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Approximate Solutions of Nonlinear Conservation Laws and Related Equations

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To Peter Lax and Louis Nirenberg on their 70^{th} birthday

Abstract

During the recent decades there was an enormous amount of activity related to the construction and analysis of modern algorithms for the approximate solution of nonlinear hyperbolic conservation laws and related problems.

To present some aspects of this successful activity, we discuss the analytical tools which are used in the development of convergence theories for these algorithms. These include classical compactness arguments (based on BV a priori estimates), the use of compensated compactness arguments (based on H^{-1}-compact entropy production), measure valued solutions (measured by their negative entropy production), and finally, we highlight the most recent addition to this bag of analytical tools – the use of averaging lemmas which yield new compactness and regularity results for nonlinear conservation laws and related equations.

We demonstrate how these analytical tools are used in the convergence analysis of approximate solutions for hyperbolic conservation laws and related equations. Our discussion includes examples of Total Variation Diminishing (TVD) finite-difference schemes; error estimates derived from the one-sided stability of Godunov-type methods for convex conservation laws (and their multidimensional analogue – viscosity solutions of demi-concave Hamilton-Jacobi equations); we outline, in the one-dimensional case, the convergence proof of finite-element streamline-diffusion and spectral viscosity schemes based on the div-curl lemma; we also address the questions of convergence and error estimates for multidimensional finite-volume schemes on non-rectangular grids; and finally, we indicate the convergence of approximate solutions with underlying kinetic formulation, e.g., finite-volume and relaxation schemes, once their regularizing effect is quantified in terms of the averaging lemma.

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1 Introduction

The construction, analysis and implementation of approximate solutions to nonlinear conservation laws and related equations were the major focus of an enormous amount of activity in recent decades. Modern algorithms were developed for the accurate computation of shock discontinuities, slip lines, and other similar phenomena which could be characterized by spontaneous evolution of change in scales. Such phenomena pose a considerable computational challenge, which is answered, at least partially, by these newly constructed algorithms. New modern algorithms were devised, that achieve one or more of the desirable properties of high-resolution, efficiency, stability — in particular, lack of spurious oscillations, etc. The impact of these new algorithms ranges from the original impetus in the field of Computational Fluid Dynamics (CFD), to the fields oil recovery, moving fronts, image processing,... [74], [137], [131], [1].

We survey a variety of these algorithms for the approximate solution of nonlinear conservation laws. The presentation is neither comprehensive nor complete — the scope is too wide for the present framework[1]. Instead, we focus our attention of the analysis part – more precisely, we discuss the analytical tools which are used to study the stability and convergence of these modern algorithms. We use these analytical issues as our 'touring guide' to provide a readers' digest on the relevant approximate methods, while studying there convergence properties.

Some general references are in order. The theory of hyperbolic conservation laws is covered in [94], [173],[155], [147]. For the theory of their numerical approximation consult [102],[57],[58],[157]. We are concerned with analytical tools which are used in the convergence theories of such numerical approximations. The monograph [49] could be consulted on recent development regarding weak convergence. The reviews of [167], [122, 123] are recommended references for the theory of compensated compactness, and [39, 40],[17] deal with applications to conservation laws and their numerical approximations. Measure-valued solutions in the context of nonlinear conservation laws were introduced in [41]. The articles [61], [52], [44] prove the averaging lemma, and [110],[111],[77] contain applications in the context of kinetic formulation for nonlinear conservation laws and related equations.

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2 Hyperbolic Conservation Laws

2.1 A very brief overview — m equations in d spatial dimensions

The general set-up consists of m equations in d spatial dimensions

$$\partial_t \rho + \nabla_x \cdot A(\rho) = 0, \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^d. \tag{2.1}$$

Here, $A(\rho) := (A_1(\rho), \ldots, A_d(\rho))$ is the d-dimensional flux, and $\rho := (\rho_1(t,x), \ldots, \rho_m(t,x))$ is the unknown m-vector subject to initial conditions $\rho(0,x) = \rho_0(x)$.

[1] Among the methods omitted from our discussion are Dafermos' polygonal method, [34], the particle method, [171], relaxation algorithms, [173], [19],[82],[124], and Boltzmann schemes, [38], [133],[136].

The basic facts concerning such nonlinear hyperbolic systems are, consult [94],[112], [35],[155],[?],[147],

- The evolution of *spontaneous* shock discontinuities which requires weak (distributional) solutions of (2.1);
- The existence of possibly infinitely *many* weak solutions of (2.1);
- To *single* out a unique 'physically relevant' weak solution of (2.1), we seek a solution, $\rho = \rho(t,x)$, which can be realized as a viscosity limit solution, $\rho = \lim \rho^\varepsilon$,

$$\partial_t \rho^\varepsilon + \nabla_x \cdot A(\rho^\varepsilon) = \varepsilon \nabla_x \cdot (Q \nabla_x \rho^\varepsilon), \quad \varepsilon Q > 0; \tag{2.2}$$

- *The entropy condition.* The notion of a viscosity limit solution is intimately related to the notion of an *entropy solution*, ρ, which requires that for all convex entropy functions, $\eta(\rho)$, there holds, [93], [88, §5]

$$\partial_t \eta(\rho) + \nabla_x \cdot F(\rho) \le 0. \tag{2.3}$$

A scalar function, $\eta(\rho)$, is *an entropy function* associated with (2.1), if its Hessian, $\eta''(\rho)$, symmetrizes the spatial Jacobians, $A_j'(\rho)$,

$$\eta''(\rho) A_j'(\rho) = A_j'(\rho)^\top \eta''(\rho), \qquad j = 1, \dots, d.$$

It follows that in this case there exists an entropy flux, $F(\rho) := (F_1(\rho), \dots, F_d(\rho))$, which is determined by the compatibility relations,

$$\eta'(\rho)^\top A_j'(\rho) = F_j'(\rho)^\top, \ j = 1, \dots, d. \tag{2.4}$$

The basic questions regarding the existence, uniqueness and stability of entropy solutions for general systems are open. Instead, the present trend seems to concentrate on special systems with additional properties which enable to answer the questions of existence, stability, large time behavior, etc. One-dimensional 2×2 systems is a notable example for such systems: their properties can be analyzed in view of the existence of Riemann invariants and a family of entropy functions, [55], [94, §6], [155], [39, 40]. The system of $m \ge 2$ chromatographic equations, [77], is another example for such systems.

The difficulty of analyzing general systems of conservation laws is demonstrated by the following negative result due to Temple, [170], which states that already for systems with $m \ge 2$ equations, there exists no metric, $\mathcal{D}(\cdot;\cdot)$, such that the problem (2.1), (2.3) is contractive, i.e.,

$$\nexists \mathcal{D}: \quad \mathcal{D}(\rho^1(t,\cdot); \rho^2(t,\cdot)) \le \mathcal{D}(\rho^1(0,\cdot); \rho^2(0,\cdot)), \quad 0 \le t \le T, \quad (m \ge 2). \tag{2.5}$$

In this context we state the following.

Theorem 2.1 *Assume the system (2.1) is endowed with a one-parameter family of entropy pairs, $(\eta(\rho;c),\ F(\rho;c)),\ c \in \mathbf{R}^m$, satisfying the symmetry property*

$$\eta(\rho;c) = \eta(c;\rho), \quad F(\rho;c) = F(c;\rho). \tag{2.6}$$

Let ρ^1, ρ^2 be two entropy solutions of (2.1). Then the following a priori estimate holds

$$\int_x \eta(\rho^1(t,x); \rho^2(t,x))dx \le \int_x \eta(\rho_0^1(x); \rho_0^2(x))dx. \tag{2.7}$$

Couple of remarks is is order.

1. Theorem 2.1 seems to circumvent the negative statement of (2.5). This is done by replacing the metric $\mathcal{D}(\cdot;\cdot)$, with the weaker topology induced by a *family* of convex entropies, $\eta(\cdot;\cdot)$. Many physically relevant systems are endowed with at least one convex entropy function (– which in turn, is linked to the hyperbolic character of these systems, [60],[51],[119]). Systems with "rich" families of entropies like those required in Theorem 2.1 are rare, however, consult [146]. The instructive (yet exceptional...) scalar case is dealt in §2.2. If we relax the contractivity requirement, then we find a uniqueness theory for one-dimensional systems which was recently developed by Bressan and his co-workers, [11]-[14]; Bressan's theory is based on the L^1-stability (rather than contractivity) of the entropy solution operator of one-dimensional systems.

2. Theorem 2.1 is based on the observation that the symmetry property (2.6) is the key ingredient for Kružkov's penetrating ideas in [88], which extends his scalar arguments into the case of general systems. I have not found a written reference of this extension (though it seems to be part of the 'folklore' familiar to some, [36],[148]). For completeness we therefore turn to

Proof of Theorem 2.1(Sketch). $\rho^1(t,x)$ being an entropy solution of (2.1) satisfies the entropy inequality (2.3). We employ the latter with the entropy pair, $(\eta(\rho^1;c), F(\rho^1;c))$ parameterized with $c=\rho^2(\tau,y)$. This tells us that $\rho^1(t,x)$ satisfies

$$\partial_t \eta(\rho^1(t,x);\rho^2(\tau,y)) + \nabla_x \cdot F(\rho^1(t,x);\rho^2(\tau,y)) \le 0. \tag{2.8}$$

Let φ_δ denotes a symmetric C_0^∞ unit mass mollifier which converges to Dirac mass in R as $\delta \downarrow 0$; set $\phi_\delta(t-\tau,x-y) := \varphi_\delta(\frac{t-\tau}{2}) \prod_j \varphi_\delta(\frac{x_j-y_j}{2})$ as an approximate Dirac mass in $R^+ \times R^d$. 'Multiplication' of the entropy inequality (2.8) by $\phi_\delta(t-\tau,x-y)$ yields

$$\partial_t(\phi_\delta \eta(\rho^1;\rho^2)) + \nabla_x \cdot (\phi_\delta F(\rho^1;\rho^2)) \le (\partial_t \phi_\delta)\eta(\rho^1;\rho^2) + (\nabla_x \phi_\delta)\cdot F(\rho^1;\rho^2). \tag{2.9}$$

A dual manipulation – this time with (τ,y) as the primary integration variables of $\rho^2(\tau,y)$ and (t,x) parameterizing $c=\rho^1(t,x)$, yields

$$\partial_\tau(\phi_\delta \eta(\rho^2;\rho^1)) + \nabla_y \cdot (\phi_\delta F(\rho^2;\rho^1)) \le (\partial_\tau \phi_\delta)\eta(\rho^2;\rho^1) + (\nabla_y \phi_\delta)\cdot F(\rho^2;\rho^1). \tag{2.10}$$

We now add the last two inequalities: by the symmetry property (2.1), the sum of the right-hand sides of (2.9) and (2.10) vanishes, whereas by sending δ to zero, the sum of the left-hand sides of (2.9) and (2.10) amounts to

$$\partial_t \eta(\rho^1(t,x);\rho^2(t,x)) + \nabla_x \cdot F(\rho^1(t,x);\rho^2(t,x)) \le 0.$$

The result follows by spatial integration. ■

2.2 Scalar conservation laws $(m=1, d\ge 1)$

The family of admissible entropies in the scalar case consists of *all* convex functions, and the envelope of this family leads to Kružkov's entropy pairs [88]

$$\eta(\rho;c) = |\rho - c|, \quad F(\rho;c) = sgn(\rho-c)(A(\rho)-A(c)), \qquad c \in R. \tag{2.11}$$

Theorem 2.1 applies in this case and (2.7) now reads

- *L^1-contraction.* If ρ^1, ρ^2 are two entropy solutions of the scalar conservation law (2.1), then
$$\|\rho^2(t,\cdot) - \rho^1(t,\cdot)\|_{L^1(x)} \le \|\rho_0^2(\cdot) - \rho_0^1(\cdot)\|_{L^1(x)}. \tag{2.12}$$

Thus, the entropy solution operator associated with scalar conservation laws is L^1-contractive (– or non-expansive to be exact), and hence, by the Crandall-Tartar lemma [32], it is also monotone
$$\rho_0^2(\cdot) \ge \rho_0^1(\cdot) \Longrightarrow \rho^2(t,\cdot) \ge \rho^1(t,\cdot). \tag{2.13}$$
Early constructions of approximate solutions for scalar conservation laws, most notably — finite-difference approximations, utilized this monotonicity property to construct convergent schemes, [30], [141]. Monotone approximations are limited, however, to first-order accuracy [71]. (We shall say more on the issue of accuracy in §3.1). At this stage we note that the limitation of first-order accuracy for monotone approximations, can be avoided if L^1-contractive solutions are replaced with (the weaker) requirement of bounded variation solutions.

