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Abstract

We consider the convergence of viscous solutions to inviscid solutions with shocks as viscosity tends to zero. Our analysis reveals rich structure of nonlinear wave interactions due to the presence of shocks and initial layers. This interaction generates four different wave patterns, initial layer, shock layers, diffusion waves and coupling waves. We study the propagation and interactions of the four wave patterns by a pointwise analysis.

1 Introduction

Consider a system of conservation laws with small artificial viscosity

$$u_t^\epsilon + f(u^\epsilon)_x - \epsilon u_{xx}^\epsilon = 0, \quad u^\epsilon \in R^n, \; t > 0, \quad \epsilon > 0, \tag{1.1}$$

and a system of hyperbolic conservation laws

$$u_t^0 + f(u^0)_x = 0, \quad u^0 \in R^n, \quad t > 0. \tag{1.2}$$

The purpose of the present paper is to study the process of zero dissipation limit $\lim_{\epsilon \to 0} u^\epsilon$ as the viscosity ϵ tends to zero with a given fixed initial data

$$u^\epsilon(x,0) = u_{in}(x).$$

This problem can be solved by a regular method, when the corresponding inviscid solution u^0 is smooth. However, when shock waves occur in the underlying inviscid solution $u^0(x,t)$, the process for analyzing the difference between the inviscid and the viscous solutions can not be done by regular methods. We obtain a pointwise description for the difference in terms of the magnitude of the viscosity. For the hyperbolic system (1.2), the initial shock waves remain a shock wave at least for a short time. On the contrary, the shock wave in the viscous solution will evolve into a smooth profile in a time scale of order *viscosity/ (strength of the shock)*^{2}. This smooth profile is called a *shock layer*, and its width is of the order *viscosity/ (strength of the shock)*. The transition from a shock wave in the initial data into a shock layer in the viscous solution is called an *initial layer*. After a shock layer develops, the viscous solution also produces relatively non-small diffusion waves moving away from the shock layer. Those diffusion waves will cope with the shock layer through characteristic curves and move toward the shock again, and thus cause the change of the wave fronts of the shock layer. In the mean time, the changes of the wave fronts will affect the diffusion waves. These wave interactions exhibit rich nonlinear wave phenomena.

The analysis is carried in two steps:

For the initial layer the formation to a shock layer is a fully nonlinear effect. In general, one can not have a direct method to analyze the formation of a shock layer. However, one can compare the compressive field with solution of Burgers' equation when the strength of the shock is sufficiently small. In this situation, one can use the nonlinearity of Burgers' equation to obtain a qualitative structure of the initial layer, since the formation of the shock layer takes place almost in the compressive field.

After the formation of the shock layer, the linearized equation around the shock layer dominates the essential wave pattern. However, this linearized equation is just neutral stable. Hence, there will be problems in analyzing the nonlinear interaction between the shock layer and diffusion waves in a time scale larger than *viscosity/ (strength of shock* $)^3$. One thus needs to introduce coupling waves and phase shifts due to the interaction between diffusion waves and shock layer. Combining the coupling wave and the phase shift together with the diffusion wave, one can introduce an anti-derivative variable to factor out the neutral stability. Finally, we can apply our pointwise estimates to establish the existence of the viscous solution and obtain a pointwise estimate for the difference between the viscous solution and the inviscid solution.

The existence of a piecewise smooth solution $u^0(x,t)$ of (1.2) is studied by [10]. The corresponding zero viscosity problem has been studied by [2] and [1] for some special 2x2 isentropic gas dynamics by the method of compensated compactness. When $f(u)$ is a scalar equation, it is studied by [16] and [17]. In [14], the zero dissipation limit of a compressible Navier-Stokes equation for isentropic solution with a shock initial data is studied, and a qualitative behavior for an initial layer evolving into a shock layer is given in details. The problem (1.1) had been studied by [5], but initial layer, diffusion waves and coupling waves are not encountered. In [11], a pointwise estimate is used to study an asymptotic stability of a viscous shock profile. In [6], [7], a reduction of an initial value problem on an infinite domain to one on finite domain is devised for relaxing smallness assumption on strength of a shock of viscous conservation laws. In [12] and [13] error waves generated at discontinuities by linear schemes approximating linear P.D.E. are found to propagate into smooth regions and the effects are analyzed. In [3], a nonlinear scheme approximating the same linear equation is used to remove the global error.

Let us briefly state the properties and mechanism of the initial layer, diffusion waves and coupling waves:

- Initial layer.
  Nonlinear wave patterns present in the time scale $O(\epsilon)\cdot$ (Strength of the shock $)^{-(2+\alpha_0)}$ for any $\alpha_0 \in (0,1/8)$.

- Diffusion wave.
  It is generated by the L^1 norm of the difference between the shock layer and shock data in the non-compressive fields. It is transversal to the shock front and of order $\frac{1}{\sqrt{\frac{t}{\epsilon}+1}}\cdot$(Strength of the shock) for any given but small time t.

- Coupling wave.
  It is generated by the interactions between diffusion waves and the shock layer.

The analysis in this paper contains a series of detailed pointwise estimates, which are motivated by those in [11]. Our analysis provides a detailed pointwise description for the difference between the inviscid solution and the viscous solution within a given space-time domain. We now give a precise statement of

our result:
The basic hypothesis is that the system (1.2) is strictly hyperbolic, that is, for each $u \in R^n$ the matrix $f'(u)$ has n real distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$:

$$\begin{aligned}\frac{\partial f(u)}{\partial u}\, r_j(u) &= \lambda_j(u)\, r_j(u),\\ l_j(u)\, \frac{\partial f(u)}{\partial u} &= \lambda_j(u)\, l_j(u),\\ (l_j(u), r_j(u)) &= \delta^i_j, \quad i,j = 1,2,\cdots,n.\end{aligned}$$

We also assume that each characteristic field is either genuinely nonlinear (g.nl.) or linearly degenerate (l.dg.), (see [8]), i.e., for each $i \in \{1, 2, \cdots, n\}$, either

$$\begin{aligned}&(g.nl.) \qquad \nabla\lambda_j(u)\cdot r_j(u) \neq 0 \ \text{ for all } u, \text{ or}\\ &(l.dg.) \qquad \nabla\lambda_j(u) \equiv 0 \ \text{ for all } u.\end{aligned}$$

For each genuinely nonlinear i-field there exists shock wave (u_-, u_+) for (1.2) which satisfies the jump (Rankine-Hugoniot) condition and entropy condition, (see [8]),

$$\begin{aligned}(R-H) \qquad & \sigma(u_- - u_+) = f(u_-) - f(u_+),\\ (E) \qquad & \begin{cases}\lambda_i(u_+) < \sigma < \lambda_i(u_-),\\ \lambda_{i-1}(u_-) < \sigma < \lambda_{i+1}(u_+),\end{cases}\end{aligned}$$

where σ is the propagation speed of the shock wave (u_-, u_+).
We also make assumptions on the solution $u^0(x,t)$ of the hyperbolic equation (1.2).

Hyp Assumption. *There exist positive constants C, T and a curve $x = \gamma(t)$ such that*

$$\sum_{k=0}^{3} \sup_{t\in[0,T]} \|\partial_t^k \gamma(t)\| + \sup_{\substack{t\in[0,T]\\ x\neq\gamma(t)}} \|\partial_x^k u^0(x,t)\| < C, \quad \gamma(0) = 0.$$

Furthermore, the states $(u^0(\gamma(t)-,t), u^0(\gamma(t)+,t))$ are shock waves in the i-field for $t \le T$.

Set

$$\delta \equiv \sup_{0\le t\le T} \| u^0(\gamma(t)-,t) - u^0(\gamma(t)+,t) \|.$$

Shock Wave Assumption. *We also assume that for $0 \le t \le T$ the curve $x = \gamma(t)$ satisfies*

$$\inf_{t\in[0,T]} \|u^0(\gamma(t)-,t) - u^0(\gamma(t)+,t)\| \ge \frac{\delta}{2}.$$

Main Theorem. *Suppose that (1.2) is strictly hyperbolic and that the Hyp Assumption and the Shock Wave Assumption are true.*

Then, there exist positive constants ϵ_0, η_0, C_0, $T_0(\delta)$ and O_0 such that if $\delta \le \eta_0$ then for each $\epsilon \in (0, \epsilon_0]$ the solution $u^\epsilon(x,t)$ of (1.1) satisfies that

$$\sup_{t\le T_0} \|u^0(x,t) - u^\epsilon(x,t)\| \le O_0 \left(\delta^2 e^{-\frac{\delta|x-\gamma(t)|}{C_0\ \epsilon}} + \epsilon + \sum_{j\neq i} \frac{\delta\sqrt{\epsilon} e^{-\frac{(x-\Xi_j(t))^2}{4C_0\epsilon t}}}{\sqrt{4\pi t}} \right)$$
$$+O_0\delta \left(\frac{\epsilon^{3/2}}{(\epsilon + |x-\gamma(t)|)\sqrt{(t+|x-\gamma(t)|)}} + \sum_{j\neq i} \frac{\epsilon^3}{\left[(x-\Xi_j(t))^2 + \epsilon t + \epsilon^2\right]^{3/2}} \right)$$

for $\delta^{-2-\alpha_0}\epsilon \le t \le T_0$, where $x = \Xi_j(t)$ is j-characteristic curve starting from 0, *and α_0 is a positive number in* $(0, 1/8)$.
Remark. The constants described in the theorem are required to satisfy

$$\epsilon \ll \delta^6 \text{ and } T_0 = O(1)\, \delta^3.$$

In section 2, we study the variation of viscous shock profiles with respect to the shock wave (u_-, u_+), and use it to construct the leading order approximate solution whose approximation error is estimated in terms of the viscosity. In section 3, the nonlinear scalar equation for the diffusion wave is given. This nonlinear equation is transformed into an integral equation, and an approximate Green's function is introduced to study its qualitative behavior. In section 4, we study a system of linear evolution equation, which is the nonlinear equation linearized around the approximate solution constructed in section 2 with some modification in the compressed field. We construct approximate Green's functions for this linear system, and give basic estimates of the coupling of all linear waves. In section 5, we construct coupling waves generated by the interaction between the diffusion waves and the shock layer, and find appropriate phase shifts of the shock layer. In section 6, we consider the situation when the initial value is a shock data. Burgers' equation is introduced to study the transition of a shock data into a shock layer in an intermediate time scale and the asymptotic stability of the viscous solution is obtained pointwise.

2 Viscous Shock Profiles and Leading Order Approximation

2.1 Structure of Shock Profile

We begin with the study of the structure of a shock layer connecting a given entropy condition satisfied shock wave (u_l, u_r) whose speed s is given by

$$s(u_l - u_r) = f(u_l) - f(u_r). \tag{2.1}$$

A shock profile for (1.1) is a travelling wave solution $u^\epsilon(x,t) = U(\frac{x-s\ t}{\epsilon})$ of (1.1) connecting (u_l, u_r). For the shock wave (u_l, u_r) and its speed s, we can treat u_r as a function of u_l and s. Therefore, there are another two extra parameters in the travelling wave solution $U(\eta; u_l, s)$. By substituting U into (1.1) and integrating it from $x = -\infty$, we have that

$$U_\eta = f(U) - f(u_l) - s(U - u_l) \text{ with } U(-\infty; u_l, s) = u_l. \tag{2.2}$$

An auxiliary system is introduced to study (2.2), (see [15]),

$$\begin{cases} \partial_\eta s = 0, \\ \partial_\eta u_l = 0, \\ U_\eta = f(U) - f(u_l) - s(U - u_l). \end{cases} \tag{2.3}$$

Let w_0 be a state whose i-characteristic speed is the same as the speed of the shock wave (u_-, u_+),

$$f'(w_0)\ r_i(w_0) \ = \ s\ r_i(w_0)$$

with a normalization condition $w_0 - u_-$ and $u_- - u_+$ are parallel. Hence, w_0 is uniquely determined since i-characteristic field is genuinely nonlinear. Thus $w_0 = w_0(u_l, s)$ is a function of u_l and s.
One can see directly that (s, w_0, w_0) is a fixed point of the dynamical system (2.3), and at this fixed point, there is an $n+2$-dimensional invariant manifold which is tangential to the plane spanned by

$$\mathbf{R}^n \times 0 \times \vec{0}, \ \ \vec{0} \times R \times \vec{0}, \ \ \vec{0} \times 0 \times r_i(w_0).$$

This invariant manifold is the center manifold $\mathscr{M}$. We can parameterize $\mathscr{M}$,

$$\begin{aligned} \mathscr{M} : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} : \ &\longmapsto \ \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n, \\ \mathscr{M}(u, \sigma, \eta) \ &\equiv \ (u, \sigma, \mathscr{M}_1(u, \sigma, \eta)), \end{aligned}$$

where the function $\mathscr{M}_1(u, \sigma, \eta)$ satisfies that

$$\mathscr{M}_1(w_0, s, 0) \ = \ w_0, \ \ \frac{\partial_\eta \mathscr{M}_1(w_0, s, 0)}{\|\partial_\eta \mathscr{M}_1(w_0, s, 0)\|} \ = \ \frac{r_i(w_0)}{\|r_i(w_0)\|}. \tag{2.4}$$

