UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Continuum Shock Profiles for Discrete Conservations Laws

Tai-Ping Liu
Shih-Hsien Yu

September 1996

CAM Report 96-35

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

Continuum Shock Profiles for Discrete Conservation Laws

Tai-Ping Liu*
Department of Mathematics, Stanford University
Shih-Hsien Yu†
Department of Mathematics, UCLA

Abstract

We construct continuum shock profiles of finite difference schemes for system of hyperbolic conservation laws through time-asymptotic analysis. Our approach differs markedly from previous ones using march mappings which yield discrete shock profiles with rational speed. Instead, we obtain continuum shock profiles of any speed through strong point-wise estimates of time-asymptotic states. For this we analyze the Green functions for the linearized evolution equations and the spectrum properties of the stationary equations. A profile satisfies the basic conservation property when it is obtained as the limit of profiles of rational speed of increasing denominators as a consequence of continuum dependence. Finally the nonlinear stability of the profiles are shown using the point-wise estimates.

1 Introduction

Consider system of hyperbolic conservation laws

$$u_t + f(u)_x = 0, \quad t \geq 0, \quad x \in \mathbf{R}, \quad u = u(x,t) \in \mathbf{R}^n, \tag{1.1}$$

and finite difference schemes

$$\begin{aligned} u^{m+1}(x) &= u^m(x) + \lambda\left(F[u^m](x+\frac{1}{2}) - F[u^m](x-\frac{1}{2})\right), \\ \lambda &= \frac{\Delta t}{\Delta x}. \end{aligned} \tag{1.2}$$

The purpose of the present paper is to study the continuum shock profiles and their stability. We construct these profiles using the time-asymptotic analysis, particularly the point-wise estimates for evolutionary equations, and also the spectrum analysis for linearized stationary equations. Previous analysis for discrete profiles uses marching mapping, and applies only to waves of rational speed. Our new approach yields continuum profiles for waves of any speed. We also obtain the basic conservation property and nonlinear stability of these profiles.

*Research supported in part by Army Basic Research Grant DAAH 4-94-GS 0045 and NSF Grant DMS-9623025.
†Research supported by ARPA/ONR grant N00014-92-J-1890

We note that the construction of discrete profiles of discrete speed p/q, $(p,q)=1$,[3], [4], [13], requires the shock strength to be much less than $1/q$. This is due to the application of center manifold theorem to the q-th iteration of the linearized scheme. In particular, such an approach cannot be applied to profiles of irration speed.

There is a qualitative difference between profiles of rational speed and those of irrational speed. Profile of rational speed p/q repeats itself after q steps, while a profile of irrational speed never repeats itself on the discrete level. Neverthless, our result shows that the states actually take values from a smooth continuum profile. That the discrete profile does not repeat itself should not be confused with numerical oscillations. Oscillations can arise due to the initial layer or the lack of numerical dissipation. For discussions of related issues see [8] and [9].

Our study of continuum profiles should form a basis for further study of general solutions, which contains interaction of shocks with smooth flows, other shocks, or with initial layers. Such a study has been carried out only for scalar equation [1].

System (1.1) is assumed to be strictly hyperbolic, that is, $f'(u)$ has real and distinct eigenvalues $\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_n(u)$ with eigenvectors normalized as follows:

$$\begin{aligned} f'(u)r_i(u) &= \lambda_i(u)r_i(u), \qquad l_i(u)f'(u)=\lambda_i(u)l_i(u), \\ l_i(u)\cdot r_j(u) &= \delta_{ij}, \qquad i,j=1,2,\cdots,n. \end{aligned} \tag{1.3}$$

The difference scheme (1.2) is in conservative form, and is explicit in that $F[u^m]$ depends only on finite values of the function u^m. We also assume the usual consistency property and (C-F-L) condition, c.f. Section 2.

A travelling wave $\phi(x-dt)$ with speed d is a shock profiles if

$$u^m(x)\equiv\phi(x-\lambda dm), \quad \phi(\pm\infty)=u_\pm \tag{1.4}$$

satisfies (1.2) and conservation property

$$\sum_{j=-\infty}^{\infty}\phi(x+j)-\phi(j)=x(u_+-u_-), \quad x\in\mathbf{R}. \tag{1.5}$$

When the i-characteristic fields is genuinely nonlinear $\nabla\lambda_i(u)\cdot r_i(u)\neq 0$, [5], and scheme (1.2) is dissipative and non-resonant, Section 2, we show that continuum shock profiles exist and are nonlinearly stable.

For works on discrete shock profiles for systems see [3], [4] and [13], for stability of those profiles see [6], [12] and for scalar equation see [2], [10] and [11].

In next section we give precise definition of shock profile and the dissipative difference scheme and derive basic equations. Our point-wise estimates require explicit construction of accurate Green functions, which are done in Section 3 and 4. The construction of continuum shock profile is carried out in Section 5 using an elaborate iteration scheme. The Green functions in near field are constructed based on the linear difference scheme and in far field on the associated viscous conservation laws. The conservation property of the scheme and the approximate profile is used crucially in estimating the tail behavior of the time-asymptotic state through the point-wise estimate of the evolution of the difference solution. This allows us to show that the time asymptotic solution of our linear difference equation depends on $x-st$ only, s the speed of the profile. To construct solutions for the nonlinear equations through iterations we need to study the far field structure of the time asymptotic solution for the linear equation. For this the cancellation effects of the conservative source is important, (2.12) and (2.13). The cancellation is easily seen for continuum viscous conservation laws . To obtain a strong cancellation property of the pointwise estimate for the discrete equations we devise a double iterations and reduce our problem to the study of schemes with constant coefficients. The cancellation property is then obtained through the spectral analysis of the stationary scheme, (5.19), and a uniqueness theorem to obtain the far field sturcture of the linear time asymptotic solution. In this analysis, our thinking is new in studying the interior differencing, (5.21), rather than the exterior differencing c.f. [3]. Further study of the profiles, particularly the conservative property, (1.5), is derived in Section 6. A profile of irrational speed is always conservative and a profile of rational speed can be reparameterize to be conservative. We show these by explicit construction of conservative profiles through a limiting process of approximateing the speed with rational speeds with increasing denominators. Our reasoning is based on the stability property of the solution's dependence on the initial data and the trucation error that an approximate profile gives raise to. This process allows us to study the continuous dependence of the profile on its speed and to make use of the increasingly fine conservative property of any given profile with rational speed of increaing denominator.

The nonlinear stability of profiles are carried out in Section 7. In Appendix we simplify and generalize the previous results on discrete profiles, which form the approximate profiles used in our construction of exact profiles in Section 5.

Acknowledge: The writting of this paper was done while the authors were visiting Institute of Mathematics, Academia Sinica, Taipei, R.O.C.. We would like to thank the Institute for the hospitality.

2 Preliminaries

Consider the finite difference approximation

$$\begin{aligned} u^{m+1}(j) &= \mathcal{L}[u^m](j), \\ \mathcal{L}[u^m](j) &\equiv u^m(j) - \lambda(F[u^m](j+\frac{1}{2}) - F[u^m](j-\frac{1}{2})), \end{aligned} \tag{2.1}$$

of the hyperbolic conservation laws

$$u_t + f(u)_x = 0. \tag{2.2}$$

Here $u^m(x) \sim u(x, m\Delta t)$ and $\lambda = \Delta t/\Delta x$. We assume that (2.2) is strictly hyperbolic:

$$f'(u)r_i(u) = \lambda_i r_i(u), \quad l_i(u)f'(u) = \lambda_i l_i(u), \; i = 1, 2, \cdots, n, \quad \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{2.3}$$

The scheme (2.1) is conservative and consistent with (2.2):

$$F[\bar{u}] = f(\bar{u}) \text{ for any constant state } \bar{u}. \tag{2.4}$$

Assume that the scheme depends on finite $2k$ grid points:

$$F[u^m](j-\frac{1}{2}) = F(u^m(j-k), \cdots, u^m(j+k-1)), \; m = 0, 1, 2, \cdots, \; j = 0, \pm 1, \pm 2, \cdots. \tag{2.5}$$

We also assume a strong (C-F-L) condition:

$$\lambda^{-1} = \frac{\Delta x}{\Delta t} > 4 \max\{|\lambda_i(u)| : i = 1, 2, \cdots, n\} \tag{2.6}$$

for all u under consideration.

Remark: In the above setting we allow the variable $x = j\Delta x = j$ (taking $\Delta x = 1$ for definiteness) as being continuous. With this, due to Peter Lax, we may define the continuous travelling waves for (2.1) as follows:

Definition 2.1. $\phi(x - dt)$ is a continuous travelling wave with speed d for the finite difference scheme (2.1) if

$$u^m(j) = \phi(j - d\lambda m + \bar{x}) \text{ satisfies (2.1) for } j = 0, \pm 1, \pm 2, \cdots, \tag{2.7}$$

$$m = 0, \pm 1, \pm 2, \cdots, \text{ and for any given } \bar{x} \in [0, 1], \text{ and}$$

$$\textstyle\sum_{i=-\infty}^{\infty}(\phi(x + i) - \phi(i)) = x(u_+ - u_-) \text{ for all } x. \tag{2.8}$$

In (2.7), note that

$$u^{m+1}(j) = \phi(j - d\lambda m - d\lambda\Delta t + \bar{x});$$

is the shift of the travelling wave to the right by $\lambda d\Delta t$ and therefore is the wave after the period Δt and thus satisfies (2.1). The conservation requirement (2.8) is consistent with (2.4) and is the discrete analogy of

$$\int_R \psi(x + y) - \psi(y) dy = x(u_+ - u_-)$$

for a travelling wave $\psi(x - dt)$ of the PDE (2.2).

For a continuum travelling $\phi(x - dt)$ of (2.1) we have form (2.8), (2.7) and (2.1), and (2.4) that

$$\begin{aligned} -\lambda d(u_+ - u_-) &= \sum_{j=-\infty}^{\infty} (\phi(j - \lambda d(m + 1) + \bar{x}) - \phi(j - d\lambda m + \bar{x})) \\ &= -\lambda \sum_{j=-\infty}^{\infty} \left(F[\phi](j - \lambda dm + \frac{1}{2}) - F[\phi](j - \lambda dm - \frac{1}{2})\right) \\ &= -\lambda(F(\phi(\infty)) - F(\phi(-\infty))) = -\lambda(f(u_+) - f(u_-)). \end{aligned}$$

Thus $\phi(x - dt)$ satisfies the jump (Rankine-Hugoniot) condition

$$d(u_- - u_+) = f(u_-) - f(u_+). \tag{2.9}$$

It follows that, [5],

$$d = \frac{\lambda_i(u_-) + \lambda_i(u_+)}{2} + O(\epsilon^2) \text{ for some } i, 1 \le i \le n,$$

where $\epsilon \sim |u_- - u_+|$ is of the same order as the strength of the shock. When the i-characteristic field is genuinely nonlinear, $\nabla\lambda_i(u) \cdot r_i(u) \neq 0$, shock (u_-, u_+) for the PDE (2.2) is physical if the following entropy condition holds:

$$\lambda_i(u_+) < d < \lambda_i(u_-). \tag{2.10}$$

For (2.1) discrete travelling waves can be constructed when the speed is rational with respect to α: $\lambda d =$ rational. In Appendix we show that accurate approximate continuum travelling wave $\bar{\phi}$ with speed not necessary rational, can be constructed with the property:

$$\bar{\phi}'(x) = O(1)\epsilon^2 e^{-\epsilon|x|}, \tag{2.11}$$

$$\begin{aligned} u^{m+1}(j) &= u^m(j) - \lambda(F[u^m](j+\frac{1}{2}) - F[u^m](j-\frac{1}{2})) + \mathcal{E}(j - d\lambda m), \quad (2.12) \\ \text{for } u^m(j) &\equiv \bar{\phi}(j - d\lambda m). \end{aligned}$$

Here the error $\mathcal{E}$ is zero if $\bar{\phi}$ is an exact travelling wave, (2.7). The approximate travelling wave is constructed accurately and conservatively:

$$\mathcal{E}(j) = O(1)\epsilon^4 e^{-\epsilon|j|}, \quad \mathcal{F}(j) \equiv \sum_{k=-\infty}^{j} \mathcal{E}(k) = O(1)\epsilon^3 e^{-\epsilon|j|}, \tag{2.13}$$

For our study of the construction and the nonlinear stability of the travelling waves we require the scheme (2.1) to be dissipative and satisfies certain non-resonance condition. We now describe these notions. Linearize (2.1) around a constant state $\bar{u}$:

$$\begin{aligned} w^{m+1}(j) = w^m(j) - \lambda \textstyle\sum_{i=1}^{2k} \bar{F}_i(w^m(j-k+i) - w^m(j-k+i-1)), \quad (2.14) \\ \bar{F}_i \equiv \frac{\partial F}{\partial w_i}(\bar{u}). \end{aligned}$$

From (2.4),

$$\begin{aligned} \sum_{i=0}^{2k} \bar{F}_i \cdot \bar{r}_l = \bar{\lambda}_l \bar{r}_l, \quad l = 1, 2, \cdots, n, \\ \bar{\lambda}_l \equiv \lambda_l(\bar{u}), \qquad \bar{r}_l \equiv r_l(\bar{u}). \end{aligned}$$

Thus, (2.14) becomes

$$\begin{aligned} w_l^{m+1}(j) &= w_l^m(j) - \lambda \sum_{i=1}^{2k} f_{il} \ (w_l^m(j-k+i) - w_l^m(j-k+i-1)), \quad (2.15) \\ w^m(j) &\equiv \sum_{l=1}^{n} w_l^m(j)\bar{r}_l, \quad f_{il} \equiv \bar{F}_i \cdot \bar{r}_l. \end{aligned}$$

Plug in the Fourier modes to the above equations, we obtain

$$\begin{aligned} A_l(\xi) &= 1 + \lambda \sum_{i=1}^{2k} f_{il} \left(e^{\sqrt{-1}(-k+i)\xi} - e^{\sqrt{-1}(-k+i-1)\xi} \right), \quad (2.16) \\ w_l^m(j) &\equiv (A_l(\xi))^m e^{-\sqrt{-1}j\xi}. \end{aligned}$$

Definition 2.2 The scheme (2.2) is dissipative if for some constants $C_1 > 0$, $C_2 > 0$,

$$\begin{aligned} 1 - C_1|\xi|^2 < |A_l(\xi)| < 1 - C_2|\xi|^2. \quad (2.17) \\ \text{for } |\xi| \ll 1, \quad l = 1, 2, \cdots, n, \end{aligned}$$

and is non-resonant if

$$|A_l(\xi)| = 1 \text{ only for } \xi = 0,\ l = 1, 2, \cdots, n. \tag{2.18}$$

Remarks: A non-resonant dissipative scheme possesses discrete travelling waves, Appendix. A sufficient condition for a scheme to be dissipative is that the diagonalized equation (2.15) has nonnegative coefficients. The random walk associated with such a scalar equation has nonzero variant, which is the effective viscosity coefficient, c.f. (3.6).