- *TV bound.* The solution operator associated with (2.1) is translation invariant. Comparing the scalar entropy solution, $\rho(t,\cdot)$, with its translate, $\rho(t,\cdot+\Delta x)$, the L^1-contraction statement in (2.12) yields the TV bound, [172],
$$\|\rho(t,\cdot)\|_{BV} \le \|\rho_0(\cdot)\|_{BV}, \quad \|\rho(t,\cdot)\|_{BV} := \sup_{\Delta x \neq 0} \frac{\|\rho(t,\cdot+\Delta x) - \rho(t,\cdot)\|_{L^1}}{\Delta x}. \tag{2.14}$$

Construction of scalar entropy solutions by TV-bounded approximations were used in the pioneering works of Oleĭnik [128], Vol'pert [172], Kružkov [88] and Crandall [28]. In the one-dimensional case, the TVD property (2.14) enables to construct convergent difference schemes with high-order (> 1) resolution; Harten initiated the construction of high-resolution TVD schemes in [69], following the earlier works [6], [98]. A whole generation of TVD schemes was then developed during the beginning of the '80s; some aspects of these developments can be found in §3.2-§3.4.

2.3 One dimensional systems ($m \ge 1, d = 1$)

We focus our attention on one-dimensional hyperbolic systems governed by
$$\partial_t \rho + \partial_x A(\rho) = 0, \quad (t,x) \in R^+ \times R, \tag{2.15}$$
and subject to initial condition, $\rho(0,x) = \rho_0(x)$. The hyperbolicity of the system (2.15) is understood in the sense that its Jacobian, $A'(\rho)$, has a complete *real* eigensystem, $(a_k(\rho), r_k(\rho)), k = 1,\dots,m$. For example, the existence of a convex entropy function guarantees the symmetry of $A'(\rho)$ (— w.r.t. $\eta''(\rho)$), and hence the complete real eigensystem. For most of our discussion we shall assume the stronger *strict hyperbolicity*, i.e, distinct real eigenvalues, $a_k(\rho) \neq a_j(\rho)$.

A fundamental building block for the construction of approximate solutions in the one-dimensional case is the solution of *Riemann's problem.*

2.3.1 Riemann's problem

Here one seeks a weak solution of (2.15) subject to the piecewise constant initial data
$$\rho(x,0) = \begin{cases} \rho_\ell, & x < 0 \\ \rho_r, & x > 0. \end{cases} \tag{2.16}$$

The solution is composed of m simple waves, each of which is associated with one (right-)eigenpair, $(a_k(\rho), r_k(\rho))$, $1 \le k \le m$. There are three types of such waves: if the k-th field is *genuinely nonlinear* in the sense that $r_k \cdot \nabla_\rho a_k \neq 0$, these are either k-shock or k-rarefaction waves; or, if the k-th field is *linearly degenerate* in the sense that $r_k \cdot \nabla_\rho a_k \equiv 0$, this is a k-th contact wave.

These three simple waves are *centered*, depending on $\xi = \frac{x}{t}$ (which is to be expected from the dilation invariance of (2.15),(2.16)). The structure of these three centered waves is as follows:

- A k-shock discontinuity of the form

$$\rho(\xi) = \begin{cases} \rho_\ell, & \xi < s \\ \rho_r, & \xi > s; \end{cases}$$

  here s denotes the shock speed which is determined by a Rankine-Hugoniot relation so that $a_k(\rho_\ell) > s > a_k(\rho_r)$.

- A k-rarefaction wave, $\rho(\xi)$, which is directed along the corresponding k-th eigenvector, $\dot\rho(\xi) = r_k(\rho(\xi))$. Here r_k is the normalized k-eigenvector, $r_k \cdot \nabla a_k \equiv 1$ so that the gap between $a_k(\rho_\ell) < a_k(\rho_r)$ is filled with a fan of the form

$$a_k(\rho(\xi)) = \begin{cases} a_k(\rho_\ell), & \xi < a_k(\rho_\ell) \\ \xi, & a_k(\rho_\ell) < \xi < a_k(\rho_r) \\ a_k(\rho_r), & a_k(\rho_r) < \xi \end{cases}$$

- A k-contact discontinuity of the form

$$\rho(\xi) = \begin{cases} \rho_\ell, & \xi < s \\ \rho_r, & \xi > s \end{cases}$$

  where s denotes the shock speed which is determined by a Rankine-Hugoniot relation so that $a_k(\rho_\ell) = s = a_k(\rho_r)$.

We are concerned with *admissible* systems — systems which consist of either genuinely nonlinear or linearly degenerate fields. We refer to [92] for the full story which concludes with the celebrated

Theorem 2.2 (Lax solution of Riemann's problem) *The strictly hyperbolic admissible system (2.15), subject to Riemann initial data (2.16) with $\rho_\ell - \rho_r$ sufficiently small, admits a weak entropy solution, which consists of shock- rarefaction- and contact-waves.*

For a detailed account on the solution of Riemann problem consult [16]. An extension to a generalized Riemann problem subject to piecewise-linear initial data can be found in [5], [99]. In this context we also mention the *approximate* Riemann solvers, which became useful computational alternatives to Lax's construction. Roe introduced in [138] a linearized Riemann solver, which resolves jumps discontinuities solely in terms of shock waves. Roe's solver has the computational advantage of sharp resolution (at least when there is one dominant wave per computational cell); it may lead, however, to unstable shocks. Osher and Solomon in [130] used, instead, an approximate Riemann solver based solely on rarefaction fans; one then achieves stability at the expense of deteriorated resolution of shock discontinuities.

2.3.2 Godunov, Lax-Friedrichs and Glimm schemes

We let $\rho^{\Delta x}(t,x)$ be the entropy solution in the slab $t^n \le t < t+\Delta t$, subject to piecewise constant data $\rho^{\Delta x}(t=t^n,x) = \sum \rho_\nu^n \chi_\nu(x)$. Here χ denotes the usual indicator function, $\chi_\alpha(x) := 1_{\{|x-\alpha\Delta x|\le \Delta x/2\}}$. Observe that in each slab, $\rho^{\Delta x}(t,x)$ consists of successive *noninteracting* Riemann solutions, at least for a sufficiently small time interval Δt, for which the CFL condition, $\Delta t/\Delta x \max|a_k(\rho)| \le \frac{1}{2}$ is met. In order to realize the solution in the next time level, $t^{n+1} = t^n + \Delta t$, it is extended with a jump discontinuity across the line t^{n+1}, by projecting it back into the finite-dimensional space of piecewise constants. Different projections yield different schemes. We recall the basic three.

Godunov Scheme. Godunov scheme [59] sets

$$\rho^{\Delta x}(t^{n+1},x) = \sum_\nu \bar\rho_\nu^{n+1}\chi_\nu(x),$$

where $\bar\rho_\nu^{n+1}$ stands for the cell-average,

$$\bar\rho_\nu^{n+1} := \frac{1}{\Delta x}\int_x \rho^{\Delta x}(t^{n+1}-0,x)\chi_\nu(x)dx,$$

which could be explicitly evaluated in terms of the flux of Riemann solution across the cell interfaces at $x_{\nu\pm\frac12}$,

$$\bar\rho_\nu^{n+1} = \bar\rho_\nu^n - \frac{\Delta t}{\Delta x}\Big\{A(\rho^{\Delta x}(t^{n+\frac12},x_{\nu+\frac12}) - A(\rho^{\Delta x}(t^{n+\frac12},x_{\nu-\frac12})\Big\}. \tag{2.17}$$

Godunov scheme had a profound impact on the field of Computational Fluid Dynamics. His scheme became the forerunner for a large class of upwind finite-volume methods which are evolved in terms of (exact or approximate) Riemann solvers. In my view, the most important aspect of what Richtmyer & Morton describe as Godunov's "ingenious method" ([140, p. 338]), lies in its *global* point of view: one does not simply evolve discrete pointvalues $\{\rho_\nu^n\}$, but instead, one evolves a globally defined solution, $\rho^{\Delta x}(t,x)$, which is realized in terms of its discrete averages, $\{\bar\rho_\nu^n\}$.

Lax-Friedrichs Scheme. If the piecewise constant projection is carried out over alternating staggered grids, $\bar\rho_{\nu+\frac12}^{n+1} := \frac{1}{\Delta x}\int_x \rho^{\Delta x}(t^{n+1}-0,x)\chi_{\nu+\frac12}(x)dx$, then one effectively integrates 'over the Riemann fan' which is centered at $(x_{\nu+\frac12},t^n)$. This recovers the Lax-Friedrichs (LxF) scheme, [91], with an explicit recursion formula for the evolution of its cell-averages which reads

$$\bar\rho_{\nu+\frac12}^{n+1} = \frac{\bar\rho_\nu^n + \bar\rho_{\nu+1}^n}{2} - \frac{\Delta t}{\Delta x}\Big\{A(\bar\rho_{\nu+1}^n) - A(\bar\rho_\nu^n)\Big\}. \tag{2.18}$$

The Lax-Friedrichs scheme had a profound impact on the construction and analysis of approximate methods for time-dependent problems, both linear problems [50] and nonlinear systems [91]. The Lax-Friedrichs scheme was and still is the stable, all purpose benchmark for approximate solution of nonlinear systems.

Both Godunov and Lax-Friedrichs schemes realize the exact solution operator in terms of its finite-dimensional cell-averaging projection. This explains the versatility of these schemes, and at the same time, it indicates their limited resolution due to the fact that waves of different families that are averaged together at each computational cell.

Glimm Scheme. Rather than averaging, Glimm's scheme, [54], keeps its sharp resolution by *randomly sampling* the evolving Riemann waves,

$$\rho^{\Delta x}(t^{n+1},x) = \sum_{\nu} \rho^{\Delta x}(t^{n+1}-0, x_{\nu+\frac{1}{2}} + r^n \Delta x)\chi_{\nu+\frac{1}{2}}(x).$$

This defines the Glimm's approximate solution, $\rho^{\Delta x}(t,x)$, depending on the mesh parameters $\Delta x \equiv \lambda \Delta t$, and on the set of random variable $\{r^n\}$, uniformly distributed in $[-\frac{1}{2},\frac{1}{2}]$. In its deterministic version, Liu [113] employs equidistributed rather than a random sequence of numbers $\{r^n\}$.

Glimm solution, $\rho^{\Delta x}(t,x)$, was then used to construct a solution for one-dimensional admissible systems of conservation laws. Glimm's celebrated theorem, [54], is still serving today as the cornerstone for existence theorems which are concerned with general one-dimensional systems, e.g. [113],[20],[144].

