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Abstract. We establish the existence of the viscous profile of an undriven divergent detonation wave for a model problem with small viscosity.

It is known that there is a sonic point inside the reaction zone of a divergent detonation wave. As a consequence, the detonation wave profile is a transonic profile and the wave speed cannot be determined before the problem is solved. The wave speed may be interpreted as a nonlinear eigenvalue. The detonations exhibiting this type of behavior are sometimes termed eigenvalue detonations. The shooting method and an asymptotic analysis are performed to prove the existence of the viscous profile for small viscosity. The condition we shoot at is the compatibility condition at the sonic point. In the construction of the viscous profile, an iteration is employed to treat the source term

arising from the chemical reaction.

It is a consequence of the constructive proof that as viscosity tends to zero the viscous profile tends to the inviscid profile of the undriven divergent detonation wave [12]. Thus the undriven divergent detonation wave found in [12] is structurally stable.

Finally, we study the nonlinear stability of the transonic profile. We prove that the solution exists globally and approaches to a shifted traveling wave solution as $t \rightarrow +\infty$ for 'large' perturbations of the transonic profile.

Key words. Detonation, curvature effect, shock wave, traveling wave, sonic point, transonic, shooting method, asymptotic analysis, asymptotic behavior, shift.

AMS(MOS) subject classifications. 35L65, 35B40, 35B50, 76L05, 76J10.

1 Introduction

For 1- D reactive Euler equations, the detonation waves predicted by the analogue of the Z-N-D (Zeldovich-von Neumann-Doring) theory has the internal structure of an ordinary precursor fluid dynamic shock wave followed by a reaction zone. Among all detonations, there is a detonation propagates at the minimum speed. This wave is called CJ (Chapman-Jouget) detonation.

In higher dimensions, however, a divergent detonation is weakened by expansion-induced rarefaction coming from behind the shock, and the end of the reaction zone moves below the CJ point yielding a detonation in which there is a sonic point of transition within the reaction zone, or a transonic profile. In order to have a smooth transition through a sonic point, certain solvability condition must be satisfied at the point. The solvability condition is a relation between the normal velocity and the curvature of the detonation front, which depends on the form of the rate and the equation of the state. So for a curved detonation wave, the detonation speed is determined by the reaction dynamics.

The effect of the divergence is similar to that of the endothermic chemical reaction, radiation effect or the effect of a fluid with a nonconvex equation of state [12] [13]. It is also an approximate analog of the standard convergent-divergent nozzle problem [4] [7].

We study existence of the viscous profile of an undriven divergent detonation through the following model

$$
\begin{aligned}
u_{\epsilon t} + (\frac{1}{2}u_\epsilon^2 - qz_\epsilon)_x &= \epsilon u_{\epsilon xx} - \sigma(u_\epsilon - u_0) & (1) \\
z_{\epsilon x} &= k\varphi(u_\epsilon)z_\epsilon^\alpha, & (2)
\end{aligned}
$$

$\varphi(y)$ has the ignition form

$$
\varphi(y) = \begin{cases} 0 & y \le u_i \\ \text{smoothly increasing} & u_i < y < 2u_i \\ 1 & y \ge 2u_i \end{cases}
$$

where $u_\epsilon = u_\epsilon(x,t)$ and $z_\epsilon = z_\epsilon(x,t)$ are scalar functions representing velocity or temperature of the combustible gas and concentration of the unburnt gas, and q, ϵ and $k > 0$ are constants representing the amount of heat released during the chemical reaction, the viscous coefficient, and the rate multiplier. $\sigma > 0$ is the weak divergent curvature constant. $u_i \ge u_0$ is a constant which refers to ignition temperature and u_0 is the quiescent fluid state ahead of the triggering shock.

