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SELF-FOCUSING IN THE PERTURBED AND UNPERTURBED NONLINEAR SCHRÖDINGER EQUATION IN CRITICAL DIMENSION

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Abstract.
The formation of singularities of self-focusing solutions of the nonlinear Schrödinger equation (NLS) in critical dimension is characterized by a delicate balance between the focusing nonlinearity and diffraction (Laplacian), and is thus very sensitive to small perturbations. In this paper we introduce a systematic perturbation theory for analyzing the effect of additional small terms on self focusing, in which the perturbed critical NLS is reduced to a simpler system of modulation equations that do not depend on the transverse variables. The modulation equations can be further simplified, depending on whether the perturbed NLS is power conserving or not. We review previous applications of the modulation theory and present several new ones that include: Dispersive saturating nonlinearities, self-focusing with Debye relaxation, the Davey Stewartson equation, self-focusing in optical fiber arrays and the effect of randomness. An important and somewhat surprising result is that various small defocusing perturbations lead to a universal form of the modulation equations, whose solutions have slowly decaying focusing-defocusing oscillations. In the special case of the unperturbed critical NLS, modulation theory leads to a new adiabatic law for the rate of blowup which is accurate from the early stages of self-focusing and remains valid up to the singularity point. This adiabatic law preserves the lens transformation property of critical NLS and it leads to an analytic formula for the location of the singularity as a function of the initial pulse power, radial distribution and focusing angle. The asymptotic limit of this law agrees with the known loglog blowup behavior. However, the loglog behavior is reached only after huge amplifications of the initial amplitude, at which point the physical basis of NLS is in doubt. We also include in this paper a new condition for blowup of solutions in critical NLS.

Keywords: self-focusing, adiabatic, nonlinear Schrödinger equation, loglog law, Davey Stewartson, Debye, fiber arrays, time-dispersion, nonparaxial

AMS subject classifications: 78-A-60, 35-Q-55

Contents.

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1. Introduction.

The nonlinear Schrödinger equation in critical dimension (CNLS)

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi = 0 \ , \quad \Delta_\perp = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ , \quad \psi(z=0,x,y) = \psi_0(x,y) \tag{1.1}$$

is the simplest model for the propagation of a laser beam in a medium with a Kerr nonlinearity. Here $\psi(z,x,y)$ is the electric field envelope, z is axial distance[1] in the direction of the wave propagation and (x,y) are the coordinates in the transverse plane. In 1965, Kelley used equation (1.1) to show that for optical beams whose power is above a critical value, "self-focusing effect ... is not compensated for by diffraction" [25]. This result was a turning point in nonlinear optics, since until that time diffraction was believed to prevent singularity formation in optics, both linear and nonlinear, much as viscosity is believed to prevent singularity formation in fluid flow. Intensive experimental work followed, in which self-focusing and the existence of a critical power, above which beams may collapse, were observed. For a review of self-focusing experiments, see [54].

Self-focusing in critical NLS attracted also the attention of mathematicians, since it serves as a simple model of nonlinear dispersive wave propagation where a solution with smooth initial conditions can become singular in finite time (i.e. z). A lot has been accomplished in the last thirty years (e.g. [11, 49, 50, 57]), but the theory for CNLS self-focusing is far from complete. For example, sharp conditions for blowup or global existence in (1.1) are still unknown.

1.1. The loglog law.

Considerable effort has been devoted to the study of the blowup rate near the singularity. Initially, self-focusing was analyzed by reducing CNLS to an ordinary differential equation for the beam width L by assuming that the solution maintains a modulated Gaussian profile. This approach was only partially successful. It predicted the existence of a critical power for self-focusing, but only up to a constant [5] and its prediction for the axial location of the singularity was quite inaccurate. There were also attempts to look for non-Gaussian self-similar solutions, but it gradually became clear that in critical transverse dimension $D = 2$ self-focusing solutions of (1.1) are only quasi self-similar and that the rate of focusing is determined by a delicate balance between the focusing nonlinearity and transverse diffraction. This delicate balance in critical self-focusing, which the Gaussian ansatz cannot capture, was at the heart of the difficulties in finding the CNLS blowup rate and is the reason why it took so long until the structure and dynamics of the function ψ near the blowup point were finally resolved by Fraiman [23], and independently (and in a different manner) by Landman, LeMesurier, Papanicolaou, Sulem and Sulem [31, 34], who showed that as the beam approaches the singularity it follows the loglog law (eq. 3.24).

1.2. The adiabatic approach.

Although with the loglog law the mathematical problem of finding the blow-up rate was finally solved, it turned out that the loglog behavior is very hard to observe numerically, even in careful simulations where the solution was amplified by more than ten orders of magnitude (e.g. Fig 3.5). However, the validity of CNLS as a physical model for beam propagation breaks down much earlier, when the field intensity reaches the threshold for material breakdown. Even at sub-threshold intensities, some small terms that are neglected in the derivation of (1.1) from Maxwell's equations (e.g. non-paraxial terms [16, 56], time-dispersion [20, 35, 47] etc.) may become important, because the delicate balance between the focusing nonlinearity and the defocusing Laplacian in critical self-focusing allows for even small terms to have a large effect on self-focusing and even to arrest it.

It is therefore clear that even though the loglog law is established, there is still a need for a description of CNLS self-focusing which is valid in the domain of physical interest and which can be extended to the analysis of the effect of small perturbations. In the last few years a new adiabatic approach was developed which achieves this. This approach is based on the main result of the analysis leading to the loglog law, which is the derivation of the reduced equation (3.13) for the slow rate of radiation losses of the focusing part of the solution. However, rather than solving this equation asymptotically, the adiabatic approach uses it only as a small correction to the adiabatic focusing-rate equation (3.5). This

[1] Since the initial condition is given at $z = 0$ for all (x, y), the variable z plays the role of 'time'

approach was first used by Malkin [38] and it lead to an adiabatic law for the blowup rate of CNLS that is accurate in physically relevant regimes. An improved adiabatic law, which becomes accurate with even less amplification and also preserves the lens transformation property of CNLS, was later obtained by Fibich [18]. In this paper we give a detailed derivation of these laws and show that near the singularity, Fibich's adiabatic law reduces to Malkin's adiabatic law, whose asymptotic limit is in turn the loglog law. Thus, the three laws agree when the amplification is very large but have different domains of validity.

An immediate consequence of the adiabatic law is an analytic formula for the location of the singularity. A previous result of this type is that of Dawes and Marburger [15, 39], derived by curve fitting values obtained from numerical simulations with Gaussian initial conditions. We give here a new curve-fitted formula for Gaussian initial conditions which is more accurate than either the adiabatic formula or the one of Dawes and Marburger.

1.3. Perturbed critical self-focusing. Since the adiabatic approach is valid in regimes of physical interest, it may be used to analyze the effect of small perturbations on critical self-focusing. The main result of this paper is an extension of this approach to a general modulation theory for analyzing the effects of small perturbations on self-focusing. In this modulation theory the perturbed CNLS is averaged over the transverse variables, leading to a simpler system of reduced equations (proposition 4.1). The analysis of the reduced system of modulation equations is further simplified by distinguishing between conservative perturbations (perturbations under which the total power (L^2 norm) is conserved) and non-conservative perturbations (Proposition 4.2). It is interesting to note that in the conservative case the modulation equations have a universal form (Proposition 4.3). The adiabatic approach was used by Malkin to study the effect of a small defocusing fifth power nonlinearity [38]. In [20], Fibich, Malkin and Papanicolaou analyzed the effect of small normal time-dispersion, using for the first time the systematic approach followed in this paper. This approach was also used by Fibich to analyze the effect of beam nonparaxiality [19] and the unperturbed CNLS [18] and by Fibich and Papanicolaou to analyze the combined effect of time-dispersion and nonparaxiality [22].

1.4. Outline. The paper is organized as follows. In section 2 we review the analytic theory of existence and blowup for CNLS and use the lens transformation of critical NLS to derive a new condition for blowup and to relate solutions of CNLS with solutions of CNLS with an additional quadratic potential term. In section 3 we present the adiabatic approach for self-focusing in the unperturbed CNLS, derive and compare the three laws for self-focusing, derive the formula for the location of the blowup point and present a new empirical formula for Gaussian initial conditions. In section 4 we develop the modulation theory for analyzing the effect of small perturbations of CNLS self-focusing, which is summarized by propositions 4.1–4.4. In section 5 we review previous applications of this approach and apply it to several new situations (see table 1.4).

Perturbed CNLS	Application	Section
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi + \epsilon\psi_{xxxx} = 0$	fiber arrays	5.1
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi - \epsilon\|\psi\|^4\psi = 0$	quintic nonlinearity	5.2
$i\psi_z + \Delta_\perp \psi + \frac{1-\exp(-2\epsilon\|\psi\|^2)}{2\epsilon}\psi = 0$	saturating nonlinearity	5.3
$i\psi_z + \Delta_\perp \psi + \frac{\|\psi\|^2}{1+\epsilon\|\psi\|^2}\psi = 0$	saturating nonlinearity	5.3
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi - \epsilon\phi_x\psi = 0\ , \quad \alpha\phi_{xx} + \phi_{yy} = -(\|\psi\|^2)_x$	Davey-Stewartson equation	5.4
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi + \epsilon\psi_{zz} = 0$	nonparaxiality	5.5
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi + \epsilon(x^2+y^2)h(z)\psi = 0, \quad h$ random	randomness	5.6
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi + \epsilon_1 x^2 h(z)\psi + \epsilon_2\psi_{xxxx} = 0, \quad h$ random	fiber arrays + randomness	5.6
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi + \epsilon_1(x^2+y^2)h(z)\psi - \epsilon_2\|\psi\|^4\psi = 0, \quad h$ random	quintic nonlinearity+ randomness	5.6
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi - \epsilon\psi_{tt} = 0$	time-dispersion	5.7.1
$i\psi_z + \Delta_\perp \psi + N\psi = 0\ , \quad \epsilon N_t + N = \|\psi\|^2$	Debye relaxation	5.7.2
$i\psi_z + \Delta_\perp \psi + \|\psi\|^2\psi + \epsilon_1\psi_{zz} + \epsilon_2\left[2i\frac{n_0 c_g}{c}(\|\psi\|^2\psi)_t - \psi_{zt}\right] - \epsilon_3\psi_{tt} = 0$	time-dispersion + nonparaxiality	5.8

TABLE 1.1
Perturbations of critical NLS which are analyzed in this paper using modulation theory.

2. Basic theory of self-focusing for CNLS. We begin with a review of the basic theory of self-focusing. More details can be found in [11, 49, 50, 57, 58]. We emphasize the importance of the lens transformation (2.13) in the analysis of critical NLS and use it to derive new results regarding blowup.

In order to understand the special character of blowup in the cubic Schrödinger equation in two-dimensions, it is instructive to begin with the two-dimensional NLS with a general power nonlinearity

$$i\psi_z + \Delta_\perp \psi + \kappa |\psi|^{2\sigma} \psi = 0 \ , \quad \kappa = \pm 1 \ , \tag{2.1}$$

where κ positive/negative corresponds to the focusing/defocusing NLS, respectively. Two important invariants of (2.1) are the power[2]

$$N = \frac{1}{2\pi} \int |\psi|^2 \, dx \, dy \equiv N(0) \tag{2.2}$$

and the Hamiltonian

$$H = \frac{1}{2\pi} \left(\int |\nabla_\perp \psi|^2 \, dx \, dy - \frac{\kappa}{\sigma + 1} \int |\psi|^{2\sigma+2} \, dx \, dy \right) \equiv H(0) \ , \tag{2.3}$$

where

$$\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \ .$$

We say that a solution exists at z if it has a finite H^1 norm

$$||\psi(z, \cdot)||_{H^1} < \infty \ , \quad ||\psi||_{H^1} = \left(\int |\psi|^2 \, dxdy + \int |\nabla_\perp \psi|^2 \, dxdy \right)^{1/2}$$

and that ψ blows up at $z = Z_c$ if it exists for $0 \le z < Z_c$ and

$$\lim_{z \to Z_c} ||\psi(z, \cdot)||_{H^1} = \infty \quad ,$$

which by (2.2) is equivalent to blowup of the gradient norm

$$\lim_{z \to Z_c} \int |\nabla_\perp \psi|^2 \, dxdy = \infty \quad .$$

From the theory for local existence of solutions of (2.1), it is known that if $||\psi(z, \cdot)||_{H^1}$ is bounded, the solution exists for all z [24]. As a result, when NLS is defocusing ($\kappa < 0$), conservation of the Hamiltonian implies that $\int |\nabla_\perp \psi|^2 \, dxdy$ is bounded and the solution exists globally. Since we are interested with singularity formation, from now on we restrict ourselves to the case of focusing NLS $\kappa = 1$.

Because of the minus sign in the Hamiltonian of the focusing NLS, conservation of the Hamiltonian does not prohibit $\int |\nabla_\perp \psi|^2 \, dxdy$ from growing to infinity. To see this can indeed happen, we note that solutions of (2.1) satisfy the variance identity [62]

$$V_{zz} \equiv 8H - \frac{8(\sigma - 1)}{2\pi(\sigma + 1)} \int |\psi|^{2\sigma+2} dxdy \ , \quad V(\psi) = \frac{1}{2\pi} \int (x^2 + y^2) |\psi|^2 \, dxdy \ . \tag{2.4}$$

From this identity (2.4), the invariance of the Hamiltonian (2.3), and the uncertainty principle

$$N(\psi) \le V(\psi) \int |\nabla_\perp \psi|^2 \, dxdy \ ,$$

[2] In the nonlinear optics context, the L^2 norm corresponds to the power of the laser beam

it follows that for $\sigma \geq 1$ the condition

$$H(0) < 0 \tag{2.5}$$

is sufficient for blowup in a finite z.

In the supercritical case $\sigma > 1$, sharper conditions for blowup can be obtained [29] and singularity formation is characterized by dominance of self-focusing over wave diffraction, resulting in a finite z blowup which is stable under small perturbations. Conversely, for $\sigma < 1$, the subcritical case, there is no finite z blowup and the solution exists globally [57], as in the case of solitons in the cubic NLS in one transverse dimension. In the physically important case of critical self-focusing $\sigma = 1$ which we study here, wave diffraction and self-focusing are nearly balanced and blowup is extremely sensitive to perturbations and to changes in the initial condition (an intuitive argument on the importance of criticality in the balance between nonlinearity and diffraction is given at the beginning of section 4). A necessary condition for blowup in critical NLS (1.1) is that

$$N(0) \geq N_c \ , \tag{2.6}$$

where $N_c \cong 1.86$ is the critical power for self-focusing. More precisely, there is no blowup when $N < N_c$ but for any $\epsilon \geq 0$, there exist solutions with $N = N_c + \epsilon$ for which there is finite z blowup [63].

We note that if in equation (2.1) the transverse Laplacian is in D dimensions, the subcritical, critical and supercritical cases correspond to the product σD being less than, equal, or greater than 2, respectively. For this reason, the case $D = 2$ for NLS with cubic nonlinearity is called 'critical dimension'.

2.1. Waveguide solutions and the Townes soliton. From now on we restrict ourselves to the critical case $\sigma = 1$ and D=2. Critical NLS (1.1) has radially-symmetric waveguide solutions

$$\psi(z,r) = \exp(iz)R(r) \ , \quad r = \sqrt{x^2 + y^2} \ , \tag{2.7}$$

where R satisfies

$$\Delta_\perp R - R + R^3 = 0 \ , \quad R'(0) = 0 \ , \quad \lim_{r \to \infty} R(r) = 0 \ , \quad \Delta_\perp = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) \ . \tag{2.8}$$

This ordinary differential equation has an enumerable set of solutions (see [49] and its references). Of most interest is the solution with the lowest power (ground state), often called the Townes soliton. The Townes soliton is positive and monotonically decreasing (Figure 2.1). In addition, it has exactly the critical power for blowup [63]

$$\int_0^\infty R^2 \, r dr = N_c \tag{2.9}$$

and its Hamiltonian is equal to zero

$$H(R) = 0 \ . \tag{2.10}$$

Therefore, the waveguide solution (2.7), being a borderline case for blowup, is unstable.

Some additional relations, which will be used later are (lemma A.1):

$$\int_0^\infty \left(\frac{dR}{dr} \right)^2 r dr = N_c \ , \quad \int_0^\infty R^4 \, r dr = 2N_c \ , \quad \int_0^\infty R^6 \, r dr = 6N_c \ . \tag{2.11}$$

The asymptotic behavior of R is given by

$$R(r) \sim A_R r^{-1/2} \exp(-r) \qquad 1 \ll r \ , \tag{2.12}$$

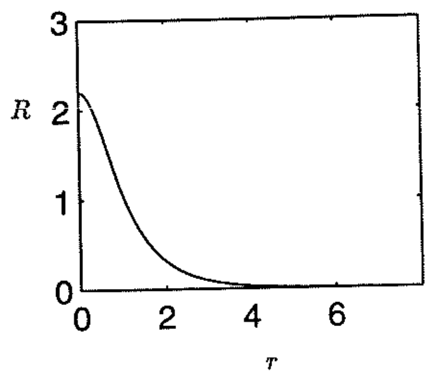

FIG. 2.1. *The Townes soliton $R(r)$.*

where

$$A_R = \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty R^3(r') I_0(r')\, r' dr' \cong 3.52$$

and I_0 is the modified Bessel function.

The Townes soliton plays an important role in CNLS theory and can be used to construct exact and approximate blowup solutions, as will be seen in the following sections. Note that although Gaussians look roughly like the Townes soliton (Figure 2.1), in the critical case there is no Gaussian that can satisfy the two conditions (2.9) and (2.10) simultaneously. Therefore, Gaussians cannot capture the delicate balance between diffraction and nonlinear focusing in critical self-focusing, which is the reason why CNLS analysis that is based on representing the solution by a modulated Gaussian is unreliable.

2.2. The lens transformation. An important tool in the analysis of critical NLS is the lens transformation: Let ψ and $\tilde{\psi}$ be related through

$$\tilde{\psi}(z,x,y) = \frac{1}{L(z)} \psi(\zeta, \xi, \eta) \exp\left(i \frac{L_z}{L} \frac{r^2}{4}\right), \quad \xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad \zeta = \int_0^z \frac{1}{L^2(z')}\, dz' . \tag{2.13}$$

Then, as noted by Talanov [59], if L depends linearly on z

$$L = 1 + \frac{z}{F}, \quad F \text{ is a constant} \tag{2.14}$$

and if ψ is a solution of (1.1) with initial condition ψ_0, then $\tilde{\psi}$ is also an exact solution of (1.1) with the initial condition

$$\tilde{\psi}_0(x,y) = \psi_0(x,y) \exp\left(i \frac{r^2}{4F}\right) . \tag{2.15}$$

The addition of a quadratic phase term to the initial condition corresponds to adding at $z = 0+$ a thin lens whose focal point is at $(z = -F, 0, 0)$. Since z and ζ are related by

$$\frac{1}{z} + \frac{1}{F} = \frac{1}{\zeta} \tag{2.16}$$

and

$$\rho := \sqrt{\xi^2 + \eta^2} = \frac{r}{L} ,$$

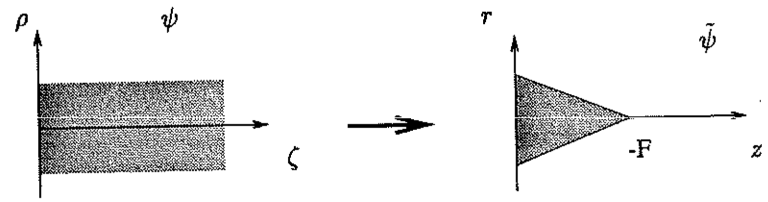

FIG. 2.2. *The lens transformation (2.13) with $L(z)$ given by (2.14) and $F < 0$ maps the values of ψ in the shaded semi-infinite strip into the corresponding values of $\tilde{\psi}$ in the shaded triangle.*

the lens transformation (2.13) shows that the effect of the lens in the diffractive case (linear, or with cubic nonlinearity) is to map the solution exactly as in ray optics (Fig. 2.2). It is interesting to note that the lens transformation is valid in the linear case in all dimensions but that the only nonlinearity for which the transformation will remain valid is the critical one.