Theorem 2.3 (Existence in the large) . *There exists a weak entropy solution, $\rho(t,\cdot) \in L^\infty[[0,T], BV \cap L^\infty(R_x)]$, of the strictly hyperbolic system (2.15), subject to initial conditions with sufficiently small variation, $\|\rho_0(\cdot)\|_{BV\cap L^\infty(R_x)} \leq \epsilon$.*

Glimm's scheme has the advantage of retaining sharp resolution, since in each computational cell, the local Riemann solution is realized by a randomly chosen 'physical' Riemann wave. Glimm's scheme was turned into a computational tool known as the Random Choice Method (RCM) in [22], and it serves as the building block inside the front tracking method of Glimm and his co-workers, [56], [21].

2.4 Multidimensional systems ($m > 1, d > 1$)

Very little rigor is known on m conservation laws in d spatial dimensions once $(m-1)(d-1)$ becomes positive, i.e., general multidimensional systems. We address few major achievements.

Short time existence. For H^s-initial data ρ_0, with $s > \frac{d}{2}$, an H^s-solution exists for a time interval $[0,T]$, with $T = T(\|\rho_0\|_{H^s})$, consult e.g, [83],[78, §5.3].

Short time existence – piecewise analytic data. An existence result conjectured by Richtmyer was proved by Harabetian in terms of a Cauchy-Kowalewski type existence result [67].

Short time stability – piecewise smooth shock data. Existence for piecewise smooth initial data where smoothness regions are separated by shock discontinuities was studied in [117],[106].

Riemann problem. Already in the $d = 2$-dimensional case, the collection of simple waves and their composed interaction in the construction of Riemann solution (– subject to piecewise constant initial data), is considerably more complicated than in the one-dimensional setup. We refer to the recent book [33] for a detailed discussion.

Compressible Euler equations. These system of $m = 5$ equations governing the evolution of density, 3-vector of momentum and Energy in $d = 3$-space variables was – and still is, the prime target for further developments in our understanding of general hyperbolic conservation laws. We refer to Majda, [117], for a definitive summary of this aspect.

3 Finite Difference Methods – TVD Schemes

We begin by covering the space and time variables with a discrete grid: it consists of time-steps of size Δt and rectangular spatial cells of size $\Delta x := (\Delta x_1, \ldots, \Delta x_d)$. Let $\mathcal{C}_\nu$ denotes the cell which is centered around the gridpoint $x_\nu = \nu \Delta x := (\nu_1 \Delta x_1, \ldots, \nu_d \Delta x_d)$, and let $\{\rho_\nu^n\}$ denote the gridfunction associated with this cell at time $t^n = n\Delta t$. The gridfunction $\{\rho_\nu^n\}$ may represent approximate gridvalues, $\rho(t^n, x_\nu)$, or approximate cell averages, $\bar{\rho}(t^n, x_\nu)$ (as in the Godunov and LxF schemes), or a combination of higher moments, e.g., [23].

To construct a finite difference approximation of the conservation law (2.1), one introduce a discrete *numerical flux*, $H(\rho^n) := (H_1(\rho^n), \ldots, H_d(\rho^n))$, where $H_j(\rho^n) = H_j(\rho^n_{\nu-p}, \ldots, \rho^n_{\nu+q})$ is an approximation to the $A_j(\rho^n)$ flux across the interface separating the cell $\mathcal{C}_\nu$ and its neighboring cell on the x_j's direction, $\mathcal{C}_{\nu+e_j}$. Next, exact derivatives in (2.1) are replaced by divided differences: the time-derivative is replaced with forward time difference, and spatial derivatives are replaced by spatial divided differences expressed in terms of $D_{+x_j}\phi_\nu := (\phi_{\nu+e_j} - \phi_\nu)/\Delta x_j$. We arrive at the finite-difference scheme of the form

$$\rho_\nu^{n+1} = \rho_\nu^n - \Delta t \sum_{j=1}^{d} D_{+x_j} H_j(\rho^n_{\nu-p}, \ldots, \rho^n_{\nu+q}). \tag{3.1}$$

The essential feature of the difference schemes (3.1) is their *conservation form*: perfect derivatives in (2.1) are replaced here by 'perfect differences'. It implies that the change in mass over any spatial domain Ω, $\sum_{\{\nu|x_\nu\in\Omega\}} \rho_\nu^{n+1}|\mathcal{C}_\nu| - \sum_{\{\nu|x_\nu\in\Omega\}} \rho_\nu^n|\mathcal{C}_\nu|$, depends solely on the discrete flux across the boundaries of that domain. This is a discrete analogue for the notion of a weak solution of (2.1). In their seminal paper [96], Lax & Wendroff introduced the notion of conservative schemes, and prove that their *strong* limit solutions are indeed weak solutions of (2.1).

Theorem 3.1 (Lax & Wendroff [96]) *Consider the conservative difference scheme (3.1), with consistent numerical flux so that $H_j(\rho, \ldots, \rho) = A_j(\rho)$. Let $\Delta t \downarrow 0$ with fixed grid-ratios $\lambda_j := \frac{\Delta t}{\Delta x_j} \equiv Const_j$, and let $\rho^{\Delta t} = \{\rho_\nu^n\}$ denote the corresponding solution (parameterized w.r.t. the vanishing grid-size). Assume that $\rho^{\Delta t}$ converges strongly, $s\lim \rho^{\Delta t}(t^n, x_\nu) = \rho(t, x)$, then $\rho(x, t)$ is a weak solution of the conservation law (2.1).*

The Lax-Wendroff theorem plays a fundamental role in the development of the so called 'shock capturing' methods. Instead of tracking jump discontinuities (– by evolving the smooth pieces of the approximate solution on both sides of such discontinuities), conservative schemes capture a discretized version of shock discontinuities. Equipped with the Lax-Wendroff theorem, it remains to prove strong convergence, which leads us to discuss the compactness of $\{\rho_\nu^n\}$.

3.1 Compactness arguments ($m = d = 1$)

We deal with scalar gridfunctions, $\{\rho_\nu^n\}$, defined on the one-dimensional Cartesian grid $x_\nu := \nu\Delta x, t^n := n\Delta t$ with fixed mesh ratio $\lambda := \frac{\Delta t}{\Delta x}$. The total variation of such gridfunction at time-level t^n is given by $\sum_\nu |\Delta\rho^n_{\nu+\frac{1}{2}}|$, where $\Delta\rho^n_{\nu+\frac{1}{2}} := \rho^n_{\nu+1} - \rho^n_\nu$. It is said to be total-variation-diminishing (TVD) if

$$\sum_\nu |\Delta\rho^n_{\nu+\frac{1}{2}}| \le \sum_\nu |\Delta\rho^0_{\nu+\frac{1}{2}}|. \tag{3.2}$$

Clearly, the TVD condition (3.2) is the discrete analogue of the scalar TV-bound (2.14). Approximate solutions of difference schemes which respect the TVD property (3.2), share the following desirable properties:

- *Convergence* – by Helly's compactness argument, the piecewise-constant approximate solution, $\rho^{\Delta x}(t^n, x) = \sum_\nu \rho_\nu^n \chi_\nu(x)$, converges strongly to a limit function, $\rho(t^n, x)$ as we refine the grid, $\Delta x \downarrow 0$. This together with equicontinuity in time and the Lax-Wendroff theorem, yield a weak solution, $\rho(t, x)$, of the conservation law (2.1).

- *Spurious oscillations* – are excluded by the TVD condition (2.14).

- *Accuracy* – is not restricted to the first-order limitation of monotone schemes. To be more precise, let us use $\rho^{\Delta t}(t, x)$ to denote a global realization (say – piecewise polynomial interpolant) of the approximate solution $\rho_\nu^n \sim \rho^{\Delta t}(t^n, x_\nu)$. The *truncation error* of the difference scheme is the amount by which the approximate solution, $\rho^{\Delta t}(t, x)$, fails to satisfy the conservation laws (2.1). The difference scheme is α-order accurate if its truncation error is, namely,

$$\|\partial_t \rho^{\Delta t} + \nabla_x \cdot A(\rho^{\Delta t})\| = \mathcal{O}((\Delta t)^\alpha). \tag{3.3}$$

  (Typically, a strong norm $\|\cdot\|$ is used which is appropriate to the problem; in general, however, accuracy is indeed a norm-dependent quantity). Consider for example, monotone difference schemes. Monotone schemes are characterized by the fact that ρ_ν^{n+1} is an increasing function of the preceding gridvalues which participate in its stencil (3.1), $\rho_{\nu-p}^n, \ldots, \rho_{\nu+q}^n$ (— so that the monotonicity property (2.13) holds) . A classical result of Harten, Hyman & Lax [71] states that monotone schemes are at most first-order accurate. TVD schemes, however, are not restricted to this first-order accuracy limitation (at least in the one-dimensional case[2]). We demonstrate this point in the context of second-order TVD difference schemes.

3.2 TVD difference schemes

We follow the presentation in [132]. The starting point is the viscosity regularization (2.2) with vanishing viscosity of order $\varepsilon = \Delta x/2\lambda$ (recall that λ denotes the fixed mesh-ratio, $\Delta t/\Delta x$),

$$\partial_t \rho + \partial_x A(\rho) = \frac{\Delta x}{2\lambda} \partial_x (Q \partial_x \rho). \tag{3.4}$$

We discretize (3.4) with the help of

(i) *An approximate flux,* $\tilde{A}_\nu^n = \tilde{A}(\rho_{\nu-p+1}^n, \ldots, , \rho_\nu^n, \ldots, \rho_{\nu+p-1}^n) \approx A(\rho_\nu^n)$;

(ii) *A numerical viscosity coefficient,* $Q_{\nu+\frac{1}{2}}^n = Q(\rho_{\nu-p+1}^n, \ldots, \rho_{\nu+p}^n)$.

These discrete quantities are used to replace the temporal and spatial derivatives in (3.4) by appropriate forward and centered divided differences. The resulting finite difference method reads

$$\rho_\nu^{n+1} = \rho_\nu^n - \frac{\lambda}{2}\Big\{ \tilde{A}_{\nu+1}^n - \tilde{A}_{\nu-1}^n \Big\} + \frac{1}{2}\Big\{ Q_{\nu+\frac{1}{2}}^n \Delta\rho_{\nu+\frac{1}{2}}^n - Q_{\nu-\frac{1}{2}}^n \Delta\rho_{\nu-\frac{1}{2}}^n \Big\}. \tag{3.5}$$

[2]Consult [64], regarding the first-order accuracy limitation for multidimensional $d > 1$ TVD schemes. This limitation is linked to the lack of a 'proper' isotropic definition for the total-variation of multidimensional gridfunctions.

Observe that (3.5) can be put into conservation form (3.1), in terms of the numerical flux

$$H_{\nu+\frac{1}{2}}(\rho^n) = \frac{1}{2}(\tilde{A}^n_{\nu+1} + \tilde{A}^n_{\nu}) - \frac{1}{2\lambda}Q^n_{\nu+\frac{1}{2}}\Delta\rho^n_{\nu+\frac{1}{2}}.$$

Also, the stencil on the right of (3.5) occupies the $(2p+1)$ neighboring gridvalues, $\rho^n_{\nu-p},\dots,\rho^n_{\nu+p}$. Thus, (3.5) is a $(2p+1)$-points conservative difference-scheme. Harten in [69], was the first to identify a useful sufficient criterion for the TVD property for such scalar difference schemes. Harten's criterion, in its reformulation from [161], states that the difference scheme (3.5) is TVD provided it contains 'enough viscosity' in the sense that

$$\lambda\left|\frac{\Delta\tilde{A}^n_{\nu+\frac{1}{2}}}{\Delta\rho^n_{\nu+\frac{1}{2}}}\right| \le Q^n_{\nu+\frac{1}{2}} \le 1. \tag{3.6}$$

We distinguish between two types of TVD schemes, depending on the size of their stencils.

