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COMPLEX SINGULARITIES FOR BURGERS EQUATION WITH COMPLEX VISCOSITY

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Abstract. A meromorphic solution to Burgers' equation with complex viscosity is analyzed. The equation is linearized via the Cole-Hopf transform which allows for a careful study of the behavior of the singularities of the solution. The asymptotic behavior of the solution as the dispersion coefficient tends to zero is derived. The time evolution of the poles is described by infinite dimensional Calogero type dynamical system. A continuum limit of the pole expansion and the Calogero system is obtained, yielding a new integral representation of the solution to the inviscid Burgers' equation.

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1. Introduction.

Many nonlinear dispersive systems exhibit rapid oscillations in their spatial-temporal dependence in the regime of small dispersion. In this paper we consider Burgers equation with an imaginary "viscosity" coefficient $\nu = i\epsilon$, given by

$$\psi_t + \psi\psi_x = i\epsilon\psi_{xx}, \quad \epsilon \geqslant 0. \tag{1.1}$$

It is perhaps the simplest example of a nonlinear dispersive equation, but has received surprisingly little attention.

Equation (1.1) can be solved using the Cole-Hopf transform which yields an integral representation involving the heat kernel. For small $|\nu| = \epsilon$, the resulting formula for ψ_ν can be approximated using the *stationary phase* method. A new method used to compute the solution is found through *pole dynamics*. This method is based on obtaining the time dependent locations of the complex poles of the function ψ_ν by solving an infinite system of coupled ODEs. Finally, in the zero-dispersion (or zero-viscosity limit) $\nu \to 0$, the poles coalesce onto a branch-cut, and the zero-dispersion solution is described by *branch-cut dynamics*. This method may be of general interest as a new (to the best of our knowledge) method for solving the inviscid Burgers equation.

These methods will be formulated in general, but illustrated for a special choice of initial data, namely the cubic polynomial

$$\psi(x, 0) = 4x^3 - x/t_* \tag{1.2}$$

which is chosen for its generic features for the inviscid equation (see [1, 6, 10]). In this initial data, t_* is positive and corresponds to the time of first singularity formation for the inviscid problem. The cube root singularity found at the origin at $t = t_*$ is known to be a generic singularity for the inviscid Burgers equation. It is due to the coalescence of two conjugate branch points of order two in the complex plane. For further details, see [1, 2, 3, 6, 13]. Moreover both cases $\nu = 0$ and $\nu \neq 0$ can be completely analyzed,

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and in the case $\nu \neq 0$, there is an instantaneous generation at $t > 0$ of a countable set of complex spatial simple poles. For this initial data, the small dispersion ($\epsilon \to 0^+$) stationary phase approximation of the solution and its zeroes can be evaluated rather explicitly, at least for $t = t_*$.

There are three main points to this work: First, in the purely dispersive case in which ν is imaginary and small, the solution ψ_ν of (1.1) develops rapid oscillations. Second, these oscillations are caused by the presence of complex poles in ψ which have moved close to the real axis. This result, which is clearly demonstrated through numerical computations in [13], is important in providing a tangible cause for the formation of the oscillations. Third, the branch cut dynamics provide a slowly varying (but incomplete) description of the pole locations.

2. Integral representation, pole expansion and pole dynamics for $\nu \in \mathbb{C}^+$. Consider a complex viscosity coefficient $\nu = \epsilon e^{i\theta}$, $\epsilon > 0$ and $|\theta| \leq \pi/2$. According to the Cole-Hopf transform $\psi_\nu = -2\nu\, \partial_x \log(\phi_\nu)$ linearizes equation (1.1) into the diffusion equation for ϕ_ν. Thus the solution for the initial data (1.2) is given by

$$(2.1) \qquad E_\nu(x,t) = E_{\epsilon e^{i\theta}}(x,t) = \int_{-\infty}^{\infty} \exp\left\{ \frac{e^{-i\theta}}{2\epsilon} \left(\frac{x}{t} y + \alpha y^2 - y^4 \right) \right\} dy.$$

A rigorous justification of this solution is presented in [11].