The lens transformation can also be used to analyze CNLS with an additional quadratic potential term

$$i\tilde{\psi}_z + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{\psi} + |\tilde{\psi}|^2\tilde{\psi} + \tilde{\gamma}(z)(x^2+y^2)\tilde{\psi} = 0 . \tag{2.17}$$

In the linear case this is the basic equation of Gaussian optics, which in the non-isotropic case can be solved by the ABCD law (e.g. [67]). Let $\tilde{\psi}(z,x,y)$ be a solution of (2.17) with $\tilde{\gamma}(z)$ given, and define $\psi(\zeta,\xi,\eta)$ by the lens transformation (2.13) with a general $L(z)$, which is not necessarily linear in z as in (2.14). Then $\psi(\zeta,\xi,\eta)$ also satisfies CNLS with a quadratic potential term

$$i\psi_\zeta + \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\psi + |\psi|^2\psi + \gamma(\zeta)(\xi^2+\eta^2)\psi = 0 , \tag{2.18}$$

where

$$\gamma(\zeta) = \left(\frac{-L^3L_{zz}}{4} + L^4\tilde{\gamma}(z(\zeta))\right) , \quad z(\zeta) = \int_0^\zeta L^2(\zeta')\,d\zeta' .$$

Therefore, the family of solutions of (2.17) with a general $\tilde{\gamma}(z)$ is closed under the lens transformation with a general $L(z)$. If $L(z)$ is chosen so that it satisfies the ordinary differential equation

$$L_{zz} = 4\tilde{\gamma}(z)L$$

and ψ is a solution of CNLS (1.1), then $\tilde{\psi}(z,x,y)$ satisfies the nonlinear Gaussian optics equation (2.17). Thus eq. (2.17) can always be reduced to (eq. 1.1).

2.3. Applications of the lens transformation. By applying the lens transformation to the CNLS waveguide solution (2.7) with $L = Z_c - z$, we get that

$$\psi_{ex}(z,r) = \frac{1}{Z_c - z}R\left(\frac{r}{Z_c - z}\right)\exp\left(i\frac{1-r^2/4}{Z_c - z}\right) \tag{2.19}$$

is an exact solution of CNLS which blows-up at Z_c:

$$\lim_{z\to Z_c} ||\psi_{ex}||_{H^1} = \infty .$$

This solution has a linear blowup rate of $L = (Z_c - z)$ and a power concentration property

$$|\psi_{ex}(z,r)|^2 \to N_c\delta(r) , \qquad \text{as } z \to Z_c .$$

However, this solution is unstable [49, 63], since $N(\psi_{ex}) = N_c$ and $H(\psi_{ex}) = 0$, and was not observed in numerical experiments.

We can also use ψ_{ex} to construct exact blowup solutions of

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi + \alpha(x^2+y^2)\psi = 0 \ , \quad \alpha \text{ constant}, \tag{2.20}$$

by defining

$$\begin{aligned} \tilde{\psi}_{ex}(z,r) &= \frac{1}{L(Z_c - \zeta(z))} R\left(\frac{r}{L(Z_c-\zeta)}\right) \exp\left(i\frac{1-r^2/4L^2}{Z_c-\zeta}\right) \\ \zeta &= \int_0^z \frac{1}{L^2(z')}\,dz' \ , \quad L = L_0 Re[\exp(\pm 2\sqrt{\alpha} z)] \ . \end{aligned}$$

Note that this construction of exact blowup solutions for (2.20) holds for both positive and negative α.

In addition to (2.5), the variance identity in the critical case

$$V_{zz} = 8H(0) \tag{2.21}$$

can also be used to derive conditions for blowup when $H(0) > 0$ which involve $V(0)$ and $V_z(0)$ [63]. However, these conditions are not sharp. Even when they do hold, blowup typically occurs well before the vanishing point for the variance [33] (but see section 3.4.1). Thus, the problem of finding sharp conditions for global existence or blowup in CNLS is still open. The following proposition rules out many potential candidates.

PROPOSITION 2.1.

Let ψ be a solution of (1.1) such that $V(\psi_0) < \infty$. Any condition which involves only the absolute value of the initial condition $|\psi_0|$ cannot be sufficient for blowup.

COROLLARY 2.2. *There is no critical threshold N_{TH} such that*

$$N(\psi_0) > N_{TH}$$

is a sufficient condition for blowup.

Proof of Proposition 2.1:

Assume that there is such a condition. Let ψ be a solution of CNLS with initial condition ψ_0 that satisfies this condition and $V(\psi_0) < \infty$. Then there exists $0 < Z_c < \infty$ such that

$$\lim_{z\to Z_c} \int |\nabla_\perp \psi|^2\,dxdy = \infty \ .$$

Let $\tilde{\psi}$ be the solution of (1.1) corresponding to the initial condition (2.15) with

$$0 < F < Z_c \ . \tag{2.22}$$

Then $\tilde{\psi}$ is given by (2.13). Since $|\tilde{\psi}_0| = |\psi_0|$, there exists $0 < Z_c^* < \infty$ such that

$$\lim_{z\to Z_c^*} \int |\nabla_\perp \tilde{\psi}|^2\,dxdy = \infty \ .$$

To see that this leads to a contradiction we first note that

$$\int |\nabla_\perp \tilde{\psi}(z,x,y)|^2\,dxdy \le \frac{2}{L^2}\int |\nabla_\perp \psi(\zeta(z),x,y)|^2\,dxdy + 2L^2 V(\psi(\zeta(z))) \ .$$

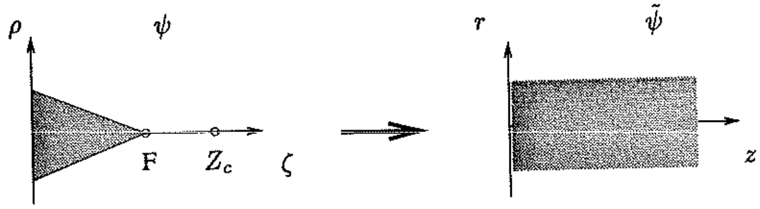

FIG. 2.3. *Sketch of proof of proposition 2.1: If ψ, a CNLS solution which blows up at Z_c, is defocused so that the point F is mapped to infinity, the defocused solution $\tilde{\psi}$ will not blowup.*

Since $F > 0$, at any finite value of z the value of L and of L^{-1} are finite. In addition, since $0 \le \zeta(z) < Z_c$, $V(\psi(\zeta(z)))$ is well defined and finite (2.4). Therefore, the right hand side can become infinite only at $\tilde{Z}_c$ such that

$$\lim_{z \to \tilde{Z}_c} \int |\nabla_\perp \psi(\zeta(z), x, y)|^2 \, dxdy = \infty \ .$$

Clearly, $0 < \tilde{Z}_c \le Z_c^*$ and $\zeta(\tilde{Z}_c) = Z_c$. From (2.16) it follows that

$$\frac{1}{\tilde{Z}_c} + \frac{1}{F} = \frac{1}{Z_c} \ , \tag{2.23}$$

which together with (2.22) implies that $\tilde{Z}_c < 0$, leading to a contradiction.

The proof shows that any initial condition, however powerful it may be, will not result in blowup if defocused at $z = 0$ by a sufficiently strong defocusing lens that maps the blowup point Z_c 'beyond infinity' (Figure 2.3). The converse, of course, is not true: If the initial power is below critical, no focusing lens can cause the solution to blowup.

2.4. Theoretical results on the nature of blowup. There is substantial numerical evidence that near the blowup point the ground state R serves as an attractor for the radial profile of the solution

$$|\psi| \sim \frac{1}{L(z)} R\left(\frac{r}{L}\right) \ , \quad z \to Z_c \tag{2.24}$$

and we will be using this assumption in the asymptotic analysis of the next section. The following theorem, due to Weinstein [65], lends partial support to (2.24):

THEOREM 2.3. *Let $\psi_0 \in H^1$ and assume that $\lim_{z \to Z_c} \int |\nabla_\perp \psi|^2 = \infty$. Then, for any sequence $z_k \to Z_c$ there is a subsequence z_{k_j} such that*

$$\frac{1}{L(z)} \psi\left(\frac{r + s(z)}{L(z)}, z\right) \exp(i\gamma(z)) \to \Psi \not\equiv 0$$

in L^p for $2 < p < \infty$, where $\gamma(z) \in [0, 2\pi)$, $s(z) \in \mathbb{R}^2$. Furthermore,

$$\int |\Psi|^2 \ge N_c \ .$$

Note that in order to make (2.24) rigorous one has to show that $\Psi \equiv R(r)$.

Relation (2.24) implies that blowup solutions of critical NLS have a unique local power concentration property

$$|\psi|^2 \sim N_c \delta(r) \ , \quad z \to Z_c \ ,$$

namely, the amount of power which goes into the singularity is always equal to the critical power for self-focusing, independent of the initial condition. Based on simulations and asymptotic arguments it is also widely believed that the rate of blowup is slightly faster than a square root

$$(Z_c - z)^{1/2+\epsilon} \ll L(z) \ll (Z_c - z)^{1/2} \qquad z \to Z_c \ , \quad \forall \epsilon > 0 \ .$$

Partial support for this can be found in the concentration theorems of Tsutsumi and Merle [42, 60] which in the radial case is as follows.

THEOREM 2.4. *Let ψ be a radially symmetric solution of (1.1) that blows up at a finite Z_c.*

1. *If $a(z)$ is a decreasing function from $[0, Z_c)$ to R^+ such that $\lim_{z \to Z_c} a(z) = 0$ and $\lim_{z \to Z_c} (Z_c - z)^{1/2}/a(z) = 0$, then*

$$\liminf_{z \to Z_c} \int_{r<a(z)} |\psi(z)|^2 \geq N_c \ .$$

2. *For any $\epsilon > 0$, there exists a $K > 0$ such that*

$$\liminf_{z \to Z_c} \int_{r<K(Z_c - z)^{1/2}} |\psi(z)|^2 \geq (1-\epsilon) N_c$$

Note that the two theorems (2.3-2.4) give only an upper bound on the amount of power that goes into the singularity ($\geq N_c$). Strictly speaking, we cannot hope to prove that the power that goes into the singularity is exactly N_c, because there are exact blowup solutions which do not satisfy it. For example, if in the focusing waveguide solution (2.19), R is any of the non ground state solutions of equation (2.8), the power going into the singularity is greater then N_c. Similarly, the concentration theorem suggests an upper bound on the blowup rate $L \ll (Z_c - z)^{1/2-\epsilon}$. However, we cannot hope to prove that for all blowup solutions $L \gg (Z_c - z)^{1/2+\epsilon}$ because the blowup rate of the focusing waveguide (2.19) is $L = Z_c - z$. Of course, these exact blowup solutions are unstable but their mere existence helps to explain the difficulty in making a completely rigorous theory for CNLS self-focusing.

The concentration theorem 2.4 illustrates the fact that blowup in critical NLS is a local phenomenon, which is why global quantities, such as N, H and the variance, cannot capture the sharp conditions for blowup. For example, if ψ_0 is composed of K well separated pulses, each of which would not blowup by itself e.g.

$$\psi_0 = \sum_{k=1}^{K} 0.8R(\sqrt{x^2 + (y - 100k)^2}) \ ,$$

then ψ will not blowup, although $N(\psi_0) > N_c$. Similarly, if

$$\psi_0 = 1.1R(\sqrt{x^2 + y^2}) + 0.8R(\sqrt{x^2 + (y-100)^2}) \ ,$$

ψ would necessarily have a finite variance when its $1.1R$ component blows-up, due to its $0.8R$ component. Nevertheless, near the singularity a 'local' variance identity does provide an accurate description of self-focusing (section 3.4.1).

3. Self-focusing in the unperturbed CNLS - an adiabatic approach. In this section we describe the local structure and dynamics of self-focusing near the blowup point. Unlike the previous section, most of the results presented in this section have not been made rigorous at present (see section 2.4).

3.1. Derivation of reduced equations - modulation theory. Self-focusing in critical dimension has the unique property that the amount of power which goes into the singularity is always equal to the

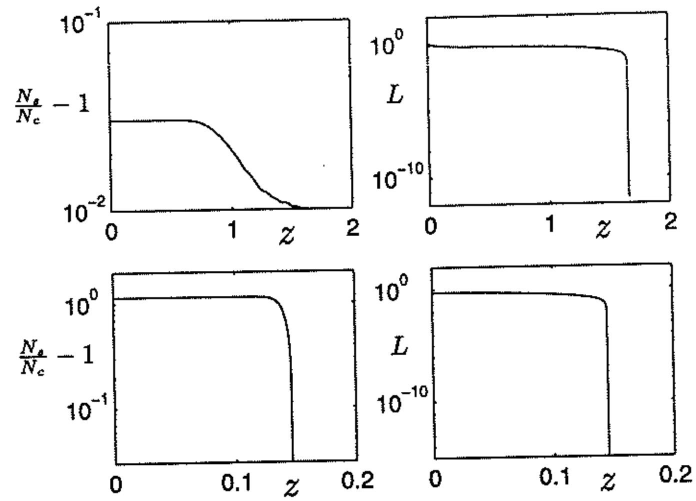

FIG. 3.1. *Most self-focusing ($L \searrow 0$) and power loss ($N_S \searrow N_c$) occur near the singularity. Initial conditions are $\psi_0 = 2.77\exp(-r^2)$ with near critical power (top), and $\psi_0 = 4\exp(-r^2)$ (bottom).*

critical power for blowup N_c. For this to happen as the total beam power is conserved (2.2), the beam separates into two components as it propagates[3]

$$\psi = \psi_s + \psi_{back} ,$$

where ψ_s is the high intensity inner core of the beam which self-focuses towards its center axis and ψ_{back} is the low intensity outer part which propagates forward following the usual linear propagation mode i.e. it diffracts and slowly diverges. This 'reorganization' stage takes place almost until the singularity in terms of the axial distance z (Figure 3.1) and is characterized by relatively slow focusing and fast power transfer from ψ_s to ψ_{back} (*non-adiabatic* self-focusing). Close enough to the singularity, ψ_s has only small excess power above the critical one and it approaches the radially symmetric asymptotic profile (Figure 3.1) [32]:

$$\psi_s(r,z) = \frac{1}{L(z)} V(\zeta, \rho) \exp\left[i\zeta + i\frac{L_z}{L}\frac{r^2}{4} \right] , \quad argV(\zeta, 0) = 0 , \tag{3.1}$$

where $L(z)$ is a yet undetermined function that is used to rescale ψ_s and the independent variables:

$$\rho = \frac{r}{L} , \quad \frac{d\zeta}{dz} = \frac{1}{L^2} . \tag{3.2}$$

Note that (3.1) can be viewed as a generalized lens transformation, in which nonlinear self-focusing is replaced by a continuum of thin lenses with a variable focal length.

Because L is a measure of the radial width of ψ_s, we can use it to give a more precise definition of

[3] If $N > 2N_c$ the beam may split into several self-focusing filaments. In this case our discussion is applicable to each filament.

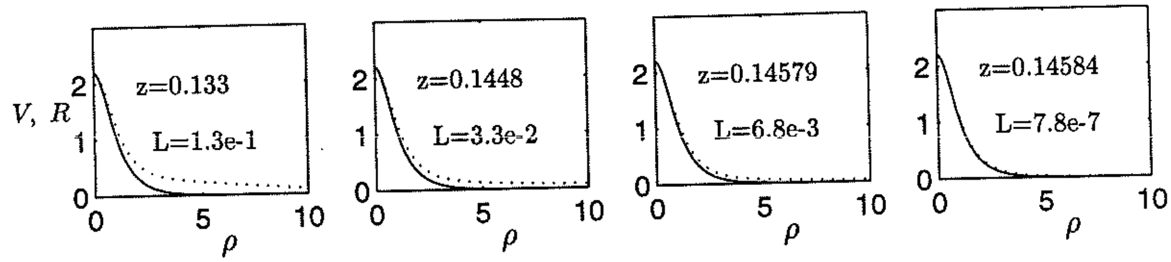

FIG. 3.2. *Convergence of the radial profile* $V(\rho) = |\psi_s(r/L)|/L$ *(dots) to the Townes soliton* R *(solid) for the initial condition* $\psi_0 = 4\exp(-r^2)$.

ψ_s and ψ_{back}. A possible definition is

$$\psi = \begin{cases} \psi_s & 0 \le r \le \rho_c L(t) \\ \psi_{back} & \rho_c L(t) \le r \end{cases} \qquad \text{with } 1 \ll \rho_c \text{ constant.} \tag{3.3}$$

This definition is based on the fact that $|\psi_s| \sim R(\rho)/L$ decays exponentially (2.12).

The resulting equation for V is

$$iV_\zeta + \Delta_\perp V - V + |V|^2 V + \frac{1}{4}\beta\rho^2 V = 0 \tag{3.4}$$

with

$$\beta(z) = -L^3 L_{zz} \ . \tag{3.5}$$

As the beam is focusing, $\beta \searrow 0$. In addition, we shall see that when $0 < \beta \ll 1$ its rate of change is exponentially small compared with that of the focusing (L). Therefore, if we expand V in an asymptotic series

$$V \sim V_0 + V_1 + \dots \ , \tag{3.6}$$

the leading order solution of (3.4) is quasi-steady i.e. $V_0 = V_0(\rho; \beta(\zeta))$. This suggests that the equation for V_0 is

$$\Delta_\perp V_0 - V_0 + |V_0|^2 V_0 + \frac{1}{4}\beta\rho^2 V_0 = 0 \ , \quad V'(0) = 0 \ , \quad V(\infty) = 0 \ . \tag{3.7}$$

However, if V_0 satisfies this real equation then $V_0 \sim \rho^{-1}\cos(\sqrt{\beta}\rho^2/4)$ for $\rho \gg \beta^{-1/2}$. Since $\sqrt{\beta} \sim -LL_z$ (H.1),

$$\psi_s \sim \frac{1}{\rho}\exp\left(i\frac{\sqrt{\beta}}{2}\rho^2\right)$$

and it is not possible to match ψ_s with ψ_{back} which has no such fast oscillations.