The model was derived by Majda and Rosales [15] under the assumptions of: weakly nonlinear, high activation energy and nearly sonic speed of the detonation wave. This model describes the motion of the combustible gas in a divergent tube. It captures the main features of solutions in the physical system. In particular, there is a sonic point in the viscous profile for the model problem and a compatibility condition must be satisfied there. As a consequence of the peculiar property of an undriven divergent detonation that there is a sonic point in the reaction zone, the detonation wave profile is a transonic profile and the wave speed cannot be determined before the problem is solved. The wave speed may be interpreted as a nonlinear eigenvalue. The detonations exhibiting this type of behavior are sometimes termed eigenvalue detonations [5].

In [12], the author established the existence of the inviscid divergent detonation profile. Furthermore, it was proved that the solution to the initial value problem exists globally and converges uniformly, away from the shock, to a shifted traveling wave solution as $t \rightarrow +\infty$ for certain 'compact support' initial data.

In the current paper, we use the shooting method, previously used by Bukiet [1] for numerical simulation of effect curvature on detonation speed, Jones [8] for an asymptotic analysis of the curvature effect on detonation speed, Hsu and Liu [7] for nonlinear singular Sturm-Liouville problems and an application transonic flow through a nozzle, the asymptotic expansions in ϵ for ϵ small and the result for the inviscid case [12] to establish the existence of the viscous profile. The condition we shoot at is the compatibility condition at the sonic point. In the construction of the viscous profile, an iteration is employed to treat the source term arising from the chemical reaction.

It is also proved that as ϵ tends to zero the viscous profile tends to the inviscid profile of the undriven divergent detonation wave [12]. Thus the undriven divergent detonation wave found in [12] is structurally stable.

Finally, we Consider the initial value problem (1) (2) with

$$u_\epsilon(x,0) = u_1(x) \tag{3}$$

$$z_\epsilon(+\infty,t) = 1 \tag{4}$$

where $u_1(x)$ satisfies

$$\int_{-\infty}^{+\infty}(u_1(x)-u_0)dx < +\infty.$$

The unique classical global solution of (1) (2) (3) (4) exists. The proof can be found in [14]. We prove that the solution approaches to a shifted traveling wave solution as $t \rightarrow +\infty$. Therefore, the transonic profile is nonlinearly stable.

In Section 2 we give a brief derivation of the model given by Majda and Rosales [15] . In Section 3 we establish the existence of the viscous profile of (1) (2) for ϵ small and the structural stability of the undriven divergent detonation wave when $\epsilon = 0$. Section 4 is devoted to the study of the Cauchy problem (1) (2) (3) (4) and the large-time behavior of its solution.

2 Derivation of the Model

Under the assumptions of: weakly nonlinear, high activation energy and nearly sonic speed of the detonation wave, Majda and Rosales [15] derived the simplified model (1) (2) in the following way.

If the multi-D reaction front is described (to leading order) by

$$\Psi(X) - T = 0,$$

then Ψ solves the eikonal equation,

$$|\nabla\Psi| = 1. \tag{5}$$

The coordinate x is defined by

$$x = \frac{(\Psi(X) - T)}{\delta},$$

and t is defined by

$$\frac{dX}{dt} = \nabla\Psi(X)$$

with the initial condition $t = 0$ at $\Psi(X) = 0$. t is proportional to the arc length along the characteristics of (5). Along characteristic rays, we have

$$\begin{aligned} \frac{d}{dt}\Psi &= 1 \\ \frac{d}{dt}\nabla\Psi &= 0. \end{aligned}$$

$\delta << 1$ is the ratio of typical absolute values of the fluid velocities and reaction front velocities.

Near the reaction front,

$$\begin{aligned} \rho &= 1 + \delta\rho_0(x, X) + ..., \\ u &= u_0(x, X) + \delta u_1(x, X) + ..., \\ T &= 1 + \delta T_0(x, X) + ..., \\ Z &= Z_0(x, X) + \delta Z_1(x, X) + ... \end{aligned}$$

Plugging into the reactive Euler equations, one shows that ρ_0, u_0 and T_0 solve a homogeneous system of equations. Therefore these quantities should be proportional to a proper eigenvector, i.e.,

$$\begin{aligned} \rho_0 &= \frac{1}{(\gamma - 1)} U(x, X) + \bar{\rho}(X), \\ u_0 &= \frac{\nabla\Psi}{(\gamma - 1)} U(x, X) + \bar{u}(X), \\ T_0 &= U(x, X) + \bar{T}(X). \end{aligned}$$