According to the Cole-Hopf transformation, the only singularities in ψ_ν come from zeroes of $E_\nu(x,t)$. These come in opposite pairs $x_n = \pm a_n(t,\nu)$, since $E_\nu(-x,t) = E_\nu(x,t)$. Furthermore, one can show that the infinite product representation of E_ν is

$$(2.2) \qquad E_\nu(x,t) = C_\nu(t) \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{a_n^2(t,\nu)} \right).$$

It follows that

$$(2.3) \qquad \psi_\nu(x,t) = \frac{x}{t} - \frac{\Psi_\nu(x,t)}{t} = \frac{x}{t} - \sum_{n=1}^{\infty} \frac{4\nu x}{x^2 - a_n^2(t,\nu)}.$$

A more symmetric expression for ψ_ν is

$$(2.4) \qquad \psi_\nu(x,t) = \frac{x}{t} - 2\nu \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{x - a_n(t,\nu)}.$$

Using partial fraction expansions, we find (see [11] for more details) the associated Calogero dynamical system for arbitrary $\nu \in \mathbb{C}^+$: Let

$$\dot{a}_n = \frac{da_n}{dt}, \quad a_{-n} = -a_n,$$

then

$$(2.5) \qquad \dot{a}_n = \frac{a_n}{t} - \frac{\nu}{a_n} - 4\nu a_n \sum_{\substack{l=1 \\ l\neq n}}^{\infty} \frac{1}{a_n^2 - a_l^2}, \qquad \forall n \in \mathbb{N}^*.$$

As in (2.4), one can express (2.5) in a more symmetric way as

$$(2.6) \qquad \dot{a}_n = \frac{a_n}{t} - 2\nu \sum_{\substack{l=-\infty \\ l\neq n,0}}^{\infty} \frac{1}{a_n - a_l}, \qquad \forall n \in \mathbb{N}^*.$$

Note finally that the pole expansion (2.4) and the dynamical system (2.6) represent a general solution to Burgers' equation which may be valid for general initial data.

One can further simplify (2.5) by multiplying both sides by a_n and introducing the variable

$$\kappa_n = \frac{a_n^2}{\nu}. \tag{2.7}$$

The corresponding system of ordinary differential equations (2.5) becomes free of ν so that

$$\frac{1}{2}\frac{d\kappa_n}{dt} = \frac{\dot{\kappa}_n}{2} = \frac{\kappa_n}{t} - 1 - 4\kappa_n \sum_{\substack{l=1 \\ l\neq n}}^{\infty} \frac{1}{\kappa_n - \kappa_l}, \quad \forall n \in \mathrm{N}^*. \tag{2.8}$$

3. Asymptotic analysis of $\Psi_\nu(x,t)$ for $\nu = i\epsilon$, as $\epsilon \to 0^+$, $t > t_*$. When $\nu = i\epsilon$, $\epsilon > 0$, we evaluate the asymptotic behavior of E_ν as $\epsilon \to 0^+$ using the method of stationary phase. We find that all three saddle points are relevant within the caustic $|x| < |x_s(t)| - \delta/2$, where $\delta > 0$ and where $\pm x_s(t)$ are the second order branch points of the inviscid solution (see [11, §6]). For a discussion on such caustics, cf. [8, 9]. When $t > t_*$, $x \in (-\infty, -x_s(t) - \delta/2) \cup (x_s(t) + \delta/2, \infty)$, $\nu = i\epsilon$, $\epsilon \to 0^+$, the same analysis holds and one recovers the characteristic solution outside of the caustic consisting of one relevant saddle point. The transition from within the caustic to outside is not uniform as the asymptotic behavior at the caustic $x = \pm x_s(t)$ is degenerate (2 saddle points have coalesced).