The difficulty is resolving the asymptotics of ψ_s for large ρ was the main reason why it took so long to determine the blowup rate of CNLS. Eventually, it was shown that in order that the leading order quasi-steady solution V_0 would have the correct behavior for large ρ, one has to add to (3.7) a term which is exponentially small in β [31, 34]:

$$\Delta_\perp V_0 - V_0 + |V_0|^2 V_0 + \frac{1}{4}\beta\rho^2 V_0 - i\frac{M}{2N_c}\nu(\beta) V_0 = 0 \ , \tag{3.8}$$

where

$$\nu(\beta) \sim \frac{2A_R^2}{M} e^{-\pi/\sqrt{\beta}} , \quad M = \frac{1}{4}\int_0^\infty r^3 R(r)\, dr \cong 0.55 .$$

The original asymptotics beyond all orders derivation of (3.8) is based on an analysis of (3.4) in the supercritical case $d > 2$. By defining $\Delta = \partial_{\rho\rho} + (d-1)/\rho\partial_\rho$ and allowing d to vary continuously, it is shown that for every $d > 2$ there is a positive limit $\lim_{\zeta\to\infty}\beta(\zeta) = \beta_*(d) > 0$. Taking the limit of $\beta_*(d)$ as $d \searrow 2$ leads to the $\nu(\beta)$ term. Parts of this derivation were later made rigorous in [27]. A clear presentation of this derivation is given in [58].

Once it is known that V_0 satisfies (3.8), we can proceed with regular perturbations and expand V_0 in an asymptotic series in β

$$V_0(\rho) \sim R(\rho) + \beta g(\rho) + O(\beta^2) , \quad g = \left.\frac{\partial V_0}{\partial \beta}\right|_{\beta=0} , \quad 0 < \beta \ll 1 . \tag{3.9}$$

The corresponding equations for R and g are (2.8) and

$$\Delta_\perp g + 3R^2 g - g = -\frac{1}{4}\rho^2 R , \quad g'(0) = 0 , \quad g(\infty) = 0 . \tag{3.10}$$

The leading order equation for V_1 follows from (3.4), (3.6), (3.8) and (3.9):

$$\Delta_\perp V_1 - V_1 + 2R^2 V_1 + R^2 V_1^* = -i\beta_\zeta g - i\frac{M}{2N_c}\nu(\beta) R . \tag{3.11}$$

The equation for the real part of V_1 is solvable, while the solvability condition for the imaginary part of V_1 is that R is perpendicular to the right-hand-side of (3.11) (lemma E.1):

$$\int_0^\infty R\left[g\beta_\zeta + \frac{M}{2N_c}\nu(\beta) R\right] \rho d\rho = 0 .$$

Using (2.9) and

$$\int_0^\infty Rg\, \rho d\rho = \frac{M}{2} \tag{3.12}$$

(lemma B.1), the solvability condition leads to the important relation

$$\beta_\zeta \sim -\nu(\beta) . \tag{3.13}$$

With this relation, the goal of reducing CNLS self-focusing to a system of equations which do not depend on the transverse variables (eqs. 3.2, 3.5, and 3.13) is achieved.

In the original derivation of (3.13) in [31, 34], this relation was written as

$$a_\zeta \sim -\frac{1}{a}\exp(-\pi/a) , \quad a = -LL_z = -\frac{L}{L_\zeta} . \tag{3.14}$$

To see that this equation agrees with (3.13) we note that $\beta = a^2 + a_\zeta$ and that $a_\zeta \ll a^2$ (3.14), so that $\beta \sim a^2$. Nevertheless, when we later extend this approach to analyze perturbed CNLS it is better to use (3.13), because with the approximation $\beta \sim a^2$ we add a constraint that $\beta > 0$, while in many cases of perturbed CNLS β becomes negative.

3.2. Adiabatic self-focusing. Malkin suggested a different way to derive (3.13) [37, 38]. Expansion of V_0 in an asymptotic series in β shows that β is related to the excess soliton power above critical (lemma B.2)

$$N_s - N_c \sim \beta M \ , \quad N_s = N(\psi_s) \ , \quad |\beta| \ll 1 \ . \tag{3.15}$$

In addition, when β is small the problem of finding the rate of power radiation of ψ_s can be formulated in analogy with the probability of penetration through a potential barrier and it can be solved using WKB (appendix C):

$$\frac{d}{d\zeta} N_s \sim -M\nu(\beta) \ . \tag{3.16}$$

If we combine (3.15)–(3.16), we again get (3.13)[4]. Thus, the small term $\nu(\beta)$ is the rate of power radiation of ψ_s. In particular, near the focal point, β is small and self-focusing is essentially *adiabatic*, that is, the beam collapses much faster than the excess power $N_s - N_c$ goes to zero.

The Hamiltonian of ψ_s can be approximated by (lemma B.2):

$$H_s := H(\psi_s) \sim \frac{M}{2}(L^2)_{zz} = M\left(L_z^2 - \frac{\beta}{L^2}\right) \ . \tag{3.17}$$

Note that relations (3.15) and (3.17), as well as adiabatic theory in general, have $O(\beta)$ accuracy, since they are based on the expansion (3.9). The rate of change of H_s is given by (appendix C):

$$\frac{d}{d\zeta} H_s \sim -\frac{M}{L^2}\nu(\beta) \ . \tag{3.18}$$

From (3.16) and (3.18) we see that as the solution approaches the blowup point,

$$\lim_{z \to Z_c} N_s = N_c \ , \quad \lim_{z \to Z_c} H_s = -\infty \ .$$

The rate at which H_s goes to infinity is given by (appendix D)

$$H_s \sim -\frac{M}{2}\frac{\nu(\beta)}{\sqrt{\beta}}\frac{1}{L^2} \ .$$

These characteristics of adiabatic self-focusing can be seen in Figure 3.2.

3.3. The loglog law. Equation (3.13) cannot be solved analytically. In order to solve it asymptotically, we rewrite it as

$$\lambda_\zeta = \frac{c_\nu}{2\pi^2}\lambda^3 \exp(-\lambda) \ , \quad \lambda = \frac{\pi}{\sqrt{\beta}} \ , \quad c_\nu = \frac{2A_R^2}{M} \ . \tag{3.19}$$

Integration by parts of $\zeta = \int_{\lambda_0}^{\lambda} \lambda'^{-1}_\zeta \, d\lambda'$ shows that

$$\zeta \sim \frac{2\pi^2}{c_\nu}\frac{\exp\lambda}{\lambda^3} \ , \quad \lambda - \lambda(0) \gg 1 \ ,$$

and

$$\lambda \sim \log\zeta \ , \quad \beta \sim \frac{\pi^2}{\log^2\zeta}, \tag{3.20}$$

[4] Recently, Pelinovsky suggested a derivation of the relation (3.13) from a multiple-scales argument which does not make use of the lens transformation [46].

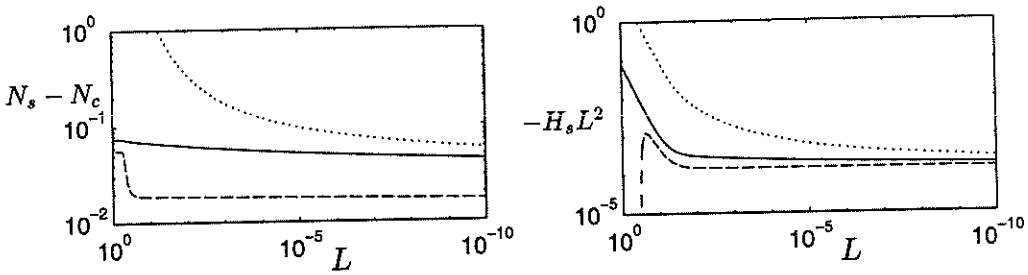

FIG. 3.3. *After an initial non-adiabatic stage with relatively fast power radiation and slow self-focusing, self-focusing become adiabatic: Changes in N_s are very slow and $H_sL^2 \sim$ constant $= o(\beta)$. Initial conditions are $\psi_0 = 1.02R(r)$ (solid line), $\psi_0 = 2.77\exp(-r^2)$ (dashed line) and $\psi_0 = 4\exp(-r^2)$ (dots).*

where now c_ν has disappeared. Using (H.1), we can rewrite (3.20) as:

$$A_{\zeta\zeta} - \frac{\pi^2}{\log^2\zeta}A \sim 0\ ,\quad A = \frac{1}{L}\ .$$

The leading order solution for this equation is

$$A \sim A_0 \exp\left(\frac{\pi\zeta}{\log\zeta}\right)\ .$$

Therefore,

$$\log\zeta \sim \log\log A \tag{3.21}$$

and

$$Z_c - z \sim \int_\zeta^\infty \frac{1}{A^2(\zeta')}\,d\zeta' \sim \frac{\log\zeta}{2\pi A^2}\ . \tag{3.22}$$

Combining (3.21) and (3.22) gives

$$\log\zeta \sim \log\log\frac{1}{Z_c - z}\ , \tag{3.23}$$

which together with (3.22) results in the loglog law

$$L \sim \left(\frac{2\pi(Z_c - z)}{\ln\ln 1/(Z_c - z)}\right)^{\frac{1}{2}}\ . \tag{3.24}$$

Although mathematically correct, it turns out that the loglog law becomes applicable only for huge and non-physical amplifications. This is because (3.20) becomes the leading order solution of (3.19) only at huge focusing factors. To see why this is true, we note that (3.20) holds when $\lambda - \lambda_0 \gg 1$. However, from (3.19) and $\lambda - \lambda_0 < \zeta\lambda_\zeta(0)$ we see that a necessary, but clearly not sufficient, condition for the loglog law to hold is that

$$\zeta \gg \frac{\beta_0^{3/2}}{\nu(\beta_0)}\ . \tag{3.25}$$

3.4. Adiabatic analysis of modulation equations. In order to derive an asymptotic law for critical self-focusing which is valid in the domain of physical interest, we note that equations (3.5) and (3.13) which govern self-focusing evolve on very different length scales

$$L_{zz} = -\frac{\beta}{L^3} \qquad \text{small scale} \tag{3.26}$$

$$\beta_z = -\frac{\nu(\beta)}{L^2} \qquad \text{large scale.} \tag{3.27}$$

The loglog law is derived by solving (3.27) to leading order and then using (3.26). However, since the length scale for power changes in (3.27) is exponentially long compared with the one for changes in the focusing rate in (3.26), we should do just the opposite: First integrate equation (3.26) while ignoring the slow changes in β (strictly adiabatic self-focusing) and only then use (3.27) in order to get the next order correction [18]. Therefore, *strictly adiabatic* self-focusing is given by:

$$L_{zz} = -\frac{\beta}{L^3}\ , \quad \beta \equiv \beta_0 := \beta(0)\ . \tag{3.28}$$

If we multiply equation (3.28) by $2L_zL^{-3}$, integrate and use (3.17), we get

$$L_z^2 = \frac{\beta}{L^2} + \frac{H_s}{M}\ , \quad H_s \equiv H_s(0)\ . \tag{3.29}$$

Comparison of (3.29) with (3.17) shows that strict power adiabaticity implies that H_s is also constant (both amount to setting $\nu(\beta) \equiv 0$). Multiplying (3.29) by L^2 gives

$$(L^2)_z = \pm 2\left(\beta + \frac{H_s}{M}L^2\right)^{1/2}\ ,$$

where the plus/minus sign correspond to the cases of defocusing/focusing at $z = 0$, respectively. Integrating one more time and using the initial condition $L(0) = L_0$ gives the adiabatic law of Fibich[5] [18]:

$$L^2(z) \sim L_0^2 \pm \left(\beta + \frac{H_sL_0^2}{M}\right)^{1/2} z + \frac{H_s}{M}z^2. \tag{3.30}$$

Note that equation (3.30) can also be derived from the adiabaticity of H_s and (3.17):

$$\frac{M}{2}(L^2)_{zz} \equiv H_s(0)\ . \tag{3.31}$$

If we set $L(Z_c) = 0$ in (3.30) we get a quadratic equation for the blowup point Z_c whose smaller positive solution is [18]:

$$Z_c \sim \begin{cases} \dfrac{L_0^2}{\sqrt{\beta} + \sqrt{\beta + H_sL_0^2/M}} & L_z(0) \leq 0 \\[2ex] \dfrac{L_0^2}{\sqrt{\beta} - \sqrt{\beta + H_sL_0^2/M}} & L_z(0) > 0 \text{ and } H_s < 0 \\[2ex] \text{no blowup} & L_z(0) > 0 \text{ and } H_s > 0\ . \end{cases} \tag{3.32}$$

[5] In [18] there is a typo in this formula.

It is instructive to compare (3.32) with the necessary and sufficient conditions for blowup (2.5–2.6). Equation (3.32) shows that the condition $\beta > 0$ (i.e. power above critical) is necessary for blowup. This condition is also sufficient when $L_z(0) \leq 0$. However, if the beam is initially defocusing, the necessary and sufficient condition for blowup is $H_s < 0$.

The expression (3.32) for Z_c inherits the lens transformation property (2.23). To see this, let us consider the case of a collimated beam (ψ_0 real). In this case $L_z(0) = 0$ and from (3.17)

$$H_s \sim -\frac{M\beta}{L_0^2} \quad . \tag{3.33}$$

Therefore, the strict adiabatic law for ψ_0 real is

$$L \sim L_0\sqrt{1 - \frac{z^2}{Z_c^2}} \ , \quad Z_c = \frac{L_0^2}{\sqrt{\beta_0}} \ . \tag{3.34}$$

If we add a lens with focal length F at $z = 0$ the initial condition becomes (2.15). Since this change does not affect the beam radius and power at $z = 0+$, $\tilde{L}_0 = L_0$ and $\tilde{\beta}_0 \sim \beta_0$ (the tildes denote the corresponding parameters for $\tilde{\psi}$). However, the Hamiltonian is changing according to (2.3)

$$\tilde{H}_s(0) \sim H_s(0) + M\frac{L_0^2}{F^2} \ .$$

Therefore, from (3.32) we see that the blowup point for $\tilde{\psi}$ is at

$$\tilde{Z}_c = \frac{L_0^2}{\sqrt{\beta_0} + L_0^2/F} \ ,$$

which is related to Z_c by

$$\frac{1}{\tilde{Z}_c} = \frac{1}{Z_c} + \frac{1}{F} \ ,$$

showing that the adiabatic law (3.30) preserves the lens transformation property of CNLS.

3.4.1. The adiabatic law and the variance identity. It has already been noted that the variance identity (2.21) does not capture the local nature of CNLS blowup, which typically occurs when the variance is still positive. However, if we apply the variance identity only to ψ_s, it turns out to be equivalent to the adiabatic law (3.30). To see this, we note that the variance of ψ_s is given by

$$V(\psi_s) \sim 4ML^2 \ .$$

Therefore, within the $O(\beta)$ accuracy of adiabatic theory, the variance identity (2.21) is equivalent to (3.31), which leads to the strictly adiabatic law (3.30). This suggests that a local variant of the variance identity may lead to sharper theoretical results.

3.5. Non-adiabatic effects. Self-focusing, as given by (3.28) or by (3.34), is strictly adiabatic i.e. radiation losses are completely neglected. Therefore, if we are interested in maintaining the $O(\beta)$ accuracy of the adiabatic law up to the blowup point, the slow scale changes in β and H_s must be included. This can be done by solving the fast equation (3.26) coupled with the slow equation (3.27), as in Fig. 3.5A. When comparing the adiabatic laws with numerical simulations of CNLS (Figures 3.5–3.6), the varying values of β and H_s can also be obtained from (3.15) and (3.17), respectively.

It may seem that we can get a more accurate asymptotic law than the strictly adiabatic one if we replace (3.28) with its Euler approximation

$$L_{zz} = -\frac{\beta}{L^3} \ , \quad \beta = \beta(0) - \zeta\nu(\beta(0)) \ , \quad \frac{d\zeta}{dz} = \frac{1}{L^2} \ . \tag{3.35}$$

However, this approximation is better than (3.28) only during the initial stage of self-focusing and eventually becomes worse than the strict adiabatic approximation (Fig. 3.5B).

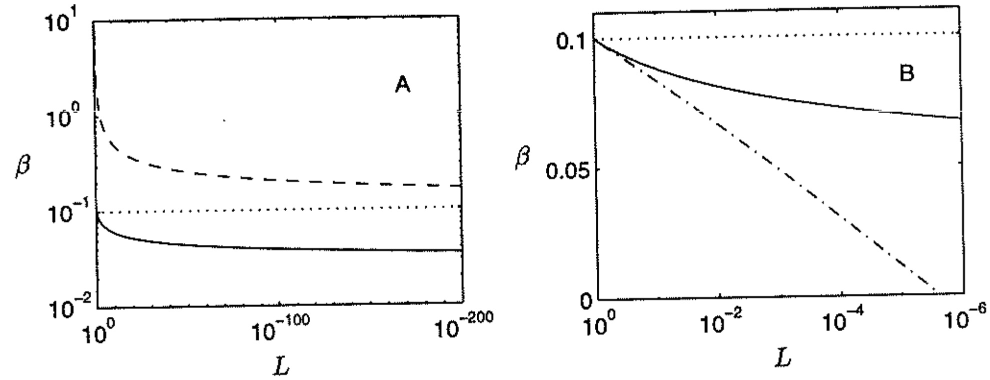

FIG. 3.4. *A: Strict adiabaticity (3.28; dotted line) is a better approximation to the evolution of β according to the reduced system (3.26–3.27; solid line) than the asymptotic approximation (3.20; dash-dot line) that leads to the loglog law, even after amplification by 200 orders of magnitude. B: Except for the initial stage of self-focusing, strict adiabaticity (dotted line) is a better approximation for the evolution of β (3.26–3.27; solid line) than the Euler 'improvement' (3.35; dash-dot line). In all cases $\beta(0) = 0.1$, $L(0) = 1$.*

3.6. Comparison of Fibich's adiabatic law, Malkin's adiabatic law and the loglog law. The adiabatic law (3.30) can be rewritten in the form

$$L(z) = \sqrt{2\sqrt{\beta}\,(Z_c - z) + \frac{H_s}{M}\,(Z_c - z)^2}\,. \tag{3.36}$$

As z approaches the singularity point, the quadratic term becomes negligible (see appendix D) and (3.36) reduces to Malkin's adiabatic law [38]:

$$L(z) = \sqrt{2\sqrt{\beta}(Z_c - z)}\,. \tag{3.37}$$

Thus, (3.36) and (3.37) agree asymptotically but (3.36) is valid earlier, since in addition to the beam power it also incorporates the focusing angle. Similarly, the asymptotic limit of (3.37) agrees with the loglog law. To see this, note that if in the derivation of the loglog law we use (3.20) instead of (3.23) in (3.22), we get (3.37). Therefore, the three laws are asymptotically equivalent; only their domains of validity differ.

