]). The idea to use coordinate relaxation applied directly for a matrix eigenvalue problem goes back to the book by Fadeev and Fadeeva [FF63] (1963) where they applied a technique similar to Gauss-Seidel method for minimizing the Rayleigh quotient. This approach was

---

1 Dept. of Math., Univ. of Calif. at Los Angeles, Los Angeles, CA 90095-1555, USA.
e-mail: chan@math.ucla.edu, isharapo@math.ucla.edu.
This work was partially supported by the NSF under contract ASC 92-01266 and ONR under contract ONR-N00014-92-J-1890.
DD9 Proceedings
Editor Petter Bjørstad, Magne Espedal and David Keyes

extended by Kaschiev [Kas88] and Malyasov [Mal92] for PDE-based problems. In this case the resulting method of minimizing the Rayleigh quotient is analogous to the block Gauss-Seidel method for linear problems.

Several domain decomposition-based methods were proposed by Lui [Lui95], in particular the method based on a nonoverlapping partitioning where the interface problem is solved either using a discrete analogue of a Steklov-Poincare operator or using Schur complement-based techniques. The former approach resembles the component mode synthesis method (cf. Bourquin and Hennezel [BH92]) which is an approximation rather than iterative technique for solving eigenproblems. The component mode synthesis was also used by Farhat and Géradin in [FG94]. Stathopoulos, Saad and Fischer [SSF95] considered iterations based on Schur complement of the block corresponding to the interface variables.

Anther way to implement the domain decomposition technique to eigenvalue problem is divide and conquer method proposed by Dongarra and Sorrensen [DS87]. An attempt to link relaxation methods (in particular SOR) to eigenvalue problem was made by Ruhe [Ruh74].

In this work we extend the results of [Mal92] and [Kas88] for the multiplicative Schwartz method by considering the two-level scheme. Convergence is proven for a more suitable class of initial approximations and asymptotic convergence rate analysis is given. We also describe the recursive implementation of the method which results in a multilevel algorithm. Finally we present an alternative variational formulation of the problem which is equivalent mathematically but is more suitable for theoretical considerations.

1.2 Subspace correction for eigenvalue problems

Let us consider the problem of finding the minimal eigenvalue λ and the corresponding eigenvector u of

$$Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{i,j} \frac{\partial u}{\partial x_j} + p(x)u = \lambda u \tag{1.1}$$

$$x \in \Omega, \qquad u\mid_{\partial\Omega} = 0 \qquad a_{i,j} > 0,$$

where Ω is a bounded region in R^2 and $a_{i,j}(x) = a_{j,i}(x)$, $p(x) \geq 0$ are piecewise smooth real functions.

To discretize the problem, we can perform a triangulation of Ω with triangles of quasi-uniform size h and use the standard finite element approach to represent (1.1) as

$$Au = \lambda M u \tag{1.2}$$

where $A = A^T > 0$ and $M = M^T > 0$ are stiffness and mass matrices respectively. The problem of finding the minimal eigenvalue of (1.2) can be viewed as a minimization of the Rayleigh quotient

$$\lambda_1 = \min_u F(u) = \min_u \frac{u^T A u}{u^T M u} \quad . \tag{1.3}$$

In order to apply domain decomposition technique to this problem we can represent Ω as a union of overlapping subdomains with Lipschitz boundaries: $\Omega = \cup_{i=1}^{J}\Omega_i$. Let $\{V_i\}_{i=1}^{J}$ be finite element subspaces corresponding to this partition and let P_i^T denote the orthogonal projection into the subspace V_i, its transpose P_i is the prolongation operator from V_i to H_h^2. We also introduce the M-norm of a vector $\|v\|_M = (v^T M v)^{1/2}$.

A scheme analogous to the multiplicative Schwartz algorithm for solving (1.2) was proposed in [Mal92] and [Kas88]:

Algorithm 1 (Multiplicative subspace correction)
Starting with u^0 for $k = 0$ until convergence
for $i = 1 : J$
find $u^{k+i/J}$ such that

$$F(u^{k+i/J}) = \min_{d_i \in V_i} F(u^{k+(i-1)/J} + P_i d_i) \tag{1.4}$$

end
end
end of the Algorithm

To have some control over the norm of the iterates we can M-normalize the approximations either after each subiteration after a loop over all subdomains is completed.

At each step the algorithm performs a subspace search minimizing the Rayleigh quotient using the correction from the current subspace. We will now show that the minimization (1.4) results in minimizing the Rayleigh quotient for the local $(n_i+1)\times(n_i+1)$ problem, where n_i is the dimension of the current subspace.