From the next order perturbation equations, one finds a compatibility condition which yields a differential equation for U (U will be denoted as u from hereon),

$$u_t + (\frac{u^2}{2} - qz)_x + \frac{1}{2}(\Delta\Psi)u = 0 \tag{6}$$

$$z_x = k\phi(u)z^\alpha \tag{7}$$

$z = z(x,t)$ represents concentration of the unburnt gas. As seen in the above asymptotic expansions, $u = u(x,t)$ plays a multiple role. The last term on left hand side of (6) accounts for the amplitude growth or decay according to the local geometric compression or expansion of areas in the wave front of geometric optics. In this paper, we study the case where $\Delta\Psi > 0$ and is a constant, see (1).

This way, the equation of propagation which governs the geometry of the wave front is decoupled from the fluid mechanical and chemical phenomena that control the local shape of the waves in the original model.

3 The Existence of the Viscous Profile

We establish the existence of the transonic profile for (1) (2). The profile is constructed based on the inviscid one in [12] for ϵ small. Since there is a sonic point in the reaction zone, a compatibility condition must be satisfied there. We prove the existence by the shooting method [1] [7] [8]. The condition we shoot at is the compatibility condition. In the construction of the viscous profile, an iteration is employed to treat the source term arising from the chemical reaction.

A traveling wave solution is a solution of the following form

$$(u_\epsilon(x,t), z_\epsilon(x,t)) = (\psi_\epsilon(x - D_\epsilon t), Z_\epsilon(x - D_\epsilon t))$$

where D_ϵ is the speed of the detonation wave.

Let $(u_\epsilon(x,t), z_\epsilon(x,t)) = (\psi_\epsilon(\xi), Z_\epsilon(\xi))$ where $\xi = x - D_\epsilon t$. Then $(\psi_\epsilon, Z_\epsilon)(\xi)$ solves the following ordinary differential equations

$$-D_\epsilon \psi_\epsilon' + \psi_\epsilon \psi_\epsilon' = \epsilon \psi_\epsilon'' + qZ_\epsilon' - \sigma(\psi_\epsilon - u_0) \tag{8}$$

$$Z_\epsilon' = k\varphi(\psi_\epsilon) Z_\epsilon^\alpha. \tag{9}$$

If we let $r = \psi_\epsilon'$ and rewrite (8) (9) as a first order system of ordinary differential equations for $(\psi_\epsilon, r, Z_\epsilon)$, then the only critical points are $(u_0, 0, 0)$ and $(u_0, 0, 1)$. They are both saddle points in (ψ_ϵ, r) plane. The boundary conditions for $(\psi_\epsilon, Z_\epsilon)$ are

$$\lim_{\xi \to -\infty} (\psi_\epsilon, Z_\epsilon)(\xi) = (u_0, 0) \tag{10}$$

$$\lim_{\xi \to +\infty} (\psi_\epsilon, Z_\epsilon)(\xi) = (u_0, 1). \tag{11}$$

Our main result is the following.

Theorem 3.1 *Let ϵ be small. There exists a unique solution $(\psi_\epsilon,\ Z_\epsilon)$ to problem (8) (9) (10) (11) provided that $\alpha > \frac{1}{2}$ and $u_i - u_0$ is small.*

Furthermore, $(\psi_\epsilon,\ Z_\epsilon)$ satisfies

$$u_0 \le \psi_\epsilon \le u_0 + \frac{qk}{\sigma}$$

and

$$0 \le Z_\epsilon \le 1.$$

The propagating speed D_ϵ satisfies

$$D_\epsilon = \frac{u_l + u_0}{2} + \frac{q - \epsilon u_l'}{u_l - u_0} - \sigma \frac{\int_{-l}^{+\infty} (\psi_\epsilon(\xi) - u_0) d\xi}{u_l - u_0}$$

and

$$\int_{-\infty}^{+\infty} (\psi_\epsilon(\xi) - u_0) d\xi = \frac{q}{\sigma}$$

where $u_l = \psi_\epsilon(-l)$ *and* $u_l' = \psi_\epsilon'(-l)$*, solution and its derivative at the end of the reaction zone.*

Proof.