In Figure 3.5 we compare the value of L from numerical simulations of CNLS (solved by the method of dynamic rescaling, see section 6) with the predictions of the three asymptotic laws. The initial conditions used are $\psi_0 = 1.02R(r)$ (power slightly above critical and close to the asymptotic profile), $\psi_0 = 2.77\exp(-r^2)$ (power slightly above critical but profile not close to the asymptotic one) and $\psi_0 = 4\exp(-r^2)$ (large excess power above critical). The values of β and H_s were obtained from (3.15) and (3.17), respectively. In all three cases, the adiabatic laws become $O(\beta)$ accurate early on and maintain this accuracy, while the loglog law is not valid even after focusing by more than ten orders of magnitude. The advantage of Fibich's law over Malkin's law during the initial stage can be seen in Figure 3.5A where the initial condition is close to the asymptotic one. In Figure 3.5B–C the initial conditions are not close to the asymptotic profile and the two adiabatic laws take longer to become valid, at which point they are already in the domain where they agree.

In order to understand why the adiabatic laws become valid quite early and the loglog law does not, we take a closer look at the point where their derivations become different. For the adiabatic laws to be applicable, β should be moderately small so that $\nu(\beta) \ll 1$. In contrast to this, a necessary condition for the loglog law to be valid is (3.25). To estimate the corresponding beam width, we apply the adiabatic

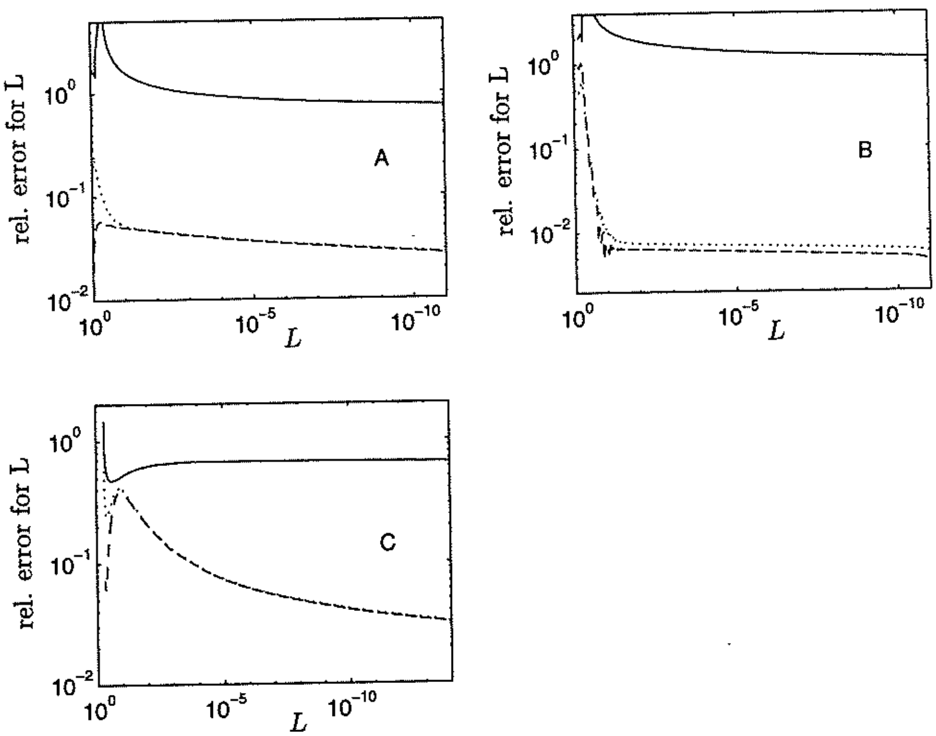

FIG. 3.5. *The relative error in the prediction for L of the adiabatic laws of Fibich (eq. 3.36; dash-dot line) and Malkin (eq. 3.37; dotted line) and of the loglog law (eq. 3.24; solid line) for the initial conditions A: $\psi_0 = 1.02R(r)$ B: $\psi_0 = 2.77\exp(-r^2)$ C: $\psi_0 = 4\exp(-r^2)$.*

approximation to $L_\zeta/L \sim -\sqrt{\beta}$ to get $L \sim \exp(-\sqrt{\beta}\zeta)$. Therefore, a necessary condition for the loglog law to hold is

$$L \ll \exp\left(-\frac{\beta_0^2}{\nu(\beta_0)}\right)$$

which shows that for $\beta(0) = 0.1$ the loglog law is not valid even when $L \sim 10^{-90}$. Indeed, in figure 3.5A it can be seen that when $\beta(0) = 0.1$, the approximation (3.20) which is used to derive the loglog law does not become valid even after amplification by 200 orders of magnitude.

As noted above, the adiabatic laws of Fibich and Malkin agree asymptotically but differ during the initial stage of focusing. Thus, if we derive the prediction for Z_c from (3.37) we would get

$$Z_c = \frac{L_0^2}{2\sqrt{\beta_0}} \,. \tag{3.38}$$

This result does not satisfy the lens relation (2.23), since it is independent of the initial focusing angle. In particular, for real initial conditions ($L_z = 0$) this prediction is off by a factor of 2 compared with (3.32).

In Figure 3.6 we compare the dynamic predictions for the distance from the singularity of (3.32)

$$Z_c - z = \frac{L^2(z)}{\sqrt{\beta(z)} + \sqrt{\beta + H_s(z)L^2/M}} \tag{3.39}$$

and of (3.38)

$$Z_c - z = \frac{L^2(z)}{2\sqrt{\beta(z)}} \,. \tag{3.40}$$

In the adiabatic regime both predictions have $O(\beta)$ relative accuracy. The advantage of (3.39) during the initial stages is again seen for the initial condition $1.02R$ (Figure 3.6A). In addition, only (3.39) will maintain the same relative accuracy for all z if we add a focusing quadratic phase factor to the initial condition.

3.7. Location of the singularity. Eq. (3.34) for the location of the singularity was derived under the assumptions that ψ_s is close to the asymptotic form (3.1) and that the excess power above critical is small ($\beta \ll 1$). However, it is desirable to have the value of Z_c for any given power, focusing angle and radial distribution. To do that, we extrapolate eq. (3.34) outside its stated domain of validity by estimating the value of β from (3.15):

$$\beta \sim \frac{N_c}{M}(p-1) \,, \quad p = \frac{N}{N_c}$$

even when β is not small. The value of L_0 is determined by looking for the modulated Townes soliton which best approximates the initial radial distribution:

$$|\psi_0(r)| \sim R_{L_0} \,, \quad R_{L_0} = \frac{1}{L_0} R\left(\frac{r}{L_0}\right) \,.$$

One possibility for matching is to set $\psi_0(0) = R_{L_0}(0)$. However, our simulations suggest that better results are obtained if matching is done by setting $\int |\nabla_\perp \psi_0|^2 = \int |\nabla_\perp R_{L_0}|^2$. From this condition and (2.11) we get

$$L_0 = N_c^{1/2} \left(\int_0^\infty |\nabla_\perp \psi_0|^2 \, r dr\right)^{-1/2} \tag{3.41}$$

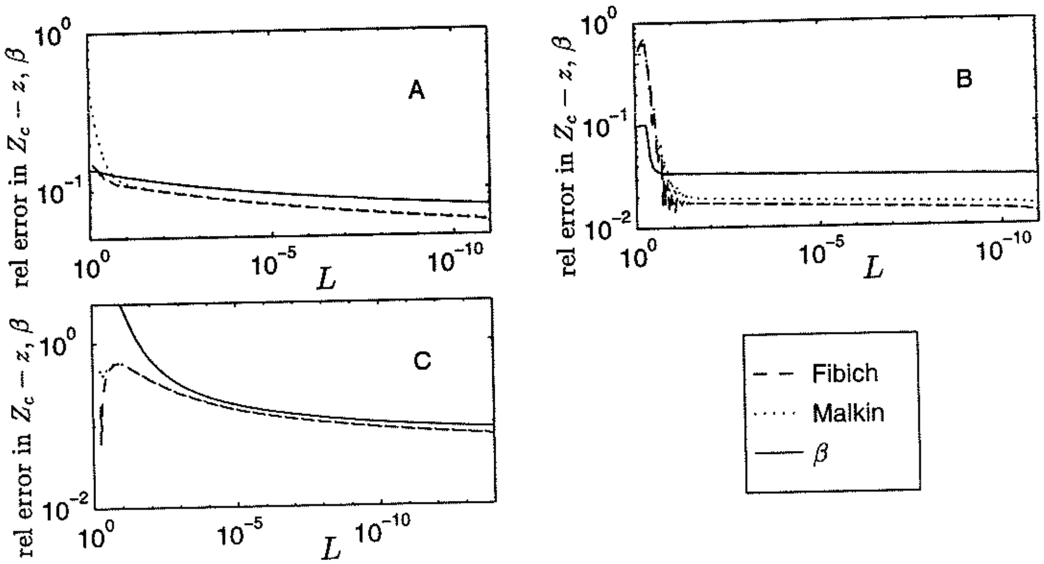

FIG. 3.6. *The relative accuracy of the dynamic prediction for the location of the singularity ($Z_c - z$) based on Fibich's adiabatic law (3.39) and on Malkin's adiabatic law (3.40) in the adiabatic regime is $O(\beta)$. Initial conditions are as in Figure 3.5.*

and

$$Z_c \sim \sqrt{\frac{MN_c}{p-1}} \left(\int_0^\infty |\nabla_\perp \psi_0|^2 \, r dr \right)^{-1} . \tag{3.42}$$

Note that equation (3.42) takes into account beam power, radial distribution and initial focusing angle. The validity of equation (3.42) for various initial conditions is shown in [18].

3.7.1. Gaussian initial conditions. The only available formula for the location of the singularity, with reasonable accuracy, is that of Dawes and Marburger [15]

$$Z_c = 0.367[(p^{1/2} - 0.852)^2 - 0.0219]^{-1/2} . \tag{3.43}$$

This formula was derived for the special case of Gaussian initial conditions $\psi_0 = c\exp(-r^2)$ by curve fitting values of Z_c obtained from simulations. For comparison, in the case of Gaussian initial conditions the theoretical formula (3.42) becomes

$$Z_c \sim \frac{1}{2}\sqrt{\frac{M}{N_c}}\sqrt{\frac{1}{p(p-1)}} . \tag{3.44}$$

Both (3.43) and (3.44) have a relative accuracy of around 10% in the range $1.05 \le p \le 2$ (Figure 3.7). Based on our numerical simulations we suggest a new empirical formula for Gaussian initial conditions

$$Z_c = 0.1585 * (p-1)^{-0.6346} \tag{3.45}$$

which has a relative accuracy of 1% in this range. (Figure 3.7).

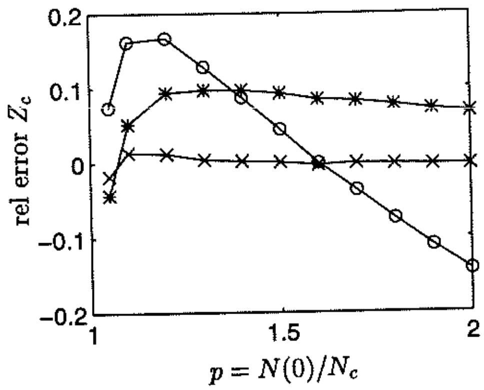

FIG. 3.7. *The relative accuracy of the predictions for the location of the singularity for Gaussian initial conditions of the formula of Dawes and Marburger (eq. 3.43, '*') and of the theoretical adiabatic formula (eq. 3.44, 'o') is around 10%. The new empirical formula (eq. 3.45, 'x') has a 1% relative accuracy.*

4. Modulation theory for self-focusing in the perturbed CNLS. In the previous sections we saw that self-focusing in critical NLS is controlled by the delicate balance between the focusing nonlinearity and defocusing Laplacian. As a result, if a small perturbation is added to CNLS it will have a large effect on self-focusing as soon as it becomes comparable to $(\Delta_\perp\psi + |\psi|^2\psi)$, even though it is small compared with each of these terms separately. This property is unique to critical focusing, which is the borderline case between subcritical self-focusing where diffraction dominates and supercritical self-focusing where nonlinear focusing dominates. Indeed, if the solution of the focusing NLS (2.1) is self-similar i.e. $\psi \sim V(r/L)/L$, then $\Delta_\perp \sim L^{-3}$ and $|\psi|^{2\sigma}\psi \sim L^{-1-2\sigma}$. Therefore, only when $\sigma = 1$ nonlinearity and diffraction can remain of the same order as $L \searrow 0$. In fact diffraction and critical nonlinearity exactly balance each other in the special case of the waveguide solution (2.19), where $V = R$ and $\beta \equiv 0$. Therefore, in critical self-focusing, given by (3.1) with $V \sim R$ and $0 < \beta \ll 1$, diffraction and critical nonlinearity almost completely balance.

4.1. Modulation theory. We have seen that the adiabatic approach is very effective in the analysis of self-focusing in CNLS. In this section we extend this approach to a modulation theory for analyzing the effects of various small perturbations on self-focusing. We consider a general perturbed critical NLS of the form:

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon F(\psi, \psi_z, \nabla_\perp\psi, \psi_t, \ldots) = 0\,, \quad |\epsilon| \ll 1\,, \tag{4.1}$$

where F is an even function in x and y. Using modulation theory, the perturbed CNLS (4.1) is replaced with a system of reduced equations which is much simpler for analysis and simulations because it is independent of the transverse variables. For example, in section 5 we apply the modulation method to the perturbations of CNLS listed in table 1.4.

Modulation theory is valid when the following three conditions hold:

Condition 1 The focusing part of the solution is close to the asymptotic profile (3.1)–(3.2)

$$\psi_s(z,x,y,\cdot) \sim \frac{1}{L(z,\cdot)} V(\zeta,\xi,\eta,\cdot) \exp\left[i\zeta(z,\cdot) + i\frac{L_z}{L}\frac{r^2}{4}\right]\,, \tag{4.2}$$

where

$$\xi = \frac{x}{L}\,, \quad \eta = \frac{y}{L}\,, \quad \zeta_z = \frac{1}{L^2}$$

and $V = R + O(\beta, \epsilon)$.

Condition 2 The power is close to critical

$$\left|\frac{1}{2\pi}\int |\psi_s(z,x,y,\cdot)|^2\, dxdy - N_c\right| \ll 1 ,$$

or equivalently,

$$|\beta(z,\cdot)| \ll 1 .$$

Condition 3 The perturbation ϵF is small compared with the other terms in equation (4.1):

$$|\epsilon F| \ll |\Delta_\perp \psi| , \quad |\epsilon F| \ll |\psi|^3 .$$

The dots in the arguments of ψ and of the modulation parameters indicate that they may depend on additional variables, such as t in the case of time-dispersion.

In general, at the onset of self-focusing only condition 3 holds. Therefore, if the power is above critical the solution will initially self-focus as in the unperturbed CNLS. As a result, near the location of the blowup point in the absence of the perturbation, conditions 1–2 will also be satisfied. It is only at this stage that the Laplacian and the nonlinearity almost completely balance each other, so that the small perturbation can have a significant effect. Therefore, one can identify at least three stages in the evolution of self-focusing in the perturbed CNLS:

Non-adiabatic self-focusing. Self-focusing is as in the non-adiabatic stage of the unperturbed CNLS. Only condition 1 holds.

Unperturbed adiabatic self-focusing. Self-focusing is as in the adiabatic stage of the unperturbed CNLS. Conditions 1–3 hold.

Perturbed adiabatic self-focusing. The perturbation is small but has a significant effect. Conditions 1–3 hold.

Note that conditions 1–3 hold in the second and third stages, both of which are therefore covered by modulation theory. In some cases (e.g. nonparaxiality, saturating nonlinearity) one can show that the reduced system remains valid for all z by showing that all three conditions remain satisfied in the reduced system. However, in other cases (e.g. small normal time-dispersion) it is unclear for how long modulation theory remains valid and self-focusing may enter a new stage which is not covered by modulation theory.

The main result of modulation theory is the following proposition.

PROPOSITION 4.1. *If conditions 1-3 hold, self-focusing in the perturbed CNLS (4.1) is given to leading order by the reduced system*

$$\beta_z + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_z - \frac{2\epsilon}{M} f_2 , \quad L_{zz} = -\frac{\beta}{L^3} . \tag{4.3}$$

The auxiliary functions f_1 and f_2 are given by

$$f_1(z,\cdot) = 2L(z,\cdot) Re\left[\frac{1}{2\pi}\int F(\psi_R)\exp(-iS)[R(\rho)+\rho\nabla_\perp R(\rho)]\, dxdy\right] \tag{4.4}$$

$$f_2(z,\cdot) = Im\left[\frac{1}{2\pi}\int \psi_R^* F(\psi_R)\, dxdy\right] \tag{4.5}$$

where

$$\psi_R = \frac{1}{L}R(\rho)\exp(iS) , \quad S = \zeta(z,\cdot) + \frac{L_z}{L}\frac{r^2}{4} , \quad \frac{\partial \zeta}{\partial z} = \frac{1}{L^2} .$$

We note that:

- Assuming that we can carry out the transverse integration, f_1 and f_2 are known functions of the modulation variables L, β, ζ and their derivatives.
- The reduced system (4.3) is much easier for analysis and simulations than (4.1) because it does not depend on the transverse variables (x,y).

A proof of proposition 4.1 is postponed until section 4.2.

4.1.1. Conservative and non-conservative perturbations. Considerable simplification is achieved if we distinguish between conservative perturbations i.e. those for which the power remains conserved in (4.1)

$$\frac{d}{dz}\int |\psi(z,x,y,\cdot)|^2\, dxdy \equiv 0$$

and non-conservative perturbations.

PROPOSITION 4.2. *Let conditions 1–3 hold.*

1. If F is a conservative perturbation i.e.

$$Im \int \psi^* F(\psi)\, dxdy \equiv 0\ ,$$

then $f_2 \equiv 0$, and to leading order (4.3) reduces to

$$-L^3 L_{zz} = \beta_0 + \frac{\epsilon}{2M} f_1\ , \quad \beta_0 = \beta(0,\cdot) - \frac{\epsilon}{2M} f_1(0,\cdot)\ , \tag{4.6}$$

where β_0 is independent of z.

2. If F is a non-conservative perturbation i.e.

$$Im \int \psi^* F(\psi)\, dxdy \not\equiv 0$$

then to leading order (4.3) reduces to

$$\beta_z = -\frac{2\epsilon}{M} f_2\ , \quad L_{zz} = -\frac{\beta}{L^3}\ . \tag{4.7}$$

Note that in both cases, non-adiabatic effects disappear from the leading order behavior of (4.3).

4.1.2. Universality of the effect of conservative perturbations. As we shall see in section 5, for various conservative perturbations f_1 turns out to have the universal form

$$f_1 \sim -\frac{C_1}{L^2}\ , \ C_1 = \text{constant}\ . \tag{4.8}$$

The following two propositions cover this canonical case. The first deals only with adiabatic effects and the second (4.4) deals with non-adiabatic effects when the conservative perturbation results in oscillatory focusing-defocusing behavior.