Examples.
(1) Lax-Friedrichs scheme

$$u^{m+1}(j) = \frac{u^m(j+1) + u^m(j-1)}{2} - \lambda \frac{f(u^m(j+1)) - f(u^m(j-1))}{2}.$$

It is dissipative, but resonant, because the even grids, $m+j$ =even, and odd grids, $m+j$ =odd, are independent. However, if one consider only one of these grids, say the even ones, then it becomes non-resonant when the scheme is iterated twice:

$$\begin{aligned} v^{m+1} &= \frac{v^m(j+1) + 2v^m(j) + v^m(j-1)}{4} \\ &-\frac{\lambda}{4}\left\{\left[f(v^m(j+1)) + f(v^m(j)) + 2f\left(\frac{v^m(j+1)+v^m(j)}{2} - \lambda\frac{f(v^m(j+1)) - f(v^m(j))}{2}\right)\right.\right. \\ &- \left.\left.\left[f(v^m(j)) + f(v^m(j-1)) + 2f\left(\frac{v^m(j)+v^m(j-1)}{2} - \lambda\frac{f(v^m(j)) - f(v^m(j-1))}{2}\right)\right]\right\}. \\ &v^m(j) \equiv u^{2m}(2j). \end{aligned}$$

(2) Godunov scheme

$$u^{m+1}(x) = u^m(x) - \lambda(F(u^m(j+1), u^m(j)) - F(u^m(j), u^m(j-1))).$$

Here $F(u_-, u_+)$ is the value $\phi(0)$ of the solution $\phi(x/t)$ of the Riemann problem (u_-, u_+). The linearized scheme is

$$\begin{aligned} w_l^{m+1}(j) &= (1 + 2\lambda\bar{\lambda})w_l^m(j) - \lambda\bar{\lambda}_l(w_l^m(j+1) + w_l^m(j) \\ &\quad \text{for } \bar{\lambda}_l \leq 0, \\ w_l^{m+1}(j) &= (1 - 2\lambda\bar{\lambda})w_l^m(j) + \lambda\bar{\lambda}_l(w_l^m(j) + w_l^m(j-1) \\ &\quad \text{for } \bar{\lambda}_l \geq 0. \end{aligned} \tag{2.19}$$

The scheme is non-resonant. It is dissipative if the characteristic speeds $\lambda_1,\ \lambda_2, \cdots, \lambda_n$ are nonzero. Godunov scheme is well-known to possess sharp stationary shocks and does not smooth out stationary waves.

(3) Modified Lax-Friedrichs scheme

$$u^{m+1}(j) = \frac{u^m(j+1) + u^m(j) + u^m(j-1)}{3} - \lambda\frac{f(u^m(j+1)) - f(u^m(j-1))}{2}. \tag{2.20}$$

It is non-resonant and is dissipative under the strong (C-F-L) condition

$$\frac{1}{\lambda} > 3\max_n(|\lambda_1(u)|,\cdots,|\lambda_n(u)|). \tag{2.21}$$

For simplicity, in the remaining of the present paper, we will carry out our analysis for this scheme.

For convenience, we make the travelling waves $\bar{\phi}$ stationary, $d=0$, with the change of variable $x \to x - dt$ so that (2.20) becomes

$$u^{m+1}(j) = \frac{u^m(x+d+1)+u^m(x+d)+u^m(x+d-1)}{3} - \frac{\lambda}{2}\left(f(u^m(j+d+1)) - f(u^m(j+d-1))\right). \tag{2.22}$$

Consider a perturbation $v^m(x)$ of an approximate travelling wave ϕ. We have from (2.22), (2.7) and (2.8) that

$$\begin{aligned}
v^{m+1}(j) &= \frac{v^m(j+d+1)+v^m(j+d)+v^m(j+d-1)}{3} \\
&\quad -\frac{\lambda}{2}\{[f(v^m(j+d+1)+\phi(j+d+1)) - f(\phi(j+d+1))] \\
&\quad -[f(v^m(j+d-1)+\phi(j+d-1)) - f(\phi(j+d-1))]\} + \mathcal{E}(j), \\
v^m(j) &\equiv u^m(j) - \phi(j)
\end{aligned}$$

or by Taylor expansion

$$\begin{aligned}
v^{m+1}(j) &= L(v^m)(j) + Q[v^m](j+d+1) - Q[v^m](j+d) + \mathcal{E}(j). \\
L(v)(x) &\equiv \frac{v(x+d+1)+v(x+d)+v(x+d-1)}{3} \qquad (2.23)\\
&\quad -\frac{\lambda}{2}[f'(\phi(x+d+1))v(x+d+1) - f'(\phi(x+d-1))v(x+d-1)], \\
Q[v](x) &= O(1)(|v(x)|^2 + |v(x-1)|^2). \qquad (2.24)
\end{aligned}$$

For the discrete anti-derivative $w^m(j)$ we have

$$\begin{aligned}
w^{m+1}(j) &= K(w^m)(j) + Q[v^m](j+d+1) + Q[v^m](j+d) + \mathcal{F}(j), \qquad (2.25\\
K[w](x) &= \frac{w(x+d+1)+w(x+d)+w(x+d-1)}{3} \\
&\quad -\frac{\lambda}{2}\left(f'(\phi(x+d+1))(w(x+d+1)-w(x+d)) - f'(\phi(x+d))(w(x+d)-w(x+d-1))\right); \\
w^m(j) &\equiv \sum_{k=-\infty}^{0} v^m(j+k).
\end{aligned}$$

This is diagonalized as follows:

$$\begin{aligned}
w_q^{m+1}(j) &= K_q(w_q^m)(j) + M_q[e^{-\epsilon|x|}w^m](j) + Q_q[v^m](j+d+1) \qquad (2.26)\\
&\quad + Q_q[v^m](j+d) + \mathcal{F}_q(j),
\end{aligned}$$

$$
\begin{aligned}
w^m(j) &\equiv \sum_{q=1}^{n} w_q^m(j) r_q(j), \\
K_q(h)(x) &= \frac{h(x+d+1)+h(x+d)+h(x+d-1)}{3} \qquad (2.27)\\
&\quad -\frac{\lambda}{2}(\lambda_q(x+d+1)(h(x+d+1)-h(x+d))) - \frac{\lambda_q}{2}(h(x+d)-h(x+d-1)), \\
\lambda_q(x) &\equiv \lambda_q(\phi(x)), \quad r_q(x) \equiv r_q(\phi(x)).
\end{aligned}
$$

Here Q_q and $\mathcal{F}_q$ satisfy the same estimates as in (2.23) and (2.13),

$$
\begin{aligned}
M_q[e^{-\epsilon|x|}w^m](x) &= -\{-(l_q(x+d)-l_q(x))\cdot w^m(x+d) \qquad (2.28)\\
&\quad +[\frac{\lambda\lambda_q(x+d+1)}{2}(l_q(x+d+1)-l_q(x+d))\cdot w^m(x+d) \\
&\quad +\frac{\lambda\lambda_q(x+d)}{2}(l_q(x+d)-l_q(x+d-1))\cdot w^m(x+d-1)]\} \\
&\quad +\frac{\lambda}{2}(-l_q(x)+l_q(x+d+1))f'(\phi(x+d+1))v^m(x+d+1) \\
&\quad -\frac{\lambda}{2}(-l_q(x-1)+l_q(x+d))f'(\phi(x+d))v^m(x+d)) \\
&\quad +\frac{1}{3}(l_q(x+d+1)-2l_q(x+d)+l_q(x+d-1))w^m(x+d) \\
&\quad -\frac{1}{3}(l_q(x+d)-l_q(x+d+1))v^m(x+d+1) \\
&\quad +\frac{1}{3}\left((l_q(x)-l_q(x+d))v^m(x+d+1)+(l_q(x)-l_q(x+d-1))v^m(x+d)\right) \\
&\quad -\frac{1}{3}(l_q(x+d)-l_q(x+d+1))v^m(x+d+1) \\
&= O(1)\{|(d-\lambda\lambda_q(x))\lambda_q'(x)|+|\lambda_q''|\}\|w^m(x+d)\| \\
&\quad +O(1)|\lambda_q'(x)|\cdot(\|v^m(x+d+1)\|+\|v^m(x+d)\|).
\end{aligned}
$$

3 Green Functions for Transverse Fields

The approximate Green function $G_q(y,m') = G_q(x,m;y,m')$ for the operator K_q of (2.27) is to satisfy

$$
G_q(y,m) = \delta(y-x) \qquad (3.1)
$$

and to minimize the expression $K_q^* G_q(\cdot,m') - G_q(\cdot,m'-1)$ for all m', $1 \le m' \le m$, where

$$
\begin{aligned}
K_q^*\ h\ (y) &= \tfrac{1}{3}(h(y-d+1)+h(y-d)+h(y-d-1)) \qquad (3.2)\\
&\quad +\tfrac{1}{2}\lambda\lambda_q(y+1)(h(y-d+1)+h(y-d)) - \tfrac{1}{2}\lambda\lambda_q(y)(h(y-d)+h(y-d+1)).
\end{aligned}
$$

From (2.26) and (2.27) we have the Duhamel's principle:

$$
\begin{aligned}
w_q^{m+1}(x) &= \int_{-\infty}^{\infty} G_q(y,0)w_q^0(y)dy + \sum_{m'=1}^{m}\int_{-\infty}^{\infty}\left(K_q^*G_q(y,m')-G_q(y,m'-1)\right)w_q^{m'}(y)dy \qquad (3.3)\\
&\quad +\sum_{m'=1}^{m}\int_R G_q(y,m')\left(M_q[e^{-\epsilon|x|}w^{m'}](y)+Q_q[w^{m'}](y+d+1)+Q_q[w^{m'}](y+d+1)+\mathcal{F}_q(y)\right)dy.
\end{aligned}
$$

The construction of $G_q(y,m')$, $q \neq i$ is divided into three regions depending on ϵ, which is equivalent to the strength $\|u_- - u_+\|$ of the shock,:

Region I: $m - \frac{1}{\epsilon} \leq m' \leq m$.
We set $G_q(y,m')$ to be the exact discrete solution $\bar{G}_q(y,m')$ of (3.1) and

$$\begin{aligned} &\bar{K_q}^*\bar{G}_q(y,m') - \bar{G}_q(y,m'-1) = 0, \\ &\bar{K}_q^* h(y) = \frac{1}{3}\left(h(y-d+1) + h(y-d)h(y-d-1)\right) \\ &\quad + \frac{1}{2}\lambda\lambda_q(x)\left\{[h(y-d+1)+h(y-d)] - [h(y-d)+h(y-d-1)]\right\} \end{aligned} \tag{3.4}$$

That is, $\bar{G}_q$ solve the adjoint problem with constant coefficient $\lambda_q(x)$ replacing $\lambda_q(y)$.
Region II: $1 \leq m' \leq 1 - \frac{2}{\epsilon}$.

The Green function $\bar{G}_q$ for this far field case is well approximated by that for the continuum equation:

$$\begin{aligned} w_{qt} + \bar{\lambda}_q(x) w_{qx} = \mu_q(x) w_{qxx} \\ \bar{\lambda}_q(x) = \lambda\lambda_q(x) - d, \quad \mu_q(x) = \tfrac{1}{3} - \tfrac{1}{2}(\lambda\lambda_q(x))^2, \end{aligned} \tag{3.5}$$

which is dervied as an approximation of (2.26), (2.27) by Taylor expansion. We adopt the procedure of constructing approximate Green function for (3.5) in [7]:

$$\begin{aligned} g_q(y,s) = (4\pi\mu_q(y)(t-s))^{-1/2}\exp -\left(\tfrac{\bar{\lambda}_q(y)(m_q(y)-m_q(x)+t-s)^2}{4\mu(y)(t-s)}\right), \\ m'(y) = (\lambda_q(y))^{-1}, \quad q \neq i. \end{aligned} \tag{3.6}$$

For the discrete equation we set

$$\bar{\bar{G}}_q(y,m') \equiv \textstyle\sum_{j=-\infty}^{\infty} g_q(j,m')\delta(y-j). \tag{3.7}$$

Finally for the Region III, $m - \frac{2}{\epsilon} \leq m' \leq m - \frac{1}{\epsilon}$ we consider the linear interpolation of $\bar{G}_q$ and $\bar{\bar{G}}_q$:

$$G_q(y,m') = \begin{cases} \bar{G}_q(y,m') & \text{for } m - \frac{1}{\epsilon} \leq m' \leq m \\ \bar{\bar{G}}_q(y,m') & \text{for } 1 \leq m' \leq m - \frac{2}{\epsilon}, \\ \epsilon(m - m' - \frac{1}{\epsilon})\bar{\bar{G}}_q(y,m') + \epsilon(\frac{2}{\epsilon} - (m-m'))\bar{G}_q(y,m') & , \\ \quad \text{for } m - \frac{2}{\epsilon} \leq m' \leq m - \frac{1}{\epsilon}. \end{cases} \tag{3.8}$$

We now study G_q and, in particular, access its accuracy by computing $K_q^* G_q(y,m') - G_q(y,m'-1)$: For $\bar{G}_q$ we take the Fourier transformation $\mathbf{T}$ over $\xi \in (-\pi,\pi)$:

$$\begin{aligned} \mathbf{T}(\bar{K}_q^* \circ S_d)(\xi) = \tfrac{1}{3}(e^{\sqrt{-1}\xi} + 1 + e^{-\sqrt{-1}\xi}) - \tfrac{1}{2}\lambda\lambda_q(x)(e^{\sqrt{-1}\xi} - e^{-\sqrt{-1}\xi}), \\ S_d\, g(x) \equiv g(x+d), \end{aligned}$$

and so there exists $C' > 0$ such that

$$\mathbf{T}(\bar{K}_q^* \circ S_d)(\xi) = 1 - \sqrt{-1}\lambda\lambda_q(x)\xi - \tfrac{1}{3}\xi^2 + O(\xi^3),$$
$$\text{for } |\xi| < |m-m'|^{-1/2+1/10},$$
$$|\mathbf{T}(\bar{K}_q^* \circ S_d)(\xi)| < 1 - C''(m-m')^{-1+\frac{1}{10}}$$
$$\text{for } |\xi| \geq |m-m'|^{-\frac{1}{2}+\frac{1}{10}}.$$

Denote by

$$\bar{\delta}(z) = \begin{cases} 1 & \text{if } z = 0, \\ 0 & \text{if } z \neq 0. \end{cases}$$

We have

$$(\bar{K}_q^*)^{m-m'}\bar{\delta}(\bar{y}-x) = \begin{cases} 0 & \text{if } \bar{y}-x+(m-m')d \notin \mathbf{Z}, \\ (\bar{K}_q^* \circ S_d)^{m-m'}\bar{\delta}(\bar{y}-x+(m-m')d) & \text{if } \bar{y}-x+(m-m')d \in \mathbf{Z}. \end{cases}$$