Let ϵ be small. We construct the viscous profile based on [7] the inviscid one in [12] through asymptotic expansions in ϵ.

First, it can be proved by applying a maximum principle to (8) that $\psi_\epsilon(\xi)$, where $\xi = x - D_\epsilon t$, is bounded

$$u_0 \leq \psi_\epsilon \leq u_0 + \frac{qk}{\sigma}. \tag{12}$$

For $\epsilon > 0$ small, there is an interior layer in solution $\psi_\epsilon(\xi)$ at 0 of width $O(\epsilon)$. Assume inside the interior layer that $\psi_\epsilon(0) = u_i$ without loss of generality. Therefore from (9) (11) we have that

$$Z_\epsilon(\xi) = 1 \tag{13}$$

for $\xi \geq 0$.

Since there is a sonic point in the reaction zone, a compatibility condition must be satisfied there. That is

$$\psi_\epsilon(\xi_0(\epsilon)) = D_\epsilon$$

and

$$-\epsilon\psi_\epsilon''(\xi_0(\epsilon)) + \sigma(\psi_\epsilon(\xi_0(\epsilon)) - u_0) = qZ_\epsilon'(\xi_0(\epsilon)). \tag{14}$$

For the inviscid solution, $\psi_0''(\xi_0(0)) > 0$ bounded and $\xi_0(0) < 0$, see [12]. So $\xi_0(\epsilon)$, which is close to $\xi_0(0)$ when ϵ is small, is outside the interior layer and $\psi_\epsilon''(\xi_0(\epsilon)) > 0$ bounded as in the inviscid case.

In order to construct the viscous profile inside the interior layer, we make a transformation $y = \dfrac{\xi}{\epsilon}$ inside the interior layer. Plugging it into (8), we have that inside the interior layer ψ_ϵ satisfies

$$\psi_{\epsilon yy} - (\psi_\epsilon - D_\epsilon)\psi_{\epsilon y} = \epsilon(\sigma(\psi_\epsilon - u_0) - qZ'_\epsilon) \tag{15}$$

and (12).

An iteration is then defined according to (15) to construct the viscous profile inside the interior layer.

Consider the following problem

$$\psi_{1yy} - (\psi_1 - D_\epsilon)\psi_{1y} = 0 \tag{16}$$

with data

$$\psi_1(+\infty) = u_0, \quad \psi_1(0) = u_i. \tag{17}$$

The exact solution of (16) (17) is

$$\psi_1(y) = D_\epsilon + \gamma \tanh(\frac{\gamma(y + y_0)}{2}) \tag{18}$$

where

$$\gamma = D_\epsilon - u_0 > 0$$

and

$$D_\epsilon + \gamma \tanh(\frac{\gamma y_0}{2}) = u_i.$$

Plugging $\psi_1(y)$ into (9) and solving it along with (13), we have $Z_1(y)$.

Since in the interior layer $\psi_1(y)$ and $Z_1(y)$ are bounded, hence they solve equation (15) with data (17) with error $O(\epsilon)$. That is

$$\psi_{1yy} - (\psi_1 - D_\epsilon)\psi_{1y} = \epsilon(\sigma(\psi_1 - u_0) - qZ'_1) + O(\epsilon).$$

Outside the layer, $Z_1(\xi)$ is obtained from (9) and $Z_1(y)$. $\psi_1(\xi)$ can then be obtained by solving equation (8), where $Z_1(\xi)$ is known, with the initial data obtained from the interior layer solution $\psi_1(y)$. It is easy to verify that $\psi_1(\xi)$ and $Z_1(\xi)$ solve equation (8) with error $O(\epsilon)$.