The most interesting part of the expansion is within the caustic $x \in (-x_s(t), x_s(t))$. The caustic $x = x_s(t)$ corresponds to the envelope of the characteristics of the inviscid Burgers solution, and is also determined by the system of equations

$$\begin{cases} 0 = w_z(z,x) = x/t + 2\alpha z - 4z^3, \\ 0 = w_{zz}(z,x) = 2\alpha - 12z^2, \end{cases} \tag{3.1}$$

where $w(z,x)$ is the phase function of the integrand in the definition of $E_\nu(x,t)$. This system represents the conditions for the phase function w to have saddle points of multiplicity two, thereby yielding a curve in the (x,t) plane on which two saddles of multiplicity one coalesce into a saddle of multiplicity two. From the second equation in (3.1), we find $z_{caustic}(t) = \pm\sqrt{\alpha/6}$, and from the first,

$$x = x_{caustic} = t(4z_{caustic}(t)^3 - 2\alpha z_{caustic}(t)) = \mp t\left(\frac{2\alpha}{3}\right)^{3/2} = \mp x_s(t), \tag{3.2}$$

where $x_s(t) = i\,(3t_*)^{-3/2}(t_* - t)^{3/2}t^{-1/2}$ is the second order branch point of the dispersionless solution described in [11, §6]. Here we are only concerned with the dominant behavior of $E_{i\epsilon}$, thus we only retain the first term:

$$E_{i\epsilon}(x,t) = \sum_{s=0,1,2} \sqrt{\frac{-4\pi i\epsilon}{w_{zz}(z_s,x)}} \exp\left(\frac{w(z_s,x)}{2i\epsilon}\right)\Big(1 + \mathcal{O}(\epsilon)\Big), \tag{3.3}$$

as $\epsilon \to 0^+$, with

$$w(z_s(x,t),x) = \frac{x}{t}z_s + \alpha z_s^2 - z_s^4 = \frac{3}{4}\frac{x}{t}z_s + \frac{\alpha}{2}z_s^2, \tag{3.4a}$$

$$w_z(z_s(x,t),x) = 0, \quad w_{zz}(z_s(x,t),x) = 2\alpha - 12z_s^2. \tag{3.4b}$$

The values of the saddle points $z_s = z_s(x,t)$ are determined by the three roots of the first equation in system (3.1), i.e. the first equation of (3.4b). They are specifically

$$\left\{\begin{array}{l} z_0 = \omega\mathcal{A} + \omega^2\mathcal{B} \\ z_1 = \omega^2\mathcal{A} + \omega\mathcal{B} \\ z_2 = \mathcal{A} + \mathcal{B} \end{array}\right. \tag{3.5}$$

with $w = e^{2\pi i/3}$ and

$$\left\{\begin{array}{l} \mathcal{A}(x,t) = (8t)^{-1/3}\cdot\sqrt[3]{x+\sqrt{x^2-x_s^2}} \\ \mathcal{B}(x,t) = (8t)^{-1/3}\cdot\sqrt[3]{x-\sqrt{x^2-x_s^2}}. \end{array}\right. \tag{3.6}$$

Note that all three saddle points are real when $x, x_s \in \mathbb{R}$ and the discriminant $\Delta = x^2 - x_s^2 < 0$, that is $|x| < |x_s(t)|$, and in this case $\mathcal{A} = \overline{\mathcal{B}}$ (see [11, Appendix B]). Therefore we have $z_s \in \mathbb{R}$, $w(z_s,x) \in \mathbb{R}$, and $w_{zz}(z_s,x) = 2\alpha - 12z_s^2 \in \mathbb{R}$. Hence all three terms in the summation signs are oscillatory and equally relevant. Note however that the expansion derived for $E_{i\epsilon}$ is only valid within $|x| < |x_s|$, and in order to get an expansion uniformly valid across $x = \pm x_s$ one needs to derive a uniform expansion as presented in [5, 8]. This analysis is similar in spirit to the one of Jin, Levermore and McLaughlin in [7, §2.2] and that of [5]. The dominant behavior of the solution $\psi_{i\epsilon}(x,t)$ is found from the Cole-Hopf representation, so that within the caustic $|x| < |x_s| - \delta/2$, following the derivation presented in [11, §3], we find

$$\begin{aligned} \Psi_{i\epsilon}(x,t) &= \frac{\sum_{s=0,1,2} z_s \cdot e^{\frac{w(z_s,x)}{2i\epsilon}} / \sqrt{w_{zz}(z_s,x)}}{\sum_{s=0,1,2} e^{\frac{w(z_s,x)}{2i\epsilon}} / \sqrt{w_{zz}(z_s,x)}} + \mathcal{O}(\epsilon) \\ &= \frac{\sum_{s=0,1,2} z_s \cdot e^{\frac{w(z_s,x)}{2i\epsilon} - \frac{i}{2}\arg(w_{zz}(z_s,x))} \cdot |w_{zz}(z_s,x)|^{-1/2}}{\sum_{s=0,1,2} e^{\frac{w(z_s,x)}{2i\epsilon} - \frac{i}{2}\arg(w_{zz}(z_s,x))} \cdot |w_{zz}(z_s,x)|^{-1/2}} + \mathcal{O}(\epsilon). \end{aligned}$$