Rewrite (1.4) as

$$\rho(u^{k+i/J}) = \min_{\tilde{d}_i} \rho(\tilde{P}_i\tilde{d}_i) = \min_{\tilde{d}_i} \frac{(\tilde{P}_i\tilde{d}_i)^T A(\tilde{P}_i\tilde{d}_i)}{(\tilde{P}_i\tilde{d}_i)^T M(\tilde{P}_i\tilde{d}_i)} = \min_{\tilde{d}_i} \frac{\tilde{d}_i^{\,T}\tilde{A}\tilde{d}_i}{\tilde{d}_i^{\,T}\tilde{M}\tilde{d}_i} ,$$

where

$$\tilde{d}_i = \begin{pmatrix} d_i \\ 1 \end{pmatrix}, \qquad \tilde{P}_i = (\, P_i \quad u^{k+(i-1)/J} \,)$$

and

$$\tilde{A} = \tilde{P}_i^{\,T} A \tilde{P}_i, \qquad \tilde{M} = \tilde{P}_i^{\,T} M \tilde{P}_i \quad . \tag{1.5}$$

Thus the subspace problem is an eigenvalue problem with $\tilde{A}$ and $\tilde{M}$.

The matrices $\tilde{A}$ and $\tilde{M}$ preserve the sparsity of the original matrices A and M except for the last row and column, therefore the minimization subproblem can be efficiently solved.

Lui [Lui96] pointed out that the convergence proof for **Alg.1** presented [Mal92] and [Kas88] has a gap and proved convergence to the smallest eigenvalue λ_1 for a modified algorithm in the case of two subdomains provided that the initial guess u^0 satisfies

$$\lambda_1 < F(u^0) < \lambda_2 \quad . \tag{1.6}$$

This condition is difficult to control unless we use the method as some refinement procedure using it after a good approximation to the lowest eigenmode was produced by some other method.

We can formulate a stronger result that **Alg.1** converges to the lowest eigenpair of (1.2) in case of any number of subdomains and with the more practical assumption that all the components of the initial approximation u^0 are of the same sign.

Theorem 1 *Vectors u^k and the corresponding Rayleigh quotients $\rho(u^k)$ produced by **Alg.1** converge to the lowest eigenmode of discretized problem (1.2) if all the components of the initial approximation u^0 satisfy $u_i^0 > 0$.*

The proof (to appear in a full version of this paper) relies on a natural assumption that the eigenvector we are looking for is not contained in any of the subspaces V_i:

Assumption 1 *For any subspace V_i there is a constant $C_i > \lambda_1$ such that*

$$v^T A v \geq C_i v^T M v \qquad \textit{for any} \qquad v \in V_i \qquad .$$

1.3 Coarse grid correction and multilevel method

In this work we modify the **Alg.1** by adding a coarse grid correction after a loop over the subdomains is completed. By doing so in the case of a linear elliptic problem with sufficient subdomain overlap, the convergence rate becomes independent of both the meshsize and the number of subdomains [BPWX91], [Xu92].

The effect of the coarse grid correction for a model problem of 2-D Laplacian is shown on the presented figures, where h and H are fine and coarse meshsizes respectively. We can see that without the coarse grid the convergence rate is dependent on both the meshsize and the number of subdomains whereas after the coarse grid correction has been added the convergence rate becomes independent of both h and H.

Since the subspace problem is of the same type as the original one i.e. a generalized eigenvalue problem, we can make the algorithm more efficient by applying it recursively. Instead of solving the eigenvalue subproblem over a subdomain by some other method we can apply several iterations of the same algorithm. Applying that recursion to the multiplicative method with coarse grid correction we can view the resulting scheme as a multilevel method and the iterations performed on each level as the smoothing of the solution (in the spirit of the multigrid method). The recursion can be stopped once the subproblems we are solving reached some small enough fixed size.

Though the presented algorithms are sequential we can add some degree of parallelism using multicoloring technique (see for example [CM94]).