Apply the discrete inverse Fourier transformation to $(\bar{K}_q^* \circ S_d)^{m-m'}$:

$$\begin{aligned}
(\bar{K}_q^*)^{m-m'}\bar{\delta}(\bar{y}-x) &= \tfrac{1}{2\pi}\textstyle\int_R e^{\sqrt{-1}(\bar{y}-x+(m-m')d)\xi}(\mathbf{T}(\bar{K}_q^* \circ S_d)^{m-m'}(\xi)d\xi \\
&\equiv \mathrm{I} + \mathrm{II}, \\
|\mathrm{I}| = \Big|\tfrac{1}{2\pi}\textstyle\int_{|m-m'|^{-\frac{1}{2}+\frac{1}{10}} \leq |\xi| \leq \pi} & e^{\sqrt{-1}(\bar{y}-x+(m-m')d)\xi}[\mathbf{T}(\bar{K}_q^* \circ S_d)]^{m-m'}(\xi)d\xi\Big| \\
&\leq \tfrac{1}{\pi}(\pi - (m-m')^{-\frac{1}{2}+\frac{1}{10}})(1-\bar{C}(m-m')^{-1+\frac{1}{10}})^{m-m'} \\
&= O(1)e^{-C(m-m')} \text{ for some } C > 0,
\end{aligned}$$

$$\begin{aligned}
|\mathrm{II}| &= \Big|\tfrac{1}{2\pi}\textstyle\int_{|\xi| \leq |m-m'|^{-\frac{1}{2}+\frac{1}{10}}} e^{\sqrt{-1}(\bar{y}-x+(m-m')d)\xi}\left[1 - \sqrt{-1}\lambda\lambda_q(x)\xi - \tfrac{1}{3}\xi^2 + O(\xi^3)\right]^{m-m'} d\xi\Big| \\
&= \Big|\tfrac{1}{2\pi\sqrt{m-m'}}\textstyle\int_{|\xi'| \leq |m-m'|^{\frac{1}{10}}} e^{\sqrt{-1}(\eta+\sqrt{(m-m')}d)\xi'}\left[1 - \sqrt{-1}\lambda\lambda_q(x)\tfrac{\xi'}{\sqrt{m-m'}} - \tfrac{1}{3}\tfrac{\xi'^2}{m-m'} + O(\tfrac{\xi'^3}{(m-m')^{3/2}})\right]^{m-m'} d\xi'\Big| \\
&= \tfrac{1}{2\pi\sqrt{m-m'}}\textstyle\int_{|\xi'| \leq |m-m'|^{\frac{1}{10}}} e^{\sqrt{-1}(\eta\xi' + (-\frac{1}{3}+\frac{(\lambda\lambda_q(x))^2}{2})\xi'^2)}(1 + O(1)\tfrac{\xi'^3}{\sqrt{m-m'}})d\xi' \\
&= (1 + O(1)(m-m')^{-\frac{1}{2}+\frac{2}{10}})\frac{1}{\sqrt{4\pi(\frac{1}{3}-\frac{1}{2}(\lambda\lambda_q(x))^2)}}e^{-\frac{\eta^2}{4(\frac{1}{3}-\frac{1}{2}(\lambda\lambda_q(x))^2)}}, \\
\eta &\equiv \tfrac{\bar{y}-x+(d-\lambda\lambda_q(x))(m-m')}{\sqrt{m-m'}}.
\end{aligned}$$

From above and (3.4) we have

$$\bar{G}_q(y,m') = (1 + O(1)(m-m')^{-1/5}) \sum_{\bar{y}-x+(m-m')d \in \mathbf{Z}} \tag{3.9}$$
$$\left\{\frac{1}{\sqrt{4\pi(\frac{1}{3}-\frac{1}{2}(\lambda\lambda_q(x))^2)(m-m')}}e^{-\frac{[\bar{y}-x+(\lambda\lambda_q(x)-d)(m-m')]^2}{4(\frac{1}{3}-\frac{1}{2}(\lambda\lambda_q(x))^2)(m-m')}}\delta(y-\bar{y})\right.$$
$$\left. + O(1)e^{-C(m-m')}char_{(x-(m-m'),x+(m-m'))}\delta(y-\bar{y})\right\}.$$

The last expression in (3.9) is to indicate that the discrete operator $\bar{K}_q^*$ has finite speed. We also have

$$\textstyle\int_R G_q(y,m')dy = 1 \quad G_q(y,m') \geq 0 \tag{3.10}$$
$$G_q(ym') = 0 \quad \text{for } |y-x| \geq m-m', \quad 0 \leq m' \leq m,$$

followed from the fact that our scheme is three points and that it is dissipative and thus $\bar{K}_q^*$ has nonnegative coefficients. The accuracy of $\bar{G}_q$ is easily assessed

$$\begin{aligned}
&K_q^*\bar{G}_q(y,m') - \bar{G}_q(y,m'-1) = (K_q^* - \bar{K}_q^*)\bar{G}_q(y,m')\\
&= \tfrac{\lambda}{2}(\lambda_q(y+1) - \lambda_q(x))(\bar{G}_q(y-d,m') + \bar{G}_q(y-d+1,m'))\\
&-\tfrac{\lambda}{2}(\lambda_q(y) - \lambda_q(x))(\bar{G}_q(y-d-1,m') + \bar{G}_q(y-d-1,m'))\\
&= \tfrac{\lambda}{2}(\lambda_q(y+1) - \lambda_q(y))(\bar{G}_q(y-d+1,m') + \bar{G}_q(y-d-1,m'))\\
&+\tfrac{\lambda}{2}(\lambda_q(y+1) - \lambda(x))(\bar{G}_q(y-d+1,m') - \bar{G}_q(y-d-1,m')).
\end{aligned}$$

Recall that $\lambda_q(y)$ is the characteristic speed of the wave $\phi(y)$, and so by (2.11) we have

$$\begin{aligned}
K_q^*G_q(y,m') - G_q(y,m'-1) = O(\epsilon^2)e^{-\epsilon|y|}(\bar{G}_q(y-d,m') + \bar{G}_q(y-d-1,m'))\\
+O(1)\epsilon(\bar{G}_q(y-d+1,m') - \bar{G}_q(y-d-1,m')),
\end{aligned}$$

where the bounded function $O(1)$ has derivative, which is of order $O(1)\epsilon$.

For the continuum $\bar{\bar{G}}_q(y,m')$ of (3.6) and (3.7) we have, (c.f. [7], where $\mu_q(y)$ is a constant. Here additional error $|m-m'|^{-3/2}$ below arises due to the discrete approximation, (3.7).)

$$\bar{K}_q^*\bar{\bar{G}}_q(y,m') - \bar{\bar{G}}_q(y,m'-1) = [O(1)\epsilon^2 e^{-\epsilon|y|} + O(1)|m-m'|^{-3/2}]\bar{\bar{G}}_q(y,m'), \quad \text{for } m' \le m - \tfrac{1}{\epsilon}. \tag{3.11}$$

$$\bar{\bar{G}}_q(u,m') = \begin{cases} O(1)\bar{\bar{G}}_q^+(y,m') \text{ for } y > 0 \\ O(1)\bar{\bar{G}}_q^-(y,m') \text{ for } y < 0, \end{cases} \tag{3.12}$$

$$\bar{\bar{G}}_q^{\pm}(y,m) \equiv \textstyle\sum_{\bar{y}-x+(m-m')d\in\mathbb{Z}}(m-m')^{-\frac{1}{2}}\left[e^{-\frac{(\bar{y}-x+\lambda_\pm(m-m')+C)^2}{D(m-m')}} + e^{-\frac{(\bar{y}-x+\lambda_\pm(m-m')-C)^2}{D(m-m')}}\right]\delta(y-\bar{y})$$

for some positive constant C and any constant $D \ge 4 \cdot \max_y(\mu_q(y))$.

Finally, we consider the linear interpolation of $\bar{G}_q$ and $\bar{\bar{G}}_q$ for the region $m - \frac{2}{\epsilon} \le m' \le m - \frac{1}{\epsilon}$, (3.8):

$$\begin{aligned}
G_q(y,m') &= \epsilon(m-m'-\tfrac{1}{\epsilon})\bar{\bar{G}}_q(y,m') + \epsilon(\tfrac{2}{\epsilon} - (m-m'))\bar{G}_q(y,m'), \qquad (3.13)\\
K_q^*G_q(y,m') - G_q(y,m'-1) &= \epsilon(m-m'-\tfrac{1}{\epsilon})[K_q^*\bar{\bar{G}}(y,m') - \bar{\bar{G}}_q(y,m'-1)]\\
&+\epsilon(\tfrac{2}{\epsilon} - (m-m'))[K_q^*\bar{G}(y,m') - \bar{G}_q(y,m'-1)]\\
&+\epsilon[\bar{\bar{G}}(y,m'-1) - \bar{G}_q(y,m'-1)],
\end{aligned}$$

In other words, the error in (3.13) is linear combination of those for $\bar{G}_q$ and $\bar{\bar{G}}_q$ plus $\epsilon[\bar{\bar{G}}_q(y,m'-1) - \bar{G}_q(y,m'-1)]$. This last expression is estimated by comparing the expressions (3.9) and (3.6), (3.7):

$$\begin{aligned}
\bar{\bar{G}}_q(y,m'-1) - \bar{G}_q(y,m'-1) \quad &= \quad O(\epsilon^{\frac{1}{2}})[\bar{\bar{G}}_q(y,m'-1) + \bar{G}_q(y,m'-1)],\\
&\text{for} \quad m - \frac{2}{\epsilon} \le m' \le m - \frac{1}{\epsilon}.
\end{aligned}$$

Here we have used the property that $\lambda_q(x) - \lambda_q(y) = O(1)\epsilon$. For notational convenience, we write

$$G_q(y, m') = \sum_{\bar{y}-x+(m-m')d\in\mathbf{Z}} g_q(y, m')\delta(y - \bar{y})$$

and the same with $\bar{G}_q$ and $\bar{\bar{G}}_q$. The Green function is function of $(x, m; y, m')$. From our explicit form of this approximate Green function, one has that

$$\begin{aligned} \Delta_x G(x, m; y, m') &= -\Delta_y G(x, m; y + 1, m') + O(1)\epsilon G(x, m; y, m'). \qquad (3.14) \\ \Delta_\xi h(\xi) &\equiv h(\xi) - h(\xi - 1). \end{aligned}$$

The Duhamel's principle (3.3) becomes:

$$\begin{aligned} w_q^{m+1}(x) = &\sum_{\bar{y}-x+md\in\mathbf{Z}} g_q(\bar{y}, 0)w_0^m(\bar{y}) \qquad (3.15) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d\in\mathbf{Z} \\ 1\le m'\le m}} g_q(\bar{y}, m')(M_q[e^{-\epsilon|x|}w^{m'}](\bar{y}) + Q_q[w^{m'}](\bar{y} + d + 1) + Q_q[w^{m'}](\bar{y} + d) + \mathcal{F}_q(\bar{y}) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d\in\mathbf{Z} \\ 1\le m'\le m-\frac{1}{\epsilon}}} \left[O(1)\epsilon^2 e^{-\epsilon|\bar{y}|} + O(1)|m - m'|^{-3/2}\right] \bar{\bar{g}}_q(\bar{y}, m')w_q^{m'}(\bar{y}) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d\in\mathbf{Z} \\ m-\frac{2}{\epsilon}\le m'\le m}} O(1)\left\{\epsilon^2 e^{-\epsilon|\bar{y}|}[\bar{g}_q(\bar{y} - d, m') + \bar{g}_q(\bar{y} - d - 1, m')]\right. \\ &\left. +\epsilon[\bar{g}_q(\bar{y} - d + 1, m') - \bar{g}_q(\bar{y} - d - 1, m')]\right\} w_q^{m'}(\bar{y}) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d\in\mathbf{Z} \\ m-\frac{2}{\epsilon}\le m'\le m-\frac{1}{\epsilon}}} O(1)\epsilon^{3/2}\left[\bar{g}_q(\bar{y}, m' - 1) + \bar{\bar{g}}_q(\bar{y}, m' - 1)\right] w_q^{m'}(\bar{y}). \end{aligned}$$

4 Green Function for Compressible Field

The construction of approximate Green function $G_i(x, m; y, m') = G_i(y, m')$ for the compressible field $\bar{\lambda}_i = \lambda\lambda_i(x) - d$, (3.5) follows the same procedure as in the last section. We set $\bar{G}_i(y, m')$ as in (3.1), (3.4) with $q = i$. The continuum function $g_i(x, t; y, s) = g_i(y, s)$, instead of (3.6), is set as follows:

$$g_i(y, s) = (4\pi\mu_i(y)(t - s))^{-\frac{1}{2}} \exp\left(-\frac{[m'(y)^{-1}(m(y) - m(x) + t - s)]^2}{4\mu_i(y)(t - s)}\right), \qquad (4.1)$$

$$(\bar{\lambda}_i A)' + (\mu_i A')' = 0, \quad m' = (\bar{\lambda}_i + \frac{2\mu_i A'}{A})^{-1}. \qquad (4.2)$$

This generalizes slightly that of [7], where the diffusion coefficient μ_i is constant. There are two solutions of (4.2), one with $A_+(\infty) = 1$, and the other $A_-(\infty) = 1$:

$$A_+(y) = \bar{\lambda}_i(\infty) \int_y^\infty \frac{1}{\mu_i(z)} e^{-\int_z^y \frac{\bar{\lambda}_i(\tau)}{\mu_i(\tau)} d\tau} dz$$

$$m'_+(y) = (\bar{\lambda}_i + \frac{2\mu_i A'_+}{A})^{-1}(y) = (-\bar{\lambda}_i + \frac{2\bar{\lambda}_i(\infty)}{A_+})^{-1}(y)$$

$$= \begin{cases} -\frac{1}{\lambda_i(-\infty)} \text{ as } y \to -\infty \\ \frac{1}{\lambda_i(\infty)} \text{ as } y \to \infty, \end{cases}$$

$$A_-(y) = \lambda_i(-\infty) \int_{-\infty}^{y} \frac{1}{\mu_i(z)} e^{-\int_z^y \frac{\bar{\lambda}_i(\tau)}{\mu_i(\tau)} d\tau} dz,$$

$$m'(y) = (\bar{\lambda}_i + \frac{2\mu_i A'_-}{A_-})(y) = (-\bar{\lambda}_i + \frac{2\bar{\lambda}_i(-\infty)}{A_-})^{-1}(y)$$

$$= \begin{cases} \frac{1}{\lambda_-} \text{ as } y \to -\infty \\ -\frac{1}{\lambda_+} \text{ as } y \to \infty. \end{cases}$$

We use A_+, m_+ for $y > 0$, A_-, m_- for $y < 0$.
From above we deduce

$$g_{is} + (\bar{\lambda}_i g_i)_y + (\mu_i(y) g_i)_{yy} = O(1)\epsilon^2(\epsilon + (t-s)^{-1/2}) e^{-\epsilon|y|}(1 + H^4) g_i.$$

The function g_i has the property that, for any fixed $k > 0$, c.f. [7],

$$\begin{aligned} \bar{\bar{g}}_i(y,s) &\equiv (1 + H^k) g_i(y,s) = \begin{cases} e^{-\epsilon|x|} g_i^-(y,s) \text{ for } xy < 0, \\ g_i^+(y,s) \text{ for } xy > 0, \end{cases} \\ g_i^{\pm}(y,s) &= (4\pi(t-s))^{-\frac{1}{2}} \left[e^{-\frac{(x-y-\lambda^{\pm}(t-s)+C)^2}{D(t-s)}} + e^{-\frac{(x-y-\lambda^{\pm}(t-s)-C)^2}{D(t-s)}} \right]. \end{aligned} \tag{4.3}$$

The Duhamel's principle (3.3) becomes:

$$\begin{aligned} w_i^{m+1}(x) = &\sum_{\bar{y}-x+md \in \mathbf{Z}} g_i(\bar{y},0) w_i^0(\bar{y}) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d \in \mathbf{Z} \\ 1 \le m' \le m}} g_i(\bar{y},m')(M_i[w^{m'}](\bar{y}) + Q_i[w^{m'}](\bar{y}+d+1) + Q_i[w^{m'}](\bar{y}+d) + \mathcal{F}_i(\bar{y}) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d \in \mathbf{Z} \\ 1 \le m' \le m - \frac{1}{\epsilon}}} O(1)\epsilon^2(\epsilon + (m-m')^{-\frac{1}{2}}) e^{-\epsilon|\bar{y}|} \bar{\bar{g}}_i(\bar{y},m') w_i^{m'}(\bar{y}) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d \in \mathbf{Z} \\ m - \frac{2}{\epsilon} \le m' \le m}} O(1)\epsilon^2 \left\{ e^{-\epsilon|\bar{y}|} [\bar{g}_i(\bar{y}-d,m') + \bar{g}_i(\bar{y}-d-1,m')] \right. \\ &\qquad \left. + [\bar{g}_i(\bar{y}-d+1,m') - \bar{g}_i(\bar{y}-d-1,m')] \right\} w_i^{m'}(\bar{y}) \\ + &\sum_{\substack{\bar{y}-x+(m-m')d \in \mathbf{Z} \\ m - \frac{2}{\epsilon} \le m' \le m - \frac{1}{\epsilon}}} O(1)\epsilon^{5/2} [\bar{g}_i(\bar{y},m'-1) + \bar{\bar{g}}_i(\bar{y},m'-1)] w_i^{m'}(\bar{y}). \end{aligned} \tag{4.4}$$