Since $w_{zz}(z_s,x) \in \mathbb{R}$, we have that $\arg(w_{zz}(z_s,x)) = \frac{\pi}{2}(1 - \operatorname{sgn}(w_{zz}(z_s,x)))$, and therefore

$$\Psi_{i\epsilon}(x,t) = \frac{\sum_{s=0,1,2} z_s \cdot e^{-\frac{i}{2\epsilon}w(z_s,x) + \frac{i\pi}{4}\operatorname{sgn}(w_{zz}(z_s,x))} \cdot |w_{zz}(z_s,x)|^{-1/2}}{\sum_{s=0,1,2} e^{-\frac{i}{2\epsilon}w(z_s,x) + \frac{i\pi}{4}\operatorname{sgn}(w_{zz}(z_s,x))} \cdot |w_{zz}(z_s,x)|^{-1/2}} + \mathcal{O}(\epsilon).$$

The asymptotic behavior of the solution is then found from the relation

$$\psi_{i\epsilon}(x,t) = \frac{x}{t} - \frac{\Psi_{i\epsilon}(x,t)}{t}.$$

Thus the presence of three competing oscillatory terms in the asymptotic behavior of $\Psi_{i\epsilon}$ is reminiscent of the oscillations observed in the solution $\psi_{i\epsilon}$. Such oscillations are also seen in computations of the pole dynamics [13].

4. Continuum limit of the pole expansion and the Calogero dynamical system. From the equation for the pole dynamic, one can obtain a set of equations for the inviscid limit which give a new representation of the solution to the inviscid burgers equation. Recall the pole expansion

$$\psi_\nu(x,t) = \frac{x}{t} - \sum_{n=1}^{\infty} \frac{4\nu x}{x^2 - a_n^2(t,\nu)} = \frac{x}{t} - 2\nu \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{x - a_n}, \tag{4.1a}$$

and the pole dynamics: $\forall n \in \mathbf{N}^*$,

$$\dot{a}_n = \frac{a_n}{t} - \frac{\nu}{a_n} - 4\nu a_n \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{1}{a_n^2 - a_l^2} = \frac{a_n}{t} - 2\nu \sum_{\substack{l=-\infty\\ l\neq n,0}}^{\infty} \frac{1}{a_n - a_l}. \tag{4.1b}$$

Assume that

$$a_n(t,\nu) = \mathcal{F}(n|\nu|,\nu,t) = f(n|\nu|,t) + e_n(t,\nu) \tag{4.2}$$

in which $e_n(t,\nu)$ is a small error term that goes to 0 as $\nu \to 0$. Now let $|\nu| \to 0$ so that $\eta = \nu/|\nu|$ remains constant. Then, at least formally,

$$\begin{aligned} 2\nu \sum_{\ell\neq n} \frac{1}{a_n(t,\nu) - a_\ell(t,\nu)} &\simeq 2\eta|\nu| \sum_{\ell\neq n} \frac{1}{f(n|\nu|,t) - f(\ell|\nu|,t)} \\ &\xrightarrow{\nu\to 0} 2\eta\, P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta,t) - f(\zeta',t)}. \end{aligned} \tag{4.3}$$

Moreover, this approximation shows that the representation (4.2) is valid for all time if it is true at $t = 0$. It then follows that

$$\frac{\partial f}{\partial t}(\zeta,t) = \frac{f(\zeta,t)}{t} - \eta f(\zeta,t)\, P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta,t) - f(\zeta',t)}, \tag{4.4a}$$

and

$$\psi(x,t) = \frac{x}{t} - \eta x \int_{-\infty}^{\infty} \frac{d\zeta'}{x^2 - f^2(\zeta',t)}, \qquad x \neq f(\zeta,t). \tag{4.4b}$$

A straightforward calculation shows that $\psi(x,t)$ is a solution to the inviscid Burgers' equation

$$\psi_t + \psi\psi_x = 0 \tag{4.5}$$

for any choice of η. We refer to the system (4.4a) and (4.4b) as the *branch cut dynamics* equations, for reasons shown next.