By applying a maximum principle to (8), we have

$$u_0 \leq \psi_1 \leq u_0 + \frac{qk}{\sigma} + O(\epsilon).$$

Now we consider the following problem for ψ_2 according to (15)

$$\psi_{2yy} - (\psi_2 - D_\epsilon)\psi_{2y} = \epsilon(\sigma(\psi_1 - u_0) - qZ_1'). \tag{19}$$

with data (17).

Integrating equation (19) from y to $+\infty$, we have

$$\psi_{2y} - \frac{1}{2}(\psi_2 - D_\epsilon)^2 = -\frac{1}{2}\gamma^2 - \epsilon\int_y^{+\infty} \sigma(\psi_1 - u_0)dy + \int_y^{+\infty} qZ_{1y}dy. \tag{20}$$

The last two terms are of order ϵ since the interior layer is of width $O(\epsilon)$ and Z_1 is continuous and $\int_0^{+\infty}(\psi_1(y) - u_0)dy < +\infty$ by (18).

Hence ψ_2 is bounded both above and below by functions of the following form

$$D_\epsilon + \gamma' \tanh(\frac{\gamma'(y + y_0')}{2}) \tag{21}$$

where $\gamma' = \gamma + O(\epsilon)$, $y_0' = y_0 + O(\epsilon)$ and $O(\epsilon)$ depends on ψ_1 and Z_1. Consequently

$$|\psi_2 - \psi_1| = O(\epsilon). \tag{22}$$

By using (13) and (18), it can be derived from (20) that $\psi_2(y) - u_0$ decays to zero with the same rate of $\psi_1(y) - u_0$ decays to zero as $y \to +\infty$. Hence $\int_y^{+\infty}(\psi_2(y) - u_0)dy < +\infty$.

Plugging $\psi_2(y)$ into (9) and solving it along with (13), we have $Z_2(y)$.

By using (22) in the interior layer, we have that $\psi_2(y)$ and $Z_2(y)$ solve equation (15) with data (17) with error $O(\epsilon^2)$. That is

$$\psi_{2yy} - (\psi_2 - D_\epsilon)\psi_{2y} = \epsilon(\sigma(\psi_2 - u_0) - qZ_2') + O(\epsilon^2).$$

Outside the layer, $Z_2(\xi)$ is obtained from (9) and $Z_2(y)$. $\psi_2(\xi)$ can then be obtained by solving equation (8), where $Z_2(\xi)$ is known, with the initial data obtained from the interior layer solution $\psi_2(y)$. It is easy to verify that $\psi_2(\xi)$ and $Z_2(\xi)$ solve equation (8) with error $O(\epsilon^2)$.

Hence, ψ_2 satisfies

$$u_0 \leq \psi_2 \leq u_0 + \frac{q}{\sigma} + O(\epsilon^2).$$

This process can be continued to any n so that $\psi_n(y)$ and $Z_n(y)$ solve equation (15) with data (17) with error $O(\epsilon^n)$. That is

$$\psi_{nyy} - (\psi_n - D_\epsilon)\psi_{ny} = \epsilon(\sigma(\psi_n - u_0) - qZ_n') + O(\epsilon^n) \tag{23}$$

and $\psi_n(\xi)$ and $Z_n(\xi)$ solve equation (8) with error $O(\epsilon^n)$ and

$$u_0 \leq \psi_n \leq u_0 + \frac{qk}{\sigma} + O(\epsilon^n). \tag{24}$$

Furthermore, ψ_n is bounded by functions of the following form

$$D_\epsilon + O(\epsilon) + \gamma' \tanh(\frac{\gamma'(y + y_0')}{2}) \tag{25}$$

where $\gamma' = \gamma + O(\epsilon)$, $y_0' = y_0 + O(\epsilon)$.

Since ψ_n is bounded (24) uniformly with respect to n, there is a weak star limit ψ_ϵ of a subsequence of ψ_n. According to (23) and (24), this limit ψ_ϵ solves equation (15) with data (17) and is bounded. ψ_ϵ must also be a classical solution since (15) is a second order ordinary differential equation.