Similar to (3.14), it holds

$$\Delta_x g_i(x,m;y,m') = -\Delta_y g_i(x,m;y+1,m') + O(1)\epsilon g_i(x,m;y,m'). \tag{4.5}$$

5 Construction of Travelling Waves

We will construct the travelling waves through time-asymptotic analysis. Consider an approximate travelling $\bar{\phi}(x - dt)$, $\bar{\phi}(\pm\infty) = u_\pm$, c.f. Appendix B, satisfying the Rankine-Hugoniot condition

$d(u_- - u_+) = f(u_-) - f(u_+)$,

$$\begin{aligned} \bar\phi - K\bar\phi &= O(1)\epsilon^3 e^{-\epsilon|x|}, && (5.1)\\ \sum_{j=-\infty}^{0} \bar\phi(x+j) &= O(1)\epsilon^2 e^{-\epsilon|x|}. \end{aligned}$$

Linearize the difference scheme (2.1) around $\bar\phi$, take anti-difference and diagonalize, c.f. Section 2, to yield

$$\begin{aligned} w_q^{m+1} &= K_q w_q^m + S_q + N_q[v^m] + M_q[e^{-\epsilon|x|}w^m], && (5.2)\\ w_q^0 &\equiv 0\\ v^m &\equiv u^m - \bar\phi,\\ w^m(x) &\equiv \sum_{j=-\infty}^{0} v^m(x+j),\\ w^m(x) &\equiv \sum_{q=1}^{n} w_q^m(x) r_q(\bar\phi(x)). && (5.3) \end{aligned}$$

It follows from (5.1) and (2.1) that

$$\begin{aligned} S_q(x) &= O(1)\epsilon^2 e^{-\epsilon|x|}, \text{ for } q \neq i, && (5.4)\\ S_i(x) &= O(1)\epsilon^3 e^{-\epsilon|x|}. \end{aligned}$$

Recall that $M_q[e^{-\epsilon|x|}w^m]$ is a small linear term and $N_q[v^m]$ is the nonlinear terms. We will construct the exact travelling wave ϕ to be $\bar\phi$ plus $v^\infty \equiv \lim_{m\to\infty} v^m$. This is due in several steps. The first step is to consider the simplified version of (5.2) when K_q, $q \neq i$, is replaced by a constant coefficient operator with $\lambda_q(\bar\phi(x))$ replaced by $\lambda_q^0 \equiv \lambda_q(u_-)$, for $q \leq i$, and $\lambda_q^0 \equiv \lambda_q(u_+)$. For the modified Lax-Friedrichs scheme, (2.27), we have

$$\begin{aligned} K_q^0 h(x) &= \frac{1}{3}(h(x+d+1) + h(x+d) + h(x+d-1)) && (5.5)\\ &\quad -\frac{\lambda\lambda_q^0}{2}(h(x+d+1) - h(x+d-1)),\\ \lambda_q^0 &\equiv \lambda_q(u_+) \text{ when } q > i,\\ \lambda_q^0 &\equiv \lambda_q(u_-) \text{ when } q < i. \end{aligned}$$

Consider the initial value problem

$$\begin{aligned} w_q^{m+1} &= K_q^0 w_q^m + S_q, \quad q \neq i, && (5.6)\\ S_q &= \tilde O(x)\epsilon e^{-\epsilon|x|},\\ w_i^{m+1} &= K_i w_i^m + S_i, && (5.7)\\ S_i(x) &= \tilde O_1(x)\epsilon^2 e^{-\epsilon|x|},\\ S_i(x) - S_i(x-1) &= \tilde O_2(x)\epsilon^3 e^{-\epsilon|x|}, && (5.8)\\ w_j^0(x) &= 0, \quad j = 1, 2, \cdots, n. && (5.9) \end{aligned}$$

Here $\tilde{O}_1(x)$ and $\tilde{O}_2(x)$ represents a bounded continuous function. Note that in (5.7), we use the original variable coefficient operator K_i.

Proposition 5.1. *There exists a solution w^m of (5.6), (5.7) and (5.9) with the following properties:*

$$
\begin{aligned}
\lim_{m\to\infty} w^m(x) &= w^\infty(x) \text{ exists,} && (5.10)\\
w_q^\infty(x) &= O(1)\begin{cases} e^{-\epsilon|x|} & x<0,\\ 1 & x>0 \end{cases} \quad \text{if } q>i. && (5.11)\\
w_i^\infty &= O(1)e^{-\epsilon|x|} && (5.12)\\
v_q^\infty &\equiv w^\infty(x) - w^\infty(x-1) && (5.13)\\
&= \begin{cases} O(1)\begin{cases} \epsilon e^{-\epsilon|x|} & x<0\\ (|x|+1)^{-\frac{1}{2}} & x>0 \end{cases} & \text{if } q>i\\ O(1)\begin{cases} \epsilon e^{-\epsilon|x|} & >0\\ (|x|+1)^{-\frac{1}{2}} & x<0 \end{cases} & \text{if } q<i, \end{cases}\\
v_i^\infty(x) &\equiv w_i^\infty(x) - w_i^\infty(x-1) = O(1)\epsilon e^{-\epsilon|x|}. && (5.14)
\end{aligned}
$$

Proof: For $q \neq i$, we apply the Duhamel's principle with exact Green function

$$
\begin{aligned}
G_q(x,m;y,m') &= \textstyle\sum_{\bar{y}-x+(m-m')d\in\mathbf{Z}} g_q(x,m;\bar{y},m')\delta(y-\bar{y})\\
w_q^{m+1}(x) &= \textstyle\sum_{\substack{\bar{y}-x+(m-m')d\in\mathbf{Z}\\ 1\le m'\le m}} g_q(x,m;\bar{y},m')S_q(\bar{y}).
\end{aligned}
$$

The Green function G_q has been studied in Section 3. It is the same as $\bar{G}_q$ of (3.4) when $\lambda_q(x)$ is replaced with λ_q^0. By (3.9), g_q essentially is the heat kernel with convective speed λ_q^0 and diffusive coefficient μ_0^q. Thus, the estimate (5.11) follows from the following Lemma 5.1 with $\beta = 1$. The existence of the limit (5.13) is shown as follows: Consider the time difference of (5.6):

$$
\begin{aligned}
z_q^{m+1} &= K_q^0 z_q^m,\\
z_q^1(x) &= \tilde{O}(x)\epsilon e^{-\epsilon|x|},\\
z_q^{m+1} &\equiv w_q^{m+1} - w_q^m, \quad m = 1,2,\cdots.
\end{aligned}
$$

Similar to Lemma 5.1 below, we can deduce easily from above that

$$
z_q^m(x) = O(1)m^{-\frac{1}{2}}e^{-\frac{(x-\lambda_q^0 m)^2}{\mu_q^0 m}} + O(1)\epsilon e^{-\epsilon C|x-\lambda_q^0 m|}
$$

for some constant $C>0$. Consequently, $w_q^m(x) = \sum_{m'=1}^m z_q^{m'}(x)$ is a convergent sequence for each fixed x. This completes the proof of (5.10), (5.11), for $q \neq i$.
For the proof of (5.10) and (5.12) with $q = i$, use the approximate Green function G_i constructed in Section 4 and also apply Lemma 5.2, $\beta = 1$, below. The solution $w_i^m(x)$ is constructed as

$$
w_i^m(x) = \lim_{j\to\infty} w_{i,j}^m(x) \qquad (5.15)
$$

through the following iteration: Replace the L.H.S. $w_i^{m+1}(x)$ and Duhamel's principle (4.4) by $w_{i,j+1}^{m+1}(x)$; the function w^m and $\mathcal{F}_i$ on the R.H.S. are replaced by $w_{i,j}^{m'}$ and S_i respectively. ($M_i[e^{-\epsilon|x|}w^m]$, $Q_i[w^{m'}]$ are zero in the present setting.) The consequence of (5.15) follows from the estimate:

$$\begin{aligned} \bar{w}_{i,j}^m(x) &= O(1)\epsilon^{j-1}e^{-\epsilon|x|}, \\ \bar{w}_{i,j}^m(x) &\equiv w_{i,j}^m(x) - w_{i,j}^{m-1}; \quad j = 1, 2, \cdots, \\ w_{i,0}^m &\equiv 0. \end{aligned} \tag{5.16}$$

From (4.4), we have

$$\begin{aligned} \bar{w}_{i,j+1}^{m+1} &= \sum_{\substack{\bar{y}-x+(m-m')d\in Z \\ 1\le m'\le m-\frac{1}{\epsilon}}} O(1)\epsilon^2(\epsilon+(m-m')^{-\frac{1}{2}})e^{-\epsilon|\bar{y}|}\bar{\bar{G}}_i(\bar{y},m')\bar{w}_{i,j}^{m'}(\bar{y}) \\ &+ \sum_{\substack{\bar{y}-x+(m-m')d\in Z \\ m-2/\epsilon\le m'\le m}} O(1)\epsilon\{\epsilon e^{-\epsilon|\bar{y}|}[\bar{g}_i(\bar{y}-d,m')+\bar{g}_i(\bar{y}-d-1,m')] \\ &\qquad +[\bar{g}_i(\bar{y}-d+1,m')-\bar{g}_i(\bar{y}-d-1,m')]\}\bar{w}_{i,j}^{m'}(\bar{y}) \\ &+ \sum_{\substack{\bar{y}-x+(m-m')d\in Z \\ m-2/\epsilon\le m'\le m-\frac{1}{\epsilon}}} O(1)\epsilon^{5/2}[\bar{g}_i(\bar{y},m'-1)+\bar{\bar{G}}_i(\bar{y},m'-1)]\bar{w}_{i,j}^{m'}(\bar{y}). \end{aligned}$$

With (3.12) on $\bar{\bar{G}}$ and the induction hypothesis (5.16), the first sum on the R.H.S. above is $O(1)\epsilon^j e^{-\epsilon|x|}$ by Lemma 5.2, $\beta = 1$ below. The second and the third sum is treated using (3.12) and (3.8) (with $q = i$); details are omitted. This shows the estimate (5.12). The convergence (5.10) for $j = i$ is shown as that of (5.10) for $q \neq i$, above. Finally (5.11) and (5.13) is shown by considering the difference of the Duhamel's principle in the above analysis and apply Lemma 5.1. When $= i$, consider difference Δ_x of the Duhamel's principle above and apply (4.5) to this difference. Finally, by using summation by parts with (5.8) and Lemma 5.2 with $\beta = 1$. **Q.E.D.**

Lemma 5.1. *Let D and λ be positive constants, and ϵ an arbitrarily small positive constant. Then for $\lambda > 0$,*

$$\begin{aligned} &\epsilon\int_0^t\int_R (t-s)^{-\beta/2}e^{-\frac{[y-x+\lambda(t-s)]^2}{D(t-s)}}e^{-\frac{4\epsilon|y|}{D}}dyds \\ &= O(1)\begin{cases} e^{-\frac{4\epsilon|x|}{D}}, & x<0, \quad \text{if } \beta=1; \\ 1, & x>0 \end{cases} \\ &= O(1)\begin{cases} \sqrt{\epsilon}e^{-\frac{4\epsilon|x|}{D}}, & x<0, \quad \text{if } \beta=2; \\ (x+1+\epsilon^{-1})^{-\frac{1}{2}}, & x>0 \end{cases} \end{aligned}$$

and for $\lambda < 0$

$$\epsilon\int_0^t\int_R (t-s)^{-\beta/2}e^{-\frac{[y-x+\lambda(t-s)]^2}{D(t-s)}}e^{-\frac{4\epsilon|y|}{D}}dyds$$

$$= O(1)\begin{cases} e^{-\frac{4\epsilon|x|}{D}}, & x>0, \quad if\ \beta=1; \\ 1, \quad x>0 \end{cases}$$

$$= O(1)\begin{cases} \sqrt{\epsilon}e^{-\frac{4\epsilon|x|}{D}}, & x<0, \qquad if\ \beta=2; \\ (x+1+\epsilon^{-1})^{-\frac{1}{2}}, & x<0 \end{cases}$$

Lemma 5.2: *Let D be a positive constant and ϵ an arbitrarily small positive constant, then for $x>0$*

$$\epsilon^2 e^{-\frac{4\epsilon x}{D}} \int_0^t \int_{-\infty}^0 (t-s)^{-\beta/2} e^{-\frac{[x-y-\epsilon(t-s)]^2}{D(t-s)}} e^{\frac{4\epsilon y}{D}} dyds$$
$$+\epsilon^2 \int_0^t \int_0^\infty (t-s)^{-\beta/2} e^{-\frac{[x-y+\epsilon(t-s)]^2}{D(t-s)}} e^{-\frac{4\epsilon y}{D}} dyds$$
$$= \begin{cases} O(1)e^{-\frac{4\epsilon x}{D}} & for\ \beta=1, \\ O(1)\sqrt{\epsilon}e^{-\frac{4\epsilon x}{D}} & for\ \beta=2, \end{cases}$$

for $x<0$

$$\epsilon^2 \int_0^t \int_{-\infty}^0 (t-s)^{-\beta/2} e^{-\frac{[x-y-\epsilon(t-s)]^2}{D(t-s)}} e^{\frac{4\epsilon y}{D}} dyds$$
$$+\epsilon^2 e^{-\frac{4\epsilon|x|}{D}} \int_0^t \int_{-\infty}^0 (t-s)^{-\beta/2} e^{-\frac{[x-y+\epsilon(t-s)]^2}{D(t-s)}} e^{\frac{4\epsilon y}{D}} dyds$$
$$= \begin{cases} O(1)e^{-\frac{4\epsilon|x|}{D}} & for\ \beta=1, \\ O(1)\sqrt{\epsilon}e^{-\frac{4\epsilon|x|}{D}} & for\ \beta=2, \end{cases}$$

The proof of Lemmas 5.1 and 5.2 are by lengthy but tedious computations. We omit the proof, c.f. Lemmas 5 to 7 of [7].

Remark 5.1.: The limit function $w_q(x) \equiv w_q^\infty(x)$ clearly satisfies

$$w_q = K_q^0 w_q + S_q, \qquad q \neq i \tag{5.17}$$
$$w_i = K_i w_i + S_i. \tag{5.18}$$

We have shown in Proposition 5.1 decay rates of $e^{-\epsilon|x|}$ or $(|x|+1)^{-\frac{1}{2}}$ for $v_q = v_\infty^\infty$, $q \neq i$. In below we will show that $v_q(x)$ decays as $e^{-\epsilon|x|}$. This is done in two steps, see Propositions 5.2 and 5.3 below for solution of (5.17), $q \neq i$.

Proposition 5.2. *There exists a solution v_q of (5.17) such that*

$$v_q(x) = w_q(x) - w_q(x-1) = O(1)\epsilon e^{-\epsilon|x|} \ as\ x \to \infty.$$

Proof: For definiteness, we consider only $q > i$, so that $\lambda_q > d$. We first prove the proposition for rational speed $d = p/q$, and shown that our analysis allows for arbitrarily large q. This would prove the proposition for any speed d. Thus (5.17) becomes, with $w = w_q$, $S = S_q$,

$$w(x) = aw(x+d+1) + bw(x+d) + cw(x+d-1) + S(x),$$

$$v(x) = av(x+d+1) + bv(x+d) + cv(x+d-1) + S(x) - S(x-1), \tag{5.19}$$

with the constants a, b, and c satisfy

i) $c - a - d > 0$, ($\lambda_q > d$; $q > i$)
ii) $a + b + c = 0$ (conservative scheme),
iii) $S(x) = O(1)\epsilon e^{-\epsilon|x|}$.