1.4 Alternative formulation

A different variational formulation for the symmetric positive definite eigenvalue problem (1.2) was recently proposed by Mathew and Reddy (94) [MR94]. They pointed

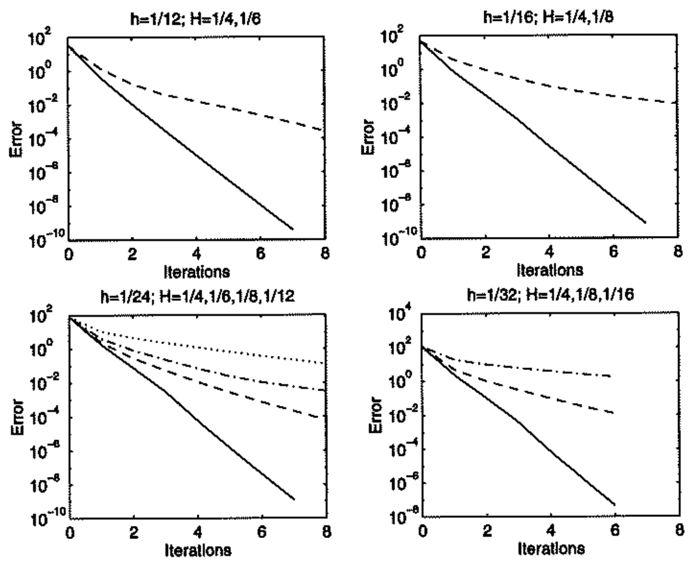

Figure 1.1 Error reduction for the model problem without the coarse grid correction. Higher curves show slow convergence for the large number of subdomains.

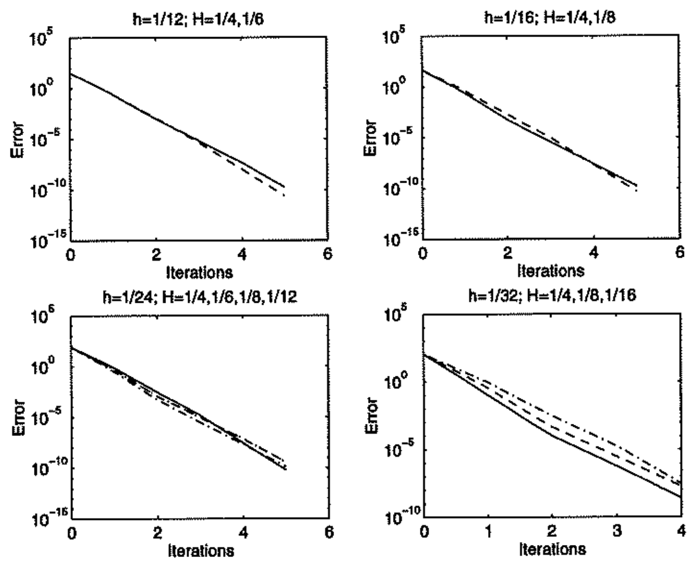

Figure 1.2 Error reduction for the the model problem with the coarse grid correction. Convergence is independent of the number of subdomains.

out that the minimal eigenpair (u_1, λ_1) can be characterized as:

$$J(u_1) = \min_v J(v) \equiv \min_v \left[v^T A v + \mu(1 - v^T M v)^2\right] \tag{1.7}$$

with

$$\lambda_1 = 2\mu - \sqrt{4\mu^2 - 4\mu J(u_1)}$$

and

$$\|u_1\|_M^2 = 1 - \frac{\lambda_1}{2\mu}$$

for any

$$\mu > \lambda_1/2 \quad . \tag{1.8}$$

Unlike the Rayleigh quotient minimization, formulation (1.7) is unconstrained. The μ-term in $J(v)$ serves as a barrier to pull the solution u away from the trivial solution 0.

The subspace problem for (1.7) is again of the same form as the original one with dimension $n_i + 1$ i.e. an eigenvalue problem of this size. For

$$\tilde{P}_i = \begin{pmatrix} P_i & u^{k+(i-1)/J} \end{pmatrix}, \qquad \tilde{d}_i = \begin{pmatrix} d_i \\ \alpha \end{pmatrix},$$

we can write the minimizaition step of the algorithm as

$$J(u^{k+i/J}) = \min_{d_i \in V_i, \alpha} J(\tilde{P}_i \tilde{d}_i)$$

$$= \min_{\tilde{d}_i} \left[(\tilde{P}_i\tilde{d}_i)^T A(\tilde{P}_i\tilde{d}_i) + \mu(1 - (\tilde{P}_i\tilde{d}_i)^T M(\tilde{P}_i\tilde{d}_i))^2 \right]$$

$$= \min_{\tilde{d}_i} \left[\tilde{d}_i^T \tilde{A}\tilde{d}_i + \mu(1 - \tilde{d}_i^T \tilde{M}\tilde{d}_i)^2 \right] \qquad ,$$

where

$$\tilde{A} = \tilde{P}_i^T A\tilde{P}_i, \qquad \tilde{M} = \tilde{P}_i^T M\tilde{P}_i$$

Therefore we can see that for any choise of μ satisfying (1.8), one subspace correction step for formulations (1.3) and (1.7) results in the same reduced generalized eigenvalue problem with matrices (1.5). The application of the multiplicative Schwarz algorithm to both formulations results in the same approximations to the lowest eigenvalue and the approximations to the eigenvector are the same up to scaling.