3.2.1 Three-point schemes

Three-point schemes ($p=1$) are the simplest ones – their stencil on the right of (3.5) occupies the three neighboring gridvalues, $\rho^n_{\nu-1},\rho^n_{\nu},\rho^n_{\nu+1}$. In this case, $\tilde{A}^n_{\nu}\equiv A(\rho^n_{\nu})$, so that three-point schemes take the form

$$\rho^{n+1}_{\nu} = \rho^n_{\nu} - \frac{\lambda}{2}\Big\{A(\rho^n_{\nu+1}) - A(\rho^n_{\nu-1}\Big\}) + \frac{1}{2}\Big\{Q^n_{\nu+\frac{1}{2}}\Delta\rho^n_{\nu+\frac{1}{2}} - Q^n_{\nu-\frac{1}{2}}\Delta\rho^n_{\nu-\frac{1}{2}}\Big\}. \tag{3.7}$$

Thus, three-point schemes are identified solely by their numerical viscosity coefficient, $Q^n_{\nu+\frac{1}{2}} = Q(\rho^n_{\nu},\rho^n_{\nu+1})$, which characterize the TVD condition (corresponding to (3.6))

$$\lambda|a^n_{\nu+\frac{1}{2}}| \le Q^n_{\nu+\frac{1}{2}} \le 1, \qquad a^n_{\nu+\frac{1}{2}} := \frac{\Delta A^n_{\nu+\frac{1}{2}}}{\Delta\rho^n_{\nu+\frac{1}{2}}}. \tag{3.8}$$

The schemes of Roe [138], Godunov [59], and Engquist-Osher (EO) [46], are canonical examples of *upwind* schemes, associated with (increasing amounts of) numerical viscosity coefficients, which are given by,

$$\begin{aligned}
Q^{\text{Roe}}_{\nu+\frac{1}{2}} &= \lambda|a^n_{\nu+\frac{1}{2}}|,\\
Q^{\text{Godunov}}_{\nu+\frac{1}{2}} &= \lambda\max_{\zeta\in\mathcal{C}_{\nu+\frac{1}{2}}}\Big[\frac{A(\rho^n_{\nu+1}) - 2A(\zeta) + A(\rho^n_{\nu})}{\Delta\rho^n_{\nu+\frac{1}{2}}}\Big],\\
Q^{\text{EO}}_{\nu+\frac{1}{2}} &= \lambda\frac{1}{\Delta\rho^n_{\nu+\frac{1}{2}}}\int_{\rho^n_{\nu}}^{\rho^n_{\nu+1}}|A'(\zeta)|d\zeta.
\end{aligned}$$

The viscosity coefficients of the three upwind schemes are the same, $Q^n_{\nu+\frac{1}{2}} = \lambda|a^n_{\nu+\frac{1}{2}}|$, except for their different treatment of sonic points (where $a(\rho^n_{\nu})\cdot a(\rho^n_{\nu+1}) < 0$). The Lax-Friedrichs (LxF) scheme (2.18) is the canonical *central* scheme. It has a larger numerical viscosity coefficient $Q^{LxF}_{\nu+\frac{1}{2}}\equiv 1$.

All the three-point TVD schemes are limited to first-order accuracy. Indeed, condition (3.8) is in fact *necessary* for the TVD property of three-point schemes, [160], and hence it excludes numerical viscosity associated with the second-order Lax-Wendroff scheme, [96], $Q^{LW}_{\nu+\frac{1}{2}} = \lambda^2(a^n_{\nu+\frac{1}{2}})^2$. Therefore, scalar TVD schemes with more than first-order accuracy require at least five-point stencils.

3.2.2 Five-point schemes

Following the influential works of Boris & Book [6], van Leer [98], Harten [69], Osher [129], Roe [138] and others, many authors have constructed second order TVD schemes, using five-point (– or wider) stencils. For a more complete account of these works we refer to the recent books by LeVeque, [102], and Godlewski & Raviart, [57]. A large number of these schemes were constructed as second-order upgraded versions of the basic three-point *upwind* schemes. The FCT scheme of Boris & Book, [6], van Leer's MUSCL scheme [98], and the ULTIMATE scheme of Harten, [69], are prototype for this trend.

We quote here a five-point TVD scheme of Nessyahu-Tadmor (NT) [125], which is a second-order upgraded version of the *central* LxF scheme (2.18): we use the same viscosity coefficient, $Q_{\nu+\frac{1}{2}} \equiv 1$, but we augmented it with a modified approximate flux, $\tilde{A}_\nu$; expressed in terms of the cell averages, $\bar{\rho}^n_\nu$, and the midvalues $\rho_\nu^{n+\frac{1}{2}} := \bar{\rho}^n_\nu - \frac{\lambda}{2}(A(\bar{\rho}^n_\nu))'$, this modified flux is given by $\tilde{A}_\nu = A(\rho_\nu^{n+\frac{1}{2}}) + (\rho^n_\nu)'/2\lambda$. Using these quantities in the viscosity form (3.5) we end up with a second-order predictor-corrector scheme, which admits a LxF-like staggered form (2.18)

$$\rho_\nu^{n+\frac{1}{2}} = \bar{\rho}^n_\nu - \frac{\lambda}{2}(A(\bar{\rho}^n_\nu))', \tag{3.9}$$

$$\bar{\rho}^{n+1}_{\nu+\frac{1}{2}} = \frac{\bar{\rho}^n_\nu + \bar{\rho}^n_{\nu+1}}{2} - \frac{(\rho^n_\nu)' - (\rho^n_{\nu+1})'}{8} - \frac{\Delta t}{\Delta x}\Big\{A(\rho^{n+\frac{1}{2}}_{\nu+1}) - A(\rho_\nu^{n+\frac{1}{2}})\Big\}. \tag{3.10}$$

Here, $\{w'_\nu\}$ denotes the *discrete numerical derivative* of an arbitrary gridfunction $\{w_\nu\}$. The choice $w'_\nu \equiv 0$ recovers the original first-order LxF scheme (2.18). Second-order accuracy requires $w'_\nu \sim \Delta x \partial_x w(x_\nu)$; a prototype example is the so called min-mod limiter,

$$w'_\nu = \frac{1}{2}(s_{\nu-\frac{1}{2}} + s_{\nu+\frac{1}{2}}) \cdot \min\{|\Delta w_{\nu-\frac{1}{2}}|, |\Delta w_{\nu+\frac{1}{2}}|\}, \quad s_{\nu+\frac{1}{2}} := sgn(\Delta w_{\nu+\frac{1}{2}}). \tag{3.11}$$

With this choice of a limiter, the central NT scheme (3.9)-(3.10) satisfies the TVD condition (3.6), and at the same time, it retains formal second order accuracy (at least away from extreme gridvalues, ρ_ν, where $\rho'_\nu = s_{\nu-\frac{1}{2}} + s_{\nu+\frac{1}{2}} = 0$).

We conclude we few additional remarks.

Limiters A variety of discrete TVD limiters like (3.11) was explored during the '80s, e.g, [159] and the references therein. For example, a generalization of (3.11) is provided by the family of min-mod limiters depending on tuning parameters, $0 < \theta_{\nu\pm\frac{1}{2}} < 1$,

$$w'_\nu(\theta) = \frac{1}{2}(s_{\nu-\frac{1}{2}} + s_{\nu+\frac{1}{2}}) \cdot \min\{\theta_{\nu-\frac{1}{2}}|\Delta w_{\nu-\frac{1}{2}}|, \frac{1}{2}|w_{\nu+1} - w_{\nu-1}|, \theta_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}|\}. \tag{3.12}$$

An essential feature of these limiters is *co-monotonicity*: they are 'tailored' to produce piecewise-linear reconstruction of the form $\sum[w_\nu + \frac{1}{\Delta x}w'_\nu(x - x_\nu)]\chi_\nu(x)$, which is co-monotone with (and hence, share the TVD property of –) the underlying piecewise-constant approximation $\sum w_\nu \chi_\nu(x)$. Another feature is the limiting property at extrema gridvalues (where $\rho'_\nu = 0$), which is necessary in order to satisfy the TVD property (3.2). In particular, these limiters are necessarily *nonlinear* in the sense of their dependence on the discrete gridfunction.

Systems (One-dimensional problems). The question of convergence for approximate solution of hyperbolic systems is tied to the question of existence of an entropy solution – in both

cases there are no general theories with $m > 1$ equations[3]. Nevertheless, the ingredients of scalar high-resolution schemes were successfully integrated in the approximate solution of system of conservation laws.

Many of these high-resolution methods for systems, employ the Godunov approach, where one evolves a global approximation which is realized as a piecewise polynomial,

$$\rho(x,t^n) = \sum_j p_j(x)\chi_j(x), \quad \bar{p}_\nu(x_\nu) = \bar{\rho}_\nu^n. \tag{3.13}$$

Typically, this piecewise polynomial approximate solution is reconstructed from the previously computed cell averages, $\{\bar{\rho}_\nu^n\}$, and in this context we may distinguish between two main classes of methods: *upwind* and *central* methods.

Upwind schemes evaluate cell averages at the center of the piecewise polynomial elements; integration of (2.15) over $\mathcal{C}_\nu \times [t^n, t^{n+1}]$ yields

$$\bar{\rho}_\nu^{n+1} = \bar{\rho}_\nu^n - \frac{1}{\Delta x}\left[\int_{\tau=t^n}^{t^{n+1}} f(\rho(\tau, x_{\nu+\frac{1}{2}},))d\tau - \int_{\tau=t^n}^{t^{n+1}} f(\rho(\tau, x_{\nu-\frac{1}{2}},))d\tau\right].$$

This in turn requires the evaluation of fluxes along the discontinuous cell interfaces, $(\tau \times x_{\nu+\frac{1}{2}})$. Consequently, upwind schemes must take into account the characteristic speeds along such interfaces. Special attention is required at those interfaces in which there is a combination of forward- and backward-going waves, where it is necessary to decompose the "Riemann fan" and determine the separate contribution of each component by tracing "the direction of the wind". These characteristic decompositions (– using exact or approximate Riemann solvers) enable to solve with high resolution the corresponding characteristic variables. At the same time, It is the need to follow these characteristic variables which greatly complicates the upwind algorithms, making them difficult to implement and generalize to complex systems. The original first-order accurate Godunov scheme (2.17) is the forerunner for all other upwind Godunov-type schemes. A variety of second- and higher-order sequels to Godunov upwind scheme were constructed, analyzed and implemented with great success during the seventies and eighties, starting with van-Leer's MUSCL scheme [98], followed by [138, 69, 129, 26]. These methods were subsequently adapted for a variety of nonlinear related systems, ranging from incompressible Euler equations, [4], [45], to reacting flows, semiconductors modeling, ... We shall say more about these methods in §3.4 below. At this point we refer to [58, 102] and the references therein a for a more complete accounts on these developments.

In contrast to upwind schemes, central schemes evaluate staggered cell averages at the breakpoints between the piecewise polynomial elements,

$$\bar{\rho}_{\nu+\frac{1}{2}}^{n+1} = \bar{\rho}_{\nu+\frac{1}{2}}^n - \frac{1}{\Delta x}\left[\int_{\tau=t^n}^{t^{n+1}} f(\tau, \rho(x_{\nu+1}))d\tau - \int_{\tau=t^n}^{t^{n+1}} f(\rho(\tau, x_\nu))d\tau\right].$$

Thus, averages are integrated over the entire Riemann fan, so that the corresponding fluxes are now evaluated at the smooth centers of the cells, (τ, x_ν). Consequently, costly Riemann-solvers required in the upwind framework, can be now replaced by straightforward quadrature rules. The first-order Lax-Friedrichs (LxF) scheme (2.18) is the

[3] There is a large literature concerning two equations – the 2×2 p-system and related equations are surveyed in [155].

canonical example of such central difference schemes. The LxF scheme (like Godunov's scheme) is based on a piecewise constant approximate solution, $p_\nu(x) = \bar{\rho}_\nu$. Its Riemann-solver-free recipe, however, is considerably simpler. Unfortunately, the LxF scheme introduces excessive numerical viscosity (already in the scalar case outlined in §3.2.1 we have $Q^{LxF} \equiv 1 > Q^{\text{Godunov}}$), resulting in relatively poor resolution. The central scheme (3.9)-(3.10) is a second-order sequel to LxF scheme, with greatly improved resolution. An attractive feature of the central scheme (3.9)-(3.10) is that it avoids Riemann solvers: instead of characteristic variables, one may use a componentwise extension of the non-oscillatory limiters (3.12).