We want to solve (5.19) with the boundary condition

$$\lim_{x\to\infty} v(x) = 0 \tag{5.20}$$

Since $d = p/q$, we can divide any grid interval into q subintervals of equal length and rewrite (5.19) as

$$\begin{aligned} v_k &= av_{k+q+p} + bv_{k+p} + cv_{k-q+p} + S_k - S_{k-q}, \\ v_k &\equiv v(y_k), \quad S_k \equiv S(y_k), \quad y_k \equiv x + \frac{k}{q}, k = 0, \pm 1, \pm 2, \cdots, \end{aligned} \tag{5.21}$$

where x is any fixed value. For definiteness take $x = 0$.
This is rewritten as

$$\vec{v}_l = \mathbf{A}\vec{v}_l + \vec{S}_{l-q} - \vec{S}_{l-2q}, \tag{5.22}$$

$$\vec{v}_l \equiv \begin{pmatrix} v_{l+1-2q} \\ v_{l+2-2q} \\ \vdots \\ v_l \end{pmatrix}_{2q\times 1}, \quad \vec{S}_l \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \\ S_l \end{pmatrix}_{2q\times 1},$$

$$\mathbf{A} \equiv \begin{pmatrix} 0 & 1 & 0 & & & \cdots & & & & 0 \\ & 0 & 1 & 0 & & & & & & \vdots \\ \vdots & & \ddots & \ddots & & & & & & \\ & & & & & & & & & \\ & & & & & & & \ddots & & \\ & & & & & & & \ddots & 1 & \\ & & & & & & & & 0 & 1 \\ A_{2q,1} & 0 & \cdots & 0 & A_{2q,q-p+1} & 0 & \cdots & 0 \;\; A_{2q,q+1} & 0 \;\; \cdots & 0 \end{pmatrix}_{2q\times 2q},$$

$$A_{2q,1} \equiv -\frac{c}{a}, \quad A_{2q,q-p+1} \equiv \frac{1}{a}, \quad A_{2q,q+1} \equiv -\frac{b}{a}.$$

Here $\vec{v}_l$ and $\vec{S}_l$ are $2q$-vectors and $\mathbf{A}$ is a $2q\times 2q$ matrix. Since $\mathbf{A}$ is of canonical form, its characteristic polynomial is

$$char_{\mathbf{A}}(x) = \frac{1}{a}(-x^{-p+q} + ax^{2q} + bx^q + c)$$

and its eigenvectors are of the form

$$\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{2q-1} \end{pmatrix}$$

for any root α of the characteristic polynomial $char_{\mathbf{A}}(\alpha) = 0$:

$$\begin{aligned} g(\beta) &\equiv -\beta^{-d} + (a\beta + b + c\beta^{-1}) \\ \beta &\equiv \alpha^q. \end{aligned}$$

$\beta = 1$ is a trivial root of $g(\beta)$. Since g is convex and $g'(1) = d + a - c < 0$ by (i), another root β_+ is greater than 1. Thus the roots of $char_{\mathbf{A}}$ are

$$\alpha \in \{e^{2\pi\sqrt{-1}j/q}, \beta^{-1/q}e^{2\pi\sqrt{-1}j/q} : j = 0, 1, \cdots, q-1\}, \quad \beta_+ > 1. \tag{5.23}$$

Thus $\mathbf{A}$ has $2q$ distinct eigenvalues and thus a complete set of eigenvectors. We consider the particular solution vv_k of (5.22):

$$\vec{v}_k \equiv \sum_{j \geq k} \mathbf{A}^{k-j}(\vec{S}_{l-q} - \vec{S}_{l-2q}).$$

This sum converges because $|A^{j-k}|$ is bounded by a constant depending on q but not on $k - j \leq 0$, and because $S_j = O(1)\epsilon e^{-\epsilon|j|/q}$. We rewrite the sum as

$$\begin{aligned} \vec{v}_k &= \sum_{j \leq k}(\mathbf{A}^{k-j} - \mathbf{A}^{k-j+q})\vec{S}_{j-q} - \sum_{l=0}^{q-1} \mathbf{A}^{-l}\vec{S}_{k-2q+l} \qquad (5.24) \\ &= \sum_{l=0}^{\infty} \mathbf{A}^{-ql} \sum_{j=0}^{q-1} \mathbf{A}^{-j}(I - A^q)\vec{S}_{j+k+(l-1)q} - \sum_{l=0}^{q-1} \mathbf{A}^{-l}\vec{S}_{k-2q+l}. \end{aligned}$$

Denote the eigenvectors of $\mathbf{A}$ by

$$e_k = \begin{pmatrix} 1 \\ \alpha_k \\ \alpha_k^2 \\ \vdots \\ \alpha_k^{2q-1} \end{pmatrix}, \quad f_k = \begin{pmatrix} 1 \\ \beta_+^{\frac{1}{q}}\alpha_k \\ \beta_+^{\frac{2}{q}}\alpha_k^2 \\ \vdots \\ \beta_+^{\frac{2q-1}{q}}\alpha_k^{2q-1} \end{pmatrix},$$

$$\alpha_k = e^{2\pi\sqrt{-1}k/q}, \quad k = 1, 2, \cdots, q.$$

and write any vector g as

$$g = \sum_{k=1}^{q} (a_k e_k + b_k f_k).$$

We have

$$\mathbf{A}^q e_k = e_k, \quad \mathbf{A}^q f_k = \beta_+ f_k,$$
$$(\mathbf{A}^q - 1)g = (\beta_+ - 1)\sum_{k=1}^{q} b_k f_k.$$

We now compare the L_∞ norm of g and $(\mathbf{A}^q - I)g$. Suppose

$$\|(\mathbf{A}^q - I)g\|_\infty = \left|(\beta_+ - 1)\sum_{k=1}^{q}(\beta_+^{\frac{1}{q}}\alpha_k)^{i_0}\right|$$

for some $i_0 \in \{0, 1, \cdots, 2q-1\}$. Consider the case $0 \le i_0 < q$; the other case $q \le i_0 < 2q$ is similar,

$$\begin{aligned}
g_{i_0} &= \sum_{k=1}^{q} (a_k + b_k \beta_+^{\frac{i_0}{q}}) e^{2\pi i_0 \sqrt{-1}/q}, \\
g_{i_0+q} &= \sum_{k=1}^{q} (a_k + b_k \beta_+^{\frac{i_0}{q}} \beta_+) e^{2\pi i_0 \sqrt{-1}/q}, \\
\|(\mathbf{A}^q - I)g\|_\infty &= (\beta_+ - 1) \left| \frac{g_{i_0+q} - g_{i_0}}{\beta_+ - 1} \right| = |g_{i_0+q} - g_{i_0}| \\
&\le 2\|g\|_\infty.
\end{aligned}$$

Thus $\mathbf{A}^q - I$ is bounded on $\mathbf{L}_\infty$, independent of q.
Note also that on the range $\{f_1, f_2, \cdots, f_q\}$ of $\mathbf{A}^q - I$, the operator $\mathbf{A}^{-ql} = \beta_q^{-l} I$ and is therefore contractive. To bound v_k, it remains to consider the term of the form as the last term in (5.24).
We have

$$\mathbf{A}^{-1} = \begin{pmatrix}
0 & 0 & & \cdots & 0 & \frac{1}{c} & 0 & \cdots & 0 & -\frac{b}{c} & 0 & \cdots & 0 & -\frac{a}{c} \\
1 & 0 & 0 & \cdots & & & & & & & & & & \\
0 & 1 & 0 & \cdots & & & & & & & & & & \\
 & 0 & 1 & & & & & & & & & & & \\
 & \vdots & & \ddots & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & \\
 & & & & & & & & & & & 1 & 0 & \\
 & & & & & & & & & & & & 1 & 0
\end{pmatrix}$$

$$\mathbf{A}^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{a}{c} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{A}^{-2} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{a}{c} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{A}^{-(q-p)} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ a_{q-p} \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{A}^{-(q-p+1)} \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ \\ \\ \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \\ a_{q-p+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$a_{q-p} = -\frac{a}{c}, \quad a_1 = \frac{a}{c^2}, \quad a_{q-p+1} = -\frac{a}{c}$$

and so

$$\sum_{l=0}^{q-1} \mathbf{A}^{-l}\vec{S}_{k+l} = \sum_{l=0}^{q-1} S_{k+l}\mathbf{A}^{-l}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= -\frac{a}{c}\begin{pmatrix} S_{k+1} \\ S_{k+2} \\ \vdots \\ S_{k+q-1} \\ 0 \\ \vdots \\ 0 \\ S_k \end{pmatrix} - \frac{a}{c^2}\begin{pmatrix} S_{k+q-p+1} \\ S_{k+q-p+2} \\ \vdots \\ S_{k+q-1} \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Consequently

$$\begin{aligned} &\|\sum_{l=0}^{q-1} \mathbf{A}^{-l}\vec{S}_{k+l}\|_\infty \leq (\frac{a}{c} + \frac{a}{c^2}) \max_{k\leq j\leq k+q-1} \|S_j\| = \max_{|x-y|\leq 1} \|S(y)\|, \\ &x \equiv k/q, \\ &v(x) = O(1) \max_{|x-y|<1} \|S(y)\|. \end{aligned}$$

This completes the proof of Proposition 5.2. **Q.E.D.**

We have thus obtained two existence results Propositions 5.1 and 5.2 for (5.17). Our next Proposition shows that these two solutions are the same.

Proposition 5.3 *(Uniqueness) Suppose that a, b, c are positive constants with $a+b+c=1$ and d is a constant with $d \leq c-a$. Then, there the difference equation*

$$\begin{aligned} &V(x) = aV(x+d+1) + bV(x+d) + cV(x+d-1) \\ &\textit{with} \lim_{|x|\to\infty} |V(x)| = 0, \end{aligned}$$

has only trivial solution.

Proof: Using the same setup as in the proof of Proposition 5.2, we have

$$V_k = aV_{k+p+q} + bV_{k+p} + cV_{k+p-q}.$$

Write:

$$\begin{aligned} &\vec{V}_l = \begin{pmatrix} V_{l-2q+1} \\ \vdots \\ V_l \end{pmatrix} = \sum_{j=1}^{q} (a_{jl}e_j + b_{jl}f_j) = E_l + F_l \\ &\vec{V}_{l+kq} = E_l + \beta_+^q F_l. \end{aligned}$$

Since $\lim_{|x|\to\infty}|V(x)| = 0$, we have $F_l = 0$ by letting $k \to \infty$, and $E_l = 0$ by letting $k \to \infty$ or $k \to -\infty$. **Q.E.D.**

Finally we conclude from Propositions 5.1 $\sim$ 5.3 that

Theorem 5.1. *Proposition 5.1 holds with the sharper estimates:*

$$v_q^\infty(x) = O(1)\epsilon e^{-\epsilon|x|} \text{ as } |x| \to \infty, \quad q \neq i.$$

We next return to the nonlinear problem (5.2). Our interest is in the large-time behavior $\vec{w}^\infty = \sum_{l=1}^n \lim_{m\to\infty} w_l^m r_l$ and $\vec{v}^\infty = \sum_{l=1}^n \lim_{m\to\infty} v_l^m r_l$. For this we set up the following iteration:

$$\begin{aligned} w_{q,j+1}^{m+1} &= K_q^0 w_{q,j+1}^m + \underbrace{(K_q - K_q^0)w_{q,j}^\infty}_{\equiv \mathbf{I}_{q,j}} + S_q + \underbrace{N_q[\vec{v}_j^\infty]}_{\equiv \mathbf{II}_{q,j}} + \underbrace{M_q[e^{-\epsilon|x|}\vec{w}_j^\infty]}_{\equiv \mathbf{III}_{q,j}}, \qquad (5.25) \\ & w_{q,0}^m = 0, \quad w_{q,j+1}^0 = 0, \\ & M_q[e^{-\epsilon|x|}\vec{w}^\infty] = \begin{cases} \|w\|_\infty \epsilon^2 e^{-\epsilon|x|} & \text{for } q \neq i, \\ \|w\|_\infty \epsilon^3 e^{-\epsilon|x|} & \text{for } q = i, \end{cases} \qquad (5.26) \end{aligned}$$

with the a priori hypothesis:

$$\begin{aligned} \bar{w}_{q,j}^\infty(x) &= \epsilon^{j+2} \begin{cases} O(1)e^{-\epsilon|x|}, & x > 0 \\ O(1), & x < 0 \end{cases} \text{ for } q < i \qquad (5.27) \\ \bar{w}_{q,j}^\infty(x) &= \epsilon^{j+2} \begin{cases} O(1)e^{-\epsilon|x|}, & x < 0 \\ O(1), & x > 0 \end{cases} \text{ for } q > i \\ \bar{w}_{i,j}^\infty(x) &= \epsilon^{j+2} e^{-\epsilon|x|}, \\ \bar{v}_{q,j}^\infty(x) &= \epsilon^{j+3} e^{-\epsilon|x|}, \quad q = 1, 2, \cdots, n \\ \bar{w}_{q,j}^\infty(x) &= w_{q,j}^\infty(x) - w_{q,j-1}^\infty(x), \quad j = 1, 2, \cdots, j. \\ \bar{v}_{q,j}^\infty(x) &= v_{q,j}^\infty(x) - v_{q,j-1}^\infty(x), \quad j = 1, 2, \cdots, j, \\ \vec{\bar{v}}_j &= \sum_{l=1}^n (\bar{v}_{l,j} - \bar{v}_{l,j-1})\, r_l, \\ \vec{\bar{w}}_j &= \sum_{l=1}^n (\bar{w}_{l,j} - \bar{w}_{l,j-1})\, r_l, \\ \mathbf{I}_q &\equiv \mathbf{I}_{q,j} - \mathbf{I}_{q,j-1} = O(1)\epsilon \max_{|y-x|\le 1} |\bar{v}_{q,j}^\infty(y)| \qquad (5.28) \\ &= O(1)\epsilon\epsilon^{j+3} \text{ for } q \neq i, \\ \mathbf{II}_q &\equiv \mathbf{II}_{q,j} - \mathbf{II}_{q,j-1}, \\ \mathbf{III}_q &\equiv \mathbf{III}_{q,j} - \mathbf{III}_{q,j-1}. \end{aligned}$$

Since $N_q[\vec{v}_j^\infty]$ is a second order nonlinear term,

$$\begin{aligned} \mathbf{II}_q(x) &= O(1)(\|\vec{v}_j^\infty\|_\infty + \|\vec{v}_{j-1}^\infty\|_\infty) \max_{|y-x|<2} \|\vec{\bar{v}}_j^{\,\infty}(y)\|_\infty \qquad (5.29) \\ &= O(1)\epsilon^2 \epsilon^{j+3} e^{-\epsilon|x|}. \end{aligned}$$

The function $M_q[e^{-\epsilon|x|}w]$ is linear in w, therefore

$$\begin{aligned} \mathbf{III}_q &= O(1)\epsilon^2 e^{-\epsilon|x|}\epsilon^{k+2}, \text{ for } q \neq i, \\ \mathbf{III}_i &= O(1)\epsilon^3 e^{-\epsilon|x|}\epsilon^{k+2}. \end{aligned} \tag{5.30}$$

From the difference between (5.25)'s with index j and $j-1$, we have from (5.27) that

$$\begin{aligned} \bar{w}_{q,j+1}^{m+1}(x) &= K_q^0 \bar{w}_{q,j+1}^m + O(1)\epsilon^{j+3}e^{-\epsilon|x|}, \\ \bar{w}_{q,j+1}^0 &= 0. \end{aligned}$$