PROPOSITION 4.3. *When self-focusing is given by (4.6) and f_1 is given by (4.8) then y satisfies the universal oscillator equation*

$$(y_z)^2 = 4\beta_0 - \frac{\epsilon C_1}{M}\frac{1}{y} + \frac{4H_0}{M} y\ , \quad y = L^2\ , \tag{4.9}$$

or equivalently

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m)\ , \tag{4.10}$$

where

$$y_M = \frac{\sqrt{\beta_0^2 + \epsilon C_1 H_0/M^2} + \beta_0}{-2H_0/M} = \frac{M\beta_0}{-H_0}\left[1 + O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right] \tag{4.11}$$

$$y_m = \frac{\epsilon C_1}{2M}\frac{1}{\sqrt{\beta_0^2 + \epsilon C_1 H_0/M^2} + \beta_0} = \frac{\epsilon C_1}{4M\beta_0}\left[1 + O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right]\ , \tag{4.12}$$

$$\beta_0 = \beta(0) + \frac{\epsilon C_1}{2ML^2(0)}\ , \quad H_0 \sim H(0) + \frac{\epsilon C_1}{4}\frac{1}{L^4(0)}\ .$$

Let us define

$$L_m := y_m^{1/2} \ , \quad L_M := y_M^{1/2} \ .$$

1. *If the perturbation is defocusing i.e.*

   (4.13) $$\epsilon C_1 > 0 \ ,$$

   then it will arrest blow-up in (4.6), i.e. L remains positive for all z.

   (a) *If in addition to (4.13),* $\beta_0 > 0$ *and* $H_0 < 0$*, then*

   $$0 < L_m < L_M$$

   and L goes through periodic oscillations between L_m *and* L_M *(Figure 4.1A). The period of the oscillations is*

   (4.14) $$\Delta Z = 2\sqrt{\frac{M y_M}{-H_0}}\, E\left(1 - \frac{y_m}{y_M}\right) \ ,$$

   where $E(m) = \int_0^{\pi/2}(1 - m\sin^2\theta)^{1/2}\, d\theta$ *is the complete elliptic integral of the second kind [2].*

   (b) *If in addition to (4.13),* $\beta_0 > 0$ *and* $H_0 > 0$ *, then*
      i. *If* $L_z(0) < 0$*, self-focusing is arrested when* $L = L_m > 0$*, after which L is monotonically defocusing to infinity (Figure 4.1B).*
      ii. *If* $L_z(0) > 0$*, L is monotonically defocusing to infinity.*

2. *If the perturbation is focusing i.e.*

   $$\epsilon C_1 < 0$$

   and if in addition $\beta_0 > 0$ *and one of the following two conditions holds (1)* $H_0 > 0$ *and* $L_z(0) < 0$ *or (2)* $H_0 < 0$*, then the solution of (4.6) will blow up in a finite distance (Figure 4.1C), i.e.*

   $$\exists Z_* \quad \text{such that} \quad 0 < Z_* < \infty \quad \text{and} \quad L(Z_*) = 0 \ .$$

3. *The location of the (first) arrest in (1a) and (1(b)i) is almost the same as that of the singularity in the unperturbed case with the same initial conditions:*

   $$z_0 = \int_{y(0)}^{y_m} z_y\, dy \sim Z_c \ , \qquad Z_c \text{ given by (3.32)} \ .$$

   In particular, if ψ_0 *is real, then*

   $$z_0 = \frac{1}{2}\Delta Z = Z_c\left(1 + O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right) \quad Z_c \text{ given by (3.34)} \ .$$

The proof of proposition 4.3 is given in appendix G.

4.1.3. Non-adiabatic effects. Proposition 4.2 shows that the exponentially small term $\nu(\beta)$, which plays such an important role in CNLS self-focusing, disappears from the leading order behavior of perturbed CNLS. In the non-conservative case the effect of $\nu(\beta)$ is even smaller than the $(f_1)_z$ term which is also ignored. However, in the conservative case when $\beta_0 > 0$ and $H_0 < 0$, the perturbation leads to periodic oscillations (proposition 4.3-1a) and $\nu(\beta)$ provides the only mechanism for the decay of the oscillations. In order to account for these non-adiabatic effects, we must use the equations

(4.15) $$\beta_z + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_z \ , \quad L_{zz} = -\frac{\beta}{L^3} \ .$$

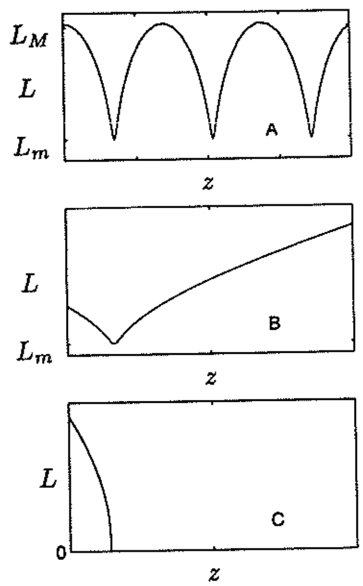

FIG. 4.1. *The leading order effect of the generic conservative perturbation (4.8) A: Defocusing perturbation and $H_0 < 0$ (proposition 4.3-1a) B: Defocusing perturbation, $H_0 > 0$ and $L_z(0) < 0$ (proposition 4.3-1(b)i) C: Focusing perturbation and $L_z(0) < 0$ (proposition 4.3-2). In all cases $\beta_0 > 0$ (i.e. power above critical).*

If in (4.15) the power loss during one oscillation is small, the oscillations are slowly decreasing and the effect of $\nu(\beta)$ can be lumped into the change in N_s over one period

$$\Delta N_s := N_s(z + \Delta Z) - N_s(z) \; .$$

In this case, the following proposition gives an estimate for ΔN_s (a detailed analysis of non-adiabatic effects is given in [38]).

PROPOSITION 4.4. *If non-adiabatic effects are included in proposition 4.3:1a and if $|\Delta N_s| \ll N_s - N_c$, the oscillations are slowly decreasing and after each cycle there is an overall power loss to radiation of*

$$\Delta N_s \sim -M \int_z^{z+\Delta Z} \frac{\nu(\beta)}{y} \, dz \quad .$$

If $y_m \ll y_M$, most radiation occurs when $y \sim y_M$ and [19]

$$\Delta N_s \sim -M\nu(\beta_M)\beta_M^{-1/4} \left(\frac{y_M}{y_m} - 1 \right)^{1/2} \quad , \tag{4.16}$$

where

$$\beta_M := \beta(y_M) \sim \frac{N_s - N_c}{M} \; .$$

The value of ΔN_s is approximated in (4.16) using Laplace's method for integrals (e.g. [9, 43]). Note that non-adiabatic radiation effects lead to slowly decaying focusing-defocusing oscillations only when the change in N_s over one oscillation is small compared with the excess power above critical ($|\Delta N_s| \ll N_s - N_c$). From proposition 4.4 we see that this holds when ϵ is moderately small, but not for a very small ϵ since as $\epsilon \to 0$, $\Delta N_s \sim \epsilon^{-1/2}$.

4.1.4. Modulation theory for multiple perturbations. In some cases, one is interested in the combined effect of several small perturbations e.g. randomness and quintic nonlinearity (section 5.6) or time-dispersion and nonparaxiality (section 5.8). Modulation theory can easily handle these cases, since the modulation equations are linear in F. Therefore, one simply adds the contribution of each perturbation to the modulation equations.

4.2. Proof of proposition 4.1. In this section we derive the reduced equation (4.3) of proposition 4.1. This derivation generalizes the one for CNLS (section 3.1).

4.2.1. Perturbation analysis. When condition 1 is satisfied, the focusing part of the solution is described by (4.2) and the corresponding equation for V is

$$iV_\zeta + \Delta_\perp V - V + |V|^2 V + \frac{1}{4}\beta\rho^2 V + \epsilon L^3 F\left(\frac{V(\rho,\zeta)}{L(z)}\exp(iS)\right)\exp(-iS) = 0\ . \tag{4.17}$$

As in the case of CNLS, we expand V asymptotically for β and ϵ small

$$V \sim V_0^\epsilon + V_1^\epsilon + \cdots \tag{4.18}$$

where V_0^ϵ is quasi-steady in z, satisfying

$$\Delta_\perp V_0^\epsilon - V_0^\epsilon + |V_0^\epsilon|^2 V_0^\epsilon + \frac{1}{4}\beta\rho^2 V_0^\epsilon - i\frac{M}{2N_c}\nu^\epsilon(\beta)V_0^\epsilon + \epsilon w(V_0^\epsilon) = 0\ , \tag{4.19}$$

$$w(V_0^\epsilon) := L^3 Re\left[F\left(\frac{V_0^\epsilon(\rho)}{L(z)}\exp(iS)\right)\exp(-iS)\right]\ .$$

When $\epsilon = 0$ this is equation (3.8), which determines V_0 and $\nu(\beta)$. We have now added the *dispersive* part of the perturbation and we assume that there is a perturbed pair V_0^ϵ and $\nu^\epsilon \sim \nu$ that satisfies (4.19). If we expand V_0^ϵ in the two small parameters ϵ and β we have

$$V_0^\epsilon \sim R(\rho) + \beta g(\rho) + \epsilon h(\zeta,\xi,\eta) + o(\beta,\epsilon)\ . \tag{4.20}$$

The equations for R and g are (2.8) and (3.10) and the equation for h is

$$\Delta_\perp h + 3R^2 h - h = -w(R)\ ,\quad (\partial_\xi, \partial_\eta)\, h(0) = 0\ ,\quad h(\infty) = 0\ . \tag{4.21}$$

Integration by parts shows that (lemma A.1):

$$\frac{1}{2\pi}\int Rh\, d\xi d\eta = -\frac{1}{4}f_1\ . \tag{4.22}$$

Equation (4.3) is obtained from the solvability condition for V_1^ϵ. The equation for the next order term V_1^ϵ is

$$\Delta_\perp V_1^\epsilon - V_1^\epsilon + 2|V_0^\epsilon|^2 V_1^\epsilon + (V_0^\epsilon)^2 (V_1^\epsilon)^* + \frac{1}{4}\beta\rho^2 V_1^\epsilon = -i\left[(V_0^\epsilon)_\zeta + \frac{M}{2N_c}\nu^\epsilon(\beta)V_0^\epsilon\right] - i\epsilon L^3 Im\left[F(\psi)\exp(-iS)\right]\ .$$

Using (4.20), to principal order in β and ϵ this equation reduces to

$$\Delta_\perp V_1 - V_1 + 2R^2 V_1 + R^2 V_1^* = -i\left[g\beta_\zeta + \epsilon h_\zeta + \frac{M}{2N_c}\nu(\beta)R\right] - i\epsilon L^3 Im\left[F(\psi_R)\exp(-iS)\right]\ .$$

From the solvability theory of Appendix E, the equation for the real part of V_1 is always solvable when h is even, and the solvability condition for the imaginary part of V_1 is that R is perpendicular to the right-hand-side of (4.23) (lemma E.1):

$$\int R\left[g\beta_\zeta + \epsilon h_\zeta + \nu(\beta)R + \epsilon L^3 Im\left[F(\psi_R)\exp(-iS)\right]\right] d\xi d\eta = 0\ .$$

Using (3.2), (3.12) and (4.22), we see that this relation is (4.3).

4.2.2. Derivation of the reduced equation (4.3) from balance of power. As in the case of CNLS (section 3.2), we can also derive the reduced equation (4.3) from balance of power. To do that, we multiply (4.1) by ψ^*, subtract the conjugate equation and integrate over the transverse variables to get an equation for the balance of power in (4.1):

$$\frac{\partial}{\partial z}\int |\psi|^2\, dxdy = -2\epsilon\, Im \int \psi^* F(\psi)\, dxdy\ . \tag{4.23}$$

The left-hand-side has two components, the focusing part ψ_s and the non-focusing one (3.3):

$$\int |\psi|^2 = \int |\psi_s|^2 + \int |\psi_{back}|^2\ .$$

The focusing part can can be approximated using

$$\frac{1}{2\pi}\int |\psi_s|^2\, dxdy \sim \frac{1}{2\pi}\int_{0\le\rho\le\rho_c} |V_0|^2\, d\xi d\eta = \int_0^\infty R^2\,\rho d\rho + 2\beta\int_0^\infty Rg\,\rho d\rho + \frac{\epsilon}{\pi}\int Rh\, d\xi d\eta + o(\beta,\epsilon)\ ,$$

and (3.12, 4.22). In addition, to leading order, the radiation rate is still given by (3.16). If we combine all the above and approximate ψ by ψ_R on the right-hand-side, equation (4.23) reduces to (4.3).

4.2.3. Derivation of the reduced equation (4.3) from a variational principle. If the perturbed CNLS equation has a Lagrangian density, then we can derive a Lagrangian density for the modulation equations by substituting the ansatz (4.2) in the action integral and integrating over the transverse variables. For example, this has already been done for the case of time-dispersive CNLS (see figure 1 in [20]).

There are several problems with this approach, which is why we do not pursue it here. For one thing, it can only be applied to perturbations of CNLS which have a variational formulation. In addition, with this approach we can only analyze the adiabatic effects of the perturbation, because the non-adiabatic term in (4.3) does not appear in the averaged Lagrangian. We finally note that when this approach is applied to the perturbed CNLS with the wrong ansatz (typically a Gaussian or a sech), the reduced equation fails to capture the delicate balance of critical self-focusing and can lead to erroneous predictions.

5. Applications of the modulation theory. In this section we apply modulation theory to various perturbations of CNLS. We include several new applications and present previous applications within the framework of modulation method.

5.1. Self-focusing in fiber arrays. In the last few years it has been suggested that faster transmission in optical fibers may be achieved by using an array of coupled optical waveguides arranged on a line in which the pulses undergo two-dimensional self-focusing. The model equation for the nth fiber is given by [3]

$$i\psi_z^n - \beta_2\psi_{tt}^n + 2\gamma|\psi^n|^2\psi^n + \delta(\psi^{n+1} - 2\psi^n + \psi^{n-1}) = 0 \tag{5.1}$$

where $\psi^n(z,t)$ is the electric field envelope in the nth fiber, δ is the coupling coefficient between neighboring fibers, β_2 is the group velocity dispersion and γ is the nonlinear coefficient. Careful analysis of (5.1) was recently carried out by Weinstein and Yeary [66].

In order to apply modulation theory to (5.1), let

$$\psi^n = \psi(z,t,nh)$$

and assume that the optical field is slowly varying over a number of fibers in the x direction. Therefore, $h \ll 1$ and one can employ the long wavelength approximation [14]:

$$\psi^{n+1} - 2\psi^n + \psi^{n-1} = h^2\left(\psi_{xx} + \frac{h^2}{12}\psi_{xxxx} + O(h^4)\right)\ .$$

When time-dispersion is anomalous ($\beta_2 < 0$) eq. (5.1) can be reduced to the nondimensional form

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi + \epsilon\psi_{xxxx} = 0\ , \quad \epsilon = \frac{h^2}{12}\ , \quad \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\ .$$

Therefore, we are interested in the effect of the perturbation

$$F = \psi_{xxxx}$$

on self-focusing. This is a conservative perturbation, as

$$f_2(z) = Im \int \psi_R^*(\psi_R)_{xxxx} \equiv 0\ .$$

The evaluation of

$$f_1(z) = \frac{1}{\pi} Re \int [R(\rho)\exp(iS)]_{xxxx} \exp(-iS)[R + \rho\nabla_\perp R]\, dxdy$$

can be simplified if we note that $[R\exp(iS)]_x \sim R_x \exp(iS)$, because $R = R(x/L)$, $S = S(x\sqrt{L_z/L})$ and $LL_z \sim \beta^{1/2} \ll 1$. Therefore,

$$f_1 \sim -\frac{2}{\pi L^2}\int (R_{\xi\xi})^2\, d\xi d\eta = -\frac{9N_c}{2L^2}\ .$$

Thus, self-focusing is covered by proposition 4.3 (the generic defocusing case) with $C_1 = 9N_c/2$:

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m)\ , \quad y_m \sim \frac{\epsilon 9N_c}{8M\beta_0}\ , \quad y_M \sim \frac{M\beta_0}{-H_0}\ .$$

From proposition 4.3 it follows that even if the initial power

$$\int |\psi_0|^2 \sim \sum_n \int |\psi^n|^2$$

is above critical, there will be no blowup. If, in addition, $H_0 < 0$, to leading order the solution of (5.1) will go through focusing-defocusing cycles with period

$$\Delta Z \sim 2\sqrt{\frac{My_M}{-H_0}}\ .$$

Gradually, non-adiabatic effects (which were neglected in proposition 4.3) will cause the oscillations in (5.1) to decay (proposition 4.4).

This qualitative picture predicted by modulation theory agrees with the simulations of (5.1) of Acheves et al. [4], where it was observed that the initial collapse towards the central fiber is arrested, and is followed by oscillations of the power between the central fiber and its neighbors.

5.2. Small defocusing fifth power nonlinearity. The case of small defocusing fifth power, dispersive nonlinearity

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi - \epsilon|\psi|^4\psi = 0\ , \quad 0 < \epsilon \ll 1 \tag{5.2}$$

was analyzed by Malkin [38]. In the notation of modulation theory we have

$$F = -|\psi|^4\psi\ ,$$

which is conservative ($f_2 \equiv 0$) and from (2.11)

$$f_1 \sim -\frac{8N_c}{L^2}\ .$$

Therefore, self-focusing is covered by Proposition 4.3 with $C_1 = 8N_c$:

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m)\ , \quad y_M \sim \frac{M\beta_0}{-H_0}\ , \quad y_m \sim \frac{2\epsilon N_c}{M\beta_0}\ . \tag{5.3}$$

5.3. Dispersive saturating nonlinearities. The use of equation (5.2) to model dispersive saturation of the nonlinearity is sometimes criticized because as $|\psi|$ increases the nonlinearity changes its sign and becomes defocusing. For this reason (5.2) is often replaced by

$$i\psi_z + \Delta_\perp \psi + \frac{1 - \exp(-2\epsilon|\psi|^2)}{2\epsilon}\psi = 0 \ , \quad 0 < \epsilon \ll 1 \ , \tag{5.4}$$

or by

$$i\psi_z + \Delta_\perp \psi + \frac{|\psi|^2}{1 + \epsilon|\psi|^2}\psi = 0 \ , \quad 0 < \epsilon \ll 1 \ . \tag{5.5}$$

Equations (5.4) and (5.5) can be viewed as regularizations of (5.2): The nonlinearity is approximately the same as in (5.2) when $\epsilon|\psi|^2 \ll 1$, but it has a finite and positive limit as $|\psi|$ goes to infinity. It turns out that these regularizations have essentially the same effect on self focusing as the unregularized case (5.2). This is true only for critical NLS and to the best of our knowledge its articulation is due to Malkin [36]:

PROPOSITION 5.1.
Self-focusing in equations (5.4) and (5.5) is the same to leading order as in equation (5.2).
Proof of Proposition 5.1:
The perturbation functions F corresponding to (5.4) and (5.5) are conservative and they satisfy

$$F = -\epsilon|\psi|^4(1 + O(\epsilon|\psi|^2) \ , \quad \text{provided that} \quad \epsilon|\psi|^2 \ll 1 \ .$$

Thus, as long as $|\psi|^2 \ll \epsilon^{-1}$, the leading order behavior of (5.4) and (5.5) is still given by (5.3). Since

$$y \geq y_m \sim \frac{\epsilon}{\beta} \ ,$$

throughout the focusing-defocusing cycle $\epsilon|\psi|^2 = O(\beta) \ll 1$ and

$$F = -\epsilon|\psi|^4(1 + O(\beta)) \ .$$

We have, therefore, the important result that all small dispersive regularizations of critical NLS lead to the same canonical focusing-defocusing effect.