From (2.3) any two states σ and u remain constant on an integral curve given on this center manifold $\mathscr{M}$, therefore the curve $\mathscr{C}(\xi) = \mathscr{M}_1(u_l, s, \xi)$ with $\xi \in \mathbf{R}$ contains the integral curve for (2.2) connecting (u_l, u_r). We re-parameterize this curve $\mathscr{C}$ by

$$\|r_i(w_0)\| \cdot (l_i(w_0), \mathscr{C}'(\eta)) = 1. \tag{2.5}$$

From (2.4) and (2.5) it yields that there exist constants C_1 and η_0 such that when $\|u_l - u_r\| \leq \eta_0$

$$\|r_i(w_0)\| \cdot (l_j(w_0), \mathscr{C}'(\xi)) \leq C_1 \|u_l - u_r\| \text{ for } j \neq i \text{ and } |\xi| \ \leq \ 2\ \|u_l - u_r\|. \tag{2.6}$$

Then, this estimate implies that there exist $\delta_\pm$ satisfying that

$$\begin{cases} \mathscr{C}(\delta_l) = u_l, \ \mathscr{C}(\delta_r) = u_r, \\ \delta_l = -\delta_r + O(\ \|u_l - u_r\|^2\ ) \ = \ \frac{1}{2}\|u_l - u_r\| \ + \ O(\ \|u_l - u_r\|^2\ ). \end{cases} \tag{2.7}$$

Since this curve $\mathscr{C}$ is an invariant curve, two vectors

$$\mathscr{C}'(\xi) \text{ and } \frac{f(\mathscr{C}(\xi)) - f(u_l) - s(\mathscr{C}(\xi) - u_l)}{\|f(\mathscr{C}(\xi)) - f(u_l) - s(\mathscr{C}(\xi) - u_l)\|}$$

are parallel. Hence we can define a function $c(\xi)$ as follows

$$c(\xi) \ \equiv \mathscr{C}'(\xi) \ / \ \frac{f(\mathscr{C}(\xi)) - f(u_l) - s(\mathscr{C}(\xi) - u_l)}{\|f(\mathscr{C}(\xi)) - f(u_l) - s(\mathscr{C}(\xi) - u_l)\|}. \tag{2.8}$$

From this definition of $c(\xi)$ there exist positive constants C_0 and η_0 such that

$$\frac{1}{C_0} < -c(\xi) < C_0 \text{ for } |\xi| < 4\|u_l - u_r\| \text{ for } \|u_l - u_r\| < \eta_0. \tag{2.9}$$

Let $U(\eta; u_l, s) \equiv \mathscr{C}(\xi(\eta))$, and substitute (2.8) into (2.2). Then, we reduce the O.D.E. for $U(\eta; u_l, s)$ into a scalar O.D.E. for ξ,

$$\partial_\eta \xi = \bar{c}(\xi), \qquad (2.10)$$
$$\xi(-\infty) = \delta_l, \quad \xi(\infty) = \delta_r,$$

where

$$\bar{c}(\xi) \equiv \frac{\|f(\mathscr{C}(\xi)) - f(u_l) - s(\mathscr{C}(\xi) - u_l)\|}{c(\xi)}.$$

From (2.10) and (2.9) there exists $C_0 > 0$ such that

$$C_0\ (\xi - \delta_l)(\xi - \delta_r) \le \bar{c}(\xi) \le \frac{1}{C_0}\ (\xi - \delta_l)(\xi - \delta_r) \text{ for } \xi \in [\delta_r, \delta_l]. \qquad (2.11)$$

Since the O.D.E. (2.10) is independent of η, there is a one-parameter family of solution. We normalize the solution by requiring that

$$\int_{-\infty}^{0} (\delta_l - \xi(\eta))d\eta + \int_0^\infty (\delta_r - \xi(\eta))d\eta = 0, \qquad (2.12)$$
$$\equiv l_i(w_0)\left(\int_{\eta<0} \{u_l - \mathscr{C}(\xi(\eta))\}d\eta + \int_{\eta>0} \{u_r - \mathscr{C}(\xi(\eta))\}d\eta\right) \ = \ 0.$$

Proposition 2.1. *Suppose that $U(\eta; u_l, s) = \mathscr{C}(\xi(\eta))$ is the solution of (2.2) normalized by (2.12). Then, there exist constants C_0, K_0 and η_0 such that when $\|u_l - u_r\| \le \eta_0$,*

$$\int_0^\infty (u_r - U(\eta; u_l, s))\, d\eta + \int_{-\infty}^0 (u_l - U(\eta; u_l, s))\, d\eta \le K\|u_l - u_r\|,$$
$$\|U(\eta; u_l, s) - u_l\| \le K\ \|u_l - u_r\|\ e^{\frac{\|u_l - u_r\|}{C_0}\eta} \quad \textit{for } \eta \le 0,$$
$$\|U(\eta; u_l, s) - u_r\| \le K\ \|u_l - u_r\|\ e^{-\frac{\|u_l - u_r\|}{C_0}\eta} \quad \textit{for } \eta \ge 0.$$

Proof. From (2.11), (2.10) and (2.12) , there exists $K > 0$ and $C_0 > 0$ such that

$$|\xi(\eta) - \delta_l| \le \frac{K}{2}\|u_l - u_r\|\ e^{-\frac{\|u_l - u_r\|}{C_0}\eta} \text{ for } \eta \le 0, \qquad (2.13)$$
$$|\xi(\eta) - \delta_r|\ \le K\ \|u_l - u_r\|\ e^{-\frac{\|u_l - u_r\|}{C_0}\eta} \text{ for } \eta \ge 0.$$

From (2.7), (2.13) and mean value theorem, we have that there exists $K > 0$ such that

$$l_j(w_0)\,(\mathscr{C}(\xi(\eta)) - u_l)$$
$$= \int_0^{\xi(\eta)-\delta_l} l_j(w_0)\cdot \mathscr{C}'(\rho)d\rho\ (\xi(\eta) - \delta_l)\ \le\ K\ \|u_l - u_r\|^2\ e^{\frac{\|u_l - u_r\|}{C_0}\eta} \text{ for } \eta \le 0, \text{ and } j \ne i; \qquad (2.14)$$
$$\left|l_j(w_0)\,(\mathscr{C}(\xi(\eta)) - u_r)\right| \le K\ \|u_l - u_r\|^2\ e^{-\frac{\|u_l - u_r\|}{C_0}\eta} \text{ for } \eta \ge 0, \text{ and } j \ne i.$$

From (2.5), it follows

$$
\begin{aligned}
&l_i(w_0)\cdot(\mathscr{C}(\xi(\eta))-u_l)\\
&\quad=\int_{\delta_l}^{\xi(\eta)} l_i(w_0)\mathscr{C}'(\rho)\,d\rho \;=\; \frac{1}{\|r_i(w_0)\|}\,(\xi(\eta)-\delta_l) \text{ for } \eta\le 0,\\
&l_i(w_0)\cdot(\mathscr{C}(\xi(\eta))-u_r)\\
&\quad=\frac{1}{\|r_i(w_0)\|}\,(\xi(\eta)-\delta_r) \text{ for } \eta\ge 0.
\end{aligned}
$$

From this and the normalization condition (2.12), we have that

$$
l_i(w_0)\left(\int_{\eta>0}\{u_l-\mathscr{C}(\xi(\eta))\}\,d\eta+\int_{\eta<0}\{u_r-\mathscr{C}(\xi(\eta))\}\,d\eta\right)=0.
$$

For $j\neq i$ from (2.14)

$$
\begin{aligned}
&\left|l_j(w_0)\left(\int_{\eta>0}\{u_l-\mathscr{C}(\xi(\eta))\}\,d\eta+\int_{\eta<0}\{u_r-\mathscr{C}(\xi(\eta))\}\,d\eta\right)\right|\\
&\quad\le 2\,K\|u_l-u_r\|^2\int_{\eta>0} e^{-\frac{\|u_l-u_r\|}{C_0}\eta}\,d\eta\le 2\,C_0\,K\,\|u_l-u_r\|.
\end{aligned}
$$

Hence, Proposition 2.1 follows. **Q.E.D.**

2.2 Leading Order Approximation and its Approximation Error

For the sake of convenience, let's denote $u(x,t)$ the viscous solution $u^\epsilon(x,t)$ in the rest of this paper, first.

Since we expect that viscosity will smear a shock into smooth shock layer with width of order ϵ, we introduce the slow variables

$$
\begin{cases}
X(x,t)\equiv \dfrac{x-\gamma(t)}{\epsilon},\\[2mm]
T(x,t)\equiv \frac{t}{\epsilon},
\end{cases}
$$

to localize the microscopic structure around the shock front. Under this new coordinate (X,T) the systems (1.2) and (1.1) become

$$
\begin{aligned}
\partial_T u^0 \;-\; s(T)\,\partial_X u^0 \;+\; \partial_X f(u^0) \;&=\; 0;\\
\partial_T u \;-\; s(T)\,\partial_X u \;+\; \partial_X f(u) \;-\; \partial_X^2 u \;&=\; 0,
\end{aligned}
\tag{2.15}
$$

with the same given initial data

$$
u^0(X,0)=u(X,0)=u_{in}(\epsilon X,0),
$$

where

$$
s(T)\equiv\gamma'(\epsilon T).
$$

For the convenience of our analysis , we will change the coordinate

$$
\begin{cases}
x\;\Rightarrow\;X(x,t),\\[2mm]
t\;\Rightarrow\;T(x,t)
\end{cases}
$$

in the rest of this paper.
In this new coordinate the Hyp Assumption implies that

$$\sup_{t\le\frac{T_0}{\epsilon}}\sup_{x\neq 0}\left(|\partial_t^j u^0(x,t)|+|\partial_x^j u^0(x,t)|\right)=O(\epsilon^j)\text{ for }j=0,1,2,3, \tag{2.16}$$

and the shock waves stay at $x=0$.
Let $U(\eta;u^0(0-,t),s(t))$ be the solution of (2.2) connecting $(u^0(0-,t),u^0(0+,t))$ and normalized by (2.12). We introduce a family of travelling waves $\phi(x,t)$ given by

$$\phi(\eta,t)\equiv U(\eta;u^0(0-,t),s(t)), \tag{2.17}$$

and a partition of unit on the domain $\mathbf{R}\times[0,T_0/\epsilon]$.
Let $\chi_-,\ \chi_0,\ \chi_+\in C^\infty(R\times[0,\infty))$ satisfy the following

$$\begin{array}{l}
\chi_-(x,t),\ \chi_0(x,t),\ \chi_+(x,t)\ge 0,\\
\chi_-(x,t)+\chi_0(x,t)+\chi_+(x,t)=1;\\
supp(\chi_0)\subset\{(x,t):|x|\le 2\},\ \text{and }\chi_0(x,t)=1\text{ for }|x|\le 1,\\
supp(\chi_-)\subset\{(x,t):x\le -1\},\ \text{and }\chi_-(x,t)=1\text{ for }x\le -2,\\
supp(\chi_+)\subset\{(x,t):1\le x\},\ \text{and }\chi_+(x,t)=1\text{ for }2\le x.
\end{array}$$

A leading order approximation to $u(x,t)$ is defined as follows.

$$\begin{aligned}
\bar u^1(x,t)\ &\equiv\ \chi_0(x,t)\cdot\phi(x,t)\\
&+\ \chi_-(x,t)\cdot\left\{[u^0(x,t)-u^0(0-,t)]+\phi(x,t)\right\}\\
&+\ \chi_+(x,t)\cdot\left\{[u^0(x,t)-u^0(0+,t)]+\phi(x,t)\right\}.
\end{aligned}$$

From Proposition 2.1 and Assumption Hyp we have that there exist constants K_0 and C_0 for $t\le T_0/\epsilon$

$$\begin{aligned}
\|\ \phi(\eta,t)-u^0(0+,t)\ \|\ &\le\ K_0\|u^0(0-,t)-u^0(0+,t)\|\ e^{-\frac{\|u^0(0-,t)-u^0(0+,t)\|}{C_0}\eta}\text{ for }\eta>0,\\
\|\ \phi(\eta,t)-u^0(0-,t)\ \|\ &\le\ K_0\|u^0(0-,t)-u^0(0+,t)\|\ e^{\frac{\|u^0(0-,t)-u^0(0+,t)\|}{C_0}\eta}\text{ for }\eta\le 0.
\end{aligned} \tag{2.18}$$

Proposition 2.2 *There exists constant C_0 such that for $t\le T_0/\epsilon$*

$$\begin{aligned}
\phi_t(\eta,t)-\partial_t u^0(0+,t)\ &=\ O(\epsilon)\ e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0}|\eta|}\quad \textit{for }\eta>0,\\
\phi_t(\eta,t)-\partial_t u^0(0-,t)&=O(\epsilon)\ e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0}|\eta|}\quad \textit{for }\eta<0.
\end{aligned}$$