From (5.28), (5.29), (5.30) and Theorem 5.1 we have that

$$\bar{v}_{q,j+1}^{\infty}(x) = O(1)\epsilon^{j+3}e^{-\epsilon|x|}$$

which yields the a priori hypothesis (5.27) for $j+1$, when $q \neq i$. **Q.E.D.**
This and (5.14) yield the following main theorem of the existence of nonlinear travelling wave of the difference scheme.
Theorem 5.2. *There exists a solution to* $w_q = K_q w_q + S_q + N_q[v] + M_q[e^{-\epsilon|x|}w]$ *with the property that*

$$\begin{aligned} v(x) &\equiv w(x) - w(x-1) \\ v(x) &= O(1)\epsilon^2 e^{-\epsilon|x|}, \end{aligned} \tag{5.31}$$

and that $u(x) \equiv v(x) + \bar{\phi}(x)$ *is an exact travelling wave of the difference scheme (2.20) satisfying that*

$$u_x(x) = O(1)\epsilon^2 e^{-\epsilon|x|}. \tag{5.32}$$

6 Continuous Dependence and Conservation Laws of Travelling Waves

In this section we show that the continuum travelling wave $\phi(x-dt)$ can be constructed to satisfy the conservation (2.8):

$$\sum_{j\in\mathbf{Z}} (\phi(x+j) - \phi(j)) = x(u_+ - u_-), \quad x \in \mathbf{R}. \tag{6.1}$$

We first consider the case the speed $d = p/q$ is rational. Let $d_{i'} = p_{i'}/2^{i'}q$, $(p_{i'}, 2q) = 1$ be such that $d_{i'} \to d$ as $i' \to \infty$, and the shocks $(u_-, u_+^{i'})$ has speed $d_{i'}$, $i' = 1, 2, \cdots$. Let $\phi_0(x-dt)$ be the travelling wave constructed in the last section

$$\phi_0(\infty) = u_+, \quad \phi_0(-\infty) = u_-.$$

Consider the conservative function

$$\phi_*(x) \equiv \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \phi_0(y)dy \tag{6.2}$$

and the travelling waves $\phi_{i'}(x-d_{i'}t)$ as constructed in the last section to be the time-asymptotic state with initial values:

$$\phi^0_{i'}(x) \equiv \phi_*(x) + (u^{i'}_+ - u_+)\frac{l_i(\phi_*(x)-u_-)}{l_i(u_+-u_-)}, \tag{6.3}$$

Similarly, with $\phi_{i'}$ given, we construct traveling waves ${}_{i'}\phi$ connecting (u_-, u_+) with an initial value:

$$_{i'}\phi^0(x) \equiv \phi_{i'}(x) + (u_+ - u^{i'}_+)\frac{l_i(\phi_{i'}(x)-u_-)}{l_i(u^{i'}_+-u_-)}. \tag{6.4}$$

Note that, due to (6.2), ϕ_* is conservative:

$$\sum_{j\in\mathbf{Z}} (\phi_*(x+j) - \phi_*(j)) = x(u_+ - u_-), \quad x \in \mathbf{R}.$$

This implies trivially that the functions $\phi^0_i(x)$ are also conservative. Consequently our construction processs in the last section can be applied to yield.
Since $d_{i'} = \frac{p_{i'}}{2^{i'}q}$, we have, by telescoping the equation (2.1), with $u^m(x) = \phi_{i'}(x - d_{i'}m)$, we have

$$\sum_{j=-\infty}^{\infty} \phi_{i'}(x+j+x_0) - \phi_{i'}(j+x_0) = x(u^{i'}_+ - u_-), \quad x = \frac{k}{2^{i'}q}, \quad k \in Z, \quad x_0 \in \mathbf{R}. \tag{6.5}$$

This implies that the initial values of (6.4) is conservative with respect to the speed $d = p/q$:

$$\sum_{j=-\infty}^{\infty} {}_{i'}\phi^0(x+j+x_0) - {}_{i'}\phi^0(j+x_0) = x(u_+ - u_-), \quad x = \frac{k}{2^{i'}q}, \quad k \in \mathbf{Z}, \quad x_0 \in \mathbf{R}. \tag{6.6}$$

The conservative property for ${}_i\phi(x-dt)$ (6.6) is sufficient for applying the construction scheme in the last section.

Set

$$v_j(x) \equiv \phi_j - \phi^0_j(x).$$

Proposition 6.1. *The functions $\{v_1, v_2, \cdots\}$ form a Cauchy sequence,*

$$\|v_j(x) - v_k(x)\|_\infty = O(1)|d_j - d_k|e^{-C\epsilon|x|}, \quad j,k = 1,2,\cdots, \quad \textit{for some } C > 0.$$

Proof:

Write down the equations for v_j and v_k:

$$\begin{aligned}
v_j(x) &= \nabla\mathcal{L}[\phi^0_j]v_j(x+d_j) + \mathbf{N}_j[v_j](x+d_j) + \mathcal{E}_j(x+d_j), && (6.7)\\
v_k(x+d_j-d_k) &= \nabla\mathcal{L}[\phi^0_k]v_k(x+d_j) + \mathbf{N}_k[v_k](x+d_j) + \mathcal{E}_k(x+d_j), && (6.8)\\
\mathbf{N}_j[v](x) &\equiv \mathcal{L}[\phi^0_j + v](x) - \mathcal{L}[\phi^0_j](x) - \nabla\mathcal{L}[\phi^0_j]v(x) && (6.9)\\
&= \int_0^1\int_0^1 \theta\nabla^2\mathcal{L}[\phi^0_j + \theta\tau v_j](v_j,v_j)(x)d\tau d\theta\\
&= \lambda\int_0^1\int_0^1 \theta\nabla^2 F[\phi^0_j + \theta\tau v_j](v_j,v_j)(x+\frac{1}{2})d\tau d\theta
\end{aligned}$$

$$
\begin{aligned}
& -\lambda \int_0^1 \int_0^1 \theta \nabla^2 F[\phi_j^0 + \theta\tau v_j](v_j, v_j)(x - \frac{1}{2}) d\tau d\theta \\
\mathcal{E}_j(x) &\equiv \mathcal{L}[\phi_j^0](x + d_j) - \phi_j^0(x), \\
\mathcal{N}_j(x) &\equiv \sum_{l\le 0} \mathrm{N}_j(x + l), \\
\mathcal{F}_j(x) &\equiv \sum_{l\le 0} \mathcal{E}_j(x + l). \\
\delta v &\equiv v_j - v_k, \\
\delta w(x) &\equiv \sum_{l\le 0} \delta v(x + l), \\
w_j(x) &\equiv \sum_{l\le 0} v_j(x + l), \text{ for } j = 1, 2, \cdots.
\end{aligned}
$$

From our construction of v_j and v_k, Section 5, we can conclude that

$$
\|v_j(x)\|_\infty = O(1)\epsilon^2 e^{-\epsilon|x|}, \quad \|v_k(x)\|_\infty = O(1)\epsilon^2 e^{-\epsilon|x|}, \tag{6.10}
$$

$$
\|v_j(x - (d_j - d_k)) - v_j(x)\| = O(1)\epsilon^2 |d_j - d_k| e^{-\epsilon|x|}. \tag{6.11}
$$

(6.10) gives the upper bound for v_j and v_k (6.11) follows from the estimate for $\frac{\partial v_j}{\partial x}$ c.f.: (5.32). This and (6.9) give the difference in the nonlinear terms $\mathcal{N}_j[v_j]$ and $\mathcal{N}_k[v_k]$:

$$
\|\mathcal{N}_j[v_j](x) - \mathcal{N}_k[v_k](x)\|_\infty = O(1)\left(\|\epsilon^2 \delta v(x)\|_\infty + \epsilon^4 |d_j - d_k| e^{-\epsilon|x|}\right), \tag{6.12}
$$

Hence, we can decompose $\mathcal{N}_j[v_j] - \mathcal{N}_k[v_k]$ as follows

$$
\mathcal{N}_j[v_j](x) - \mathcal{N}_k[v_k](x) = \mathcal{N}_{jk}[\delta v](x) + O(1)\epsilon^4 |d_j - d_k| e^{-\epsilon|x|}. \tag{6.13}
$$

Similarly, we also have

$$
\|\mathcal{F}_j(x) - \mathcal{F}_k(x)\|_\infty = O(1)\epsilon^2 |d_j - d_k| e^{-\epsilon|x|}. \tag{6.14}
$$

From (6.7), (6.8), (6.12) and (6.14), we have the linear equation for δv:

$$
\begin{aligned}
\delta v(x - d_j) &= \nabla\mathcal{L}[\phi_j^0]\delta v(x) + v(x + d_j - d_k) - v(x) \qquad (6.15) \\
&\quad + \mathcal{N}_{j,k}[\delta v](x) - \mathcal{N}_{j,k}[\delta v](x - 1) + O(1)\epsilon^2 |d_j - d_k| e^{-\epsilon|x|} \\
&= \nabla\mathcal{L}[\phi_j^0]\delta v(x) + \mathcal{N}_{j,k}[\delta v](x) - \mathcal{N}_{j,k}[\delta v](x - 1) + O(1)\epsilon^2 |d_j - d_k| e^{-\epsilon|x|}. \\
\delta w(x - d_j) &= \mathcal{K}[\delta w](x) + \mathcal{N}_{j,k}[\delta v](x) + O(1)\epsilon |d_j - d_k| e^{-\epsilon|x|}, \qquad (6.16) \\
\mathcal{K}[h](x) &\equiv h(x) + \nabla F[\phi_i^0] \cdot (h - \sigma \circ h)(x), \quad (\sigma \circ h)(x) = h(x - 1).
\end{aligned}
$$

Similar to construction of the discrete shock profile in the previous section, for solution δv of (6.15) it satisfies that

$$
\|\delta v(x)\|_\infty = O(1)|d_j - d_k| e^{-\epsilon|x|}. \tag{6.17}
$$

Q.E.D.

From a similar reasoning to Proposition 6.1. it yields the following estimate for the function (6.4)

$$
\|{}_j\phi^0(x) - {}_k\phi^0(x)\| = O(1)|d_j - d_k| e^{-\epsilon C|x|}
$$

for some constant $C > 0$; details are omitted. By the same reasoning we have

$$\|{}_j\phi(x) - {}_k\phi(x)\| = O(1)|d_j - d_k|e^{-C\epsilon|x|}.$$

This implies that $\{{}_1\phi(x), {}_2\phi(x), \cdots,\}$ is a Cauchy sequence with limit

$$\phi(x) \equiv \lim_{i' \to \infty} {}_{i'}\phi(x).$$

Clearly $\phi(x - dt)$ is a travelling wave, $\phi(\pm\infty) = u_\pm$. The conservation laws (6.6) and the above estimates yield the approximate conservation laws:

$$\sum_{j \in \mathbf{Z}} {}_{i'}\phi(x + j + x_0) - {}_{i'}\phi(j + x_0) = x(u_+ - u_-) + O(1)|d_i - d|,$$
$$x = \frac{k}{2^{i'}q}, \quad k \in \mathbf{Z}, \quad x_0 \in \mathbf{R}.$$

This yields, in the limit ${}_{i'}\phi \to \phi$ as $i' \to \infty$ the conservation law (6.1) for ϕ.

When the speed d is irrational, we may apply the conservation law (6.1) to travelling waves with nearby rational speed. We also need to make use of the above reasoning of continuous dependence of the travelling waves (as time-asymptotic state) on the initial data. This establish the conservation laws (6.1) for any speed d. We now summarize our main result on the continuum profile as follows:

Theorem 6.1. *Suppose that a shock (u_-, u_+) of the hyperbolic conservation laws (1.1) is sufficiently weak, $\|u_- - u_+| \le \epsilon_0$, ϵ_0 depending only on the function $f(u)$ and the scheme. Then the continuum shock profile, (2.7) and (2.8), of a dissipative, non-resonant, (2.17), (2.18), difference scheme exists. Moreover, the profile with strength $O(1)\epsilon_0$ is unique up to a phase shift.*

Actually, the uniqueness in the above theorem has yet to be shown and is a consequence of the nonlinear stability of the profile to be shown in the next section.

7 Nonlinear Stablity of Shock profiles

Consider the perturbation of a shock profile $\phi(x - dt)$, $\phi(\pm\infty) = u_{\pm\infty}$:

$$u^0(i) = \phi(i) + \bar{u}^0(i), \quad i = 0, \pm1, \pm2, \cdots. \tag{7.1}$$

We first extend the initial value $u^0(i)$ conservatively as follows: Let $\tilde{u}^0(x)$ be any smooth function satisfying

$$\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \tilde{u}^0(x)dx = u^0(x), \quad i = 0, \pm1, \pm2, \cdots. \tag{7.2}$$

The extension is then set as:

$$u^0(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \tilde{u}^0(y)dy, \quad x \in \mathbf{R}. \tag{7.3}$$

From (7.2) and (7.3) the following is an extension of (7.1),

$$u^0(x) = \phi(x) + \bar{u}^0(x), \quad x \in \mathbf{R}. \tag{7.4}$$

This extension, along with the conservative property (2.8) for ϕ, yields the conservative property:

$$\begin{aligned} u^m(x) &= \phi(x-\lambda dm)+\bar{u}^m(x), &(7.5)\\ &\sum_{i\in\mathbf{Z}}\bar{u}^m(x+j)=\sum_{i\in\mathbf{Z}}\bar{u}^0(i), \quad x\in\mathbf{Z}, \quad m=0,1,2,\cdots. &(7.6)\end{aligned}$$

These conservation laws are usual to determine the shift x_0 of ϕ and to the strength C_q, $q\neq i$ of the diffusion waves that the perturbation gives rise to:

$$\sum_{i\in\mathbf{Z}}\bar{u}^0(i)=x_0(u_+-u_-)+\sum_{q<i}C_qr_q(u_-)+\sum_{q>i}C_qr_q(u_+). \tag{7.7}$$

Since our interest is in the time asymptotic behavior of the solution, the diffusion waves θ_q is accurately approximated by the diffusion waves for the parabolic PDE:

$$\theta_{qt}+\lambda_q\theta_{qx}+\frac{1}{2}(\theta_q^2)_x=\mu_q\theta_{qxx}, \tag{7.8}$$

if q-field is g.nl. (normalized by $\nabla\lambda_q\cdot r_q\equiv 1$)

$$\theta_{qt}+\lambda_q\theta_{qx}=\mu_q\theta_{qxx}, \tag{7.9}$$

if field is linear degenerated.

$$\begin{aligned} &\int_R\theta_q(x,t)=C_q, \quad q\neq i,\\ &\mu_q\equiv\begin{cases}\mu_q(u_-) & \text{for } q<i,\\ \mu_q(u_+) & \text{for } q>i,\end{cases}\\ &\lambda_q\equiv\begin{cases}\bar{\lambda}_q(u_-) & \text{for } q<i,\\ \bar{\lambda}_q(u_+) & \text{for } q>i,\end{cases} \end{aligned} \tag{7.10}$$

and, to have conservative property, we set

$$\theta_q^m(x)=\int_{x-\frac{1}{2}}^{x+\frac{1}{2}}\theta_q(y,m)dy, \tag{7.11}$$

By a change of variable we may assume, without loss of generality, that the sfift x_0 is zero and so we have from above

$$\begin{aligned} u^m(x) &\equiv \phi(x-d\lambda m)+\sum_{q<i}\theta_q(x,m)r_q(u_-)+\sum_{q>i}\theta_q(x,m)r_q(u_+)+v^m(x), &(7.12)\\ &\sum_{j\in\mathbf{Z}}v^m(x+j)=0, \quad x\in\mathbf{R}, m=0,1,2,\cdots. &(7.13)\end{aligned}$$