Multidimensional systems There are basically two approaches.

One approach is to reduce the problem into a series of one-dimensional problems. Alternating Direction (ADI) methods and the closely related dimensional splitting methods, e.g., [140, §8.8-9], are effective, widely used tools to solve multidimensional problems by piecing them from one-dimensional problems – one dimension at a time. Still, in the context of nonlinear conservation laws, dimensional splitting encounters several limitations, [31]. A particular instructive example for the effect of dimensional splitting errors can be found in the approximate solution of the weakly hyperbolic system studied in [48],[81, §4.3].

The other approach is 'genuinely multidimensional'. There is a vast literature in this context. The beginning is with the pioneering multidimensional second-order Lax-Wendroff scheme, [97]. To retain high-resolution of multidimensional schemes without spurious oscillations, requires one or more of several ingredients: a careful treatment of waves propagations ('unwinding'), or alternatively, a correctly tuned numerical dissipation which is free of Riemann-solvers ('central differencing'), or the use of adaptive grids (which are not-necessarily rectangular), ... Waves propagation in the context of multidimensional upwind algorithms were studied in [25, 103, 139, 154] .... Another 'genuinely multidimensional' approach can be found in the positive schemes of [95]. The pointwise formulation of ENO schemes due to Shu & Osher, [151, 152], is another approach which avoids dimensional splitting: here, the reconstruction of cell-averages is bypassed by the reconstruction *pointvalues* of the fluxes in each dimension; the semi-discrete fluxed are then integrated in time using non-oscillatory ODEs solvers (which are briefly mentioned in §3.4.2 below). Multidimensional non-oscillatory *central* scheme was presented in [81], generalizing the one-dimensional (3.9)-(3.10); consult [105],[89] for applications to the multidimensional incompressible Euler equations. Finite volume methods, [85, 86, 24, 29]... , and finite-element methods (the streamline-diffusion and discontinuous Galerkin schemes, [76, 79, 80, 146, 121]...) have the advantage of a 'built-in' recipe for discretization over general triangular grids (we shall say more on these methods in §7 below). Another 'genuinely multidimensional' approach is based on a relaxation approximation was introduced in [82]. It employs a central scheme of the type (3.9)-(3.10) to discretize the relaxation models models, [173], [19], [124],....

3.3 TVD filters

Every discretization method is associated with an appropriate finite-dimensional projection. It is well known that *linear* projections which are monotone (or equivalently, positive), are at most first-order accurate, [59]. The lack of monotonicity for higher order projections is reflected by spurious oscillations in the vicinity of jump discontinuities. These are evident with the second-order (and higher) centered differences, whose dispersive nature is responsible

to the formation of binary oscillations [63],[104]. With highly-accurate spectral projections, for example, these $\mathcal{O}(1)$ oscillations reflect the familiar Gibbs phenomena.

TVD schemes avoid spurious oscillations — to this end they use the necessarily *nonlinear* projections (expressed in terms of nonlinear limiters like those in (3.12)). TVD filters, instead, suppress spurious oscillations. At each time-level, one post-process the computed (possibly oscillatory) solution $\{\rho^n\}$. In this context we highlight the following.

• **Linear filters**. Consider linear convection problems with discontinuous initial data. Approximate solutions of such problems suffer from loss of accuracy due to propagation of singularities and their interference over domain of dependence of the numerical scheme. Instead, one can show, by duality argument, that the numerical scheme retains its original order of accuracy when the truncation in (3.3) is measured w.r.t. sufficiently large negative norm, [120]. Linear filters then enable to accurately recover the exact solution in any smoothness region of the exact solution, bounded away from its singular support. These filters amount to finite-order mollifiers [120], or spectrally accurate mollifiers, [118], [66], which accurately recover pointvalues from high-order moments. (We outline such technique in §4.2).

• **Artificial compression**. Artificial compression was introduced by Harten [68] as a method to sharpen the poor resolution of contact discontinuities. (Typically, the resolution of contacts by α-order schemes diffuses over a fan of width $(\Delta t)^{(\alpha)/(\alpha+1)}$). The idea is to enhance the focusing of characteristics by adding an anti-diffusion modification to the numerical fluxes: if we let $H_{\nu+\frac{1}{2}}$ denote the numerical flux of a three-point TVD scheme (3.7), then one replaces it with a modified flux, $H_{\nu+\frac{1}{2}} \longrightarrow H_{\nu+\frac{1}{2}} + \tilde{H}_{\nu+\frac{1}{2}}$, which is expressed in terms of the min-mod limiter (3.11)

$$\tilde{H}_{\nu+\frac{1}{2}} := \frac{1}{\lambda}\{\rho'_\nu + \rho'_{\nu+1} - sgn(\Delta\rho_{\nu+\frac{1}{2}})|\rho'_{\nu+1} - \rho'_\nu|\}. \tag{3.14}$$

Artificial compression can be used as a second-order TVD filter as well. Let $Q_{\nu+\frac{1}{2}}$ be the numerical viscosity of a three-point TVD scheme (3.7). Then, by adding an artificial compression modification (3.14) which is based on the θ-limiters (3.12), $\rho'_\nu = \rho'_\nu(\theta)$ with $\theta_{\nu+\frac{1}{2}} := Q_{\nu+\frac{1}{2}} - \lambda^2 a^2_{\nu+\frac{1}{2}}$, one obtains a second-order TVD scheme, [69], [132]. Thus, in this case the artificial compression (3.14) can be viewed as a second-order anti-diffusive TVD filter of first-order TVD schemes

$$\rho^{n+1}_\nu \longleftarrow \rho^{n+1}_\nu - \{\tilde{H}_{\nu+\frac{1}{2}}(\rho^n) - \tilde{H}_{\nu-\frac{1}{2}}(\rho^n)\}. \tag{3.15}$$

• **TVD filters**. A particularly useful and effective, general-purpose TVD filter was introduced by Engquist et. al. in [47]; it proceeds in three steps.

{i} (Isolate extrema). First, isolate extrema cells where $\Delta\rho^n_{\nu-\frac{1}{2}} \cdot \Delta\rho^n_{\nu+\frac{1}{2}} < 0$.

{ii} (Measure local oscillation). Second, measure local oscillation, osc_ν, by setting

$$osc_\nu := \min\{m_\nu, \frac{1}{2}M_\nu\}, \qquad \left\{ \begin{matrix} m_\nu \\ M_\nu \end{matrix} \right\} = \left\{ \begin{matrix} \min \\ \max \end{matrix} \right\}(\Delta\rho^n_{\nu-\frac{1}{2}}, \Delta\rho^n_{\nu+\frac{1}{2}})$$

{iii} (Filtering). Finally, oscillatory minima (respectively – oscillatory maxima) are increased (and respectively, increased) by updating $\rho^n_\nu \to \rho^n_\nu + sgn(\Delta\rho^n_{\nu+\frac{1}{2}})osc_\nu$, and the corresponding neighboring gridvalue is modified by subtracting the same amount to retain conservation. This post-processing can be repeated, if necessary, and one may use a local maximum principle, $\min_j \rho^n_j \le \rho^n_\nu \le \max_j \rho^n_j$ as a stopping criterion. In this case, the above filter becomes TVD once the binary oscillations are removed, [153].

3.4 TVB approximations

3.4.1 Higher resolution schemes (with three letters acronym)

We have already mentioned the essential role played by nonlinear limiters in TVD schemes. The mechanism in these nonlinear limiters is switched on in extrema cells, so that the zero discrete slope $\rho' = 0$ avoids new spurious extrema. This, in turn, leads to deteriorated first-order local accuracy at non-sonic extrema, and global accuracy is therefore limited to second-order[4].

To obtain an improved accuracy, one seeks a more accurate realization of the approximate solution, in terms of higher (than first-order) piecewise polynomials

$$\rho^{\Delta x}(t^n, x) = \sum_\nu p_\nu(x)\chi_\nu(x), \quad p_\nu(x) = \sum_j \rho_\nu^{(j)}(\frac{x - x_\nu}{\Delta x})^j/j!. \tag{3.16}$$

Here, the exact solution is represented in a cell $\mathcal{C}_\nu$ in terms of an r-order polynomial p_ν, which is reconstructed from the its neighboring cell averages, $\{\bar\rho_{\nu_\mu}\}$. If we let $\rho^{\Delta x}(t \geq t^n, \cdot)$ denote the entropy solution subject to the reconstructed data at $t = t^n$, $P^{\Delta x}\rho(t^n, \cdot)$, then the corresponding Godunov-type scheme governs the evolution of cell averages

$$\bar\rho_\nu^{n+1} := \frac{1}{\Delta x}\int_x \rho^{\Delta x}(t^{n+1} - 0, x)\chi_\nu(x)dx. \tag{3.17}$$

The properties of Godunov-type scheme are determined by the polynomial reconstruction should meet three contradicting requirements:

{i} *Conservation*: $p_\nu(x)$ should be cell conservative in the sense that $\fint_{\mathcal{C}_\nu} p_\nu(x) = \fint_{\mathcal{C}_\nu} \rho_\nu(x)$. This tells us that $P^{\Delta x}$ is a (possibly nonlinear) projection, which in turn makes (3.17) a conservative scheme in the sense of Lax-Wendroff, (3.1).

{ii} *Accuracy*: $\rho_\nu^{(j)} \sim (\Delta x \partial_x)^j \rho(t^n, x_\nu)$.

At this stage, we have to relax the TVD requirement. This brings us to the third requirement of

{iii} *TVB bound*: we seek a bound on the total variation on the computed solution. Of course, a bounded variation, $\|\rho^{\Delta x}(t^n, \cdot)\|_{BV} \leq$ Const. (and in fact, even the weaker $(\Delta x)^\theta \|\rho^{\Delta x}\|_{BV} \leq$ Const.) will suffice for convergence by L^1-compactness arguments.

The (re-)construction of non-oscillatory polynomials led to new high-resolution schemes. In this context we mention the following methods (which were popularized by their trade-mark of three-letters acronym ...): the Piecewise-Parabolic Method (PPM) [26], the Uniformly Non-Oscillatory (UNO) scheme [73], and the Essentially Non-Oscillatory schemes (ENO) of Harten et. al. [70]. There is large numerical evidence that these highly-accurate methods are TVB (and hence convergent), at least for a large class of piecewise-smooth solutions. We should note, however, that the convergence question of these schemes is open. (It is my opinion that new characterizations of the (piecewise) regularity of solutions to conservation laws, e.g., [37], together with additional tools to analyze their compactness, are necessary in order to address the questions of convergence and stability of these highly-accurate schemes).

There are alternative approach to to construct high-resolution approximations which circumvent the TVD limitations. We conclude by mentioning the following two.