Proof: The situation for $x<0$ and $x>0$ are the same, therefore we consider the situation $x\le 0$ only. From the definition of $\phi(x,t)$ and (2.10), the equation for $\phi(x,t)$ is

$$\phi_x\ =\ F(\phi,t), \tag{2.19}$$

$$F(\phi,t)\ \equiv\ f(\phi)-f(u^0(0-,t))-s(t)(\phi-u^0(0-,t)), \tag{2.20}$$

with a normalization condition

$$l_i(w_0(u(0-,t),s(t)))\cdot\left(\int_0^\infty\phi(\eta,t)-u^0(0+,t)d\eta\ +\int_{-\infty}^0\phi(\eta,t)-u^0(0-,t)d\eta\right)\ =\ 0.$$

Take a partial t derivative on (2.19). Then, we have that

$$\partial_\eta \phi_t = F_\phi(\phi,t)\phi_t + F_t(\phi,t), \tag{2.21}$$

From the definition of $F(y,s)$ (2.20), we have that

$$F_{ty}(y,t) = -s'(t) = O(\epsilon), \tag{2.22}$$
$$F(u^0(0-,t),t) = 0 \text{ for } t \in \mathbf{R}. \tag{2.23}$$

$$F_\phi(u^0(0-,t),t)\partial_t u^0(0-,t) + F_t(u^0(0-,t),t) = 0.$$

From (2.21), (2.22) and (2.23) we have that

$$\begin{aligned}
&\partial_\eta(\phi_t - \partial_t u^0(0-,t)) \\
&= \partial_\eta \phi_t \\
&= F_\phi(\phi,t)\left(\phi_t - \partial_t u^0(0-,t)\right) \\
&\quad +(F_\phi(\phi,t) - F_\phi(u^0(0-,t),t))\partial_t u^0(0-,t) + (F_t(\phi,t) - F_t(u^0(0-,t),t)).
\end{aligned} \tag{2.24}$$

On the other hand, from (2.22) we have that

$$\begin{aligned}
&F_t(\phi,t) - F_t(\bar{u}^0(0-,t),t) \\
&= \int_0^1 F_{t\eta}(\theta(\phi - u^0(0-,t)) + u^0(0-,t),t)\ d\theta\ |\phi - u^0(0-,t)| \\
&= O(\epsilon)|\phi - u^0(0-,t)| \\
&= O(\ \epsilon\|u^0(0-,t) - u^0(0+,t)\|\ )e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0}\ |\eta|}.
\end{aligned} \tag{2.25}$$

Combine (2.24) and (2.25),

$$\left(\phi_t - \partial_t u^0(0-,t)\right)_\eta + F_\phi(\phi,t)\left(\phi_t - \partial_t u^0(0-,t)\right) = \mathscr{S}(\eta), \tag{2.26}$$
$$\text{where } \mathscr{S}(\eta) = O(\epsilon)\ (\phi(\eta,t) - u(0-,t)\ ) \tag{2.27}$$
$$= O(\ \epsilon\|u^0(0-,t) - u^0(0+,t)\|\ )e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0}\ |\eta|}.$$

By taking the η-derivative on (2.19), it yields that $V(\eta) = \phi_\eta(\eta)$ is a solution of

$$V_\eta = F_\phi(\phi,t)V.$$

Hence, the general solution $V(\eta)$ of this O.D.E. is of the form

$$V(\eta) = k\ \phi_\eta(\eta) \text{ for any } k \in R. \tag{2.28}$$

On the other hand, for the system (2.26) we have a particular solution bounded by

$$\begin{aligned}
&\int_{-\infty}^{\eta} e^{-k_0(\eta-\rho)}\|\mathscr{S}(\rho)\|\ d\rho + \int_\eta^0 e^{k_0(\eta-\rho)}\|\mathscr{S}(\rho)\|\ d\rho + \int_{-\infty}^{\eta} \|\mathscr{S}(\rho)\| \cdot \left\|\frac{\phi_\eta(\eta)}{\phi_\eta(\rho)}\right\| d\rho \\
&= O(\epsilon)\ \left(\int_R e^{-k_0|(\eta-\rho)|}\|\phi(\rho,t) - u(0-,t)\|\ d\rho + \int_{-\infty}^{\eta} \|\phi_\eta(\eta)\| \cdot \frac{\|\phi(\eta,t) - u(0-,t)\|}{\|\phi_\eta(\rho)\|} d\rho\right) \\
&= O(1)\epsilon\ e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0}\ |\eta|} \text{ for } \eta < 0,
\end{aligned}$$

where k_0 is a positive constant independent of $\|u(0-,t) - u(0+,t)\|$. Therefore, from this and (2.28) we have that there is a constant B_0 such that

$$\phi_t(\eta, t) - \partial_t u^0(0-, t) = B_0 \phi_\eta \; + \; O(1)\epsilon \; e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0} |\eta|} \text{ for } \eta < 0. \tag{2.29}$$

Similar to (2.29), one could have that there is a constant B_1 such that

$$\phi_t(\eta, t) - \partial_t u^0(0+, t) = B_1 \phi_\eta + O(\epsilon) e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0} \eta} \text{ for } \eta \geq 0. \tag{2.30}$$

Now we match $\phi_t(\eta, t)$ in (2.29) and (2.30) at $\eta = 0$. From (2.29) and (2.30) we have that

$$(B_0 - B_1)\phi_\eta(0, t) = O(\epsilon) + \partial_t \left(u^0(0-, t) - u^0(0+, t)\right) = O(\epsilon).$$

This yields that

$$(B_0 - B_1) = O(1) \, \frac{\epsilon}{\|u(0-,t) - u(0+,t)\|^2}. \tag{2.31}$$

Since w_0 is a smooth of $u(0-, t)$ and $s(t)$,

$$\frac{d}{dt} w_0(u(0-, t), s(t)) = O(1)\epsilon. \tag{2.32}$$

Applying ∂_t to (2.12), it yields that

$$\begin{aligned} &-\frac{dl_i(w_0(u(0-,t), s(t)))}{dt} \cdot \left(\int_{x<0} \phi(x,t) - u^0(0-,t)dx + \int_{x>0} \phi(x,t) - u^0(0+,t)dx \right) \\ &= l_i(w_0(u(0-,t), s(t))) \cdot \left(\int_{x<0} \phi_t(x,t) - u_t^0(0-,t)dx + \int_{x>0} \phi_t(x,t) - u_t^0(0+,t)dx \right). \end{aligned} \tag{2.33}$$

Substitute (2.32) and Proposition 2.1 into (2.33), then we can conclude that

$$\begin{aligned} O(1)\epsilon \;\; &= \;\; \|u(0-,t) - u(0+,t)\| \left(\frac{(B_0 + B_1)}{2} + O(1) \; \|u(0-,t) - u(0+,t)\| \; (|B_0| + |B_1|) \right) \\ &\quad + O(1) \; \frac{\epsilon}{\|u(0-,t) - u(0+,t)\|}. \end{aligned} \tag{2.34}$$

(2.31) and (2.34) yield that

$$B_0, \; B_1 = O(1) \frac{\epsilon}{\|u(0-,t) - u(0+,t)\|^2} \tag{2.35}$$

Substitute (2.35) and $\phi_\eta = O(1) \; \|u(0+,t) - u(0-,t)\|^2 \; e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0} |\eta|}$ into (2.29), (2.30), then Proposition 2.2 follows. **Q.E.D.**

We define the approximate error $E_1(x, t)$ of this leading approximation $\bar{u}^1$,

$$E_1(x,t) \equiv \partial_t \bar{u}^1 - s(t)\partial_x \bar{u}^1 + \partial_x f(\bar{u}^1) - \partial_x^2 \bar{u}^1. \tag{2.36}$$

Lemma 2.1 *For $t \leq T_0/\epsilon$ the approximation error $E_1(x,t)$ satisfies that*

$$E_1(x,t) = O(1)(\epsilon^2 + \epsilon^2 |\log \epsilon| \; \chi_{I_a} + \epsilon \; \chi_{I_b}),$$

where

$$I_a \equiv \left\{(x,t) : |x| \le 4\frac{|\log\epsilon|}{C_0\|u^0(0-,t)-u^0(0+,t)\|},\ \ t \le \frac{T_0}{\epsilon}\right\},$$
$$I_b \equiv \left\{(x,t) : |x| \le 2,\ t \le \frac{T_0}{\epsilon}\right\},$$

the constant C_0 is given in Proposition 2.1, and the functions χ_{I_a} and χ_{I_b} are the characteristic functions of the domains I_a and I_b, respectively.

Proof.

When $x \ \ge (\le -)\ 4C_0|\log\epsilon| \ / \ \|u^0(0-,t)-u^0(0+,t)\|$, from (2.18) we can expand $\bar{u}^1(x,t)$ as $u^0(x,t) + (\phi(x,t)-u^0(0\pm,t))$. Hence we have that

$$\bar{u}^1 = u^0 + O(\epsilon^4) \text{ for } |x| \ge \ 4C_0\frac{|\log\epsilon|}{\|u(0-,t)-u(0+,t)\|}.$$

This estimate implies that

$$E_1(x,t) = -\partial_x^2 u^0(x,t) + O(\epsilon^4) = O(\epsilon^2) \quad \text{for } |x| \ge \ 4C_0\frac{|\log\epsilon|}{\|u(0-,t)-u(0+,t)\|}.$$

When $2 \le x \le 4C_0\frac{|\log\epsilon|}{\|u(0-,t)-u(0+,t)\|}$, we expand $\bar{u}^1(x,t)$ as

$$\begin{aligned}\bar{u}^1(x,t)-\phi(x,t) &= u(x,t)-u(0+,t)\\ &= x\int_0^1 u_x^0(x\rho,t)\,d\rho \ = \ O(\ \epsilon\ |x|\ ).\end{aligned}$$

From this expansion, it yields that for $2 \le x \le 4C_0\frac{|\log\epsilon|}{\|u(0-,t)-u(0+,t)\|}$

$$\begin{aligned}&E_1(x,t)\\ &= \partial_t\left\{\phi(x,t)+[u^0(x,t)-u^0(0+,t)]\right\} - s(t)\partial_x(u^0(x,t)-u^0(0+,t))\\ &\quad +\partial_x\left\{f\left(\phi(x,t)+u^0(x,t)-u^0(0+,t)\right)-f(\phi)\right\} - \partial_x^2u^0(x,t)\\ &= (\phi_t(x,t)-u_t^0(0+,t)\ ) + [f'(\bar{u}^1)-f'(\phi)]\phi_x + [f'(\bar{u}^1)-f'(\phi)]u_x^0 + O(\epsilon^2)\\ &= \left(\phi_t(x,t)-\partial_t u^0(0+,t)\right)\\ &\quad +O(1)\left(\ \epsilon\ |x|\cdot\|u(0+,t)-u(0-,t)\|^2 e^{-\frac{\|u(0+,t)-u(0-,t)\|}{C_0}|x|}\ +\ \epsilon^2\ |x| + \epsilon^2\ \right).\end{aligned} \tag{2.37}$$

From Proposition 2.1, Proposition 2.2 and (2.37), we have that $2 \le x \le 4C_0\frac{|\log\epsilon|}{\|u(0-,t)-u(0+,t)\|}$

$$E_1(x,t) = O(1)\left(\epsilon\ e^{-\frac{\|u(0+,t)-u(0-,t)\|}{2\ C_0}|x|}\ +\ |\log\epsilon|\ \epsilon^2\ \right). \tag{2.38}$$

When $|x| \le 2$, we expand $\bar{u}^1(x,t)$ as follows

$$\bar{u}^1(x,t) = \phi(x,t) + \chi_-(x,t)(u^0(x,t)-u^0(0-,t)) + \chi_+(x,t)(u^0(x,t)-u^0(0+,t)).$$

Substituting this into the definition of $E_1(x,t)$ we have that

$$E_1(x,t) = O(\epsilon) \text{ for } |x| \le 2. \tag{2.39}$$

Hence, Lemma 2.1 follows. **Q.E.D.**

Note. Without loss of generality we may the constant C_0 in Proposition 2.2 and Lemma 2.1 is one, and replace the factor $\|u(0-,t)-u(0+,t)\|$ in the exponent by δ, that is,

$$e^{-\frac{\|u(0-,t)-u(0+,t)\|}{2C_0}|x|}=O(1)e^{-\delta\ |x|}$$

in the rest of this paper.

3 Asymptotic Nonlinear Diffusion Waves

Though the system (2.15) is a parabolic system, there are hyperbolic type waves, which carry masses of the difference between shock wave initial data and the shock layer away from the shock front along characteristic curves. They are called the nonlinear diffusion waves. In this section our primary interest is to construct those nonlinear diffusion waves.