The objective function in (1.7) is convex near the solution so for the local analysis we can use apply the theory of Multiplicative Schwarz methods for minimization problems (Tai, Espedal, 96) [TE96]. Equivalence of the formulations gives the following assymptotic result

Theorem 2 *The iterates produced by **Alg.1** with coarse grid correction applied to formulations (1.3) and (1.7) (in SPD case) satisfy for k large enough*

$$\|u_1 - u^{k+1}\| \leq (1-\delta)\|u_1 - u^k\| \qquad ,$$

where (u_1, λ_1) is the mininmal eigenpair of (1.2) with $\|u_1\|_M = 1$ for (1.3) and $\|u_1\|_M = 1 - \frac{\lambda_1}{2\mu}$ for (1.7) and the value of $\delta > 0$ is independent of the meshsize h and the number of subdomains J.

The proof of this theorem will be given in a larger version of this paper.

Acknowledgment: The authors wish to thank Jinchao Xu and Xue-Cheng Tai for their helpful suggestions during discussions on the topic of this paper.

References

[BH92] Bourquin F. and Hennezel F. (1992) Application of domain decomposition techniques to modal synthesis for eigenvalue problems. In *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991)*, pages 214–223. SIAM, Philadelphia.

[BPWX91] Bramble J., Pasciak J., Wang J., and Xu J. (1991) Convergence estimates for product iterative methods with applications to domain decomposition. *Math. Comp.* 57(195): 1–21.

[CM94] Chan T. and Mathew T. (1994) Domain decomposition algorithms. In *Acta Numerica*, pages 61–143. Cambridge Univ. Press, Cambridge.

[DS87] Dongarra J. and Sorensen D. (1987) A fully parallel algorithm for symmetric eigenvalue problem. *SIAM J. Sci. Statist. Comput.* 8(2): 139–154.

[FF63] Fadeev D. and Fadeeva V. (1963) *Computational Methods of Linear Algebra.* W.H. Freeman and Company, San Francisco.

[FG94] Farhat C. and Géradin M. (1994) On a component mode synthesis method and its application to incompatible substructures. *Computers and Structures* 51: 459–473.

[Hac84] Hackbusch W. (1984) *Multigrid Methods.* Springer-Verlag.

[Kas88] Kaschiev M. (1988) An iterative method for minimization of the Rayleigh-Ritz functional. In *Computational processes and systemss, No. 6 (Russian)*, pages 160–170. Nauka, Moscow.

[Lui95] Lui S. (1995) Domain decomposition for eigenvalue problems (preprint). Hong Kong Univ. of Science and Tech.

[Lui96] Lui S. (1996) On two Schwarz alternating methods for the symmetric eigenvalue problem (preprint). Hong Kong Univ. of Science and Tech.

[Mal92] Malyasov S. (1992) On the analog of Schwartz method for spectral problems. In *Numerical methods and mathematical modeling (Russian)*, pages 70–79. Otdel Vychisl. Mat., Moscow.

[McC92] McCormick S. (1992) *Multilevel Projection Methods for Partial Differential Equations.* SIAM, Philadelphia.

[MR94] Mathew G. and Reddy V. (1994) Development and analysis of a neural network approach to Pisarenko's harmonic retrieval method. *IEEE Trans. Sig. Proc.* 42(3): 663 – 673.

[Ruh74] Ruhe A. (1974) SOR-methods for the eigenvalue problem with large sparse matrices. *Math. Comput.* 28: 695–710.

[SSF95] Stathopoulos A., Saad Y., and Fischer C. (1995) A Schur complement method for eigenvalue problems (preprint). Comp. Science Dep., Univ. of Minnesota.

[TE96] Tai X.-C. and Espedal M. (1996) Rate of convergence of a space decomposition method and applications liear and nonlinear ellicptic problems (preprint). Univ. of Bergen.

[Xu92] Xu J. (1992) Iterative methods by space decomposition and subspace correction. *SIAM Rev.* 34: 581–613.

DD9 Proceedings
Editor Petter Bjørstad, Magne Espedal and David Keyes