One approach is to evolve more than one-piece of information per cell. This is fundamentally

[4]The implicit assumption is that we seek an approximation to *piecewise-smooth* solutions with finitely many oscillations, [165]. The convergence theories apply to general BV solutions. Yet, general BV solutions cannot be resolved in *actual* computations in terms of 'classical' macroscopic discretizations – finite-difference, finite-element, spectral methods, etc. Such methods can faithfully resolve piecewise smooth solutions.

different from standard Godunov-type schemes where only the cell average is evolved (and higher order projections are *reconstructed* from these averages – one per cell). In this context we mention the quasi-monotone TVB schemes introduced in [23]. Here, one use a TVD evolution of cell averages together with additional higher moments. Another instructive example for this approach is found in the third-order TVB scheme, [142]: in fact, Sanders constructed a third-order non-expansive scheme (circumventing the first-order limitation of [71]), by using a 2×2 system which governs the first two moments of the scalar solution. More recently, Bouchut et. al. [8], constructed a second-order MUSCL scheme which respects a discrete version of the entropy inequality (2.3) w.r.t *all* Kružkov's scalar entropy pairs in (2.11); this circumvents the second-order limitation of Osher & Tadmor [132, Theorem 7.3], by evolving *both* – the cell average and the discrete slope in each computational cell.

Another approach to enforce a TVB bound on higher(> 2)-resolution schemes, makes use of gridsize-dependent limiters, $\rho^{(j)} = \rho^{(j)}\{\bar{\rho}^n, \Delta x\}$, such that the following holds, e.g., [149],

$$\|\rho^{\Delta x}(t^{n+1}, \cdot)\|_{BV} \leq \|\rho^{\Delta x}(t^{n+1}, \cdot)\|_{BV} + \text{Const} \cdot \Delta x.$$

Such Δx-dependent limiters fail to satisfy, however, the basic dilation invariance of (2.15)-(2.16), $(t, x) \to (ct, cx)$.

3.4.2 Time discretizations

One may consider separately the discretization of time and spatial variables. Let P_N denote a (possibly nonlinear) finite-dimensional spatial discretization of (2.1); this yields an N-dimensional approximate solution, $\rho_N(t)$, which is governed by the system of N nonlinear ODEs

$$\frac{d}{dt}\rho_N(t) = P_N(\rho_N(t)). \tag{3.18}$$

System (3.18) is a *semi-discrete* approximation of (2.1). For example, if we let $P_N = P^{\Delta x}$, $N \sim (\Delta x)^{-d}$, to be one of the piecewise-polynomial reconstructions associated with Godunov-type methods in (3.16), then one ends up with a semi-discrete finite-difference method, the so called method of lines. In fact, our discussion on streamline-diffusion and spectral approximations in §5 and §6 below will be primarily concerned with such semi-discrete approximations.

An explicit time discretization of (3.18) proceeds by either a multi-level or a Runge-Kutta method. A CFL condition should be met, unless one accounts for wave interactions, consult [101]. For the construction of non-oscillatory schemes, one seeks time discretizations which retain the non-oscillatory properties of the spatial discretization, P_N. In this context we mention the TVB time-discretizations of Shu & Osher, [150],[151, 152]. Here, one obtains high-order multi-level and Runge-Kutta time discretizations as *convex combinations* of the standard forward time differencing, which amounts to the first-order accurate forward Euler method. Consequently, the time discretizations [151, 152] retain the nonoscillatory properties of the low-order forward Euler time differencing — in particular, TVD/TVB bounds, and at the same time, they enable to match the time accuracy with the high-order spatial accuracy.

4 Godunov Type Methods

4.1 Compactness arguments cont'd — one-sided stability estimates

We prove convergence and derive error bounds using one-sided stability estimates. The one-sided stability estimates restrict our discussion to scalar equations – one-dimensional convex

conservation laws in §4.2 and multidimensional convex Hamilton-Jacobi equation in §4.3. (We refer to [100] for a recent contribution concerning the one-sided stability of one-dimensional systems). We begin with the case $d = 1$.

Let $\{\rho^\varepsilon(t,x)\}$ be a family of approximate solutions tagged by their small-scale parameterization, ε. To upper-bound the convergence rate of such approximations, we shall need the usual two ingredients of stability and consistency.

- *Lip^+-stability.* The family $\{\rho^\varepsilon\}$ is Lip^+-stable if

$$\|\rho^\varepsilon(t,\cdot)\|_{Lip^+} := \sup_x \partial_x \rho^\varepsilon(t,x) \leq Const. \tag{4.1}$$

This notion of Lip^+-stability is motivated by Oleĭnik's One-Sided Lipschitz Condition (OSLC), $\rho_x(t,\cdot) \leq Const$, which uniquely identifies the entropy solution of *convex* conservation laws, (2.15), with scalar $A'' > 0$. Since the Lip^+-(semi)-norm dominates the total-variation,

$$\|\rho^\varepsilon(t,\cdot)\|_{BV} \leq Const.\|\rho^\varepsilon(t,\cdot)\|_{Lip^+} + \|\rho_0^\varepsilon(\cdot)\|_{L^1}, \quad Const = 2|\mathrm{supp}_x \rho^\varepsilon(t,\cdot)|,$$

$\{\rho^\varepsilon\}$ have bounded variation and convergence follows. Equipped with Lip^+-stability, we are able to *quantify* this convergence statement. To this end, we measure the local truncation error in terms of

- *Lip'-consistency.* The family $\{\rho^\varepsilon\}$ is Lip'-consistent of order ε if

$$\|\partial_t \rho^\varepsilon + \partial_x A(\rho^\varepsilon)\|_{Lip'(t,x)} \sim \varepsilon. \tag{4.2}$$

It follows that the stability+consistency in the above sense, imply the convergence of $\{\rho^\varepsilon\}$ to the entropy solution, ρ, and that the following error estimates hold [163], [126],

$$\|\rho^\varepsilon(t,\cdot) - \rho(t,\cdot)\|_{W^s(L^p(x))} \sim \varepsilon^{\frac{1-sp}{2p}}, \qquad -1 \leq s \leq 1/p. \tag{4.3}$$

The case $(s,p) = (-1,1)$ corresponds to a sharp Lip'-error estimate of order ε — the Lip'-size of the truncation in (4.2); the case $(s,p) = (0,1)$ yields an L^1-error estimate of order one-half, in agreement with Kuznetsov's general convergence theory, [90]. (We shall return to it in §7.3). Moreover, additional *local* error estimates follow, and we illustrate this in the context of Godunov-type schemes.

4.2 Godunov type methods revisited ($m = d = 1$)

Godunov type schemes form a special class of transport projection methods for the approximate solution of nonlinear conservation laws, [72].

Let $E(t_2 - t_1)$ denote the entropy solution operator associated with the convex conservation law (2.15). A Godunov-type method yields a globally defined approximate solution, $\rho^{\Delta x}(t,x)$, which is governed by iterating the evolution-projection cycle,

$$\rho^{\Delta x}(\cdot,t) = \begin{cases} E(t - t^{n-1})\rho(\cdot,t^{n-1}), & t^{n-1} < t < t^n = n\Delta t, \\ \\ P^{\Delta x}\rho(\cdot,t^n - 0), & t = t^n, \end{cases} \tag{4.4}$$

subject to initialization step, $\rho(t=0,\cdot) = P^{\Delta x}u_0(\cdot)$.
Here, $P^{\Delta x}$ is an *arbitrary, possibly nonlinear* conservative projection, which depends on a small spatial scale Δx. For example, the piecewise polynomial projection (3.16), $P^{\Delta x}\rho(x) =$

$\sum p_\nu(x)\chi_\nu(x)$, where the χ_ν's are the characteristic functions of cells $\mathcal{C}_\nu$ with possibly variable sizes, $\Delta x \le |\mathcal{C}_\nu| \le \text{Const.}\Delta x$.

The question of Lip'-consistency for Godunov-type schemes based on cell-conservative projections, $P^{\Delta x}$, could be answered in terms of the L^1-size of $I - P$ over all BV functions [127]. Together with Lip^+-stability we conclude

$$\|\rho^{\Delta x}(t,\cdot) - \rho(t,\cdot)\|_{W^{s,p}} \le Const.\|I - P\|^{\frac{1-sp}{2p}}_{BV\to L^1}. \tag{4.5}$$

The last error bound, (4.5), tells us that the convergence rate of a Godunov-type scheme depends solely on the properties of $P^{\Delta x}$. First, Lip^+ stability is guaranteed if $P^{\Delta x}$ retains the OSLC of the exact solution operator; the OSLC property of such projections was studied in [128],[10], [126]. Second, the convergence rate depends on measuring $P^{\Delta x}$ as an approximate identity. Typically, $\|I - P^{\Delta x}\|_{BV\to L^1}$ is of order $\mathcal{O}(\Delta x)$, and (4.5) yields the familiar L^1 rate of order $\mathcal{O}(\sqrt{\Delta x})$, [114], [30], [141], [145],... (and [24, 27, 85, 86, 7] in the multidimensional case). Moreover, one can interpolate between the weak $W^{-1}(L^1)$-error estimate of order $\mathcal{O}(\Delta x)$, and the one-sided Lipschitz bounds of ρ and $\rho^{\Delta x}$ to conclude, [163]

$$|\rho^{\Delta x}(t,x) - \rho(t,x)| \le Const.[1 + \max_{\Omega_x}|\rho_x(t,x)|](\Delta x)^{1/3}.$$

This shows a pointwise convergence which depends solely on the *local* smoothness of the entropy solution in $\Omega_x := \{y|\ |y-x| \le C(\Delta x)^{1/3}\}$.

4.3 Hamilton-Jacobi equations ($m = 1, d \ge 1$)

We consider the multidimensional Hamilton-Jacobi (HJ) equation

$$\partial_t\rho + H(\nabla_x\rho) = 0, \quad (t,x) \in R^+ \times R^d, \tag{4.6}$$

with convex Hamiltonian, $H'' > 0$. Its unique viscosity solution is identified by the one-sided concavity condition, $D_x^2\rho \le$ Const., consult [87], [108]. Given a family of approximate HJ solutions, $\{\rho^\varepsilon\}$, we make the analogous one-sided stability requirement of

- *Demi-concave stability.* The family $\{\rho^\varepsilon\}$ is demo-concave stable if

$$D_x^2\rho^\varepsilon \le Const. \tag{4.7}$$

We then have the following.

Theorem 4.1 ([107]) *Assume $\{\rho_1^\varepsilon\}$ and $\{\rho_2^\varepsilon\}$ are two demi-concave stable families of approximate solutions. Then*

$$\begin{aligned}\|\rho_1^\varepsilon(t,\cdot) - \rho_2^\varepsilon(t,\cdot)\|_{L^1(x)} \quad &\le \quad Const.\|\rho_1^\varepsilon(0,\cdot) - \rho_2^\varepsilon(0,\cdot)\|_{L^1(x)} + \\ &\quad + Const.\sum_{j=1}^{2}\|\partial_t\rho_j^\varepsilon + H(\nabla_x\rho_j^\varepsilon)\|_{L^1(t,x)}.\end{aligned} \tag{4.8}$$

If we let $\rho_1^\varepsilon \equiv \rho^1$, $\rho_2^\varepsilon \equiv \rho^2$ denote two demi-concave viscosity solutions, then (4.8) is an L^1-stability statement (compared with the usual L^∞-stability statements of viscosity solutions, [29]). If we let $\{\rho_1^\varepsilon\} = \{\rho^\varepsilon\}$ denote a given family of demi-concave approximate HJ solutions, and let ρ_2^ε equals the exact viscosity solution ρ, then (4.8) yields the L^1-error estimate

$$\|\rho^\varepsilon(t,\cdot) - \rho(t,\cdot)\|_{L^1(x)} \le Const.\|\partial_t\rho^\varepsilon + H(\nabla_x\rho^\varepsilon)\|_{L^1(t,x)} \sim O(\varepsilon). \tag{4.9}$$

This corresponds to the Lip'-error estimate of (4.3) with $(s,p) = (-1,1)$. One can then interpolate from (4.9) an L^p-error estimates of order $\mathcal{O}(\varepsilon^{\frac{1+p}{2p}})$. For a general L^∞-convergence theory for approximate solutions to HJ equations we refer to [3] and the references therein.