First, we introduce a modified background profile u_j^0 from $u^0(x,t)$ for constructing the diffusion waves in the j-characteristic field. For each $j<i$ ($j>i$) the function $\bar{u}_j^0$ is a function satisfying that

$$\begin{aligned} \bar{u}_j^0(x,t) &= u^0(x,t) \quad \text{for } x<0, (x>0) \\ \partial_x^k\bar{u}_j^0(x,t) &= O(\epsilon^k) \quad \text{for } k=0,1,2,3. \end{aligned} \tag{3.1}$$

We linearize the system (2.15) around the background profile $\bar{u}_j^0$ to obtain a linear system for the variable $\mathscr{V}_j$, and then add a non-coupling nonlinear source term $\partial_x\ \mathscr{Q}_j(\mathscr{V}_j)$ given in (3.3), to the LHS of this linear system. The resulting nonlinear system is

$$\partial_t\mathscr{V}_j - s(t)\ \partial_x\mathscr{V}_j + \partial_x\ f'(\bar{u}_j^0)\mathscr{V}_j - \partial_x^2\mathscr{V}_j + \partial_x\mathscr{Q}_j(\mathscr{V}_j)\ =\ 0. \tag{3.2}$$

Introduce the variables associated to $\bar{u}_j^0$,

$$\begin{aligned} \lambda_{j,k}(x,t) &\equiv \lambda_k(\bar{u}_j^0(x,t)) - s(t), \\ r_{j,k}(x,t) &\equiv r_k(\bar{u}_j^0(x,t)), \\ l_{j,k}(x,t) &\equiv l_k(\bar{u}_j^0(x,t)), \\ \mathscr{V}_{j,k}(x,t) &\equiv l_{j,k}(x,t)\mathscr{V}_j(x,t), \\ C_{j,pq}^m(x,t) &\equiv l_{j,m}(x,t)f''(\bar{u}_j^0(x,t))(r_{j,p}(x,t),r_{j,q}(x,t)). \end{aligned}$$

Then, we decompose the vector $\mathscr{V}_j$ and related variables in terms of the above variables,

$$\begin{aligned} \mathscr{V}_j(x) &\equiv \sum_{k=1}^n \mathscr{V}_{j,k}(x,t)\ r_{j,k}(x,t); \\ f'(\bar{u}_j^0(x,t))\ r_{j,k} &\equiv (\lambda_{jk}(x,t)+s(t))\ r_{j,k}, \\ l_{j,k}(x,t)\ f'(\bar{u}_0^j(x,t)) &\equiv (\lambda_{j,k}(x,t)+s(t))\ l_{j,k}(x,t), \\ \mathscr{Q}_j(\mathscr{V}_j) &\equiv \frac{1}{2}\ C_{j,jj}^m(x,t)\ \mathscr{V}_{j,j}(x,t)^2\ r_{j,j}(x,t), \end{aligned} \tag{3.3}$$

Let $\mathscr{V}_j(x,t)=\theta_j(x,t)r_{j,j}(x,t)$ and substitute it into RHS of (3.2), then we consider the following initial value problem,

$$\begin{aligned} l_{j,j}(x,t)\cdot\left[\partial_t\ \theta_j(x,t)r_{j,j} + \partial_x\ \lambda_{j,j}\theta_j r_{j,j} - \partial_x^2\ \theta_j r_{j,j} + \partial_x\ \mathscr{Q}_j(\theta_j r_{j,j})\ \right] &= 0, \\ \theta_j(x,0) = \delta_j K(x,1), & \end{aligned} \tag{3.4}$$

where $K(x,t)$ is the heat kernel

$$K(x,t) \equiv \frac{1}{\sqrt{4\pi t}} Exp\left(-\frac{x^2}{4t}\right),$$

and the constants δ_k are given by the following decomposition

$$\begin{aligned} &\sum_{j<i} \delta_j r_j(u^0(0-,0)) + \sum_{i<j} \delta_j r_j(u^0(0+,0)) + \delta_i(u^0(0+,0) - u^0(0-,0)) \\ &= \int_0^\infty \phi(x,0) - \bar{u}(0+,0)dx + \int_{-\infty}^0 \phi(x,0) - \bar{u}(0-,0)dx. \end{aligned} \tag{3.5}$$

By adjusting a suitable wave fronts of $\phi(\eta,0)$ one can assume that $\delta_i = 0$, (see [11]); and from Proposition 2.1 we have that

$$\delta_j \;=\; O(1)\ (\|u(0-,t) - u(0+,t)\|) \;=\; O(1)\ \delta.$$

Expanding (3.4) directly, we have the nonlinear scalar equation

$$\begin{aligned} \partial_t\theta_j + \lambda_{j,j}(x,t)\ \partial_x\theta_j - \partial_x^2\theta_j &= -\frac{1}{2}\partial_x\ C^j_{j,jj}(x,t)\ \theta_j^2 + O(\epsilon)\ \theta_j, \\ \theta_j(x,0) &= \delta_j\ K(x,t). \end{aligned} \tag{3.6}$$

3.1 Generalized Burgers' Equation and Duhamel's Principle

If $\epsilon = 0$, then $\lambda_{j,j}$ and $C^j_{j,jj}$ are constants because of their definition and (3.1). In this case, (3.6) is a Burgers' equation. Then, by Cole-Hopf transformation one can have the precise behavior of the solution. However, in our case ϵ is a nonzero positive constant, therefore the functions $\lambda_{j,j}$ and $C^j_{j,jj}$ are almost constant functions in the sense

$$|\partial_x^i\lambda_{j,j}|,\ |\partial_x^i C^j_{j,jj}| = O(\epsilon^i) \text{ for } i = 1,2,3. \tag{3.7}$$

Lemma 3.1 *There exist positive constants T_0 and η_0 such that for any $\delta_j \le \eta_0$ the solution of (3.6) satisfies that for $t \le T_0/\epsilon$*

$$\theta(x,t) \le \delta_j K\left(\frac{M_j(x,t)}{2}, t\right).$$

Proof. We will proceed an iteration procedure to construct the solution and its qualitative behavior for this generalized Burgers' equation.
Let's rewrite (3.6) in terms of the variables (y,s).

$$\begin{aligned} \partial_s\theta_j + \lambda_{j,j}(y,s)\ \partial_y\theta_j - \partial_y^2\theta_j &= \epsilon\ \theta_j(y,s) - \frac{1}{2}\partial_y\ C^j_{j,jj}\theta_j^2(y,s), \\ \theta(y,0) &= \delta_j K(y,1). \end{aligned} \tag{3.8}$$

Next, we want to transform this equation into an integral equation for the $\theta_j(x,t)$.

For any given (x,t) we multiply (3.8) by a function $g(y,s)$, and integrate it over the domain $R \times [0,t]$. Here, we make the *basic assumption* on $g(y,t)$,

$$g(y,t) = \delta(x-y). \tag{3.9}$$

Then, by integration by parts we have that

$$\begin{aligned}\theta_j(x,t) &= \int_R g(y,0)\theta_j(y,0)dy \\ &+ \int_0^t \int_R \{g_s(y,s) + (\lambda_{j,j}\ g)_y(y,s) + g_{yy}(y,s)\} \cdot \theta_j(y,s)dyds \\ &+ \int_0^t \int_R \left\{\epsilon\ g(y,s)\ \theta_j(y,s) + \frac{1}{2}\ g_y(y,s)\ C^j_{j,jj}(y,s)\ \theta_j^2(y,s)\right\} dyds.\end{aligned} \tag{3.10}$$

For the first double integral in R.H.S. of (3.10), we introduce a function $M_j(y,s)$ to approximate the solution of of dual equation $(\partial_s + \partial_y\ \lambda_{j,j} + \partial_y^2)\ g(y,s) = 0$, first.
Let $\{\mathscr{S}_j(t';x') : t \in R\}$ be integral curves given by

$$\begin{aligned}\frac{d}{dt'}\mathscr{S}_j(t';x') &= \lambda_{j,j}(\mathscr{S}_j(t';x'),t'); \\ \mathscr{S}_j(0;x') &= x'.\end{aligned}$$

A real-valued function $M_j(y,s)$ is defined by the implicit function theorem,

$$\mathscr{S}_j(s;M_j(y,s)) = y \text{ for } y \in \mathbf{R},\ \ s \in \mathbf{R}^+. \tag{3.11}$$

From this definition (3.11) there exists a constant T_0 such that for $s \in (0, T_0/\epsilon)$

$$\partial_s M_j(y,s) + \lambda_{j,j}(y,s)\ \partial_y M_j(y,s) = 0; \tag{3.12}$$

$$\|\partial_y M_j(\cdot,s)\|_\infty = O(1) \text{ and } \|\partial_{ys} M_j(\cdot,s)\|_\infty,\ \ \|\partial_{yy} M_j(\cdot,s)\|_\infty = O(\epsilon). \tag{3.13}$$

For each given (x,t) set

$$g(y,s;x,t) \equiv K\left(\frac{M_j(y,s) - M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right). \tag{3.14}$$

Note. Without confusion, we simply use $g(y,s)$ in stead of $g(y,s;x,t)$ in the rest of this section.

From this definition (3.14) the function $g(y,t)$ is a delta function at time $s = t$, therefore (3.9) the *basic assumption* on $g(y,s)$ is true.

Now we proceed to construct the iteration solutions $\{\ \psi_l(x,t)\ \}_{l\geq 1}$ for showing the existence of the nonlinear solution $\theta_j(x,t)$.
When $l = 1$,

$$\psi_1(x,t) \equiv \int_R g(y,0)\ \theta_j(y,0)dy. \tag{3.15a}$$

When $l \geq 2$,

$$\begin{aligned}\psi_l(x,t) &\equiv \int_R g(y,0)\ \theta_j(y,0)dy \\ &+ \int_0^t \int_R \{g_s(y,s) + (\lambda_{j,j}\ g)_y(y,s) + g_{yy}(y,s)\} \cdot \psi_{l-1}(y,s)dyds \\ &+ \int_0^t \int_R \left\{\epsilon\ g(y,s)\ \psi_{l-1}(y,s) + \frac{1}{2}\ g_y(y,s)\ C^j_{j,jj}(y,s)\ \psi_{l-1}(y,s)^2\right\} dyds.\end{aligned} \tag{3.15b}$$

It remains to show that $\{\psi^l\}_{l\geq 1}$ is a Cauchy sequence for some given norm space. This is equivalent to show that the sequence,

$$\{\Delta\psi^l\}_{l\geq 2} \;\equiv\; \{\psi^l - \psi^{l-1}\}_{l\geq 2},$$

is a geometric sequence for some given norm space.

In order to find the suitable norm space, we need a good pointwise estimate of the function $\psi_1(x,t)$. From (3.11) the definition of $M_j(y,s)$ we have that $M_j(y,0) = y$ and substitute this into the definition of $g(y,s)$. Then, (3.15a) becomes

$$\begin{aligned}\psi_1(x,t) &= \delta_j \int_R K(M_j(x,t)-y,t)\; K(y,1)dy \\ &= \delta_j K(M_j(x,t), t+1).\end{aligned} \tag{3.16}$$

The estimate (3.16) suggests a priori assumption on ψ_l for $l \geq 2$ and $t \in [0, T_0/\epsilon]$,

$$\begin{aligned}\psi_l(x,t) &\leq \delta_j\; \Psi_0(x,t), \\ \Psi_0(x,t) &\equiv 2\; K\left(\frac{M_j(x,t)}{2}, t+1\right).\end{aligned} \tag{3.17}$$

From this a priori estimate we introduce a pointwise norm $\|\cdot\|_{\Psi_0,T}$ for functions in $L^\infty(\mathbf{R}\times[0,T/\epsilon])$,

$$\|\mathscr{F}\|_{\Psi_0,T} \;\equiv\; \sup_{t\leq\frac{T}{\epsilon}}\sup_{x\in R}\left|\frac{\mathscr{F}(x,t)}{\Psi_0(x,t)}\right|_\infty \quad \text{for } \mathscr{F}\in L^\infty(\mathbf{R}\times[0,T/\epsilon]).$$

From this definition we rewrite (3.16) and the a priori assumption (3.17) as

$$\|\psi_1\|_{\Psi_0,T_0} \;\leq\; \frac{1}{2}\delta_j, \tag{3.18}$$

$$\|\psi_j\|_{\Psi_0,T_0} \;\leq\; \delta_j \text{ for } j\geq 2. \tag{3.19}$$

From this and (3.15b) it yields that

$$\begin{aligned}\Delta\psi_2(x,t) = &\int_0^t\int_R \left\{\partial_s g(y,s) + \partial_y\;\lambda_{j,j}(y,s)\; g(y,s) + \partial_y^2 g(y,s)\right\}\;\psi_1(y,s)dyds \\ &+\int_0^t\int_R\left\{\epsilon\; g(y,s)\psi_1(y,s) + \frac{1}{2}\; g_y(y,s)\; C^j_{j,jj}(y,s)\;\psi_1(y,s)^2\right\}dyds \\ \leq\; &\frac{1}{2}\,\delta_j\int_0^t\int_R \left|\partial_s g(y,s) + \partial_y\;\lambda_{j,j}(y,s)\; g(y,s) + \partial_y^2 g(y,s)\right|\Psi_0(y,s)dyds \\ &+\frac{1}{2}\,\delta_j\;\epsilon\;\int_0^t\int_R g(y,s)\;\Psi_0(y,s)dyds \\ &+O(1)\;\delta_j^{\,2}\;\int_0^t\int_R |g_y(y,s)|\;\Psi_0(y,s)^2dyds.\end{aligned} \tag{3.20a}$$