With (7.13) we consider the anti-difference:

$$w^m(x)\equiv\sum_{j=-\infty}^{0}v^m(x+j). \tag{7.14}$$

The equations for w^m follows from the difference scheme and (7.8),

$$\begin{aligned} w_q^{m+1}(x) &= K_q[w_q^m](x)+M_q[e^{-\epsilon|x|}w^m](x)+Q_q[v^m](x+d+1) \\ &+ Q_q[v^m](x+d)+F_q^m(x) \end{aligned} \tag{7.15}$$

where we have followed the notation of (2.26) expect for F_q, which represents the error created by the diffusion waves (instead of due to the approximate profile as in (2.26):

$$F_q^m(x) = \sum_{j\neq q,i} C_{jq}\theta_j^2(x) + \sum_{\substack{j\neq i \\ k}} D_{qjk}\theta_{qjk}^m v_k^m + O(1)|\theta^m|^3(x). \tag{7.16}$$

We apply the Duhamel's principles in Sections 3 and 4 when the source terms M_q and Q_q are assumed to be known through the following a priori estimates:

$$\begin{aligned}
|w_q^m(x)| &\le M\delta\psi_q(x,m)^{\frac{1}{2}}, \quad q\neq i, && (7.17)\\
|w_i^m(x)| &\le M\delta\psi_i(x,m)^{\frac{1}{2}}, && (7.18)\\
|v_q^m(x)| &\le M\delta[\psi_q(xm)^{3/2} + \sum_{j\neq q,i}\bar{\psi}_j(x,m)^{3/2} && (7.19)\\
&\quad +(|x|+1)^{-1}(m+|x|+1)^{-\frac{1}{2}} + \chi_q(x,m)], \quad q\neq i, &&\\
|v_i^m(x)| &\le M\delta[\psi_i(x,m)^{3/2} + \sum_{j\neq i}\bar{\psi}_j(x,m)^{3/2} && (7.20)\\
&\quad +(|x|+1)^{-1}(m+|x|+1)^{-\frac{1}{2}}], \quad q\neq i, &&\\
|v_i^m(x) - v_i^m(x-1)| &\le M\delta[\psi_i(x,m)^2 + \sum_{j\neq i}\bar{\psi}_j(x,m)^2], && (7.21)\\
|w_q^{m+1}(x) - w_q^m(x)| &\le M\delta[\psi_q(x,m)^{3/2} + \sum_{j\neq q}\bar{\psi}_j(x,m)^{3/2} && (7.22)\\
&\quad +\theta_q(x,m) + \bar{\chi}_q(x,m)], \quad q\neq i, &&\\
|w_i^{m+1} - w_i^m(x) &\le M\delta[\psi_i(x,m)^{3/2} + \sum_{j\neq i}\bar{\psi}_j(x,m)^{3/2} + \chi_i(x,m)], &&
\end{aligned}$$

where the various algebraic-decaying function are defined as follows:

$$\begin{aligned}
\theta_q(x,m) &\equiv (m+1)^{-\frac{1}{2}} e^{-\frac{(x-\lambda\lambda_q^0(m+1))^2}{\mu_q(m+1)}}, && (7.23)\\
\psi_q(x,m) &\equiv \left[(x-\lambda\lambda_q^0(m+1)\ )^2 + m+1\right]^{-\frac{1}{2}},\\
\psi_i(x,m) &\equiv \left[(|x|+\epsilon(m+1)\ )^2 + m+1\right]^{-\frac{1}{2}},\\
\bar{\psi}_q(x,m) &\equiv \left[|x-\lambda\lambda_q^0(m+1)|^3 + (m+1)^2\right]^{-\frac{1}{3}},\\
\chi_q(x,m) &\equiv \min(\bar{\chi}_q(x,m), (m+1)^{-\frac{1}{2}}(|x|+1)^{-\frac{1}{2}}\ ),\\
\chi_i(x,m) &\equiv (m+1)^{-\frac{1}{2}}(1+\epsilon(|x|+\epsilon m)\ )^{-1}e^{-\epsilon|x|}\cdot \begin{cases} 1 & \text{for } |x|\le C(m+1),\\ 0 & \text{otherwise,}\end{cases}\\
&\quad \text{for some constant } C>0,\\
\bar{\chi}_q(x,m) &\equiv |x-\lambda\lambda_q^0 m|^{-1}(1+\epsilon^2(|x|+1)^{-\frac{1}{2}}\ )\cdot \begin{cases} 1 & \text{for } 0<x<\lambda\lambda_q^0(m+1)-\sqrt{m+1}, \quad q>i\\ 1 & \text{for } 0>x>\lambda\lambda_q^0(m+1)-\sqrt{m+1}, \quad q<i\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The estimates of (7.17) ∼(7.22) are done by using the Duhamel's principle, (3.15), (4.4) derived in Section 3 and 4. We assume that the functions $w^m(x)$ and $v^m(x)$ in the M_q and Q_q satisfy the estimate (7.17) ∼(7.22). The initial perturbation is assumed to satisfy

$$
\begin{aligned}
\bar{u}^0(i) &= O(1)\delta[i^2+1]^{-3/4}, \\
\bar{u}^0(i+1)-\bar{u}^0(i) &= O(1)\delta[i^2+1]^{-5/4}, \quad i=0,\pm1,\pm2,\cdots,
\end{aligned} \tag{7.24}
$$

with these we may compute the R.H.S. of (3.15) and (4.4) for $w^m(x)$, $m = 1,2,\cdots, x \in \mathbf{R}$. The computation involves the convolution of the approximate Green functions with power of the power of functions in (7.). The approximate Green functions $G(x,m;y,m')$ are accurately approximated by those for the associated viscouse conservation laws, (3.12), (4.3), except for m' close m, where they are close the heat kernels, (3.9). Thus we may apply the stability analysis for viscous shocks [7], to obtain the stability of the shock profiles for difference schemes; details are omitted. We have the following stability results:

Theorem: Suppose theat the perturbation of a shock profile $\phi(x-dt)$ of a dissipative, non-resonant difference scheme satisfying (7.24) for δ small. Then the difference shceme has a global solution (7.12) satisfying (7.17), (7.18), (7.19) and (7.20).

A Existence and Structure of Discrete Shock Profile by a Marching Map

In this appendix, we will simplify and generalize the existence theormem of discrete shock profiles given by [3], [4] and [13].

Let u_0 be a state such $\lambda\lambda_i(u_0) = p/q$ and normalize the nonlinearity by setting

$$
\frac{l_i(u_0)\cdot\nabla^2 f(u_0)(r_i(u_0), r_i(u_0))}{l_i(u_0)F_1(u_0,u_0)r_i(u_0)} = -1.
$$

We write the equation of a discrete travelling wave ϕ with a discrete p/q as a recurrancy relationship:

$$
\begin{aligned}
\phi(x-p) = \mathcal{L}^q(\phi(x+q),\cdots,\phi(x-q)), \qquad \text{for } x\in R, \\
\lim_{j\to\pm\infty}\phi(j+x) = u_{\pm}.
\end{aligned} \tag{A.1}
$$

In particular, this scheme $\mathcal{L}$ is the modified Lax-Friedrichs scheme, hence the recurrancy relationship can be transformed into a nonlinear march mapping $\mathcal{T}$ in an open set $U \subset \mathbf{R}^{2q\times n}$ around a point $\vec{u}_0 \equiv \underbrace{(u_0,u_0,\cdots,u_0)^t}_{2q}$:

$$
\mathcal{T}: \quad \mathbf{R}^{2q\times n} \longmapsto \mathbf{R}^{2q\times n},
$$

$$
\mathcal{T}(\vec{u}) \equiv \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ u_{2q} \\ t \end{pmatrix}, \quad \vec{u} \equiv \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2q-1} \\ u_{2q} \end{pmatrix},
$$

where t solves

$$
u_{q-p} = \mathcal{L}^q(t, u_{2q}, \cdots, u_1).
$$

Then, the limit conditions in (A.1) for ϕ become to find a state $\vec{\phi}_0 \equiv (\phi(1), \cdots, \phi(2q))^t$ such that

$$\lim_{j \to \pm\infty} \mathcal{T}^j(\vec{\phi}_0) = \vec{u}_\pm, \tag{A.2}$$

$$\vec{u}_\pm \equiv \begin{pmatrix} u_\pm \\ u_\pm \\ \vdots \\ u_\pm \end{pmatrix}.$$

The existence of such a state $\vec{\phi}_0$ is reduced into a state on the center manifold $\mathcal{M}$ of $\mathcal{T}$ at its fixed point $\vec{u}_0$ c.f. [3], [4]. On the other hand, for any state satisfying one of the limit conditions in (A.1) will be on the set defined by the following equation

$$\begin{aligned} \mathcal{C} &\equiv \left\{(u_1, u_2, \cdots, u_{2q})^t : H(u_1, \cdots, u_{2q}) = pu_- - q\lambda f(u_-)\right\}; \qquad (A.3) \\ H(u_1, \cdots, u_{2q}) &\equiv \sum_{l=1}^{p} u_{q-p+l} - \lambda \sum_{l=0}^{q-1} F(\mathcal{L}^l[u.](q+1), \mathcal{L}^l[u.](q)). \end{aligned}$$

Hence the existence of such a state is reduced into that on $\mathcal{J} \equiv \mathcal{C} \cap \mathcal{M}$ c.f. [4]. (This set $\mathcal{J}$ is an invariant curve for $\mathcal{T}$, since both $\mathcal{C}$ and $\mathcal{M}$ are invariant under $\mathcal{T}$.) One can show that this set $\mathcal{J}$ is a 1-dimensional curve connecting $\vec{u}_+$ and $\vec{u}_-$ for this modified Lax-Rriedrichs scheme. On this curve, one could easily show the existence of the state $\vec{\phi}_0 \in \mathcal{J}$, c.f. [4].

Next, we will establish a parameter for $\phi(x)$ such that

$$\|\phi'(x)\| = O(1)\epsilon^2 e^{-(1+O(\epsilon))\ \epsilon|k_1 x|}, \quad \|\phi''(x)\| = O(1)\epsilon^3 e^{-(1+O(\epsilon))\ \epsilon|k_1 x|}, \tag{A.4}$$

$$k_1 \equiv \frac{l_i(u_0) \cdot \nabla^2 f(u_0)(r_i(u_0), r_i(u_0))}{l_i(u_0) F_1(u_0, u_0) r_i(u_0)} < 0. \tag{A.5}$$

The tangent space $\mathcal{M}_0 \equiv d T_{\vec{u}_0} \mathcal{M}$ at $\vec{u}_0$ is spanned the vectors

$$\begin{array}{ll} \vec{E}_1 = (r_1(u_0), \cdots, r_1(u_0))^t, & \vec{E}_i = (r_{i+1}(u_0), \cdots, r_{i+1}(u_0))^t, \\ \vdots & \vdots \\ \vec{E}_{i-2} = (r_{i-2}(u_0), \cdots, r_{i-2}(u_0))^t, & \vec{E}_{n-1} = (r_n(u_0), \cdots, r_n(u_0))^t, \\ \vec{E}_{i-1} = (r_{i-1}(u_0), \cdots, r_{i-1}(u_0))^t, & \vec{E}_n = (r_i(u_0), \cdots, r_i(u_0))^t, \\ & \vec{E}_{n+1} = (0, r_i(u_0), 2r_i(u_0), \cdots, (2q-1) r_i(u_0))^t. \end{array} \tag{A.6}$$

In the terms of the basis $\{\vec{E}_1, \vec{E}_2, \cdots, \vec{E}_{n+1}\}$ the matrix $d\mathcal{T}_{\vec{u}_0}(\vec{u}_0)$ is represented by

$$d\mathcal{T}(\vec{u}_0)_{|\mathcal{M}_0} \equiv \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}_{\{\vec{E}_1, \vec{E}_2, \cdots, \vec{E}_{n+1}\}}. \tag{A.7}$$

Now, we need to parameterize the set $\mathcal{J}$ in terms of the coordinates of $\vec{E}_1, \cdots, \vec{E}_{n+1}$. Next, by a straight calculation we have the following

$$\begin{array}{ll} \nabla_{\vec{E}_1} H(\vec{u}_0) = (p - \lambda q \lambda_1(u_0)) r_1(u_0), \quad \cdots, & \nabla_{\vec{E}_{i-1}} H(\vec{u}_0) = (p - \lambda q \lambda_{i-1}(u_0)) r_{i-1}(u_0), \\ \nabla_{\vec{E}_{i+1}} H(\vec{u}_0) = (p - \lambda q \lambda_{i+1}(u_0)) r_{i+1}(u_0), \quad \cdots, & \nabla_{\vec{E}_{n-1}} H(\vec{u}_0) = (p - \lambda q \lambda_n(u_0)) r_n(u_0), \\ \nabla_{\vec{E}_n} H(\vec{u}_0) = 0, & \nabla_{\vec{E}_{n+1}} H(\vec{u}_0) = q F_1(u_0, u_0) r_i(u_0). \end{array} \tag{A.8}$$

Then, by using Taylor's expansion to compute the coordinates $(e_1, \cdots, e_{n+1})^t \equiv \sum_{j=1}^{n+1} e_j \vec{E}_j$ of $\mathcal{J}$, that is, we expand the function H in term of the coordinate of $\mathcal{M}_0$

$$pu_- - qf(u_-) = H(\vec{u}_0) + \sum_{k\neq n} \nabla_{\vec{E}_k} H(\vec{u}_0) e_k + \frac{1}{2}\nabla^2 H(\vec{u}_0)(\vec{E}_n, \vec{E}_n) e_n^2 + \cdots \tag{A.9}$$

From (A.9), on the curve $\mathcal{J}$ the coordinate $e_1, \cdots, e_{n-1}, e_{n+1}$ can be expressed in terms of e_n. We are particularly interested in the coordinate e_{n+1}, which is a function of e_n:

$$e_{n+1}(e_n) \;=\; \frac{k_1}{2}(1 + O(\epsilon))(\epsilon_- - e_n)(-\epsilon_+ - e_n), \tag{A.10}$$

where k_1 is defined in (A.5) and where ϵ_- and ϵ_+ are the $\vec{E}_n$-components of the points $\vec{u}_-$ and $\vec{u}_+$ and they satisfy that

$$\epsilon_- + \epsilon_- = O(\epsilon^2), \quad \epsilon_- = O(1)\epsilon.$$

Therefore, we choose the $\vec{E}_n$-component as a parameter for the curve $\mathcal{J}$. This curve can be represented as an interval $[\epsilon_+, \epsilon_-]$. Denote $\mathcal{J}(e_n)$, $e_n \in [\epsilon_+, \epsilon_-]$, the invariant curve.