Outside the layer, $\psi_\epsilon(\xi)$ can then be obtained by solving equation (8), where $Z_\epsilon(\xi)$ is known, with the initial data obtained from the interior layer solution $\psi_\epsilon(y)$. Hence ψ_ϵ and Z_ϵ solve (8).

Now we shoot for the compatibility condition (14) by adjusting the parameter D_ϵ. By taking $D_\epsilon \to +\infty$ and noting that $\psi_\epsilon''(\xi_0(\epsilon)) > 0$ bounded, the equal sign in (14) is replaced by $>$.

Using the assumption that $u_i - u_0$ is small and taking $D_\epsilon - u_0 > \dfrac{u_i - u_0}{2} > 0$ small, we conclude that the equal sign in (14) is replaced by $<$.

There must be a D_ϵ such that the compatibility condition (14) is satisfied. We have completed the construction of the viscous profile.

■

It can be shown that in the limit that ϵ tends to zero the viscous profile tends to the inviscid profile of the undriven divergent detonation wave [12]. Thus the undriven divergent detonation wave found in [12] is structurally stable.

Theorem 3.2 *The undriven divergent detonation wave is structurally stable. Moreover, we have the following rate of convergence in L^1 norm:*

$$||\psi_\epsilon - \psi||_1 \leq C\epsilon|\ln \epsilon| \tag{26}$$

where C is a constant independent of ϵ.

Proof. The equation satisfied by $\psi_\epsilon - \psi$ is

$$-(D - \psi_\epsilon)(\psi_\epsilon - \psi)' + \psi'(\psi_\epsilon - \psi) + (D - D_\epsilon)\psi_\epsilon'$$

$$= \epsilon\psi_\epsilon'' - \sigma(\psi_\epsilon - \psi) + q(Z_\epsilon - Z)' \tag{27}$$

where $\psi_\epsilon(0) = \max_x \psi_\epsilon(x)$ and $\psi(0) = \max_x \psi(x)$.

Integrating (27) over $(-\infty, +\infty)$, we have that

$$\int_{-\infty}^{+\infty} (\psi_\epsilon - \psi)(x)dx = 0. \tag{28}$$

From the proof of Theorem 3.1, we know that

$$|\psi_\epsilon - \psi|(x) \leq C\epsilon$$

except in the interior layer which is of width $O(\epsilon)$.

¿From the construction of the viscous profile ψ_ϵ, we have that

$$(\psi_\epsilon - \psi)(x) > 0$$

for $x > 0$ and that

$$\int_0^{+\infty} (\psi_\epsilon - \psi)(x)dx = \int_0^{+\infty} |\psi_\epsilon - \psi|(x)dx = O(\epsilon) > 0.$$

Since both ψ_ϵ and ψ decay exponentially to u_0 as $x \to -\infty$, therefore

$$\int_{-\infty}^{L} |\psi_\epsilon - \psi|(x)dx = O(\epsilon)$$

for some $L = C \ln \epsilon < 0$.

For $L < x < 0$ which is outside the interior layer, using the above consequence of Theorem 3.1, we have

$$\int_L^0 |\psi_\epsilon - \psi|(x)dx = O(\epsilon |\ln \epsilon|).$$

Adding the three integrals together, we have the desired results (26).

■

4 Nonlinear Stability of the Transonic Profile

In this section, we study the time-asymptotic limit of solutions of initial value problem (1) (2) (3) (4). The unique classical global solution of (1) (2) (3) (4) exists. The proof can be found in [14]. We prove that the solution approaches to a shifted traveling wave solution as $t \to +\infty$ for 'large' perturbation of the transonic profile. That is that the transonic profile is nonlinearly stable.

We establish a bound for the solution by a maximum principle.

Lemma 4.1 *If u_ϵ is a solution to problem (1) (2) (3) (4), then it is bounded:*

$$u_0 \leq u_\epsilon(x,t) \leq \max\{u_0 + \frac{qk}{\sigma},\ \ M_0\} \tag{29}$$

where M_0 is the maximum value of the initial data.