Similarly, for $l\geq 3$

$$\begin{aligned}&\Delta\psi_l(x,t) \\ &\leq\;\|\Delta\psi_{l-1}\|_{\Psi_0,T}\int_0^t\int_R\left|\partial_s g(y,s) + \partial_y\;\lambda_{j,j}(y,s)\; g(y,s) + \partial_y^2 g(y,s)\right|\;\Psi_0(y,s)dyds\end{aligned} \tag{3.20b}$$

$$+ O(1)\ \epsilon\ \|\Delta\psi_{l-1}\|_{\Psi_0,T} \int_0^t \int_R g(y,s)\ \Psi_0(y,s)\ dyds$$
$$+ O(1)\ \delta_j\ \|\Delta\psi_{l-1}\|_{\Psi_0,T} \int_0^t \int_R |g_y(y,s)|\Psi_0^2(y,s) dyds.$$

Before we continue the above estimates (3.20), let's introduce the functions,

$$\begin{aligned}
\mathscr{I}_1(x,t) &\equiv \int_0^t \int_R g(y,s)\Psi_0(y,s)dyds,\\
\mathscr{I}_2(x,t) &\equiv \int_0^t \int_R |g_y|(y,s)\Psi_0(y,s)dyds,\\
\mathscr{I}_3(x,t) &\equiv \int_0^t \int_R |g_y(y,s)|\Psi_0^2(y,s)dyds,\\
\mathscr{I}_4(x,t) &\equiv \int_0^t \int_R (\partial_s g + \lambda_{j,j}\partial_y g + \partial_y^2 g)\ \Psi_0(y,s)dyds.
\end{aligned}$$

Then, we can rewrite (3.20) as

$$|\Delta\psi_2(x,t)| = O(1)\ \delta_j\ \left(\mathscr{I}_4(x,t) + \epsilon\ \mathscr{I}_1(x,t)\ +\ \delta_j\ \mathscr{I}_3(x,t)\ \right), \tag{3.21a}$$
$$|\Delta\psi_l(x,t)| = O(1)\ \|\Delta\psi_{l-1}\|_{\Psi_0,T}\ \left(\mathscr{I}_4(x,t)\ +\ \epsilon\ \mathscr{I}_1(x,t)\ +\ \delta_j\mathscr{I}_3(x,t)\ \right)\ \text{for } l \geq 3. \tag{3.21b}$$

From this estimate we need to show that $\|\mathscr{I}_4\|_{\Psi_0,T} \ll 1$, before showing that $\{\|\Delta\psi^l(x,t)\|_{\Psi_0,T}\}_{l\geq 2}$ is a geometric sequence. Hence, from the definition of $\mathscr{I}_4$ we need to evaluate the function $\partial_s g + \lambda_{j,j}\ \partial_y g + \partial_y^2 g$ first. Substitute the express (3.14) into this function, then it yields that

$$\begin{aligned}
&\partial_s g + \lambda_{j,j}\partial_y g + \partial_y^2 g \\
&= K_X\left(\frac{M_j(y,s) - M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right) \cdot \frac{\{M_j(x,t) - M_j(y,s)\}(M_{jys}(y,s) + M_{jyy}(y,s))}{M_{jy}(y,s)^2}\\
&-K_X\left(\frac{M_j(y,s) - M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right) \cdot \left\{\frac{(M_j(x,t) - M_j(y,s))M_{jyy}(y,s)}{M_{jy}(y,s)^2}\right\}_y\\
&+K_{XX}\left(\frac{M_j(y,s) - M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right) \cdot M_{jyy}\frac{M_j(y,s) - M_j(x,t)}{M_{jy}(y,s)^2}\left(M_{jyy}\frac{M_j(y,s) - M_j(x,t)}{M_{jy}(y,s)^2} + 1\right).
\end{aligned} \tag{3.22}$$

Plugging the condition (3.13) into the RHS of (3.22), it follows

$$\partial_s g + \lambda_{j,j}\partial_y g + \partial_y^2 g \tag{3.23}$$
$$= O(\epsilon)\ K_X\left(\frac{M_j(y,s) - M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right)$$
$$+O(\epsilon)\ K_{XX}\left(\frac{M_j(y,s) - M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right) \cdot (M_j(y,s) - M_j(x,t)) \tag{3.24}$$
$$+O(\epsilon^2)\ K_{XX}\left(\frac{M_j(y,s) - M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right) \cdot (M_j(y,s) - M_j(x,t))^2.$$

Set

$$\begin{aligned}
A_+^j &\equiv \sup_{t\leq T/\epsilon} \|\partial_y M_j(\cdot,t)\|_\infty,\\
A_-^j &\equiv \inf_{t\leq T/\epsilon} \|\partial_y M_j(\cdot,t)\|_\infty.
\end{aligned} \tag{3.25}$$

From the structure of the characteristic curve $\mathscr{S}_j(t,x)$ it follows

$$A^j_\pm \;=\; 1 \;+\; O(T).$$

Hence, we can choose T so small that

$$|A^j_\pm| \;\leq\; \frac{3}{2}. \tag{3.26}$$

The definition of $M_j(y,s)$ (3.11) gives that $M_j(\mathscr{S}(t;0),t) = 0$. This and (3.25) imply that

$$\begin{aligned}
|M_j(x,t)| &= |M_j(x,t) - M_j(\mathscr{S}_j(t,0),t)| \\
&= \left| \int_{\mathscr{S}_j(t,0)}^{x} \partial_y M_j(y,t)dy \right| \\
&\leq A^j_+ \, |x - \mathscr{S}_j(t,0)|, \\
|M_j(x,t)| &\geq A^j_- \, |x - \mathscr{S}_j(t,0)|.
\end{aligned}$$

From the definition of $g(y,s)$ (3.14) it yields that for $t \leq 1/\epsilon$

$$|\partial_y g(y,s)| = O(1)\frac{1}{\sqrt{t-s}}K\left(\frac{M_j(x,t)-M_j(y,s)}{\frac{5}{4}A^j_+}, t-s\right), \tag{3.27a}$$

$$\left|K_{YY}\left(\frac{M_j(y,s)-M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right)\cdot(x-y)\right| = \frac{O(1)}{\sqrt{t-s}}K\left(\frac{M_j(x,t)-M_j(y,s)}{\frac{5}{4}A^j_+}, t-s\right), \tag{3.27b}$$

$$\left|K_{YY}\left(\frac{M_j(y,s)-M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right)\cdot(x-y)^2\right| = O(1)\;K\left(\frac{M_j(x,t)-M_j(y,s)}{\frac{5}{4}A^j_+}, t-s\right), \tag{3.27c}$$

$$\left|K_{YY}\left(\frac{M_j(y,s)-M_j(x,t)}{\partial_y M_j(y,s)}, t-s\right)\cdot(t-s)\right| = O(1)K\left(\frac{M_j(x,t)-M_j(y,s)}{\frac{5}{4}A^j_+}, t-s\right). \tag{3.27d}$$

Substitute (3.27) into (3.23), then we obtain that

$$\begin{aligned}
&\partial_s g + \lambda_{j,j}\partial_y g + \partial_y^2 g \\
&= O(\epsilon)\left(1 + \frac{1}{\sqrt{t-s}}\right)K\left(\frac{M_j(x,t)-M_j(y,s)}{\frac{5}{4}A^j_+}, t-s\right).
\end{aligned} \tag{3.28}$$

Combining this, (3.17) and the definition of Ψ_0 (3.17) we have that for $t \;\leq\; T/\epsilon$

$$\begin{aligned}
&\int_0^t \int_R \left(\partial_s g(y,s) + \lambda_{j,j}\partial_y g(y,s) + \partial_y^2 g(y,s)\right)\Psi_0(y,s)dyds \\
&= \; O(1)\left(T \;+\; \sqrt{\epsilon}\,T\right)\;\Psi_0(x,t), \\
\text{that is, } \|\mathscr{I}_4\|_{\Psi_0,T} &= \; O(1)\left(T \;+\; \sqrt{\epsilon}\,T\right).
\end{aligned} \tag{3.29}$$

For the function $\mathscr{I}_1$

$$\int_R g(y,s)\Psi_0(y,s)dy$$

$$
\begin{aligned}
&= O(1) \int_R K\left(\frac{M_j(y,s)-M_j(x,t)}{\frac{5}{4}A_+^j}, t-s\right) K\left(\frac{M_j(y,s)}{2}, s\right) dy \\
&= O(1) K\left(\frac{M_j(x,t)}{2}, t\right);
\end{aligned}
$$

$$
\|\mathscr{I}_1\|_{\Psi_0,T} = O(1)\ \frac{T}{\epsilon}. \tag{3.30a}
$$

For the function $\mathscr{I}_2(x,t)$

$$
\begin{aligned}
&\int_R |\partial_y g(y,s)|\ \Psi_0(y,s)dy \\
&\quad = O(1) \int_R \frac{1}{\sqrt{t-s}} K\left(\frac{M_j(y,s)-M_j(x,t)}{\frac{5}{4}A_+^j}, t-s\right) K\left(\frac{M_j(y,s)}{2}, s\right) dy \\
&\quad = O(1) \frac{1}{\sqrt{t-s}} K\left(\frac{M_j(x,t)}{2}, t\right);
\end{aligned}
$$

$$
\|\mathscr{I}_2\|_{\Psi_0,T} = O\left(1\right)\ \frac{\sqrt{T}}{\sqrt{\epsilon}}. \tag{3.30b}
$$

For the function $\mathscr{I}_3(x,t)$

$$
\begin{aligned}
&\int_R |\partial_y g(y,s)|\ \Psi_0^2(y,s)dy \\
&\quad = O(1) \int_R \frac{1}{\sqrt{t-s}\sqrt{s}} K\left(\frac{M_j(y,s)-M_j(x,t)}{\frac{5}{4}A_+^j}, t-s\right) K\left(\frac{M_j(y,s)}{2}, s\right) dy \\
&\quad = O(1) \frac{1}{\sqrt{t-s}\sqrt{s}} K\left(\frac{M_j(x,t)}{2}, t\right),
\end{aligned}
$$

$$
\|\mathscr{I}_3(x,t)\|_{\Psi_0,T} = O(1). \tag{3.30c}
$$

Substitute the estimates in (3.30) into (3.21), then we have that

$$
\|\Delta\psi_l\|_{\Psi_0,T} = O(1)\ \left(T + \sqrt{\epsilon\, T} + \delta_j\right)\ \|\Delta\psi_{l-1}\|_{\Psi_0,T}.
$$

Hence when both T and δ_j are sufficiently small, the sequence $\{\psi_l\}_{l\geq 2}$ is a geometric sequence under the norm $\|\cdot\|_{\Psi_0,T}$. This concludes that there exists T_0 and η_0 such for any $\delta_j < \eta_0$ and $T \leq T_0$

$$
\theta_j(x,t) = O(\delta_j) K\left(\frac{M_j(x,t)}{2}, t\right) \text{ for } t \leq \frac{T}{\epsilon}.
$$

Lemma 3.1 follows. **Q.E.D.**

4 Higher Order Linear Diffusion Waves, I

In the previous section, we have established the nonlinear diffusion wave θ_j in the non-compressive characteristic fields, that is, the j-characteristic field with $j \neq i$. From those nonlinear diffusion waves we introduce a diffusion wave system $\Theta_1(x,t)$,

$$\Theta_1(x,t) \equiv \sum_{j\neq i} \theta_j(x,t) r_{j,j}(x,t). \tag{4.1}$$

This system is constructed from waves leaving the shock front without interacting with the the shock layer. We want to use the diffusion system to study interaction of the nonlinear diffusion waves and the shock layer.