5 Streamline Diffusion Finite Element Schemes

5.1 Compensated compactness ($m \leq 2, d = 1$)

We deal with a family of approximate solutions, $\{\rho^\varepsilon\}$, such that

(i) *It is uniformly bounded*, $\rho^\varepsilon \in L^\infty$, with a weak* limit, $\rho^\varepsilon \rightharpoonup \rho$;

(ii) *The entropy production*, for all convex entropies η, lies in a compact subset of $W^{-1}_{loc}(L^2(t,x))$,

$$\forall \eta'' > 0: \qquad \partial_t \eta(\rho^\varepsilon) + \partial_x F(\rho^\varepsilon) \hookrightarrow W^{-1}_{loc}(L^2(t,x)). \tag{5.1}$$

The conclusion is that $A(\rho^\varepsilon) \rightharpoonup A(\rho)$, and hence ρ is a weak solution; in fact, there is a strong convergence, $\rho^\varepsilon \to \rho$, on any nonaffine interval of $A(\cdot)$. For a complete account on the theory of compensated compactness we refer to the innovative works of Tartar [167] and Murat [123]. In the present context, compensated compactness argument is based on a clever application of the div-curl lemma. First scalar applications are due to Murat-Tartar, [122],[167], followed by extensions to certain $m = 2$ systems by DiPerna [39] and Chen [17].

The current framework has the advantage of dealing with L^2-type estimates rather than the more intricate BV framework. How does one verify the $W^{-1}_{loc}(L^2)$-condition (5.1)? we illustrate this point with canonical viscosity approximation (2.2). Multiplication by η' shows that its entropy production amounts to $\varepsilon(\eta' Q \rho^\varepsilon_x)_x - \varepsilon \eta'' Q (\rho^\varepsilon_x)^2$. By entropy convexity, $\varepsilon \eta'' Q > 0$[5], and space-time integration yields

- *An entropy production bound*

$$\sqrt{\varepsilon} \| \frac{\partial \rho^\varepsilon}{\partial x} \|_{L^2_{loc}(t,x)} \leq Const. \tag{5.2}$$

Though this bound is too weak for strong compactness, it is the key estimate behind the $W^{-1}_{loc}(L^2)$-compactness condition (5.1). We continue with the specific examples of streamline-diffusion in §5.2 and spectral viscosity methods in §6.

5.2 The streamline diffusion method

The Streamline Diffusion (SD) finite element scheme, due to Hughes, Johnson, Szepessy and their co-workers [76], [79], [80], was one of the first methods whose convergence was analyzed by compensated compactness arguments. (Of course, finite-element methods fit into L^2-type Hilbert-space arguments). In the SD method, formulated here in several space dimensions, one seeks a piecewise polynomial, $\{\rho^{\Delta x}\}$, which is uniquely determined by requiring for all piecewise polynomial test functions $\psi^{\Delta x}$,

$$\langle \partial_t \rho^{\Delta x} + \nabla_x \cdot A(\rho^{\Delta x}),\ \psi^{\Delta x} + |\Delta x| \cdot \boxed{(\psi^{\Delta x}_t + A'(\rho^{\Delta x}) \psi^{\Delta x}_x)} \rangle = 0. \tag{5.3}$$

Here, Δx denotes the spatial grid size (for simplicity we ignore time discretization). The expression inside the framed box on the left represents a diffusion term along the streamlines,

[5] Observe that the viscosity matrix is therefore required to be positive w.r.t. the Hessian η''.

$\dot{x} = A'(\rho^{\Delta x})$. Setting the test function, $\psi^{\Delta x} = \rho^{\Delta x}$, (5.3) yields the desired entropy production bound

$$\sqrt{\Delta x}\|\partial_t \rho^{\Delta x} + \nabla_x \cdot A(\rho^{\Delta x})\|_{L^2_{loc}(t,x)} \leq Const. \tag{5.4}$$

Thus, the spatial derivative in (5.2) is replaced here by a streamline-directional gradient. This together with an L^∞-bound imply $W^{-1}_{loc}(L^2)$-compact entropy production, (5.1), and convergence follows [79],[80],[158]. We note in passing that the extension of the SD method for systems of equations is carried out by projection into entropy variables, [119], which in turn provide the correct interpretation of (5.4) as an entropy production bound.

5.3 TVD schemes revisited ($d = 1$)

We replace the streamline diffusion term inside the framed box on the left of (5.3) by a *weighted* spatial diffusion expression,

$$\langle \partial_t \rho^{\Delta x} + \partial_x A(\rho^{\Delta x}), \psi^{\Delta x} \rangle + \Delta x \cdot \boxed{\langle \rho^{\Delta x}_x, \psi^{\Delta x}_x \rangle_Q} = 0.$$

This yields a semi-discrete finite-difference scheme in its viscosity form (3.5), and one may carry an alternative convergence analysis based on compensated compactness arguments [169].

6 Spectral Viscosity Approximations

6.1 Compensated compactness cont'd ($m \leq 2, d = 1$)

Let P_N denote an appropriate spatial projection into the space of N-degree polynomials,

$$P_N \rho(t,x) = \sum_{|k| \leq N} \hat{\rho}_k(t) \phi_k(x);$$

here $\{\phi_k\}$ stands for a given family of orthogonal polynomials, either trigonometric or algebraic ones, e.g., $\{e^{ikx}\}$, $\{L_k(x)\}$, $\{T_k(x)\}$, etc. The corresponding N-degree approximate solution, $\rho_N(t,x)$, is governed by the spectral viscosity (SV) approximation

$$\partial_t \rho_N + \partial_x P_N A(\rho_N) = \frac{1}{N} \partial_x (Q * \partial_x \rho_N). \tag{6.1}$$

The left hand side of (6.1) is the standard spectral approximation of the conservation law (2.1). The expression on the right represents the so called spectral viscosity introduced in [162]. It contains a minimal amount of high-modes regularization which retains the underlying spectral accuracy of the overall approximation,

$$\frac{1}{N} \partial_x (Q * \partial_x \rho_N) := \frac{1}{N} \sum_{|k| > N^\theta} \hat{Q}_k \hat{\rho}_k(t) \phi''_k(x).$$

It involves a viscous-free zone for the first N^θ modes, $0 < \theta < \frac{1}{2}$. High modes diffusion is tuned by the viscosity coefficients $\hat{Q}_k$.

Spurious Gibbs oscillations violate the strict TVD condition in this case. Instead, an entropy production estimate, analogous to (5.2) is sought,

$$\frac{1}{\sqrt{N}} \| \frac{\partial \rho_N}{\partial x} \|_{L^2_{loc}(t,x)} \leq Const.$$

This together with an L^∞-bound carry out the convergence analysis by compensated compactness arguments, [162], [116]. Extensions to certain $m = 2$ systems can be found in [143].

6.2 Hyper-viscosity approximations

The second-order high-modes diffusion on the right of (6.1) is replaced by higher $2s$-order diffusion,

$$\frac{(-1)^{s+1}}{N^{2s-1}}\partial_x^s(Q * \partial_x^s \rho_N) := \frac{(-1)^{s+1}}{N^{2s-1}} \sum_{|k|>N^\theta} \hat{Q}_k \hat{\rho}_k(t) \phi_k^{(2s)}(x). \tag{6.2}$$

This allows for a larger viscosity-free zone of size N^θ, with $0 < \theta < \frac{2s-1}{2s}$ (with possibly $s = s_N \leq \sqrt{N}$), consult [164]. The underlying hyper-viscosity approximation (for say $s = 2$) reads

$$\partial_t \rho^\varepsilon + \partial_x A(\rho^\varepsilon) + \varepsilon^3 \partial_x^4 \rho^\varepsilon = 0. \tag{6.3}$$

The solution operator associated with (6.3) is not monotone, hence L^1-contraction and the TVD condition fail in this case. Instead, compensated compactness arguments show, under the assumption of an L^∞- bound[6], the hyper-viscosity approximation (6.3) and its analogous spectral-viscosity approximations, converge to the entropy solution.

7 Finite Volume Schemes ($d \geq 1$)

7.1 Measure-valued solutions ($m = 1, d \geq 1$)

We turn our attention to the multidimensional scalar case, dealing with a families of uniformly bounded approximate solutions, $\{\rho^\varepsilon\}$, with weak* limit, $\rho^\varepsilon \rightharpoonup \rho$. DiPerna's result [41] states that if the entropy production of such a family tends weakly to a negative measure, $m \leq 0$,

$$\forall \eta'' > 0: \qquad \partial_t \eta(\rho^\varepsilon) + \nabla_x \cdot F(\rho^\varepsilon) \rightharpoonup m \leq 0, \tag{7.1}$$

then the measure-valued solution ρ coincides with the entropy solution, and convergence follows. This framework was used to prove the convergence of multidimensional finite-difference schemes [27], streamline diffusion method [79],[80], spectral-viscosity approximations [18] and finite-volume schemes [24], [86],[85]. We focus our attention on the latter.

7.2 Finite-volume schemes

We are concerned with finite-volume schemes based on possibly *unstructured* triangulation grid $\{T_\nu\}$ (for simplicity we restrict attention to the $d = 2$ case). The spatial domain is covered by a triangulation, $\{T|\nu\}$, and we compute approximate averages over these triangles, $\bar{\rho}_\nu^n \sim \frac{1}{|T_\nu|}\int_{T_\nu} \rho(t^n, x)dx$, governed by the finite volume (FV) scheme

$$\bar{\rho}_\nu^{n+1} = \bar{\rho}_\nu^n - \frac{\Delta t}{|T_\nu|} \sum_\mu \tilde{A}_{\nu_\mu}(\rho_\nu^n, \rho_{\nu_\mu}^n). \tag{7.2}$$

Here $\tilde{A}_{\nu_\mu}$ stand for approximate fluxes across the interfaces of T_ν and its neighboring triangles (identified by a secondary index μ).

Typically, the approximate fluxes, $\tilde{A}_{\nu_\mu}$ are derived on the basis of approximate Riemann solvers across these interfaces, which yield a monotone scheme. That is, the right hand side of (7.2) is a monotone function of its arguments $(\rho_\nu^n, \rho_{\nu_\mu}^n)$, and hence the corresponding FV scheme is L^1-contractive. However, at this stage one cannot proceed with the previous compactness arguments which apply to TVD schemes over fixed Cartesian grid: since the grid

[6] The L^∞ boundedness of (6.3) is to the best of my knowledge, an open question, [62].

is unstructured, the discrete solution operator is not translation invariant and L^1-contraction need not imply a TV bound. Instead, an entropy dissipation estimate yields

$$\sum_n \Delta t \sum_{\nu,\mu} |\rho^n_\nu - \rho^n_{\nu_\mu}|(\Delta x)^\theta \le Const, \quad 0 < \theta < 1. \tag{7.3}$$

Observe that (7.3) is weaker than a TV bound (corresponding to $\theta = 0$), yet it suffices for convergence to a measure-valued solution, consult [24], [85].

7.3 Error estimates — compactness arguments revisited

Kuznetsov [90] was the first to provide error estimates for scalar approximate solutions, $\{\rho^\varepsilon\}$, of *multidimensional* scalar conservation laws. Subsequently, many authors have used Kuznetsov's approach to prove convergence + L^1-error estimates; we refer for the detailed treatments of [141], [115], [166],... . A more recent treatment of [24] employs the entropy dissipation estimate (7.3), which in turn, by Kuznetsov arguments, yields an L^1-convergence rate estimate of order $(\Delta x)^{\frac{1-\theta}{2}}$ (independently of the BV bound).