Since $\mathcal{J}$ is an invariant curver, $\mathcal{T}$ induce 1-dimensional dynamical system on $\mathcal{J}$. From the matrix representation (A.7), the change of its $\vec{E}_n$-component mostly contributed from its $\vec{E}_{n+1}$ component. Then, this induced dynamical system $\mathbf{t}$ can be represented as

$$\mathbf{t} : e_n \longmapsto e_n + (1 + O(\epsilon)) e_{n+1} = e_n + \frac{k_1}{2}(1 + O(\epsilon))(-\epsilon_+ - e_n)(\epsilon_- - e_n). \tag{A.11}$$

From (A.11), one has that

$$\begin{aligned} \mathbf{t}'(e) &= 1 + \frac{k_1}{2}(1 + O(1)\epsilon)\left((-e + \epsilon_+) + (-e + \epsilon_-)\right), \\ \mathbf{t}''(e) &= k_1 + O(\epsilon). \end{aligned} \tag{A.12}$$

Taking the following rescale for the system (A.11):

$$\begin{cases} e_n = \epsilon \bar{e}_n, \\ \mathbf{t}(e_n) = \epsilon \bar{\mathbf{t}}(\bar{e}_n), \end{cases}$$

it results

$$\frac{\Delta \bar{\mathbf{t}}_n}{\Delta e_n} \equiv \frac{\bar{\mathbf{t}}(\bar{e}_n) - e_n}{-k_1 \epsilon} = -\frac{(\bar{e}_n - \frac{\epsilon_-}{\epsilon})(\bar{e}_n - \frac{\epsilon_+}{\epsilon})}{2}. \tag{A.13}$$

(A.13) concludes that this dynamical system is a discretization of $\dot{e} = \frac{1}{2}(e - \frac{\epsilon_-}{\epsilon})(e - \frac{\epsilon_+}{\epsilon})$. From this, one can conclude that

$$|\mathbf{t}^j(0) - \mathbf{t}^{j-1}(0)| = O(1)\epsilon^2 e^{k_1 \epsilon |j|}, \tag{A.14}$$

$$\begin{cases} |\mathbf{t}^j(0) - \epsilon_-| = O(1)\epsilon e^{-k_1 \epsilon |j|} & \text{for } j \leq 0, \\ |\mathbf{t}^j(0) - \epsilon_+| = O(1)\epsilon e^{-k_1 \epsilon |j|} & \text{for } j \geq 0. \end{cases} \tag{A.15}$$

Now, we parameterize the discrete shock profile $\phi(x)$ as the following:

$$\phi(j) \;=\; \mathcal{J}(\mathbf{t}^j(0)), \quad j = 0, \pm 1, \pm 2, \cdots, \tag{A.16}$$

$$\phi(j + \theta) \;=\; \mathcal{J}(\mathbf{t}^j(0 + m(\theta))), \quad j = 0, \pm 1, \pm 2, \cdots, \quad \theta \in [0, 1], \tag{A.17}$$

where $m(\theta)$ is a function defined satisfying that

$$\begin{cases} m'(\theta) = O(1)\epsilon^2, \\ m''(\theta) = O(1)\epsilon^3, \\ \mathbf{t}'(0) \cdot m'(0) = m'(1). \end{cases} \tag{A.18}$$

This gives a C^1 parameterization for ϕ; and $\{\phi(j + x_0)\}_j$ solves the discrete shock profile for any constant x_0. Since

$$\phi'(x) = \mathcal{J}'(\mathbf{t}^k(m(x-k))) \cdot \mathbf{t}^k(m(x-k))' = O(1)\mathbf{t}^k(m(x-k))', \text{ with } k = [x].$$

we need to evaluate $\mathbf{t}^k(m(\theta))'$ for any $k \in \mathbf{Z}$ and $\theta \in [0,1)$.

$$\begin{aligned} \mathbf{t}^k(m(\theta))' &= \mathbf{t}'(\mathbf{t}_{k-1}) \cdot \mathbf{t}^{k-1}(m(\theta))' = \cdots \\ &= \mathbf{t}'(\mathbf{t}_{k-1}) \cdot \mathbf{t}'(\mathbf{t}_{k-2}) \cdot \mathbf{t}'(\mathbf{t}_{k-3}) \cdots \mathbf{t}'(\mathbf{t}_0) \cdot m'(\theta), \\ &\text{here } \mathbf{t}_k \equiv \mathbf{t}^k(m(\theta)). \end{aligned} \tag{A.19}$$

Substitute (A.15) into (A.12), one conclude that

$$\begin{aligned} &\log\left(|\mathbf{t}'(\mathbf{t}_{k-1}) \cdot \mathbf{t}'(\mathbf{t}_{k-2}) \cdot \mathbf{t}'(\mathbf{t}_{k-3}) \cdots \mathbf{t}'(\mathbf{t}_0)|\right) \\ &= \sum_{j=1}^{k} \log|\mathbf{t}'(\mathbf{t}_j)| = \sum_{j=1}^{k} \log\left(1 + \frac{(1+O(1)\epsilon)k_1}{2}\{(\epsilon_+ - \mathbf{t}^j(m(\theta))) + (\epsilon_- - \mathbf{t}^j(m(\theta)))\}\right) \\ &= -\frac{|k_1|}{2}(1+O(\epsilon))(\epsilon_- - \epsilon_+)k = -k_1(1+O(1)\epsilon)\ \epsilon k. \end{aligned}$$

From above and (A.18), we have that

$$\mathbf{t}^k(\theta)' = O(1)\epsilon^2 e^{-\epsilon k_1 |k|}. \tag{A.20}$$

Take one more derivative for (A.19), then it follows

$$\begin{aligned} \mathbf{t}^k(m(\theta))'' &= \mathbf{t}'(\mathbf{t}_{k-1}) \cdot \mathbf{t}'(\mathbf{t}_{k-2}) \cdot \mathbf{t}'(\mathbf{t}_{k-3}) \cdots \mathbf{t}'(\mathbf{t}_0) \cdot m''(\theta) \\ &+ \sum_{l=0}^{k} \mathbf{t}'(\mathbf{t}_{k-1}) \cdot \mathbf{t}'(\mathbf{t}_{k-1}) \cdots \mathbf{t}'(\mathbf{t}_{l+1}) \cdot \mathbf{t}''(\mathbf{t}_l) \cdot (\mathbf{t}'(\mathbf{t}_l))^2 \cdot (\mathbf{t}'(\mathbf{t}_{l-1}))^2 \cdots (m'(\theta))^2. \end{aligned} \tag{A.21}$$

Substitute (A.20) and (A.17) into (A.21), then it follows that

$$\mathbf{t}^k(m(\theta))'' = O(1)\epsilon^3 e^{-(1+O(\epsilon))\epsilon |k_1||k|}. \tag{A.22}$$

(A.20) and (A.22) conclude (A.4).

B Construction of an Approximate Discrete Shock Profile

Let $\phi_0(x)$ be a discrete shock connecting (u_-, u_+^0) with a discrete shock speed $d_0 = \frac{p_0}{q_0}$ such that

$$\begin{aligned} d_0 &\equiv \lambda s_0, \\ s_0(u_+^0 - u_-) &= f(u_+^0) - f(u_-). \end{aligned}$$

Let (u_-, u_+) be an entropy condition satisfied shock and

$$\begin{aligned} &\|u_+ - u_+^0\| \ll \epsilon^4. \\ &|d - d_0| \ll \epsilon^4, \quad d \equiv \lambda s, \quad s(u_- - u_+) = f(u_-) - f(u_+). \end{aligned} \tag{B.1}$$

Set

$$\tilde{\phi}(x) \equiv \phi_0(x) + (u_+ - u_+^0)\frac{l_i(u_0)(\phi_0(x) - u_-)}{l_i(u_0)(u_+^0 - u_-)}. \tag{B.2}$$

Then we define conservative functions:

$$\begin{aligned} \hat{\phi}(x) &\equiv \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \phi_0(\tau)d\tau, \\ \bar{\phi}(x) &\equiv \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \tilde{\phi}(\tau)d\tau = \tilde{\phi}(x) + \int_0^{\frac{1}{2}}\int_0^1 \tau(\phi'(x+\tau\theta) - \phi'(x-\tau\theta))d\theta d\tau \\ &= \tilde{\phi}(x) + 2\int_0^{\frac{1}{2}}\int_0^1\int_0^1 \tilde{\phi}''(x + (2\rho - 1)\tau\theta)\tau^2\theta d\rho d\theta d\tau. \end{aligned} \tag{B.3}$$

Since $\phi_0''(x) = O(1)\epsilon^3 e^{-\epsilon|x|}$,

$$\begin{aligned} \bar{\phi}(x) &= \tilde{\phi}(x) + O(1)\epsilon^3 e^{-\epsilon|x|}, \\ \hat{\phi}(x) &= \phi_0(x) + O(1)\epsilon^3 e^{-\epsilon|x|}. \end{aligned} \tag{B.4}$$

The conservation laws clearly hold:

$$\sum_{j\in\mathbf{Z}} \bar{\phi}(x+j) - \bar{\phi}(j) = \int_R \bar{\phi}(x+\tau) - \bar{\phi}(\tau)d\tau = x(u_+ - u_-). \tag{B.5}$$

From (B.5), it follows that

$$\sum_{j\in\mathbf{Z}} \left[\bar{\phi}(j+x-d) - \bar{\phi}(j+x) + \lambda\left(F[\bar{\phi}](x+j+\frac{1}{2}) - F[\bar{\phi}](x+j-\frac{1}{2})\right)\right] = 0. \tag{B.6}$$

We consider the equation error for the anti-difference function. We consider only $x \le 0$, the case $x \ge 0$ is similar,

$$\begin{aligned} I &\equiv \sum_{j\le 0}\left(-\bar{\phi}(x-d+j) + \bar{\phi}(x+j) - \lambda\left(F[\bar{\phi}](x+j+\frac{1}{2}) - F[\bar{\phi}](x+j-\frac{1}{2})\right)\right) \\ &= \int_{x-d+\frac{1}{2}}^{x-d_0+\frac{1}{2}} \left(\tilde{\phi}(\tau) - u_-\right)d\tau + \sum_{j\le 0}\left(-\bar{\phi}(x-d_0+j) + \bar{\phi}(x+j)\right) - \lambda\left(F[\bar{\phi}](x+\frac{1}{2}) - f(u_-)\right) \\ &= \int_{x-d+\frac{1}{2}}^{x-d_0+\frac{1}{2}} \left(\tilde{\phi}(\tau) - u_-\right)d\tau + \int_{x+\frac{1}{2}-d_0}^{x+\frac{1}{2}} (\phi_0(\tau) - u_-)\,d\tau - \lambda\left(F[\bar{\phi}](x+\frac{1}{2}) - f(u_-)\right). \end{aligned} \tag{B.7}$$

From (B.7),

$$\begin{aligned} I &= \int_{x-d+\frac{1}{2}}^{x-d_0+\frac{1}{2}} \left(\tilde{\phi}(\tau) - u_-\right)d\tau - \lambda\left(F[\bar{\phi}](x+\frac{1}{2}) - f(u_-)\right) + O(1)\epsilon|d-d_0|e^{-\epsilon|x|} \\ &= \int_{x-d+\frac{1}{2}}^{x-d_0+\frac{1}{2}} \left(\tilde{\phi}(\tau) - u_-\right)d\tau - \lambda\left(F[\hat{\phi}](x+\frac{1}{2}) - f(u_-)\right) + O(1)\epsilon|d-d_0|e^{-\epsilon|x|}. \end{aligned} \tag{B.8}$$

For the discrete shock profile ϕ_0, we have

$$\frac{1}{q_0}\left\{\sum_{0\le k\le p_0-1}\left(\phi_0(x-\frac{k}{q_0}+\frac{1}{2})-u_-\right)-\lambda\sum_{0\le k\le q_0-1}\left(F[\phi_0](x+1-\frac{k}{q_0})-f(u_-)\right)\right\}=0. \quad \text{(B.9)}$$

Comparing (B.8) with (B.9), we have

$$\begin{aligned} I_1 \equiv & \frac{1}{q_0}\sum_{0\le k\le p_0-1}\left(\phi_0(x-\frac{k}{q_0}+\frac{1}{2})-u_-\right)-\int_{x+\frac{1}{2}-d_0}^{x+\frac{1}{2}}(\phi_0(\tau)-u_-)\,d\tau \\ & -\frac{1}{q_0}\lambda\sum_{0\le k\le q_0-1}\left(F[\phi_0](x+1-\frac{k}{q_0})-f(u_-)\right)+\int_x^{x+1}F[\phi_0](\tau)d\tau. \\ = & \sum_{0\le k\le p_0-1}\frac{1}{q_0^2}\int_0^1\int_0^1\rho\phi_0'(x-\frac{k+1}{q_0}+\frac{1}{2}+\frac{\theta\rho}{q_0})d\theta d\rho \\ & -\lambda\sum_{0\le k\le q_0-1}\frac{1}{q_0^2}\int_0^1\int_0^1 F'[\phi_0]\cdot\phi_0'(x+1-\frac{k}{q_0}+\frac{\tau\theta}{q_0})\tau d\theta d\tau. \end{aligned} \quad \text{(B.10)}$$

Substitute $F'[\phi_0]\cdot\phi_0'(x)=(1+O(\epsilon))\phi'(x)$ and $\phi_0''(x)=O(1)\epsilon^3e^{-\epsilon|x|}$ into (B.10) to yields

$$I_1=O(1)\epsilon^3e^{-\epsilon|x|}. \quad \text{(B.11)}$$

Comparing the following identities

$$\begin{aligned} \int_x^{x+1}F[\phi_0](\tau)d\tau &= F[\phi_0](x+\frac{1}{2})+O(1)\nabla_x^2(F[\phi_0]\cdot\phi_0(x)) \quad \text{(B.12)} \\ &= F[\phi_0](x+\frac{1}{2})+O(1)\epsilon^3e^{-\epsilon|x|}; \\ F[\hat{\phi}](x+\frac{1}{2}) &= F[\phi_0+O(1)\epsilon^3e^{-\epsilon|x|}](x+\frac{1}{2})=F[\phi_0](x+\frac{1}{2})+O(1)\epsilon^3e^{-\epsilon|x|}, \quad \text{(B.13)} \end{aligned}$$

and with (B.11), (B.12) and (B.13) we conclude that the equation error for the anti-difference is

$$\sum_{j\le 0}\left(-\bar{\phi}(x-d+j)+\bar{\phi}(x+j)-\lambda\left(F[\bar{\phi}](x+j+\frac{1}{2})-F[\bar{\phi}](x+j-\frac{1}{2})\right)\right)=O(1)\epsilon^3e^{-\epsilon|x|}. \quad \text{(B.14)}$$

Apply difference in x to (B.14) to yield, c.f. (B.14),

$$-\bar{\phi}(x-d)+\bar{\phi}(x)-\lambda\left(F[\bar{\phi}](x+\frac{1}{2})-F[\bar{\phi}](x-\frac{1}{2})\right)=O(1)\epsilon^4e^{-\epsilon|x|}. \quad \text{(B.15)}$$

References

[1] B. Engquist & S.-H. Yu, *Convergence of a finite difference scheme.*

[2] Gray Jennings, *Discrete Shocks,* Comm. Pure Appl. Math. v27, 1974, pp 25-37

[3] A. Madja & J. Ralston, *Discrete Shock Profiles for System of Conservation Laws,* Comm. Pure Appl. Math., v32, 1979, pp 445 - 482

[4] Michaelson, *Discrete shocks for difference approximations to system of conservation laws,* Adv. in Appl. Math., v5, 1984, pp. 433-469.

[5] P. Lax, *Hyperbolic Systems of Conservation Laws II,* Comm. Pure & Appl. Vol 10, pp 537-566, 1957

[6] J.-G. Liu & Z. Xin, L^1*−stability of stationary discrete shocks,* Math. Comp., v60, 1993 pp 233-244

[7] T.-P. Liu, *Pointwise Convergence to Shock Waves for Viscous Conservation Laws,* to appear

[8] P. Roe, *Private communication.*

[9] D. Serre, *Remarks about the Discrete Profiles of Shock Waves,* Preprint.

[10] Y. Smylris, *Existence and Stability of Stationary Profiles of the LW scheme,* Comm. Pure Appl. v42, 1990, pp509-545

[11] Y. Smylris & S.-H. Yu, *On existence and stability of a discrete shock profiles for a finite difference scheme*, to appear

[12] L.-A. Ying, *Private communication.*

[13] S.-H. Yu, *Existence of Discrete Shock Profiles for the Lax-Wendroff Scheme,* to appear in JDDE.