First, we introduce a formal approximation error system $E_2(x,t)$,

$$E_2 \equiv \sum_{j\neq i} \left\{ \partial_t\ \theta_j\ r_{j,j} + \partial_x\ \lambda_{j,j}\ \theta_j\ r_{j,j} - \partial_x^2\ \theta_j\ r_{j,j} + \frac{1}{2}\ \partial_x\ \left[C^j_{j,jj}\ \theta_j^2\ r_{j,j}\right] \right\}. \tag{4.2}$$

From (3.4) we have that for each $j \neq i$

$$\begin{aligned}
&\partial_t\ \theta_j\ r_{j,j} + \partial_x\ \lambda_{j,j}\ \theta_j\ r_{j,j} - \partial_x^2\ \theta_j\ r_{j,j} + \frac{1}{2}\ \partial_x\ C^j_{j,jj}\ \theta_j^2\ r_{j,j} \\
&= \sum_{k\neq j} \left\{ l_{j,k} \cdot \left[\partial_t\ \theta_j\ r_{j,j} + \partial_x\ \lambda_{j,j}\ \theta_j\ r_{j,j} - \partial_x^2\ \theta_j\ r_{j,j} + \frac{1}{2}\ \partial_x\ C^j_{j,jj}\ \theta_j^2\ r_{j,j} \right] \right\}\ r_{j,k} \\
&= \sum_{k\neq j} \left\{ l_{j,k} \cdot \left(O(1)(|\theta_{jx}| + |\theta_j|) r_{j,jx} + O(1)\theta_j\ r_{jt} + O(1)\theta_j\ r_{j,jxx} \right) \right\}\ r_{j,k} \\
&= \sum_{k\neq j} O(\epsilon)(\ \theta_j + \theta_{jx})\ r_{j,k}.
\end{aligned}$$

This yields that

$$E_2(x,t) = \sum_{j\neq i}\sum_{k\neq j}\ \left[O(\epsilon)(\ \theta_j(x,t)\ \ +\ \ \theta_{jx}(x,t)\ )\right]\ r_{j,k}(x,t). \tag{4.3}$$

Next, we want to analyze the interaction between E_2 and the structure of the shock layer, therefore we introduce the functions around the leading order approximation $\bar{u}^1(x,t)$,

$$\begin{aligned}
\bar{r}_k(x,t) &\equiv r_k(\bar{u}^1(x,t)), \\
\bar{l}_k(x,t) &\equiv l_k(\bar{u}^1(x,t)), \\
\bar{\lambda}_k(x,t) &\equiv \lambda_k(\bar{u}^1(x,t)) - s(t), \\
C^k_{pq}(x,t) &\equiv \bar{l}_k(x,t) f''(\bar{u}^1(x,t))(\bar{r}_p(x,t), \bar{r}_q(x,t)) \text{ for } 1 \leq k,p,q \leq n.
\end{aligned}$$

We rewrite the formal approximation error system $E_2(x,t)$ in the terms of the functions around the approximation solution $\bar{u}^1$,

$$\begin{aligned}
&\partial_t\Theta_1 - s(t)\ \partial_x\Theta_1 + \partial_x\ f'(\bar{u}^1)\ \Theta_1 - \partial_x^2\Theta_1 + \frac{1}{2}\ \partial_x \sum_{j\neq i} C^j_{j,jj}\theta_j^2\ \bar{r}_j \\
&= E_2(x,t) + \partial_x \left\{ \sum_{j\neq i} \theta_j \cdot (f'(\bar{u}^1) - f'(\bar{u}^0_j)) r_{j,j}(x,t) + \frac{1}{2} \sum_{j\neq i} C^j_{j,jj}\theta_j^2\ (\bar{r}_j - r_{j,j}) \right\}.
\end{aligned} \tag{4.4}$$

We need to find the coordinate of the vector $E_2(x,t)$ with respect to the basis $\{\bar{r}_1(x,t), \cdots, \bar{r}_n(x,t)\}$, that is,

$$E_2(x,t)\ \equiv\ \sum_{m=1}^{n} E_2^m(x,t)\ \bar{r}_m(x,t).$$

First, we compare the vectors $r_j(x,t)$ and $r_{j,j}(x,t)$. They are the normalized right eigenvectors of $f'(\bar{u}^1)$ and $f(\bar{u}_j^0)$, The limit of their difference $\lim_{\text{sgn}(j-i)\cdot x\to\infty} r_j(x,t) - r_{j,j}(x,t) = 0$ with the same rate as $\lim_{\text{sgn}(j-i)\cdot x\to\infty} \bar{u}^1(x,t) - \bar{u}_j^0(x,t)=0$, because both $r_i(x,t)$ and $r_{j,j}(x,t)$ are normalized eigenvector of the matrices $f'(\bar{u}^1(x,t))$ and $f'(u_j^0)$. We obtain that

$$\begin{aligned}
&\|\bar{r}_j(x,t) - r_{j,j}(x,t)\| \\
&= O(1)\|u^0(x,t) - \bar{u}_j^0(x,t)\| \\
&= O(\delta)\ \min\left(1, e^{-\text{sgn}(j-i)\ \delta\ |x|}\right).
\end{aligned}$$

From this and (4.3) we have that for each $j \neq i$

$$E_2^j(x,t) = O(\ \epsilon\ \delta\ )(\theta_j + \theta_{jx})\ \min\left(1, e^{-\text{sgn}(j-i)\ \delta\ |x|}\right) + O(\epsilon)\sum_{k\neq i,j} \theta_k(x,t). \tag{4.5a}$$

From Lemma 3.1 we can replace $\theta_k(x,t)$ to obtain that for $j \neq i$

$$E_2^j(x,t) = O(\epsilon\delta^2)K\left(\frac{M_j(x,t)}{2}, t\right)\ \min\left(1, e^{-\text{sgn}(j-i)\ \delta\ |x|}\right) + O(\epsilon)\sum_{k\neq i,j}\delta_k K\left(\frac{M_k(x,t)}{2}, t\right). \tag{4.5b}$$

For $j = i$

$$E_2^i(x,t) = O(\epsilon\ \delta)\sum_{k\neq i} K\left(\frac{M_k(x,t)}{2}, t\right). \tag{4.5c}$$

Introduce a reaction wave system $\Theta_2(x,t)$ to correct the error vector $E_2(x,t)$ generated by the diffusion wave system $\Theta_1(x,t)$. We need to express the equation for $\Theta_2(x,t)$ in terms of its coordinate with respect to the basis $\{\bar{r}_1(x,t), \cdots, \bar{r}_n(x,t)\}$,

$$\Theta_2(x,t) \equiv \sum_{j=1}^n \Theta_2^j(x,t)\bar{r}_j(x,t).$$

The equation for the system is

$$\begin{aligned}
\partial_t\ \Theta_2 - s(t)\ \partial_x\ \Theta_2 + \partial_x\ f'(\bar{u}^1)\ \Theta_2 - \partial_x^2\Theta_2 - \bar{\lambda}_{ix}\ \Theta_2^i\ \bar{r}_i &= -E_2, \\
\Theta_2(x,0) &\equiv 0.
\end{aligned} \tag{4.6}$$

We rewrite (4.6) Component-wise.
For the transversal fields, $j \neq i$,

$$\partial_t\Theta_2^j + \partial_x\ \bar{\lambda}_j\ \Theta_2^j - \partial_x^2\Theta_2^j = O\left(\delta^2 e^{-\delta\ |x|} + \epsilon\ \right)\ \|\Theta_2(x,t)\| - E_2^j(x,t). \tag{4.7a}$$

For the compressive wave

$$\begin{aligned}
\partial_t\Theta_2^i + \bar{\lambda}_i\ \partial_x\Theta_2^i - \partial_x^2\Theta_2^i &= O\left(\ \delta^2\ e^{-\delta|x|} + \epsilon\ \right)\ \bar{\lambda}_i\ \Theta_2^i \\
&\quad + O\left(\delta^2\ e^{-\delta|x|} + \epsilon\ \right)\sum_{j\neq i}\Theta_2^j(x,t) - E_2^i(x,t).
\end{aligned} \tag{4.7b}$$

Before we continue the estimates of Θ_2, we will introduce approximate Green's functions for equations (4.7) and their basic properties.

4.1 Approximate Green's Functions for Transversal Fields

Since we are interested in waves crossing the shock layer, we introduce a modified j-characteristic curve $\{(c_j(t,y),t) : t \in \mathbf{R}\}$ related the leading approximation $\bar{u}^1$. For $j \neq i$

$$\begin{aligned} \frac{dc_j(t,y)}{dt} &= \bar{\lambda}_j(c_j(t,y),t), \\ c_j(0,y) &= y. \end{aligned} \tag{4.8}$$

From this family of the characteristic curve we can define an implicit function $m_j(x,t)$ satisfies that

$$\begin{aligned} \partial_s\, m_j(y,s) \;+\; \bar{\lambda}_j(y,s)\,\partial_y\, m_j(y,s) &= 0; \\ \|\, \partial_y\, m_j(\cdot,s)\,\|_\infty &= O(1) \text{ for } s \leq \delta/\epsilon. \end{aligned} \tag{4.9}$$

Proposition 4.1. *There exist a function $m_j(x,t)$ satisfying (4.9) and a constant $T_0 > 0$ such that for $t \leq T_0/\epsilon$*

$$\begin{aligned} m_{jx}(x,t) &= O(1), \\ m_{jxx}(x,t) &= O(1)(\,\epsilon \;+\; \delta^2 e^{-\frac{1}{2}\,\delta\,|x|}\,). \end{aligned}$$

Proof: Let's assume that $j > i$; and define a shock zone $\mathscr{X}$,

$$\mathscr{X} \equiv \left\{ (x,t) : |x| \leq \frac{2|\log\epsilon|}{\delta},\; t \leq T_0 \right\}.$$

Then, we separate the space-time domain into three regions, $\mathfrak{I}_1$, $\mathfrak{I}_2$ and $\mathfrak{I}_3$.

$$\begin{aligned} \mathfrak{I}_1 &\equiv \{\, (x,t) \;:\; x \in (-\infty,\; c_j(t,z_0)\,],\;\; t \in [\,0,\; \delta/\epsilon\,]\,\}, \\ \mathfrak{I}_2 &\equiv \{\, (x,t) \;:\; x \in [\,c_j(t,z_0),\; c_j(t,-z_0)\,],\;\; t \in [\,0,\; \delta/\epsilon\,]\,\}, \\ \mathfrak{I}_3 &\equiv \{\, (x,t) \;:\; x \in [\,c_j(t,-z_0),\; \infty\,),\; t \in [\,0,\; \delta/\epsilon]\,\}. \end{aligned}$$

We define the function $m_j(y,s)$. For $(x,t) \in \mathfrak{I}_1$

$$m_j(x,t) \;\equiv\; y \text{ where } y \text{ is given by } c_j(t,y) = x. \tag{4.10}$$

For $(x,t) \in \mathfrak{I}_2$ the value of $m_j(x,t)$ is given by the problem (4.9) with a restriction on $\{\, x = -z_0, \} \cap \mathfrak{I}_2 \;\equiv\; \{\, (0,s) \;: s \in [0,\; s_0)\,\}$

$$m_i(-z_0,t) \;\equiv\; \frac{z_0 t}{s_0} \;+\; \mathfrak{C}(t), \tag{4.11}$$

where $\mathfrak{C}(t)$ is a function defined on $[0,s_0]$ such that

$$\begin{aligned} \max_{s\in[0,s_0],\; i=0,1,2} |\partial_s^i\, \mathfrak{C}(s)| &= \epsilon \\ \mathfrak{C}(s_0) \;=\; 0, \quad \partial_x^q \mathfrak{C}(0) &= 0 \text{ for } q = 0,1,2, \end{aligned}$$

as well as $m_j(x,t)$ is C^2 for $(x,t) \in \mathfrak{I}_1 \cup \mathfrak{I}_2$. For $(x,t) \in \mathfrak{I}_3$ one needs to consider a boundary value problem (4.9) on the line $\{t = 0\} \cap \mathfrak{I}_3$ with a boundary value of the form

$$m_j(x,0) \;=\; \frac{-(x+z_0)\,z_0}{s_0\,\bar{\lambda}_j(-z_0,0)} \;+ \mathfrak{C}_1(x),$$

where $\mathfrak{C}_t(x)$ is a function chosen such that

$$\sup_{x\geq -z_0, q=0,1,2} |\partial_x^q \, \mathfrak{C}_1(x)| \;=\; O(1)\;\epsilon$$

as well as the function $m_j(x,t)$ is C^2 in $\mathfrak{I}_1 \cup \mathfrak{I}_2 \cup \mathfrak{I}_3$.
For this proposition we only need to consider the situation $(x,t) \in \mathfrak{I}_1$. For the other situations $(x,t) \in \mathfrak{I}_2 \cup \mathfrak{I}_3$ the proposition can be obtained by the same method.