The oscillatory behavior of solutions of (5.4) and (5.5), in accordance with (5.3), was observed in numerical simulations of LeMesurier et al. [34]. In [61], special attention was given to the non-adiabatic power radiation in (5.4).

5.4. Davey-Stewartson equation. The Davey-Stewartson equation (DS)

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi - \epsilon\phi_x\psi = 0 \ , \quad \alpha\phi_{xx} + \phi_{yy} = -(|\psi|^2)_x \tag{5.6}$$

arises in the study of gravity-capillary surface waves [1]. When $0 < \epsilon \ll 1$, equation (5.6) can be viewed as a perturbation of CNLS with

$$F = -\phi_x\psi \ .$$

This is a conservative perturbation and

$$f_1 = -\frac{1}{\pi} Re \left[\int (\phi_R)_x R(\rho)(R + \rho\nabla_\perp R) \, dx dy \right] \ ,$$

where ϕ_R is the solution of

$$\alpha(\phi_R)_{xx} + (\phi_R)_{yy} = -(|\psi_R|^2)_x \ .$$

Let $\tilde{\phi}_R(\xi,\eta)$ be the solution of

$$\alpha(\tilde{\phi}_R)_{\xi\xi} + (\tilde{\phi}_R)_{\eta\eta} = -(R^2)_\xi(\xi,\eta) .$$

Then $\phi_R(x,y) = L^{-1}\tilde{\phi}_R(\xi,\eta)$ and

$$f_1 = -\frac{1}{\pi}\int (\tilde{\phi}_R)_\xi R(\rho)(R+\rho\nabla_\perp R)\, d\xi d\eta = \text{constant} .$$

Therefore, to leading order (4.15) reduces to

$$\beta_\zeta = -\nu(\beta) ,$$

as in the case of self-focusing in the unperturbed CNLS (3.13). It follows that self-focusing in DS follows the adiabatic law for CNLS self-focusing (3.36), which ultimately reduces to the loglog law. It is remarkable that this perturbation has no effect on the blow up rate, as was first shown by Papanicolaou et al. [44], who derived the asymptotically equivalent equation (3.14) for self-focusing in DS and concluded that self-focusing in DS is given by the loglog law.

5.5. Nonparaxiality. In the standard derivation of CNLS as the model equation for laser beam propagation through a Kerr medium, the vectorial Maxwell equations for the propagation of a laser beam (the time-harmonic case) are first reduced to the scalar Helmholtz equation

$$\left(\Delta_\perp + \frac{\partial^2}{\partial z^2}\right) E + k^2 E = 0 , \quad k^2 = k_0^2(1 + \frac{2n_2}{n_0}|E|^2) .$$

Introducing the slowly varying envelope form $E = \psi\exp(ik_0 z)$ for the field leads to the nondimensional form of the Helmholtz equation [19]

$$\epsilon\psi_{zz} + i\psi_z + \Delta_\perp\psi + |\psi|^2\psi = 0 , \quad \epsilon = \left(\frac{\lambda}{4\pi r_0}\right)^2 . \tag{5.7}$$

Since the beam wavelength λ is much smaller than the initial beam radius r_0,

$$0 < \epsilon \ll 1 .$$

This suggests that $\epsilon\psi_{zz}$ can be neglected, in which case (5.7) reduces to CNLS.

Neglecting $\epsilon\psi_{zz}$ is called the *paraxial approximation* or the *parabolic approximation* and it is a valid approximation for rays which propagate almost parallel to the z axis. Mathematically, this is a problematic approximation, because a boundary value problem (Helmholtz) is replaced with an initial value problem (NLS). Moreover, the paraxial approximation breaks down near the focal point, as was pointed out by Feit and Fleck [16]. Indeed, from the asymptotic form of CNLS solution (3.1), we see that the magnitudes of $\Delta_\perp\psi$ and $|\psi|^2\psi$ are $O(L^{-3})$, and that of the nonparaxial term is $O(\epsilon L^{-5})$. This suggests that the paraxial approximation breaks down when $L = O(\sqrt{\epsilon})$. In fact, we will now show that the nonparaxial term does not even get to be of the same size as the other terms, because it arrests self-focusing when it is still $O(\beta)$ small compared with the CNLS terms.

We analyze the effect of small beam nonparaxiality by applying modulation theory with the perturbation

$$F = \psi_{zz} .$$

This perturbation is non-conservative and

$$f_2 \sim N_c\left(\frac{1}{L^2}\right)_z .$$

Therefore, equation (5.7) reduces to [19]

$$\beta_z = -\frac{2\epsilon N_c}{M}\left(\frac{1}{L^2}\right)_z . \tag{5.8}$$

Although this is a nonconservative perturbation, in light of (5.8) we can still apply the results of proposition 4.3 with $C_1 = -4N_c$ to get:

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m) , \quad y_M \sim \frac{M\beta_0}{-H_0} , \quad y_m \sim \frac{2\epsilon N_c}{M\beta_0} . \tag{5.9}$$

It is remarkable that this non-conservative perturbation leads to the same generic reduced equation as in the previous examples of conservative perturbations.

From (5.9) and proposition 4.3 we see that even when $\beta_0 > 0$ (i.e. initial power above critical), the solution of (5.9) does not blow up. If in addition $H_0 < 0$, the behavior is given by focusing-defocusing oscillations which gradually decay because of non-adiabatic effects. Note that throughout the focusing-defocusing cycle the relative magnitude of the nonparaxial term is

$$\frac{[\epsilon\psi_{zz}]}{[|\psi|^2\psi]} = \frac{\epsilon}{L^2} \le \frac{\epsilon}{y_m} = O(\beta) ,$$

providing an a-posteriori justification for treating it as a small perturbation.

The prediction of modulation theory of decaying focusing-defocusing oscillations is in qualitative agreement with the simulations of the Helmholtz equation [16] and with the studies of [6, 56]. This suggests that there is no blowup in the nonlinear Helmholtz equation and that the non-physical singularity formation in CNLS is due to the paraxial approximation. However, at present the role of back scattering, as well as vectorial effects [13], is not included in the analysis and a fuller picture of self-focusing in the nonlinear Helmholtz equation is still lacking.

5.6. Effect of randomness. The propagation of a narrow laser beam in a medium with impurities can be modeled by

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon_1(x^2+y^2)h(z)\psi = 0 , \quad 0 < \epsilon_1 \ll 1 , \tag{5.10}$$

where $h(z)$ is a real-valued random function. The perturbation

$$F = (x^2+y^2)h(z)\psi$$

is conservative and

$$f_1 = 8ML^4h(z) .$$

Therefore, in this case the reduced equation (4.6) becomes

$$L_{zz} = -\frac{\beta_0}{L^3} - 4\epsilon L(z)h(z) ,$$

showing that the effect of this random perturbation becomes negligible as $L \searrow 0$.

Random inhomogeneities can become important if they act in conjunction with an additional defocusing perturbation which leads to oscillatory behavior. One example is defocusing quintic nonlinearity and randomness

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon_1(x^2+y^2)h(z)\psi - \epsilon_2|\psi|^4\psi = 0 , \tag{5.11}$$

whose reduced equation is

$$- L^3 L_{zz} = \beta_0 + 4\epsilon_1 L^4 h(z) - \frac{4\epsilon_2 N_c}{M} \frac{1}{L^2} \,. \tag{5.12}$$

Another example is self-focusing in fiber array with axial random imperfections, for which the approximate continuum nonlinear Schrödinger equation is (see section 5.1):

$$i\psi_z + \Delta_\perp \psi + |\psi|^2 \psi + \epsilon_1 x^2 h(z) \psi + \epsilon_2 \psi_{xxxx} = 0 \,, \tag{5.13}$$

with $\Delta_\perp = \partial_t^2 + \partial_x^2$. In this case the reduced equation is

$$- L^3 L_{zz} = \beta_0 + 4\epsilon_1 L^4 h(z) - \frac{9\epsilon_2 N_c}{4M} \frac{1}{L^2} \,. \tag{5.14}$$

The random inhomogeneities have, in general [10], the form $h(z, x, y)\psi$, or $h(z, x)\psi$ in (5.13). However, when the beam is narrow we can expand h about the beam axis

$$h = h_0(z) + (x, y) \cdot \nabla_\perp h + \frac{1}{2}(x, y) \cdot \nabla\nabla h \cdot (x, y) + \cdots \,.$$

The linear terms can be eliminated by preliminary transformations of the transverse coordinates and the phase [40]. If we also assume, for simplicity, that the inhomogeneities are transversely isotropic then we get equations (5.11) and (5.13). We will also assume that $h(z)$ is stationary with mean zero $< h(z) >= 0$, where $<>$ is ensemble average.

The reduced equation (5.14) can be written as a nonlinear oscillator equation with a parametrically random, linear term:

$$L_{zz} + 4\epsilon_1 h(z) L(z) + U'(L) = 0 \,, \quad U(L) = \frac{9\epsilon_1 N_c}{16 M L^4} - \frac{\beta_0}{2L^2} \,. \tag{5.15}$$

The effects of randomness in (5.15) are not easy to assess and will be analyzed elsewhere [21]. In the following we present some preliminary results. The potential $U(L)$ has a minimum at $L = 2\sqrt{\epsilon_2 N_c / M\beta_0}$. For small oscillations about this minimum we can linearize (5.15) by writing $L = L_{min} + \delta L$, with $0 \le \delta L \ll L_{min}$, to get for δL the randomly forced linear oscillator equation

$$\delta L_{zz} + \omega^2 \delta L = \tilde{h}(z) \,, \tag{5.16}$$

where the frequency ω is given by

$$\omega = \frac{\beta_0^{3/2} M}{2\sqrt{2}\epsilon_2 N_c}$$

and the random forcing by

$$\tilde{h}(z) = -8\epsilon_1 \sqrt{\frac{\epsilon_2 N_c}{M \beta_0}} h(z) \,.$$

Note that the frequency of the small oscillations decreases with β_0 but increases as $\epsilon_2 \to 0$. The random forcing will make the energy of the small oscillations increase on the average as z increases:

$$\frac{d}{dz} < \frac{1}{2}(\delta L)_z^2 + \frac{\omega^2}{2}(\delta L)^2 >= \int_0^z \cos(\omega s) \tilde{R}(s) ds$$

where $\tilde{R}(z) =< \tilde{h}(z+s)\tilde{h}(s) >$ is the covariance of the random force $\tilde{h}(z)$. For large z the energy of the small oscillations grows linearly

$$< \frac{1}{2}(\delta L)_z^2 + \frac{\omega^2}{2}(\delta L)^2 >\sim \frac{z}{2}\hat{R}(\omega)$$

where $\hat{R}(\omega) \geq 0$ is the power spectral density [45] of the random forcing $\tilde{h}$

$$\hat{R}(\omega) = \int_{-\infty}^{\infty} e^{i\omega s}\tilde{R}(s)ds \ .$$

Ultimately, the growth of the energy will make the linearization invalid and the full nonlinear equation (5.15) should be considered. An important issue is to estimate the probability of escape (i.e. $L \to +\infty$) by the random inhomogeneities. This is done in [21] in a manner similar to the one used in [26]. The main result is that the amplitude of the focusing-defocusing oscillations grows until there is no more focusing.

5.7. Temporal effects. In nonlinear optics CNLS (1.1) is derived for time-harmonic laser beams propagating in a medium with an instantaneous nonlinear polarization response. However, temporal effects, such as time-dispersion and Debye relaxation, can become important in the propagation of ultrashort laser pulses. Since in these non-stationary cases the initial condition is given at the medium interface $z = 0$ for all (x, y, t), time behaves like a third spatial variable and z plays the role of 'time'. As a result, the reduced equations and the modulation variables L, β and ζ depend on both z ('time') and t ('space').

5.7.1. Small time-dispersion. The nonlinear Schrödinger equation with small time-dispersion

$$i\psi_z + \Delta_\perp\psi - \epsilon\psi_{tt} + |\psi|^2\psi = 0 \ , \quad \psi(z=0,x,y,t) = \psi_0(x,y,t) \ , \quad |\epsilon| \ll 1 \tag{5.17}$$

arises in the study of the propagation of ultrashort laser pulses in media with an instantaneous Kerr nonlinearity. The correct expression for ϵ is[6]

$$\epsilon = \frac{r_0^2 k_0 k_{\omega\omega}}{T^2}$$

where r_0 is the initial pulse radius, $k = \omega n_0(\omega)/c$, n_0 is the linear index of refraction, c is speed of light and T is the pulse duration. Time-dispersion is called normal if $\epsilon > 0$ and anomalous if $\epsilon < 0$.

If time-dispersion is anomalous, (5.17) is supercritical NLS, which has solutions that undergo 3D collapse. However, the dynamics in the case of normal time-dispersion is more complicated. Zharova et al. [68] were the first to show, both numerically and asymptotically, that in the case of small normal time-dispersion the pulse undergoes a temporal split into two pulses. They went on to conjecture that the new peaks would split again. Pulse splitting was observed numerically in [12, 35, 52] and was recently observed experimentally [48].

If we apply modulation theory to (5.17), then

$$F = -\psi_{tt}$$

is non-conservative and

$$f_2 = -\frac{1}{2\pi} Im \int \psi^* \psi_{tt}\, dxdy \sim -N_c\zeta_{tt} \ .$$

Using (4.7), we get that (5.17) can be reduces to [20]:

$$\beta_z = \frac{2\epsilon N_c}{M}\zeta_{tt} \ , \quad L_{zz} = -\frac{\beta}{L^3} \ , \quad \zeta_z = \frac{1}{L^2} \ . \tag{5.18}$$

[6] We would like to thank B.A. Rockwell [51] for pointing out to us the error in the expression for ϵ in [20]

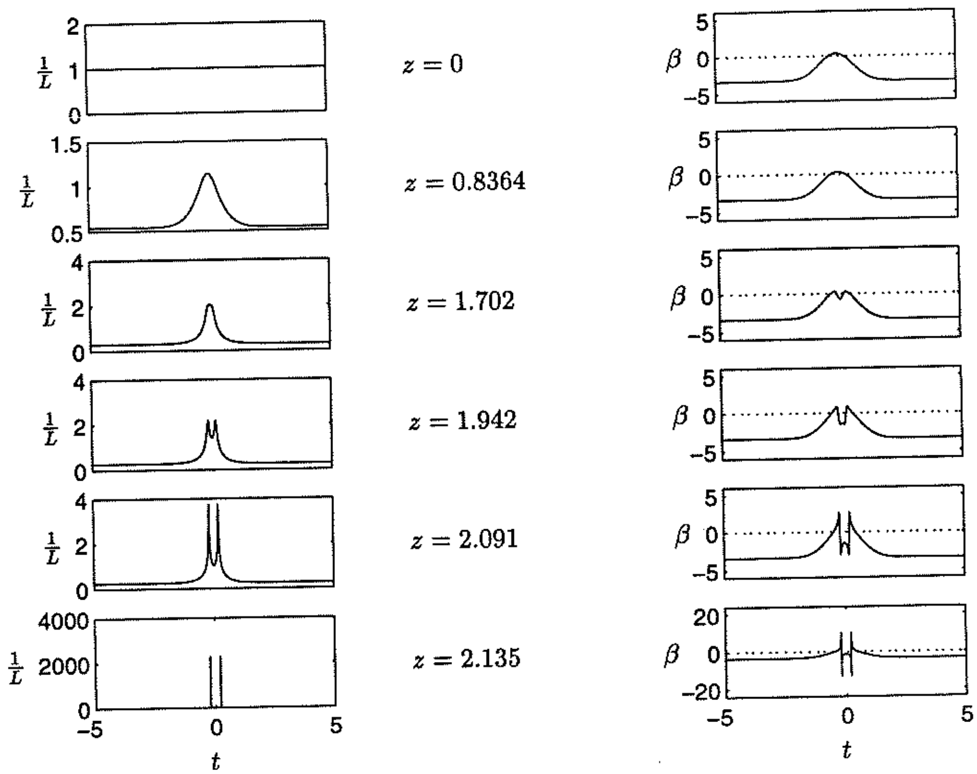

FIG. 5.1. *As a result of small normal time-dispersion the power (β) radiates away from the center, leading to the formation of two symmetrical peaks which continue to self-focus. Results are shown for the reduced system (5.18) with the initial conditions $L(0,t) \equiv 1$, $L_z(0,t) \equiv 0$ and $\beta(0,t) = (1.1\exp(-t^2) - 1)N_c/M$ and $\epsilon = 0.01$. Although we do not observe secondary peak splitting, the large values of β and the sharp t gradients suggest that the validity of (5.18) breaks down at some point.*

The numerical agreement of (5.18) with (5.17) was demonstrated in [20]. Analysis of (5.18) shows that it captures the peak splitting phenomena (Figure 5.1), and that peak splitting is associated with the transition from self-similar 2D collapse to full 3D dynamics. Therefore, it was suggested in [20] that the new peaks would not necessarily split again. However, this issue is still open, as is the question of whether the solution becomes singular or not and whether the reduced system (5.18) remains valid for all z.

5.7.2. Debye relaxation. In models for the propagation of an intense laser beam, the nonlinear cubic term in CNLS represents an instantaneous nonlinear material polarization response. If the mechanism for the induced nonlinear polarization is molecular orientation, then for sufficiently long pulses the frictional drag between the molecules tends to make the rotation lag behind the torque induced by the electric field. The resulting model equation in this case is CNLS with Debye relaxation:

$$i\psi_z + \Delta_\perp \psi + N\psi = 0 \quad . \tag{5.19}$$

$$\epsilon N_t + N = |\psi|^2 \ , \quad \epsilon = \frac{\tau_D}{T} > 0 \ , \tag{5.20}$$

where t is retarded time $(t - z/c_g)$, τ_D is the characteristic response time for dipole reorientation ($\sim 10^{-11} sec$ for water) and T is pulse duration.

In this section we use modulation theory to address the question of whether Debye relaxation can arrest self-focusing when $0 < \epsilon \ll 1$. The Debye perturbation $\epsilon F = (N - |\psi|^2)\psi$ is conservative ($f_2 = 0$).