Proof. Noting the initial data (3) and applying the maximum principle to equation (1), we arrive at our conclusion.

∎

We now prove a useful comparison principle for solution u_ϵ of (1) and (2) and the viscous profile ψ_ϵ constructed in Theorem 3.1.

In the following theorem the initial value has the following form

$$u_\epsilon(x,0) = \begin{cases} u_1(x) & x \leq s(0) \\ u_0 & x > s(0) \end{cases}$$

where $u_1(x) \geq u_0$ and $s(0)$ is the initial position of the ignition.

Lemma 4.2 *(A Comparison Principle)*

Suppose that $u_\epsilon(x,t)$ is the solution of (1) and (2) with initial data $u_1(x)$ and the position of the ignition $s(t)$. If

$$s(0) > D_\epsilon t|_{t=0}$$

and

$$u_1(x) \leq \psi_\epsilon(x),\ x \leq D_\epsilon t|_{t=0},$$

then there is a $T > 0$ such that for $0 < t < T$,

$$u_\epsilon(x,t) \leq \psi_\epsilon(x - D_\epsilon t),\ x \leq x_m$$

where $x_m < D_\epsilon t$ is the maximum point of ψ_ϵ.

Proof. Consider the equation for the difference $\psi_\epsilon - u_\epsilon$,

$$(\psi_\epsilon - u_\epsilon)_t + (\psi_\epsilon - u_\epsilon)\psi_{\epsilon x} + u_\epsilon(\psi_\epsilon - u_\epsilon)_x - q(Z_\epsilon - z_\epsilon)_x = -\sigma(\psi_\epsilon - u_\epsilon) + \epsilon(\psi_\epsilon - u_\epsilon)_{xx}.$$

Noticing that $s(0) > D_\epsilon t|_{t=0}$, there is a $T > 0$ such that for $0 < t < T$

$$s(t) > D_\epsilon t$$

and the maximum point of u_ϵ is bigger than that of ψ_ϵ.

Hence, from (2) we have

$$q(Z_\epsilon - z_\epsilon)_x > 0$$

for $x < x_m$, where $x_m < D_\epsilon t$ is the maximum point of ψ_ϵ.

¿From Theorem 3.1 and the results of [12], we have that $\psi_{\epsilon x} > 0$ for $x < x_m$. Noting that $u_1(x) \leq \psi_\epsilon(x)$ and applying a maximum principle to the above equation for $\psi_\epsilon - u_\epsilon$, we have that there is no negative minimum of $\psi_\epsilon - u_\epsilon$ for $x < x_m$. Since $(\psi_\epsilon - u_\epsilon)(x,t) \to 0$ as $x \to -\infty$, we conclude at the following

$$u_\epsilon(x,t) \leq \psi_\epsilon(x - D_\epsilon t),\ \ x \leq x_m.$$

$\blacksquare$

Similar to the inviscid case [12], there is an asymptotic conservative property for the solution of (1) (2) (3) (4).

Lemma 4.3 *If $u_\epsilon(x,t)$ is a solution of (1), (2), (3) and (4), then*

$$\frac{d}{dt}(e^{\sigma t}\int_{-\infty}^{+\infty}(u_\epsilon(x,t)-\psi_\epsilon(x-D_\epsilon t))dx)=0$$

Or

$$\int_{-\infty}^{+\infty}(u_\epsilon(x,t)-\psi_\epsilon(x-D_\epsilon t))dx=e^{-\sigma t}\int_{-\infty}^{+\infty}(u_\epsilon(x,0)-\psi_\epsilon(x))dx.$$

We now study the convergence of the solution u_ϵ to the traveling wave ψ_ϵ as $t \rightarrow +\infty$.