Case 1. $x < -2|\log\epsilon|/\delta$.
Since $j > i$, the curve $\{(c_j(\tau, m_j(x,t)), \tau) : 0 \leq \tau \leq t\}$ travels from left to right, and reaches x at time $\tau = t$. In the case the point x is in the left side of the shock zone $\mathscr{X}$, therefore the characteristic curve $\{c_j(\tau, m_j(x,t)), \tau : 0 \leq \tau \leq t\}$ never intersects with the shock zone $\mathscr{X}$ for $\tau \leq t$. Hence we can assume that

$$\frac{d}{d\tau} c_{jy}(\tau,y) = \bar{\lambda}_{jx}(c_j(\tau,y),\tau) = O(\epsilon), \quad c_{jy}(0,y) = 1 \text{ for } y \in [m_j(t,x), x] \tag{4.12}$$

Hence it follows

$$c_{jy}(\tau,y) = O(1) \text{ for } \tau \leq t \leq \frac{1}{\epsilon}. \tag{4.13}$$

On the other hand from the implicit function (4.10) we have that

$$m_{jxx}(c_j(t,y),t) = -\frac{m_{jx}(c_j(t,y),t)\partial_y^2 c_j(t,y)}{(\partial_y c_j(t,y))^2}. \tag{4.14}$$

In this expression we have obtained the estimate of the term $c_{jy}(t,y)$ in (4.13), $c_{jy}(t,y) = O(1)$. This also implies that

$$m_{jx}(c_j(t,y),t) \;=\; O(1).$$

It remains to estimate $\partial_y^2 c_j(\tau,y)$ for $\tau \leq t$. By differentiating (4.8) twice

$$\begin{aligned} \frac{dc_{jyy}(\tau,y)}{d\tau} &= \bar{\lambda}_{jx}(c_j(\tau,y),\tau)c_{jyy}(\tau,y) + \bar{\lambda}_{jxx}(c_j(\tau,y),\tau)c_{jy}^2(y,\tau); \\ & c_{jyy}(0,y) = 0. \end{aligned} \tag{4.15}$$

Substitute $y = m(x,t)$, $\bar{\lambda}_{xx}(c_j(\tau,y),\tau) = O(\epsilon^2)$ and (4.13) into (4.15), we have that

$$c_{jyy}(t, m_j(x,t)) \;=\; O(1)\;t\;\epsilon^2.$$

From this and (4.14) there exists a constant T_0 such

$$m_{jyy}(x,t) = O(1)\;\epsilon \text{ for } t \leq T_0/\epsilon.$$

Case 2. $m_j(x,t) < -z_0$ and $x \in \mathscr{X}$.
Let τ_0 be the time at which characteristic curve $\{(c_j(\tau, m_j(x,t)), \tau) : \tau \in R\}$ interacts the shock zone $\mathscr{X}$, that is,

$$\begin{aligned} c_j(\tau_0, m_j(x,t)) &= z_0, \\ z_0 \;\equiv\; -2\frac{|\log\epsilon|}{\delta}, \quad \{z_0, -z_0\}\times\mathbf{R} &= \partial\mathscr{X}. \end{aligned} \tag{4.16}$$

Clearly, from this implicit relationship τ_0 is function of x, and denote it $\tau_0(x)$. Since this characteristic curve travels with a positive speed, it will spend $O(|\log\epsilon|/\delta)$ time to travel from $\partial\mathscr{X}$ to the point $x \in \mathscr{X}$. Hence,

$$|t - \tau_0(x)| = O(1)\frac{|\log\epsilon|}{\delta}. \tag{4.17}$$

Consider the comparison equation for studying this characteristic curve in the shock zone,

$$\begin{aligned} \frac{1}{\bar{\lambda}_j(\bar{C}_j, \tau_0(x))}\frac{d\,\bar{C}_j(\tau, x)}{d\tau} &= 1, \\ \bar{C}_j(\tau_0(x), x) &= z_0. \end{aligned}$$

We integrate this to obtain that

$$\begin{aligned} H(\bar{C}_j(\tau), x) &= (\tau - \tau_0(x)), && (4.18)\\ H(y, x) &\equiv \int_{z_0}^{y} \frac{1}{\bar{\lambda}_j(\rho, \tau_0(x))} d\rho. && (4.19) \end{aligned}$$

From this we introduce a more accurate equation to approximate the characteristic curve,

$$\begin{aligned} \frac{d}{d\tau}C_j &= \bar{\lambda}_j(C_j, \tau_0 + H(C_j, x)); && (4.20)\\ C_j(\tau_0, x) &= c_j(\tau_0, m_j(x, t)). \end{aligned}$$

Since we have uniform bounds on $\bar{\lambda}_j(y, t)$ and $\bar{\lambda}_{jy}(y, t)$, we may assume that for $\tau \in [\tau_0(x), t]$

$$C_j(\tau, x) - C_j(\tau_0, x) = O(1)(\tau - \tau_0) = O(1)\frac{|\log\epsilon|}{\delta}. \tag{4.21}$$

From (4.17) and (4.18) we may make a hypothesis that for $\tau \in [\tau_0, t]$

$$H(C_j(\tau, x), x) = O(1)\frac{|\log\epsilon|}{\delta^2}. \tag{4.22}$$

On the other hand we can write (4.20) as

$$\begin{aligned} \frac{\frac{d}{d\tau}C_j}{\bar{\lambda}_j(C_j, \tau_0)} &= 1 + O(1)\frac{\bar{\lambda}_{js}\, H(C_j, x)}{\bar{\lambda}_j} \\ &= 1 + O(1)\,\epsilon\,\frac{|\log\epsilon|}{\delta^2}. \end{aligned}$$

Integrating this equation we have that

$$H(C_j(\tau, x), x) = \tau - \tau_0 + O(1)\,\epsilon\frac{|\log\epsilon|^2}{\delta^3} \text{ for } \tau \in [\tau_0, t]. \tag{4.23}$$

Comparing $C_j(\tau, x)$ and $c_j(\tau, m_j(x, t))$, set

$$\Delta C_j(\tau) \equiv C_j(\tau, x) - c_j(\tau, m_j(x, t)).$$

Let's make a hypothesis for ΔC_j

$$|\Delta C_j(\tau, x)| \leq 2 \text{ for } \tau \in [\tau_0, t]. \tag{4.24}$$

The equation for ΔC_j is

$$\begin{aligned}
\frac{d}{d\tau}\Delta C_j &= \bar{\lambda}_j(C_j, \tau_0 + H(C_j, x)) - \bar{\lambda}_j(c_j, \tau) \qquad (4.25)\\
&= \int_0^1 \bar{\lambda}_{jx}(\rho\Delta C_j + c_j, \rho(\tau-\tau_0)+\tau_0)\, d\rho \cdot \Delta C_j \\
&+ \int_0^1 \bar{\lambda}_{js}(\rho\Delta C_j + c_j, \rho(\tau-\tau_0)+\tau_0) d\rho \cdot (H - (\tau - \tau_0)\ ) \\
&= O(1)\left(e^{-\frac{1}{2}\delta|C_j|}\cdot \Delta C_j \ + \ \epsilon^2 \frac{|\log\epsilon|}{\delta^3}\right); \\
\Delta C_j(\tau_0, x) &= 0.
\end{aligned}$$

From this and (4.21) there is a constant τ^* such that

$$\frac{d}{d\tau}\Delta C_j \ = \ O(1)\left(e^{-\frac{1}{2}\delta|\tau-\tau_*|}\cdot \Delta C_j \ + \ \epsilon^2 \frac{|\log\epsilon|}{\delta^3}\right).$$

Then it yields the estimate for $\tau \in [\tau_0, t]$

$$\Delta C_j(\tau, x) = O(1)\epsilon^2\ \frac{|\log\epsilon|^2}{\delta^3}. \qquad (4.26)$$

This justifies the hypothesis (4.24).
Applying the ∂_x and ∂_x^2 to (4.25), then we need estimate that $H_x(\tau, x)$ and $H_{xx}(\tau, x)$ for $\tau \in [\tau_0, t]$. From (4.19), the uniform L^∞ bounds of c_{jx} and m_{jx} in the shock zone $\mathscr{X}$, and Proposition 2.2 we have that for $t \in [\tau_0, t]$

$$H_{yy}(\tau, x),\ \ H_x(\tau, x) \ = \ O(1)\ \epsilon\ \frac{|\log\epsilon|}{\delta}.$$

Similarly, under the same hypothesis as ΔC_j we have that for $\tau \in [\tau_0, t]$

$$\partial_x \Delta C_j(\tau, x) = O(1)\epsilon^2\ \frac{|\log\epsilon|^2}{\delta^3}; \qquad (4.27)$$

$$\partial_x^2 \Delta C_j(\tau, x) = O(1)\epsilon^2\ \frac{|\log\epsilon|^2}{\delta^3}. \qquad (4.28)$$

It remains to estimate $\partial_x^i C_j(\tau, x)$ for $i = 0, 1, 2$.
The solution $C_j(x, \tau)$ can be represented as

$$\int_{z_0}^{C_j} \frac{1}{\bar{\lambda}_j(r, H(\rho, x) + \tau_0)} d\rho = \tau - \tau_0.$$

Applying ∂_x and ∂_x^2 to this identity, we obtain that for the functions $C_j(\tau, x)$ and $H(C_j(\tau, x), \tau)$

$$\begin{aligned}
\frac{C_{jx}}{\bar{\lambda}_j(C_j, H+\tau_0)} - \int_{z_0}^{C_j} \frac{\bar{\lambda}_{jt}(\rho, H(\rho, x)+\tau_0)\cdot(H_x(\rho, x) + \tau_{0x})}{\bar{\lambda}_j(\rho, H(\rho, x)+\tau_0)^2} d\rho &= -\partial_x \tau_0; \\
\frac{C_{jxx}}{\bar{\lambda}_j(C_j, H+\tau_0)} - \frac{C_{jx}\,\bar{\lambda}_{jx} \ + \ \bar{\lambda}_{jt}\,[H_\tau\, C_{jx} + H_x]}{\bar{\lambda}_j(C_j, H+\tau_0)^2} & \\
-\partial_x \int_{z_0}^{C_j} \frac{\bar{\lambda}_{jt}(r, H(r, x)+\tau_0)\cdot(H_x(r, x) + \tau_{0x})}{\bar{\lambda}_j(r, H(r, x)+\tau_0)^2} dr &= -\partial_x^2 \tau_0.
\end{aligned}$$

From this and that $H_x,\ \tau_{0x} = O(1)$ we have that

$$\begin{aligned} C_{jx} &= O(1), \\ C_{jxx}(\tau, x) &= O(1)(|\bar{\lambda}_{jx}(C,\tau)| + |\bar{\lambda}_{js}(C,\tau)|)|C_{jx}| + O(1)|\tau_{0xx}| \\ &= O(\delta^2)e^{-\delta|C_j|} + O(\tau_{0xx}). \end{aligned} \tag{4.29}$$

We need to estimate τ_{0x} and τ_{0xx}. Applying ∂_x^2 to (4.16) it yields that

$$c_{jx}m_{jx} + c_{j\tau}\tau_{0x} = 0, \tag{4.30a}$$

$$c_{jxx}m_{jx}^2 + 2c_{jx\tau}\tau_{0x}m_{jx} + c_{jx}\cdot m_{jxx} + c_{j\tau\tau}\tau_{0x}^2 + c_{j\tau}\tau_{0xx} = 0. \tag{4.30b}$$

Since the curve $\{(c_j(\tau, m_j(x,t)), \tau) : 0 < \tau < \tau_0\}$ is completely outside the shock zone $\mathscr{Z}$, from Case 1 we have that

$$|c_{jxx}(\tau_0, m_j(x,t))| = O(\epsilon).$$

Similarly, we can have that

$$\partial_x^i\partial_\tau^j c_j(\tau_0, m_j(x,t)),\ \partial_x^i\partial_\tau^j m_j(z_0, \tau_0) = O(\epsilon) \text{ for } i+j=2,\ i,j \geq 0.$$

Applying this and the uniform L^∞ upper bounds $c_{j\tau},\ c_{jx}, m_{j\tau}, m_{jx} = O(1)$ to (4.30b) we have that

$$\tau_{0xx} = O(\epsilon). \tag{4.31}$$

This and (4.29) yield that for $\tau \in [\tau_0, t]$

$$C_{jxx} \ = \ O(1)\left(\delta^2 e^{-\delta|C_j|} + \epsilon\right).$$

Combine this and (4.28) it yields that

$$\begin{aligned} \partial_x^2 c_j(\tau, m_j(x,t)) &= O(1)\ (\epsilon + e^{-\delta|x|/2}\ ) \text{ for } \tau \in [\tau_0, t], \\ m_{jxx}(x,t) &= O(1)\ (\ \epsilon\ +\ e^{-\delta\ |x|\ /\ 2}. \end{aligned}$$

and by using this estimate one show that the hypothesis (4.22) and (4.24) are true.
Hence, in this case Proposition 4.1 is true.
Case 3: $m_j(x,t) < -z_0$ and $-z_0 \leq x$.
Similar to Case 2, we define $\tau_1(x)$ be the leaving time at which the characteristic curve $\{(c_j(\tau, m_j(x,t)), \tau) : \tau \in \mathbf{R}\}$ leaves the shock zone $\mathscr{Z}$, that is,

$$c_j(\tau_1(x), m_j(x,t)) = -z_0. \tag{4.32}$$

Since this curve is not in the shock zone $\mathscr{Z}$ for for $\tau \in [\tau_1, t]$, it is true that for $\tau \in [\tau_1, t]$

$$\begin{aligned} \bar{\lambda}_{jxx}(c_j(\tau, m_j(x,t)), \tau) &= O(\epsilon^2); \\ \bar{\lambda}_{jx}(c_j(\tau, m_j(x,t)), \tau) &= O(\epsilon);\ \ c_{jx}(\tau, m_j(x,t)) = O(1). \end{aligned} \tag{4.33}$$

In Case 2 we have estimated that $c_{jxx}(m_j(\tau_1, m_j(x,t)) = O(\epsilon)$. Therefore, substituting this and (4.33) into the O.D.E. (4.15) we have that

$$\begin{aligned} c_{jxx}(\tau, m_j(x,t)) &= O(\epsilon) \text{ for } \tau \in [\tau_1, t], \\ m_{jxx}(x,t) &= O(1)\epsilon. \end{aligned}$$

Case 4. $m_j(x,t) > z_0$.
This characteristic curve $\{(c_j(\tau, m_j(x,t)), \tau) : \tau \in \mathbf{R}\}$ never intersects with the shock zone, therefore the method for Case 1 can be applied. Thus, we have that

$$m_{jxx}(x,t) = O(\epsilon).$$

Q.E.D.