These equations can be rephrased in a second, non-parametric formulation involving a moving curve $\Gamma(t)$ in the complex plane (which may consist of several disconnected parts) and a density function $\rho(z,t)$ defined for $z \in \Gamma(t)$. In particular $\Gamma(t)$ is the image of $f(\zeta,t)$ for ζ varying over the real line. The density function $\rho(z,t)$ is defined by (see [12, §5])

$$\rho(z,t) = \frac{1}{f_\zeta(\zeta,t)}, \tag{4.6}$$

in which $z = f(\zeta,t)$. Then $d\zeta' = \rho(z',t)\,dz'$ and

$$\psi(x,t) = \frac{x}{t} - 2\eta \int_{\Gamma(t)} \frac{\rho(z',t)}{x - z'}\, dz'. \tag{4.7}$$

This formula can be extended into the complex x-plane but is discontinuous across the curve $\Gamma(t)$, i.e. $\Gamma(t)$ is a branch cut for the function ψ. Variations in the arbitrary complex parameter η correspond to variations in the branch cut $\Gamma(t)$ for ψ, without change in the branch point. An application of the Plemejl formulas at a point z on $\Gamma(t)$ shows that limiting values ψ_+ and ψ_- from the right and left, respectively, are

$$\psi_\pm(z,t) = \frac{z}{t} - 2\eta \oint \frac{\rho(z',t)}{z-z'}\,dz' \mp 2\eta\pi i\rho(z,t). \tag{4.8}$$

It follows that the difference of $\psi_\pm$ is

$$\psi_-(z,t) - \psi_+(z,t) = 4\eta\pi i\rho(z,t), \tag{4.9}$$

and the average of $\psi_\pm$ is

$$\begin{aligned}\tilde{\Psi}(z,t) &\equiv \frac{1}{2}(\psi_+(z,t)+\psi_-(z,t)) \\ &= \frac{z}{t} - 2\eta \oint_{\Gamma(t)} \frac{\rho(z',t)}{z-z'}\,dz' \\ &= \frac{x}{t} - 2\eta\, P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta,t)-f(\zeta',t)} = \frac{\partial f}{\partial t}(\zeta,t).\end{aligned} \tag{4.10}$$

Since $0 = \psi_t + \psi\psi_z = \psi_t + \left(\frac{1}{2}\psi^2\right)_z$ for both $\psi = \psi_+$ and ψ_-, it follows that ρ satisfies the conservation equation

$$\rho_t + (\tilde{\Psi}\rho)_z = 0. \tag{4.11}$$

Therefore the branch cut dynamics equations (4.4a) and (4.4b) are equivalent to the motion of $\Gamma(t)$ by the velocity $\tilde{\Psi}(z,t)$, and the evolution of the density $\rho(z,t)$ through (4.11).

The usefulness of this method in the present context is its relation to the pole dynamics for the viscous (or dispersive) Burgers equation. An interesting equivalent form of the branch cut dynamics equations (4.4a) and (4.4b) is found by considering the change of time variable

$$\begin{aligned}\tau &= t^{-1} - t_0^{-1} \\ g(\zeta,\tau) &= t^{-1} f(\zeta,t)\end{aligned} \tag{4.12}$$

for any constant t_0. The resulting equation for g is

$$\frac{\partial g}{\partial \tau}(\zeta,\tau) = 2\eta\, P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{g(\zeta,\tau)-g(\zeta',\tau)}. \tag{4.13}$$

If $\eta = 1/(4\pi i)$, and if the left hand side was replaced by its complex conjugate $\partial\bar{g}/\partial\tau$, this equation would be identical to the Birkhoff-Rott equation for a vortex sheet [4].

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