Theorem 4.4 *The unique classical solution (u_ϵ, z_ϵ) of (1) (2) (3) (4) satisfies*

$$\lim_{t\longrightarrow+\infty}\|(u_\epsilon,z_\epsilon)(\cdot,t)-(\psi_\epsilon,Z_\epsilon)(\cdot-D_\epsilon t+C_\epsilon)\|_1=0$$

for some C_ϵ and provided that the initial data satisfies

$$\int_{-\infty}^{+\infty}(u_\epsilon(x,0)-u_0)dx<+\infty.$$

Proof. The proof consists of two steps. In the first step, we prove shape convergence to the traveling wave. In the second one, we determine the shift in the resulting traveling wave.

First, take the initial data $u_0 \le u_\epsilon(x,0) \le \psi_\epsilon(x)$ and nondecreasing.

Step 1.

For each time $t \ge 0$, let $s(t)$ be the maximum point of $u_\epsilon(x,t)$. That is

$$u_\epsilon(s(t),t)=\max_x u_\epsilon(x,t).$$

Choose $C_{\epsilon,t}$ such that

$$\psi_\epsilon(s(t) - D_\epsilon t + C_{\epsilon,t}) = \max_x \psi_\epsilon(x - D_\epsilon t + C_{\epsilon,t}).$$

Using the result of the comparison Lemma 4.2, we have

$$u_0 \leq u_\epsilon(x,t) \leq \psi_\epsilon(x - D_\epsilon t + C_{\epsilon,t}).$$

Hence

$$\begin{aligned} & \|u_\epsilon(\cdot,t) - \psi_\epsilon(\cdot - D_\epsilon t + C_{\epsilon,t})\|_1 \\ = & \ |\int_{-\infty}^{+\infty} (u_\epsilon(x,t) - \psi_\epsilon(x - D_\epsilon t + C_{\epsilon,t}))dx|. \end{aligned}$$

Now using the asymptotic conservative property of the solutions proved in Lemma 4.3, we have,

$$\|u_\epsilon(\cdot,t) - \psi_\epsilon(\cdot - D_\epsilon t + C_{\epsilon,t})\|_1 = d_t e^{-\sigma t} \rightarrow 0 \tag{30}$$

as $t \rightarrow +\infty$ where d_t satisfies

$$|\int_{-\infty}^{+\infty} (u_\epsilon(x,0) - \psi_\epsilon(x + C_{\epsilon,t}))dx| = d_t < +\infty.$$

Step 2.

In this step, we obtain estimates of $C_{\epsilon,t}$. Our goal is to determine a shift C_ϵ which is independent of t such that the solution converges to $\psi_\epsilon(x - D_\epsilon t + C_\epsilon)$ as $t \rightarrow +\infty$.

¿From (30) in Step 1 and the smoothness of the solutions, we conclude at that

$$u_\epsilon(x,t) - \psi_\epsilon(x - D_\epsilon t + C_{\epsilon,t}) = O(e^{-\sigma t}).$$

In particular,

$$u_\epsilon(s(t),t) - \psi_\epsilon(s(t) - D_\epsilon t + C_{\epsilon,t}) = O(e^{-\sigma t}).$$

That is the strength of the viscous shock wave in the solution of the initial value problem approaches to that of the viscous shock wave in the traveling wave exponentially fast. Hence the speed of the viscous shock wave approaches to that of the viscous shock wave in the traveling wave exponentially fast. That is

$$s'(t) - D_\epsilon = O(e^{-\sigma t})$$

which implies that

$$s(t) - D_\epsilon t + C_\epsilon = O(e^{-\sigma t})$$

for some C_ϵ. That is

$$C_{\epsilon,t} = C_\epsilon + O(e^{-\sigma t}).$$

The proof of the conclusion in this case is finished.

For the initial data $u_\epsilon(x,0) \geq \psi_\epsilon(x)$, nondecreasing and satisfies

$$\int_{-\infty}^{+\infty} (u_\epsilon(x,0) - u_0)dx < +\infty,$$

we can prove the same result using the same method.

By using the comparison principle Lemma 4.2, for any initial data that is in between two initial data of the above kinds, we can prove the same result.

■

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