Kuznetsov's approach employs a *regularized* version of Kružkov's entropy pairs in (2.11), $\eta^\delta(\rho;c) \sim |\rho - c|$, $F^\delta(\rho;c) \sim sgn(\rho - c)(A(\rho) - A(c))$. Here, one measures by how much the entropy dissipation rate of $\{\rho^\varepsilon\}$ fails to satisfy the entropy inequality (2.3), with Kružkov's regularized entropies. Following the general recent convergence result of [7], we consider a family of approximate solutions, $\{\rho^\varepsilon\}$, which satisfies

$$\partial_t|\rho^\varepsilon - c| + \nabla_x \cdot \{sgn(\rho^\varepsilon - c)(A(\rho^\varepsilon) - A(c))\} \le \partial_t R_0(t,x) + \nabla_x \cdot R(t,x), \tag{7.4}$$

with

$$\|R_0(t,x)\|_{\mathcal{M}_{t,x}} + \|R(t,x)\|_{\mathcal{M}_{t,x}} \le \text{Const} \cdot \varepsilon. \tag{7.5}$$

Then, the convergence rate proof proceeds along the lines of Theorem 2.1: Using the key property of symmetry of the regularized entropy pairs, $(\eta^\delta := \phi_\delta \eta, F^\delta := \phi_\delta F)$, one finds $\int_x \eta^\delta(\rho^\varepsilon;\rho)dx \le \text{Const.}\varepsilon/\delta$. In addition, there is a regularization error, $\|\eta^\delta - \eta\|_{L^1(x)}$, of size $\mathcal{O}(\delta)$, and an L^1 error estimate of order $\mathcal{O}(\sqrt{\varepsilon})$ follows (under reasonable assumptions on the L^1-initial error w.r.t. BV data), consult [7]

$$\|\rho^\varepsilon(t,\cdot) - \rho(t,\cdot)\|_{L^1_{\text{loc}}(x)} \le \text{Const.}\sqrt{\varepsilon}.$$

Observe that this error estimate, based on (7.4)-(7.5) is the multidimensional analogue of the Lip'-consistency requirement we met earlier in (4.2).

8 Kinetic Approximations

8.1 Velocity averaging lemmas ($m \ge 1, d \ge 1$)

We deal with solutions to transport equations

$$a(v) \cdot \nabla_x f(x,v) = \partial_v^s g(x,v). \tag{8.1}$$

The averaging lemmas, [61], [52], [44], state that in the generic non-degenerate case, averaging over the velocity space, $\bar{f}(x) := \int_v f(x,v)dv$, yields a gain of *spatial* regularity. The prototype statement reads

Lemma 8.1 ([44], [110]) . *Let $f \in L^p(x,v)$ be a solution of the transport equation (8.1) with $g \in L^q(x,v), 1 \le q < p \le 2$. Assume the following non-degeneracy condition holds*

$$meas_v\{v|\ |a(v)\cdot\xi| < \delta\}_{|\xi|=1} \le Const\cdot\delta^\alpha, \quad \alpha\in(0,1). \tag{8.2}$$

Then $\bar f(x) := \int_v f(x,v)dv$ belongs to Sobolev space $W^\theta(L^r(x))$,

$$\bar f(x)\in W^\theta(L^r(x)), \qquad \theta < \frac{\alpha}{\alpha(1-\frac{p'}{q'})+(s+1)p'}, \quad \frac{1}{r}=\frac{\theta}{q}+\frac{1-\theta}{p}. \tag{8.3}$$

Variants of the averaging lemmas were used by DiPerna and Lions to construct global weak (renormalized) solutions of Boltzmann, Vlasov-Maxwell and related kinetic systems, [42], [43]; in Bardos et. al., [2], averaging lemmas were used to construct solutions of the incompressible Navier-Stokes equations. We turn our attention to their use in the context of nonlinear conservation laws and related equations.

8.1.1 Scalar conservation laws

The following result, adapted from [110], is in the heart of matter.

Theorem 8.1 ([110]) *Consider the scalar conservation law (2.1) whose flux satisfies the non-degeneracy condition (consult (8.2))*

$$\exists\alpha\in(0,1):\ meas_v\{v|\ |\tau+A'(v)\cdot\xi|<\delta\}\le Const\cdot\delta^\alpha, \quad \forall\tau^2+|\xi|^2=1. \tag{8.4}$$

Let $\{\rho^\varepsilon\}$ be a family of approximate solutions satisfying the entropy condition (2.3),

$$\partial_t\eta(\rho^\varepsilon)+\nabla_x\cdot F(\rho^\varepsilon)\le 0, \qquad \forall\eta''>0. \tag{8.5}$$

Then $\rho^\varepsilon(t,x)\in W^{\frac{\alpha}{\alpha+4}}_{loc}(L^r(t,x))$, $r=\frac{\alpha+4}{\alpha+2}$.

Proof. To simplify notations, we use the customary 0^{th} index for time direction,

$$x=(t\leftrightarrow x_0,x_1,\dots,x_d), \qquad A(\rho)=(A_0(\rho)\equiv 1, A_1(\rho),\dots,A_d(\rho)).$$

The entropy condition (8.5) with Kružkov entropy pairs (2.1), reads

$$\nabla_x\cdot[sgn(\rho^\varepsilon-v)(A(\rho^\varepsilon)-A(v))]\le 0.$$

This defines a family of non-negative measures, $m^\varepsilon(x,v)$,

$$\nabla_x\cdot[sgn(v)A(v)-sgn(\rho^\varepsilon-v)(A(\rho^\varepsilon)-A(v))] =: m^\varepsilon(x,v). \tag{8.6}$$

Differentiate (8.6) w.r.t. v: one finds that the indicator function, $f(x,v)=\chi_{\rho^\varepsilon}(v)$, where

$$\chi_{\rho^\varepsilon}(v) := \begin{cases} +1 & 0<v<\rho^\varepsilon \\ -1 & \rho^\varepsilon<v<0 \\ 0 & |v|>\rho^\varepsilon \end{cases}, \tag{8.7}$$

satisfies the transport equation (8.1) with $g(x,v)=m^\varepsilon(x,v)\in\mathcal{M}_{x,v}$ [7]. We now apply the averaging lemma with $(s=q=1,p=2)$, which tells us that $\rho^\varepsilon(t,x)=\int\chi_{\rho^\varepsilon}(v)dv\in W^{\frac{\alpha}{\alpha+4}}_{loc}(L^r(t,x))$ as asserted. ■

Couple of remarks is in order.

[7] Once more, it is the symmetry property (2.6) which has a key role in the derivation of the transport kinetic formulation (8.1).

1. The last theorem quantifies the regularity of entropy satisfying approximate solutions, $\{\rho^\varepsilon\}$, in terms of the non-degeneracy (8.5). In particular $\{\rho^\varepsilon\}$ is compact and strong convergence follows.
   In fact more can be said if the solution operator associated with $\{\rho^\varepsilon\}$ is translation invariant: a bootstrap argument yields an improved regularity, [110],

$$\rho^\varepsilon(t>0,\cdot)\in W^{\frac{\alpha}{\alpha+2}}(L^1(x)). \tag{8.8}$$

   This shows that due to nonlinearity, (8.4), the corresponding solution operator has a *regularization* effect beyond the initial layer at $t=0$.

2. In particular, Theorem 8.1 provides an alternative route to analyze the entropy stable multi-dimensional schemes whose convergence proof was previously accomplished by measure-valued arguments; here we refer to finite-difference, finite-volume, streamline-diffusion and spectral approximations ..., which were studied in [29],[24],[85],[86], [79],[80],[18],....
   Indeed, the feature in the convergence proof of all these methods is the $W^{-1}_{loc}(L^2)$-compact entropy production, (8.11). Hence, if the underlying conservation law satisfies the nonlinear degeneracy condition,

$$meas_v\{v|\ \tau+A'(v)\cdot\xi=0\}=0,$$

   then the corresponding family of approximate solutions, $\{\rho^\varepsilon(t>0,\cdot)\}$ becomes compact. Moreover, if the entropy production is bounded measure, then there is actually a *gain* of regularity indicated in Theorem 8.1 (– and in (8.8) for the translation invariant case).

8.1.2 Degenerate parabolic equations

The above results can be extended in several directions, consult [110] (and [111] for certain $m=2$ systems). As an example one can treat convective equations together with (possibly degenerate) diffusive terms

$$\partial_t\rho^\varepsilon+\nabla_x\cdot A(\rho^\varepsilon)=\nabla_x\cdot(Q\nabla_x\rho^\varepsilon),\quad Q\geq 0. \tag{8.9}$$

Assume the problem is not linearly degenerate, in the sense that

$$meas_v\{v|\ \tau+A'(v)\cdot\xi=0,\ \langle Q(v)\xi,\xi\rangle=0\}=0. \tag{8.10}$$

Let $\{\rho^\varepsilon\}$ be a family of approximate solutions of (8.1) with $W^{-1}_{loc}(L^2)$-compact entropy production,

$$\partial_t\eta(\rho^\varepsilon)+\nabla_x\cdot F(\rho^\varepsilon)\hookrightarrow W^{-1}_{loc}(L^2(t,x)),\quad \forall\eta''>0. \tag{8.11}$$

Then $\{\rho^\varepsilon\}$ is compact in $L^2_{loc}(t,x)$, [110].
The case $Q=0$ corresponds to a multidimensional extension of Tartar's compensated compactness arguments in §5.1, and it quantifies the regularity of DiPerna's measure-valued solutions in §7.1. The case $A=0$ correspond possibly degenerate parabolic equations (consult [84] and the references therein, for example). According to (8.10), satisfying the ellipticity condition, $\langle Q(v)\xi,\xi\rangle>0$ on a set of non-zero measure, guarantees regularization, compactness ...

8.2 Kinetic schemes

We restrict our attention to the scalar case (— and refer to [15],[109], [133] for a comprehensive rigorous treatment of Boltzmann equation). Here, we demonstrate an application of Theorem 8.1 in the context of the BGK-like relaxation model introduced in [135] following the earlier works [9],[53],

$$\partial_t f^\varepsilon + A'(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(\chi_{\rho^\varepsilon} - f^\varepsilon\right). \tag{8.12}$$

As before, the indicator function $\chi_{\rho^\varepsilon(t,x)}(v)$ denotes the 'pseudo-Maxwellian' (8.8) associated with $\rho^\varepsilon := \bar{f}^\varepsilon$. The relaxation term on the right of (8.12) belongs to $W^{-1}(\mathcal{M}_{t,x})$, [9], and the averaging lemma 8.1 applies with $(s = q = 1,\ \ p = 2)$. It follows that if the conservation law is linearly non-degenerate in the sense that (8.5) holds, then $\{\rho^\varepsilon\}$ is compact – in fact $\{\rho^\varepsilon(t > 0, \cdot)\}$ gains Sobolev regularity of order $\frac{\alpha}{\alpha+2}$, [110]. The relaxation model (8.12) was analyzed previously by BV-compactness arguments, e.g., [53], [135].

There is more than one way to convert microscopic kinetic formulations of nonlinear equations, into macroscopic algorithms for the approximate solution of such equations. We conclude by mentioning the following three (in the context of conservation laws). Brenier's transport collapse method, [9], is a macroscopic projection method which preceded the BGK-like model (8.12) (see also [53]). Another approach is based on Chapman-Enskog asymptotic expansions, and we refer to [145], for an example of macroscopic approximation other than the usual Navier-Stokes-like viscosity regularization, (3.4). Still another approach is offered by Godunov-type schemes, (4.4), based on projections of the Maxwellians associated with the specific kinetic formulations. These amount to specific Riemann solvers which were studied in [38], [134], [136].

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