From (5.20), we get that

$$N \sim |\psi|^2 - \epsilon(|\psi|^2)_t \ .$$

Therefore, we approximate (5.19)–(5.20) by (4.1) with

$$F = -(|\psi|^2)_t\psi \ .$$

Evaluation of f_1 yields

$$f_1 \sim \frac{C_D L_t}{L} \ , \quad C_D = \int (\nabla_\perp R^2)^2 \rho^3 \, d\rho \cong 6.43 \ ,$$

showing that self-focusing in the presence of Debye relaxation is given by (4.6):

$$-L^3 L_{zz} = \beta_0 + \frac{\epsilon C_D}{2M}\frac{L_t}{L} \ . \tag{5.21}$$

From this equation we see that Debye relaxation slows focusing at times earlier than the pulse peak and enhances it at later times. As a result, self-focusing becomes temporally asymmetrical, with the peak moving towards later times (Figure 5.2), as can be expected from a delay mechanism and as was observed in numerical simulations of (5.19)–(5.20) [55].

In order to further analyze the initial effect of Debye relaxation, we note that during the non-adiabatic self-focusing and unperturbed adiabatic self-focusing stages (see section 4.1), the effect of Debye relaxation is negligible and each t cross-section (i.e. the plane $t = constant$ in the (x, y, t) space) focuses independently in a 2D self-similar fashion

$$L(z, t) = L(Z_c(t) - z)$$

with $Z_c(t)$ given by (3.32). If we use this self-similar form in (5.21), we get:

$$-L^3 L_{zz} = \beta_0 - \frac{\epsilon C_D \dot{Z}_c(t)}{2M}\frac{L_z}{L} \ , \quad \dot{} := \frac{d}{dt} \ .$$

Making a change of variable, we can rewrite this equation as

$$A_{\zeta\zeta} = \beta_0 A + \frac{\epsilon C_D \dot{Z}_c(t)}{6M}(A^3)_\zeta \ , \quad A = \frac{1}{L} \ .$$

If the peak power is initially at $t = t_0$, then $Z_c'(t) > 0$ for $t > t_0$ and $Z_c'(t) < 0$ for $t < t_0$. Therefore, we see that if the power is above critical ($\beta_0 > 0$), there is blowup ($A \nearrow +\infty$) for $t > t_0$ and arrest of blowup for $t < t_0$. However, one cannot apply this conclusion to (5.19–5.20) or even to (5.21), because as self-focusing starts to deviate from that the unperturbed CNLS, the validity of the 2D self-similar argument breaks down and the dynamics become fully 3D (i.e. (x, y, t)), as manifested by the shift of the peak towards later times. At present, the question whether solutions of (5.19–5.20) can become singular is still open.

5.8. Time-dispersion and nonparaxiality. We have seen that both normal time-dispersion and nonparaxiality may lead to self-focusing arrest. This raises the question of determining which of these two mechanisms is dominant in self-focusing of ultrashort pulses. Similarly, if time-dispersion is anomalous, it is enhancing self-focusing as nonparaxiality is slowing it down, and we would like to know which of the two effects will ultimately prevail. Therefore, we are interested in analyzing self-focusing in the presence of both time-dispersion and nonparaxiality.

It may seem that all we need to do is add the separate contribution of each mechanism in the corresponding reduced equation (5.8) and (5.18). However, more careful examination of the derivation

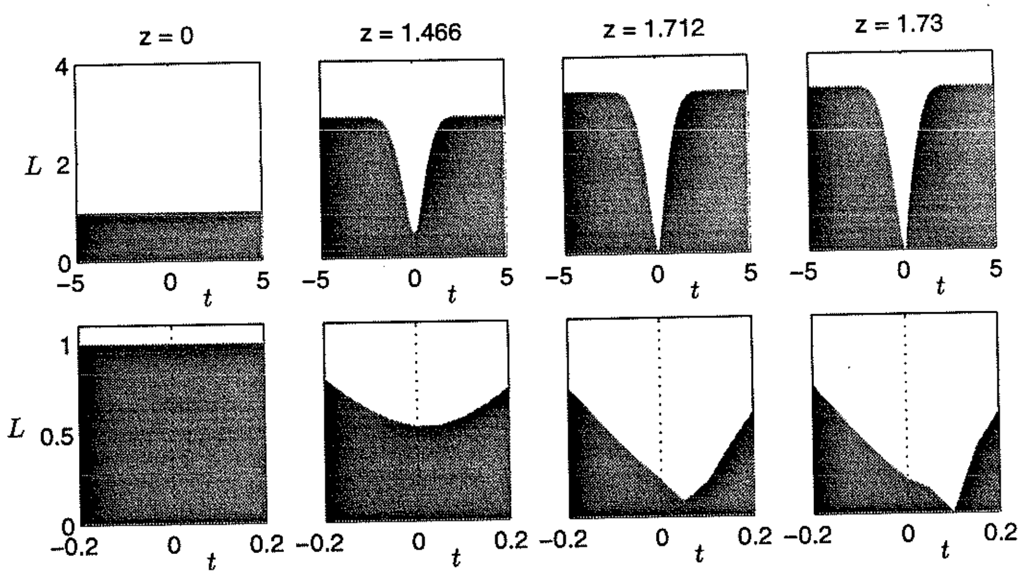

FIG. 5.2. *Self-focusing in the presence of Debye relaxation according to the reduced equation (5.21) with the initial conditions $L_0 \equiv 1$ and $\beta(0) = (1.1\exp(-t^2) - 1)Nc/M$. While most of the pulse is defocusing (top), an asymmetric self-focusing takes place in the center, with the peak moving towards later times.*

of CNLS shows that if one retains both time-dispersion and nonparaxiality in the model, then the model equation contains additional terms [22]:

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi + \epsilon_1 \psi_{zz} + \epsilon_2 \left[2i\frac{n_0 c_g}{c}(|\psi|^2\psi)_t - \psi_{zt} \right] - \epsilon_3 \psi_{tt} = 0 \tag{5.22}$$

where

$$\epsilon_1 = \frac{1}{4r_0^2 k_0^2} \ , \quad \epsilon_2 = \frac{1}{c_g k_0 T} = \frac{1}{\omega_0 T}\frac{c}{n_0 c_g} \ , \quad \epsilon_3 = \frac{k_0 r_0^2 k_{\omega\omega}}{T^2} \tag{5.23}$$

and c_g is the group velocity. The dimensionless parameter $\epsilon_1 \sim$ (wavelength/radial pulse width)^2, $\epsilon_2 \sim$ (period of one oscillation/pulse duration), and ϵ_3 is a dimensionless measure of group velocity dispersion (GVD). Note that ϵ_2 is proportional to the geometric mean of ϵ_1 and ϵ_2:

$$\epsilon_2^2 = \epsilon_1 \epsilon_3 q \ , \quad q = \frac{4}{c_g^2 k_0 k_{\omega\omega}} \ .$$

Therefore, if one retains time-dispersion and nonparaxiality, the mixed term and the shock term (ψ_{zt} and $|\psi|^2\psi)_t$, respectively) should be also included in the model. Moreover, in the visible spectrum $q \gg 1$, and the ϵ_2 terms can dominate over both time-dispersion and nonparaxiality [22].

The reduced system corresponding to (5.23) is

$$\beta_z(z,t) = -\gamma_1 \left(\frac{1}{L^2}\right)_z - \gamma_2 \left(\frac{1}{L^2}\right)_t + \gamma_3 \zeta_{tt} \ , \quad \zeta_z(z,t) = \frac{1}{L^2} \ , \quad L_{zz}(z,t) = -\frac{\beta(z,t)}{L^3} \ , \tag{5.24}$$

where

$$\gamma_1 = 2\epsilon_1 N_c/M \ , \quad \gamma_2 = \epsilon_2(6c_g n_0/c - 2)N_c/M \ , \quad \gamma_3 = 2\epsilon_3 N_c/M \ .$$

Following [20], we can analyze the initial effect of the three terms in (5.22) by looking at special solutions of (5.24). Away from the focal point, the three perturbing terms in (5.22) are small and each t cross-section of the pulse (i.e. the 2D plane $t =$ const in the (x, y, t) space) focuses independently with

$$L(z,t) = L(Z_c(t) - z) \ , \quad \beta(z,t) = \beta(Z_c(t) - z) \ , \quad \zeta(z,t) = \zeta(Z_c(t) - z) \ . \tag{5.25}$$

Here $Z_c(t)$ is the location of the focus in the (z, t) plane when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ (3.32). Therefore, eq. (5.24) becomes

$$\beta_z = -\gamma_1 \left(\frac{1}{L^2}\right)_z + \gamma_2 \dot{Z}_c \left(\frac{1}{L^2}\right)_z + \gamma_3(-\ddot{Z}_c \zeta_z + \dot{Z}_c^{\,2} \zeta_{zz}) \ , \quad \dot{} = \frac{d}{dt} \ . \tag{5.26}$$

This equation can be transformed into a nonlinear Airy equation [20]

$$g_{ss} = sg + \kappa g^3 \ , \quad \text{with} \quad g = L^{-1} > 0 \ . \tag{5.27}$$

Here

$$\begin{aligned} s &= \left(\beta_0 - \gamma_3 \ddot{Z}_c \zeta\right) \left(\gamma_3 \ddot{Z}_c\right)^{-2/3} \ , \quad \beta_0 \sim \beta(0,t) \ , \\ \kappa &= -(\gamma_1 - \gamma_2 \dot{Z}_c - \gamma_3 \dot{Z}_c^{\,2}) \left(\gamma_3 \ddot{Z}_c\right)^{-2/3} \end{aligned}$$

The initial conditions for eq. (5.27) are given at

$$s_0(t) := s(z = 0, t) \sim \beta(0,t) \left(\gamma_3 \ddot{Z}_c\right)^{-2/3} \ .$$

At the time t_0 of the initial peak power of the pulse, $Z_c(t)$ attains its minimum, $\dot{Z}_c(t_0) = 0$ and the evolution is given by (5.27) with $\kappa = -\gamma_1 \left(\gamma_3 \ddot{Z}_c\right)^{-2/3} < 0$. Because $\ddot{Z}_c(t_0) > 0$, as $z \to Z_c$ and $\zeta \to +\infty$, $s \to -\infty$ for normal time-dispersion ($\epsilon_3 > 0$), and both time-dispersion and nonparaxiality (first and second terms on the right-hand-side of (5.27), respectively) contribute to the arrest of the blowup by preventing g from becoming infinite. When time-dispersion is anomalous ($\epsilon_3 < 0$), it enhances blowup ($s \to +\infty$) while nonparaxiality opposes it. Eventually, as $s \to +\infty$ nonparaxiality prevails and the solution of (5.27) will decay (no blowup).

In the case of normal time-dispersion and $\epsilon_1 = \epsilon_2 = 0$, blowup is arrested only in an exponentially small neighborhood of t_0 [20], where pulse splitting occurs. In order to assess the added effects of nonparaxiality and the mixed term, we note that the condition for blowup [20] in (5.27) as $s \to -\infty$ is $\kappa > 2L^2(0,t)Ai^2(s_0)$ or

$$\gamma_3 \dot{Z}_c^{\,2} > \gamma_1 - \gamma \dot{Z}_c + 2L^2(0,t) Ai^2(s_0) \left(\gamma_3 \ddot{Z}_c\right)^{2/3}$$

where $Ai(s)$ is the Airy function. Therefore, if nonparaxiality dominates, arrest of blowup occurs over a much larger region (possible everywhere). If the ϵ_2 term dominates, blowup will occur when $\epsilon_3 > -\epsilon_2/\dot{Z}_c$, i.e. only for $t > t_0$. Note that as the solution starts to deviate from that of the unperturbed CNLS, the 2D self-similar structure (5.25) will gradually break down. Therefore, for later z this 2D self-similar argument becomes invalid and the full 3D nature of (5.24) has to be considered.

From eq. (5.26) we see that the effect of the ϵ_2 term on a self-focusing pulse is a temporal power transfer towards later times (recall that β is proportional to the excess power above critical). This will result in an asymmetric temporal development of the pulse, with a greatly enhanced trailing portion and a suppressed leading part, in agreement with previous results on the effect of the shock term [8] and of the linear component of the ϵ_2 term [53].

6. Numerical methods. Numerical integration of self-focusing in CNLS (1.1) requires a code that can handle the ever increasing gradients near the singularity. In the method of *dynamic rescaling* [41], the independent variables and of the function are dynamically rescaled in a way which is based on the asymptotic form of the solution (3.1). In the rescaled variables the function is smooth and the problem can be solved on a fixed grid using standard techniques. Then, the solution of CNLS is recovered from that of the rescaled problem. Subsequent improvements to this method include the use of approximate boundary conditions [28] and extension to the non-isotropic case [32]. For example, by applying this method to CNLS we reached focusing factors of 10^{15} (Fig 3.5C).

Dynamic rescaling was also applied to perturbed CNLS: Saturating nonlinearity ([34]), the Davey-Stewartson equation [44] and small normal time-dispersion [20]. However, in these cases the method becomes less successful, since it is inherently based on the special rescaling of CNLS self-focusing. Some of the difficulties which arise are instabilities due to the use of the approximate boundary conditions during defocusing stages and the need for (a yet unknown) additional rescaling in the t direction in the non-stationary cases.

Another approach is to apply a split-step method (e.g. [47]): The linear parts are solved by a Fourier transform in space, and the nonlinear part is solved by an appropriate nonlinear solver. A different approach was taken in [7], were CNLS was solved by a Galerkin finite-element method.

6.1. Numerical comparison of CNLS and adiabatic theory. In order to compare the numerical solution of a perturbed CNLS with its corresponding reduced system, one needs to be able to recover the values of L, β and ζ from ψ. In the case of dynamic rescaling, one solves for the rescaled function u and for $\bar{L}$, which are related to ψ through

$$\psi = \frac{u(\bar{\zeta}, \cdot)}{\bar{L}} , \quad u \sim \exp\left(i\lambda^2(\bar{\zeta})\bar{\zeta} + i\frac{\bar{L}_z}{\bar{L}}\frac{r^2}{4}\right)\lambda R(\lambda\bar{\rho}) .$$

The bars denote the (numerical) values of L, ζ etc in dynamic rescaling. In general, these values are different from the ones used in the asymptotic theory where $\lambda \equiv 1$. The modulation variables can be recovered using [17]:

$$\beta \sim \frac{1}{M}\left(\int_0^{\rho_c} |u|^2 \rho d\rho - N_c\right) , \quad \zeta = \arg u(\rho = 0) , \quad L = \bar{L}\frac{R(0)}{|u(0)|} ,$$

where the integration domain in the radial variable is $0 \le \rho \le \rho_c$.

When we apply modulation theory for non-stationary perturbations of CNLS, the question arises as to how to represent t cross-sections whose power is much smaller than N_c, since modulation theory was derived for $|\beta| \ll 1$. The simplest approach is to use (3.15) for all t cross-sections. If we do that, then

$$\lim_{t\to\pm\infty} \beta(t) = \frac{-N_c}{M} \cong -3.38 .$$

Fortunately, this approximation is quite reasonable, since as $t \to \pm\infty$ the propagation is determined only by linear diffraction, in which case

$$L_{zz} = \frac{4}{L^3}$$

(see, for example, eq. (39) of [5]), corresponding to

$$\lim_{t\to\pm\infty} \beta(t) = -4 .$$

A related question is which value to use for $L_0(t)$ for $|t|$ large. We cannot use (3.41), since then

$$\lim_{t\to\pm\infty} L_0(t) = \infty .$$

One possibility is to set $L_0(t) \equiv 1$.

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A. Perturbation analysis for $P \sim R + \epsilon h$.

In this lemma we use regular perturbations to evaluate several integrals which arise when we average over the transverse variables. In the case of perturbed CNLS, the results of this lemma are applied with

$$P = V_0(\xi, \eta; \beta = 0, \epsilon) \; .$$

In appendix B we apply this lemma for the case of unperturbed CNLS with $P = V_0(\rho; \beta)$, but in that case we have to be more careful with the domains of integration.

LEMMA A.1.

1. Let $R(\rho)$ be the solution of (2.8). Then the following identities hold:

$$\int_0^\infty R^2 \, \rho d\rho = \int_0^\infty (\nabla_\perp R)^2 \, \rho d\rho = \frac{1}{2} \int_0^\infty R^4 \, \rho d\rho \qquad (A.1)$$

2. Let $P(\xi, \eta) \in H^1$ satisfy the equation

$$\Delta P - P + P^3 + \epsilon w(P) = 0 \qquad (A.2)$$

with $w(P)$ real. Then

$$H(P) = \frac{\epsilon}{2\pi} \int w(P)[P + (\xi, \eta) \cdot \nabla_\perp P] \, d\xi d\eta \; . \qquad (A.3)$$

In addition, if we expand

$$P(\xi, \eta) = R(\rho) + \epsilon h(\xi, \eta) + O(\epsilon^2) \; , \quad |\epsilon| \ll 1 \; ,$$

then the equation for h is (4.21) and

$$\int Rh \, \rho d\xi d\eta = \int (R^3 h - \nabla_\perp R \nabla_\perp h) \, d\xi d\eta = -\frac{1}{2} \int w(R)[R + \rho \nabla_\perp R] \, d\xi d\eta \; . \qquad (A.4)$$

Proof:

If we multiply (A.2) by P and integrate by parts, we get:

$$-\int (\nabla_\perp P)^2 - \int P^2 + \int P^4 + \epsilon \int w(P)P = 0 \qquad (A.5)$$

Similarly, if we multiply (A.2) by $(\xi, \eta) \cdot \nabla_\perp P$ and integrate by parts, we get:

$$\int P^2 - \frac{1}{2} \int P^4 + \epsilon \int w(P)(\xi, \eta) \cdot \nabla_\perp P = 0 \qquad (A.6)$$

Adding (A.5) and (A.6) gives (A.3).

If we multiply (2.8) by P and integrate by parts, we get

$$-\int \nabla_\perp R \nabla_\perp P - \int PR + \int PR^3 = 0 \qquad (A.7)$$

The $O(1)$ and $O(\epsilon)$ equations in (A.5) are respectively:

$$-\int (\nabla_\perp R)^2 - \int R^2 + \int R^4 \;\; = \;\; 0 \qquad (A.8)$$

$$-2 \int \nabla_\perp R \nabla_\perp h - 2 \int Rh + 4 \int R^3 h \;\; = \;\; -\int w(R)R \; . \qquad (A.9)$$

The $O(1)$ and $O(\epsilon)$ equations in (A.6) are respectively:

$$\int R^2 - \frac{1}{2}\int R^4 = 0 \tag{A.10}$$

$$2\int Rh - 2\int R^3 h = -\int w(R)(\xi,\eta)\cdot\nabla_\perp R\,. \tag{A.11}$$

The $O(1)$ and $O(\epsilon)$ equations in (A.7) are respectively:

$$-\int(\nabla_\perp R)^2 - \int R^2 + \int R^4 = 0 \tag{A.12}$$

$$-\int \nabla_\perp R\nabla_\perp h - \int Rh + \int R^3 h = 0 \tag{A.13}$$

From (A.8–A.13), we get (A.1) and (A.4).

B. Perturbation analysis for $V_0 \sim R + \beta g$. In this appendix we derive the modulation approximations to various integrals which arise when we derive the reduced equations from balance of power. Modulation theory for CNLS is based on the ansatz for the focusing part of the solution

$$\psi_s(r,z) \sim \frac{1}{L}V_0(\rho;\beta)\exp(i\zeta + i\frac{L_z}{L}\frac{r^2}{4})\,, \quad \rho = \frac{r}{L}\,,$$

where V_0 is quasi-steady. As we have seen, if V_0 is assumed to satisfy (3.7), then it is not possible to match ψ_s with ψ_{back}. There are two ways to take care of this problem. One possibility is to have V_0 defined for all ρ, in which case one has to add a small term to (3.7) which will correct its behavior for large ρ. In this case, the equation for V_0 is (3.8). Alternately, if we consider the equation for V_0 to be (3.7), then to the right of the turning point at $\rho_b = 2\beta^{-1/2}$, V_0 is oscillatory

$$V_0 \sim \frac{1}{\rho}\cos\left(\frac{\beta\rho^2}{4}\right)\,.$$

Therefore, with this definition $\int_0^\infty |V_0|^2\,\rho d\rho$ diverges. In order to take care of that, we consider ψ_s to be defined only for $0 \le \rho \le \rho_c$, where $1 \ll \rho_c < \rho_b$ is constant (eq. 3.3). In this domain the expansion (3.9) is uniform in ρ, since V_0, R and g are all exponentially decreasing.

LEMMA B.1. *Let $V_0(\rho)$ be the solution of*

$$\Delta V_0 - V_0 + V_0^3 + \frac{1}{4}\beta\rho^2 V_0 = 0\,, \quad 0 \le \rho \le \rho_c\,. \tag{B.1}$$

Then

$$H(V_0) := \int_0^{\rho_c}|\nabla V_0|^2\,\rho d\rho - \frac{1}{2}\int_0^{\rho_c}|V_0|^4\,\rho d\rho = -\frac{\beta}{4}\int_0^{\rho_c}\rho^2 V_0^2\,\rho d\rho\,.$$

In addition, if we expand

$$V_0 \sim R(\rho) + \beta g(\rho) + O(\beta^2)\,, \quad |\beta| \ll 1\,, \tag{B.2}$$

the equations for R and g are (2.8) and (3.10) and:

$$\int_0^\infty Rg\,\rho d\rho = \frac{M}{2}\,, \quad N(V_0) := \int_0^{\rho_c}|V_0|^2\,\rho d\rho = N_c + \beta M + O(\beta^2)$$

Proof:
Use

$$\int_0^{\rho_c} V_0^2 = \int_0^\infty R^2 + 2\beta \int_0^\infty Rg + O(\beta^2)$$

and lemma A.1 with $P = V_0$, $\epsilon = \beta$, $w = (1/4)\rho^2 V_0$ and $h = g$. Note that the error of replacing ρ_c with infinity in integrals involving R and g is exponentially small in β.

LEMMA B.2. *Let*

$$\psi_s(r,z) = \frac{1}{L} V_0(\rho;\beta) \exp(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}) \ , \quad \rho = \frac{r}{L} \ .$$

Then:

$$(B.3) \qquad N(\psi_s) := \int_0^{L\rho_c} |\psi_s|^2 \, r dr = N_c + \beta M + O(\beta^2)$$

$$(B.4) \qquad H(\psi_s) := \int_0^{L\rho_c} |\nabla \psi_s|^2 \, r dr - \frac{1}{2}\int_0^{L\rho_c} |\psi_s|^4 \, r dr = M\left(L_z^2 - \frac{\beta}{L^2}\right)(1 + O(\beta))$$

$$= \frac{M}{2}(L^2)_{zz}(1 + O(\beta))$$

Proof:

This follows from lemma B.1,

$$H(\psi_s) = \frac{1}{4}\left(L_z^2 - \frac{\beta}{L^2}\right)\int_0^{\rho_c} \rho^2 V_0^2$$

and (3.5).

C. WKB calculation of the rate of power and Hamiltonian radiation. In this appendix we derive (3.16) and (3.18). Let us rewrite equation (3.4) as

$$(C.1) \qquad iV_\zeta + \Delta_\rho V - UV = 0 \ , \quad U = 1 - |V|^2 - \frac{1}{4}\beta\rho^2 \ .$$

The radiation rates for the power and Hamiltonian of ψ_s are given by

$$(C.2) \qquad \frac{d}{dz}N(\psi_s) = \frac{d}{dz}\int_0^{L\rho_c} |\psi|^2 \, r dr$$

$$\frac{d}{dz}H(\psi_s) = \frac{d}{dz}\int_0^{L\rho_c} \left[|\psi_r|^2 - \frac{1}{2}|\psi|^4\right] r dr \ .$$

When $0 < \beta \ll 1$,

$$(C.3) \qquad V \sim R(\rho) \ , \qquad 0 \le \rho \ll \beta^{-1/2} \ ,$$

and the potential U has two turning points: $\rho_a = O(1)$ and $\rho_b \sim 2/\sqrt{\beta}$. Since in the classically inaccessible region $[\rho_a, \ \rho_b]$ the solution V has an exponential decay, if we set ρ_c in (C.2) to be just past the second turning point to the right i.e. $0 < \rho_c - \rho_b \ll 1$ (rather than $1 \ll \rho_c < \rho_b$, as in appendix B), this would result in an exponentially small change in the values of N_s and H_s.

If we differentiate (C.2), use (1.1) and integrate by parts, we get:

$$\frac{d}{dz}N(\psi_s) = |\psi|^2 L L_z \rho_c^2 + (i\psi^* \psi_r L\rho_c + c.c. \ )$$

$$\frac{d}{dz}H(\psi_s) = |\psi_r|^2 L L_z \rho_c^2 - \frac{1}{2}|\psi|^4 L L_z \rho_c^2 + [iL\rho_c(\psi_r^* \psi_{rr} - |\psi|^2 \psi^* \psi_r) + c.c. \ ] \ .$$

Using (3.1), these equations can be rewritten in terms of V:

(C.4) $$\frac{d}{dz}N(\psi_s) = \frac{1}{L^2}(i\rho_c V^* V_\rho + c.c.\ )$$

and

(C.5) $$\begin{aligned}\frac{d}{dz}H(\psi_s) \ &= \ -\frac{L_z\rho_c^2}{L^3}|V_\rho|^2 - \frac{L_z^2\rho_c^3}{4L^2}(iVV_\rho^* + c.c\ ) + \frac{L_z\rho_c^2}{2L^3}|V|^4 + \frac{\rho_c}{L^4}(iV_\rho^* V_{\rho\rho} + c.c\ ) \\ &\quad -\frac{L_z\rho_c}{2L^3}(|V|^2)_\rho + \frac{L_z\rho_c^2}{2L^3}(V^* V_{\rho\rho} + c.c\ ) - \frac{\rho_c}{L^4}(i|V|^2 V^* V_\rho + c.c\ )\ .\end{aligned}$$

In order to find the asymptotic behavior of V for $\rho > b$, we rewrite (C.1) as

(C.6) $$iV_\zeta + \delta^2\Delta_s V - UV = 0\ , \quad U = 1 - |V|^2 - s^2\ , \quad s = \delta\rho\ , \quad \delta = \frac{\beta^{1/2}}{2} \ll 1\ .$$

Since for CNLS $V_\zeta = o(\beta)$, we can use the stationary version of (C.6):

(C.7) $$\delta^2\Delta_s V - UV = 0$$

In terms of the new independent variable s, the turning points are at $s_a = O(\delta)$ and $s_b \sim 1$. Using (C.3) and

$$R(\rho) \sim A_R \exp(-\rho)\rho^{-1/2}\ , \qquad \rho \gg 1\ ,$$

we get that

(C.8) $$V \sim A_R \exp(-s/\delta)(s/\delta)^{-1/2}\ , \qquad \delta \ll s \ll 1\ .$$

When $s \gg \delta$, the nonlinearity becomes negligible. Application of WKB to (C.7) shows that

(C.9) $$V \sim \frac{C_w}{s^{1/2}p^{1/2}}\exp\left(+\frac{i}{\delta}\int_1^s p(r)\,dr\right)\ , \quad p = (-2U)^{1/2} \sim \sqrt{s^2-1}\ , \qquad \delta^{2/3} \ll s-1\ ,$$

from which it follows that

(C.10) $$V \sim \frac{C_w}{s}\exp\left(+i\frac{s^2}{2\delta}\right)\ , \quad s \gg 1\ .$$

Only the term with the plus sign in the exponent was used in (C.9) and (C.10) in order to ensure that $\psi_s \sim V/L\exp(ir^2 L_z/4L)$ has no rapid oscillations as it connects to ψ_{back}. The connection formula for (C.9) beyond the turning point at s_b (e.g. [9, chapter 10], [30, chapter 7]) gives

$$V \sim \frac{C_w\exp(-i\pi/4)}{s^{1/2}|p|^{1/2}}\exp\left(-\frac{1}{\delta}\int_1^s |p(s')|\,ds'\right)\ , \quad |p| \sim \sqrt{1-s^2}\ , \qquad \delta \ll s < 1\ , \quad s-1 \ll \delta^{2/3}\ .$$

In particular, when $\delta \ll s \ll 1$, $|p| \sim 1$ and

(C.11) $$V \sim \frac{C_w\exp(-i\pi/4)}{s^{1/2}}\exp\left(-\frac{1}{\delta}\left(\int_1^0 \sqrt{1-s^2}\,ds + s\right)\right)\ , \qquad \delta \ll s \ll 1\ .$$

The value of C_w is determined by matching (C.11) with (C.8) and using $\int_0^1 \sqrt{1-s^2}\,ds = \pi/4$:

(C.12) $$C_w = A_R\delta^{1/2}\exp\left(-\frac{\pi}{4\delta}\right)\exp\left(+i\frac{\pi}{4}\right)\ .$$

Combining (C.6), (C.10) and (C.12) gives

$$V \sim 2^{1/2} A_R \beta^{-1/4} \rho^{-1} \exp\left(-\frac{\pi}{2\sqrt{\beta}} + i\frac{\pi}{4} + i\frac{\beta^{1/2}}{4}\rho^2\right) , \quad \rho \gg \beta^{-1/2} . \tag{C.13}$$

If we substitute (C.13) into (C.4), we get that in the domain of validity of (C.13) the rate of power radiation is independent of ρ:

$$\frac{d}{dz} N_s \sim -\frac{2A_R^2}{L^2} \exp(-\pi/\sqrt{\beta}) ,$$

which is (3.16). However, if we substitute (C.13) into (C.5), the result will depend on ρ. Therefore, in order to estimate (C.5) we need the asymptotic behavior of V just to the right of the second turning point $s_b = 1$ (C.9):

$$V \sim \frac{C_w}{2^{1/4}(s-1)^{1/4}} \exp\left(+i\frac{2^{1/2}}{\delta}\int_1^s (s'-1)^{1/2}\, ds'\right) , \quad \delta^{2/3} \ll s-1 \ll 1 . \tag{C.14}$$

If we substitute (C.14) into (C.5) and use $\delta \sim -LL_z/2$ and $\delta^{2/3} \ll s-1$, we get that for leading order

$$\frac{d}{dz} H_s \sim -\frac{2A_R^2}{L^4} \exp(-\pi/\sqrt{\beta}) ,$$

which is (3.18).

D. Asymptotic growth of H_s. In order to estimate the rate at which H_s growths in the adiabatic regime, we use (3.18) and the adiabaticity of β to write

$$H_s \sim -M\nu(\beta) \int^z \frac{1}{L^4(z')}\, dz' .$$

If we use (3.37) and integrate, we get

$$H_s \sim -\frac{M\nu(\beta)}{4\beta} \frac{1}{Z_c - z}$$

or

$$H_s \sim -\frac{M\nu(\beta)}{2\sqrt{\beta}} \frac{1}{L^2} .$$

Note that this implies that $H_s(Z_c - z)^2$ becomes exponentially small compared with $\sqrt{\beta}(Z_c - z)$, which is consistent with the approximation of (3.36) by (3.37).

E. Solvability conditions for V_1. In the derivation of the reduced equations from a solvability condition for V_1 we use the following result:

LEMMA E.1. *Let $V_1 = S + iT$ be the solution of*

$$\Delta_\perp V_1 - V_1 + 2R^2 V_1 + R^2 V_1^* = p(x,y) + iq(x,y) , \tag{E.1}$$

where S, T, p, q are real and $R(r)$ is the positive solution of $\Delta_\perp R + R^3 - R = 0$. Then the solvability condition for S is that $\int S \nabla_\perp R = 0$ and the solvability condition for T is that $\int TR = 0$.

From lemma E.1 it immediately follows that

COROLLARY E.2. *If p is an even function, the equation for the real part of V_1 in (E.1) is always solvable.*

The proof of lemma E.1 follows from the the following result, which is given in [64], but not proved there for L_+ for the 2d case. Here we give a proof which can be generalized to all dimensions and powers of nonlinearity.

LEMMA E.3. *Let*

$$L_+ = (\Delta_\perp + 3R^2 - 1)\ , \quad L_- = (\Delta_\perp + R^2 - 1)\ ,$$

where

$$\Delta_\perp = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

be operators on

$$B = \{ f \in C^2[0\ \infty) \mid f_r(0) = 0,\ f(\infty) = 0 \}\ .$$

Then

1. L_+ is a self-adjoint operator with null space $N(L_+) = span\{R_r\}$.

2. L_- is a self-adjoint operator with null space $N(L_-) = span\{R\}$

Proof:

We can easily see that R_r is in the null space of L_+ by differentiating the equation for R. Hence, we can use R_r to find the second independent solution u by considering

$$u = vR_r$$

From $L_+u = 0$, the equation for v is:

$$2v_r(R_r)_r + v_{rr}R_r + \frac{d-1}{r}v_rR_r = 0$$

This equation can be easily solved, and we get that

$$u = R_r \int^r \frac{1}{(r')^{d-1}[R_r]^2}\, dr'$$

For large r, $R \sim r^{-1/2}e^{-r}$ and u diverges. Hence, u is not in $N(L_+)$.

The proof for L_- is similar.

F. Proof of proposition 4.2. When $f_2 \not\equiv 0$, dimensional argument shows that

$$\frac{[(f_1)_z]}{[f_2]} = \frac{[L^2]}{[Z]} \sim \beta^{1/2} \ll 1\ .$$

Therefore, the leading order behavior of (4.3) is given by (4.7). Since in this case we the accuracy of the approximation is $O(\beta^{1/2})$, there is no point in keeping the $\nu(\beta)$ radiation term.

When $f_2 \equiv 0$ (4.3) becomes (4.15). For leading order we can neglect the $\nu(\beta)$ term and integrate (4.15) to get equation (4.6).

G. Proof of proposition 4.3. If (4.8) holds, we can multiply (4.6) by $-2L_z/L^3$, use (3.5) and integrate to get

$$L_z^2 = \frac{\beta_0}{L^2} - \frac{\epsilon C_1}{4M}\frac{1}{L^4} + D\ , \quad D = \text{constant} \tag{G.1}$$

or

$$y_z^2 = 4\beta_0 - \frac{\epsilon C_1}{M}\frac{1}{y} + 4Dy\ . \tag{G.2}$$

Although the value of D can be obtained directly from (G.1), it is more instructive to obtain it by deriving (G.2) from Hamiltonian balance. To do so, we multiply (4.1) by ψ_z^*, add the conjugate equation and integrate, to get an equation for balance of Hamiltonian in (4.1):

$$\text{(G.3)} \qquad \frac{\partial}{\partial z} H(\psi) = \frac{\epsilon}{2\pi} \int [\psi_z^* F(\psi) + \text{c.c.}]\, dxdy \; .$$

The right-hand-side of (G.3) can be approximated using (3.12), (3.18) and (4.22):

$$\text{(G.4)} \qquad \frac{1}{2\pi} \int [\psi_z^* F(\psi) + \text{c.c.}]\, dxdy \sim \left(\frac{1}{2L^2}\right)_z f_1 + \frac{2}{L^2} f_2 \; .$$

In order to simplify the left-hand-side of (G.3), we note that

$$H(\psi) = H(\psi_s) + H(\psi_{back}) \; , \quad H(\psi_s) \sim ML_z^2 + \frac{H(V_0)}{L^2} \; .$$

In addition, from lemmas A.1 and B.1 we have

$$H(V_0) \sim -\beta M + \frac{1}{2}\epsilon f_1 \; .$$

Therefore,

$$\text{(G.5)} \qquad H(\psi_s) \sim \frac{M}{2}(L^2)_{zz} + \frac{\epsilon}{2L^2} f_1 \; .$$

In the conservative case (G.3–G.4):

$$H_z \sim \frac{\epsilon}{2}\left(\frac{1}{L^2}\right)_z f_1 \; .$$

Using (4.8) and integrating we get

$$H = H_0 - \frac{\epsilon C_1}{4}\frac{1}{L^4} \; , \quad H_0 \sim H(0) + \frac{\epsilon C_1}{4}\frac{1}{L_0^4} \; .$$

Using (4.8) and (G.5), multiplying by $4y_z/M$ and integrating again gives

$$y_z^2 = -\frac{\epsilon C_1}{M}\frac{1}{y} + \frac{4H_0}{M} y + \text{constant} \; .$$

Comparison of this equation with (G.2) gives (4.9), which can be rewritten as (4.10).

1. From (4.9) we see that if $\epsilon C_1 > 0$ then y (or L) cannot go to zero.
   (a) When $\beta_0 > 0$ and $H_0 > 0$, (4.9) can be written as

   $$y_z^2 = \frac{4|H_0|}{M}\frac{1}{y}(y + |y_M|)(y - y_m) \; .$$

   (b) If $\beta_0 > 0$ and $H_0 < 0$ then $0 < y_m < y_M$. To evaluate ΔZ we note that

   $$\Delta Z = \sqrt{\frac{M}{-H_0}} \int_{y_m}^{y_M} \sqrt{\frac{y}{(y_M - y)(y - y_m)}}\, dy \; .$$

   Substituting $(y - y_m)/(y_M - ym) = \cos^2 u$ gives (4.14).
2. In this case $y_m > 0$.
3. The location of (first) arrest is

$$z_0 = -\int_{y(0)}^{y_m} \frac{1}{2}\sqrt{\frac{M}{-H_0}}\sqrt{\frac{y}{(y_M - y)(y - y_m)}}\, dy \sim \frac{1}{2}\int_0^{y(0)} \left(\beta_0 + \frac{H_0}{M} y\right)^{-1/2} dy = Z_c \; .$$

When ψ_0 is real, then $L_z(0) = 0$, and $y(0) = y_M$.

H. Useful relations. The following relations are useful in analysis of the reduced modulation equations:

(H.1)
$$\beta = -L^3 L_{zz} = \frac{1}{4}y_z - \frac{1}{2}yy_{zz} = \frac{A_{\zeta\zeta}}{A} = a^2 + a_\zeta \ ,$$

(H.2)
$$\beta_z = -\frac{1}{2}yy_{zzz} \ ,$$

where

$$y = L^2 \ , \quad A = \frac{1}{L} \ , \quad a = -LL_z = \frac{-L_\zeta}{L} \ .$$

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