Now, we define the approximate Green's function for the transversal fields in terms of the function $m_j(y,s)$,

$$g_j(y,s;x,t) \equiv K\left(\frac{m_j(y,s) - m_j(x,t)}{m_{jy}(y,s)},\ t-s\right). \tag{4.34}$$

From Proposition 4.1 we have that

$$\begin{aligned} &\partial_s g_j + \partial_y \bar{\lambda}_j g_j + \partial_y^2 g_j \\ &= O(1)\left(\epsilon + \delta^2 e^{-\delta\frac{|y|}{2}}\right) \cdot |m_j(x,t) - m_j(y,s)| \cdot \left|K_{XX}\left(\frac{m_j(y,s) - m_j(x,t)}{m_{jy}(y,s)}, t-s\right)\right| \\ &+ O(1)\left(\epsilon + \delta^2 e^{-\delta\frac{|y|}{2}}\right)^2 \cdot |m_j(x,t) - m_j(y,s)|^2 \cdot \left|K_X\left(\frac{m_j(y,s) - m_j(x,t)}{m_{jy}(y,s)}, t-s\right)\right| \\ &+ O(1)\left(\epsilon + \delta^2 e^{-\delta\frac{|y|}{2}}\right)\left|K_X\left(\frac{m_j(y,s) - m_j(x,t)}{m_{jy}(y,s)}, t-s\right)\right|. \end{aligned} \tag{4.35}$$

Since we have uniform bound $m_{jy}(y,s) = O(1)$ for $s \le T/\epsilon$, we can have two constants $B_\pm^j$ of order $O(1)$,

$$\begin{aligned} B_-^j &\equiv \inf_{s \le T_0/\epsilon} \|m_{jy}(\cdot, s)\|_\infty, \\ B_+^j &\equiv \sup_{s \le T_0/\epsilon} \|m_{jy}(\cdot, s)\|_\infty, \\ B_\pm^j &= 1 + O(1)\ (\delta + T). \end{aligned} \tag{4.36}$$

Set

$$\bar{g}_j(y,s;x,t) \equiv K\left(\frac{m_j(y,s) - m_j(x,t)}{2B_+^j}, t-s\right). \tag{4.37}$$

Hence

$$g_j(y,s;x,t) \le \bar{g}_j(y,s;x,t). \tag{4.38}$$

Substitute this into (4.35), then it follows

$$\partial_s g_j + \partial_y \bar{\lambda}_j g_j + \partial_y^2 g_j = O(1)\ \left(\epsilon + \delta^2 e^{-\delta\frac{|y|}{2}}\right) \frac{\bar{g}_j(x,t;y,s)}{\sqrt{t-s}}. \tag{4.39}$$

4.2 Approximate Green's Function for the Compressive Field

We continue to construct the approximate Green's function for the compressive field. This approach is adopted from [11].

We consider the case $x > 0$, only.

The approximate Green's function $g_i(y,\tau;x,t)$ for the compressive field is still of the same form as that for the transversal waves,

$$g_i(y,\tau;x,t) \equiv \frac{A(y,\tau)}{A(x,t)} K\left(\frac{N(y,\tau)-N(x,t)}{N_y(y,\tau)}, t-\tau\right), \tag{4.40}$$

but it has an extra factor $A(y,\tau)/A(x,t)$ to correct the compressive effect, where the functions $N(y,\tau)$ and $A(y,\tau)$ will be given in the following.
When $y \geq z_0$, the equations for $A(y,\tau)$ and $m_i(y,\tau)$ are

$$\partial_y\{[\lambda_i(\phi(y,\tau)) - s(\tau)]A(y,\tau)\} + \partial_y^2 A(y,\tau) = 0, \tag{4.41a}$$

$$\lim_{y\to\infty} A(y,\tau) = 1; \ \left(\lim_{y\to-\infty} A(y,\tau) = 1 \text{ for the case } x<0\right);$$

$$\partial_y^2 m_i + \left(\bar{\lambda}_j(\phi(y,\tau)) + \frac{2A_y}{A}\right)\partial_y m_i = 0; \tag{4.41b}$$

$$m_i(-z_0,\tau) = 0. \tag{4.41c}$$

By (4.11), (4.12) and (4.13) in [11] or a direct calculation, we have that

$$A(x,t) = 1 + e^{-\lambda_-|x|} + O(1)\delta e^{-\delta|x|} \text{ for } x>0; \tag{4.42a}$$

$$\begin{aligned} A(x,t) &= \left(\frac{A(0,t)}{4} + \frac{\lambda_-}{2|\lambda_-|}\right)\left(1+e^{-\lambda_- x}\right) + O(1)\left(\frac{A(0)}{4} - \frac{|\lambda_+|}{2\lambda_-}\right)\left(1+e^{\lambda_- x}\right) \\ &\quad + O(1)\delta\left(1-e^{-\delta|x|}\right)e^{\lambda_- x} \text{ for } x<0; \end{aligned} \tag{4.42b}$$

where $\lambda_\pm \equiv \lambda_i(u(0\pm,t)) - s(t)$; and

$$\partial_y m_i(x,t) = \begin{cases} -\frac{1}{\lambda_-}\left(1+O(1)\delta e^{-\delta|x|}\right) & \text{for } x<0, \\ \frac{1}{\lambda_+}\left(1+O(1)\delta e^{-\delta|x|}\right) & \text{for } x>0 \end{cases} \tag{4.43a}$$

$$\partial_y^2 m_i(x,t) = O(\delta)e^{-\delta|x|}. \tag{4.43b}$$

We define the function $N(y,s)$ on the domain $|y| \leq -z_0$ in terms of the function $m_j(y,s)$,

$$N(y,s) \equiv m_i(y,s) - s. \tag{4.44}$$

Introduce a modified i-characteristic field $\tilde{\lambda}_i(y,s)$ on the domain $y > -z_0$,

$$\tilde{\lambda}_i(y,s) \equiv \bar{\lambda}_i(y,s) - \left(\bar{\lambda}_i(0,s) - \bar{\lambda}_i(-z_0,s)\right). \tag{4.45}$$

We extend the function $N(y,s)$ to the domain $y > -z_0$ by considering the boundary value problem on $y \geq -z_0$,

$$\begin{aligned} N_t(y,s) + \tilde{\lambda}_i(y,s)N_y(y,s) &= 0; \\ N(-z_0,s) &= -s. \end{aligned} \tag{4.46}$$

From the definition (4.45), we have that

$$\partial_s^k \tilde{\lambda}_i(-z_0,s) = O(1)\|\partial_s^k \bar{\lambda}_i\|_\infty = O(1)\epsilon^k.$$

Therefore, from this and (4.46) it yields that

$$N_{yy}(y,s) = O(\epsilon/\delta^2). \tag{4.47}$$

Now, we have constructed $N(x,t)$ and $A(y,s)$ on the domain $y \geq z_0$. Substitute those functions into (4.40) to evaluate the approximate error $\left(\partial_s + \partial_y\ \bar{\lambda}_i + \partial_y^2\right) g_i(y,s)$ for $y \geq z_0$.

Case 1. $|y| \leq -z_0$.

From a direct calculation shown in (4.19) of [11], it shows that for the compressive field $\lambda_i(\phi(x))$ the approximation error the function $g_i(y,s)$ is

$$\left(\partial_s - s(t)\ \partial_y + \partial_y\ \lambda_i(\phi(y,s)) + \partial_y^2\right)\ g_i(y,s) = O(\delta^2)\left(\delta + \frac{1}{\sqrt{t-s}}\right)\bar{g}_i, \tag{4.48a}$$

where

$$\bar{g}_i(y,s;x,t)\ \equiv\ K\left(\frac{N(y,s)-N(x,t)}{2N_y(y,s)}, t-s\right).$$

We use this estimate to obtain the approximate error for the compressive field $\bar{\lambda}_i(y,s)$ for $|y| \leq -z_0$

$$\begin{aligned}&\left(\partial_s + \partial_y\bar{\lambda}_i(y,s) + \partial_y^2\right) g_i(y,s) \qquad (4.48\text{b})\\ &= \left(\partial_s - s(t)\partial_y + \partial_y\lambda_i(\phi(y,s)) + \partial_y^2\right) g_i(y,s) + \partial_y\left\{(\bar{\lambda}_i(y,s) - [\lambda_i(\phi(y,s)) - s(t)])g_i\right\}\\ &= O(\delta^2)\left(\delta + \frac{1}{\sqrt{t-s}}\right)e^{-\frac{\delta|y|}{2}}\bar{g}_i + O\left(\frac{\epsilon|\log\epsilon|}{\sqrt{t-s}}\right)\bar{g}_i + O\left(\epsilon\right)g_i.\end{aligned}$$

Case 2. $y \geq -z_0$.

The level set of $N(y,s)$ defines the i-characteristic curve. Hence, $N(y,s)$ shares this property with the function $M_j(y,s)$, which is used to define the approximate green's function $g(y,s;x,t)$ in (3.14). Besides, we still have the estimate $N_{yy}(y,s) = O(1)\epsilon/\delta^2$, (4.47). Thus, the estimate (3.28) for $g(y,s;x,t)$ is still valid for $g_i(x,t;y,s)$ with a modification,

$$\left(\partial_s + \partial_y\bar{\lambda}_i + \partial_y^2\right) g_i(y,s)\ =\ O(1)\ \frac{\epsilon}{\delta^2}\ \bar{g}_i(y,s). \tag{4.49}$$

Case 3. $y \leq z_0$.

In this case we will alternate the form the approximate green's function, (4.40).

Let's begin the construction of the approximate green's function for $y \leq z_0$ from introducing a pseudo-image.

For any given (x,t) and s with $x > 0$ and $s \in [0,t]$, a pseudo-image $\bar{x}(s)$ of the point x is defined by the implicit function,

$$m_i(\bar{x}(s),s) = N(x,t) + t. \tag{4.50}$$

From the definition of $N(x,t)$ (4.44), we have that

$$\bar{x}(t) = x \text{ for } |x| \leq -z_0.$$

When $x \geq -z_0$, we may assume that

$$\bar{x}(s) \geq -\frac{1}{2}x \text{ for } s \in [0,t]. \tag{4.51}$$

Then, we use an uniformly lower bound of $|m_y(y,s)|$ for $y \geq -\frac{1}{2}z_0$, $s \leq \delta/\epsilon$ in (4.43a), and implicit function theorem,

$$\bar{x}'(s)\ m_{iy}(\bar{x}(x),s)\ +\ m_{is}(\bar{x}(x),s)\ =\ 0,$$

to obtain that

$$\partial_s \bar{x}(s) = O(\epsilon). \tag{4.52}$$

Let's rewrite $N(x,t) = m_i(\bar{x}(s), s) - s$ and substitute it into $g_i(z_0, s; x, t)$ in (4.40).
Then, use (4.42b), (4.43a) with a straightforward calculation to expand it. Then, we have that, (see estimates for (4.16) in [11]), for $w \in [z_0, 0]$

$$\begin{aligned} g_i(w,s;x,t) &= \frac{A(w,s)}{A(x,t)} K\left(\frac{m_i(w,s) - m_i(\bar{x},s) + t - s}{m_{iy}(w,s)}, t-s\right) \\ &= \frac{1}{\sqrt{4\pi(t-s)}} \frac{A(z_0,s)}{A(x,s)} e^{\frac{-(m_i(,s)-m_i(0,s))+m_i(\bar{x},s)-m_i(0,s)}{m_{iy}(w,s)^2}} e^{-\frac{[m_i(z_0,s)-m_i(\bar{x},s)-(t-s)]^2}{4m_{iy}(z_0,s)^2(t-s)}} \\ &= O(1) e^{-\delta\bar{x}} K\left(\frac{m_i(w,s) - m_i(\bar{x},s) - (t-s)}{m_y(w,s)}, t-s\right). \end{aligned}$$

This form suggests the boundary value problem for $y \le z_0$,

$$\begin{aligned} &\partial_s \bar{N}_i(y,s) + \bar{\lambda}_i^-(y,s;x)\ \partial_x \bar{N}_i(y,s) = 0, \\ &\frac{\bar{N}_i(z_0,s)}{\bar{N}_{iy}(z_0,s)} = \frac{m_i(z_0,s) - m_i(\bar{x},s) - (t-s)}{m_{iy}(z_0,s)}, \end{aligned} \tag{4.53}$$

and use it to construct the approximate green function in the domain $y \le z_0$ by matching the value $g_i(y,s;x,t)$ at (z_0, s),

$$\begin{aligned} g_i(y,s;x,t) &\equiv \frac{A(z_0,s)}{A(x,s)} e^{\frac{-(m_i(z_0,s)-m_i(0,s))+m_i(\bar{x},s)-m_i(0,s)}{m_{iy}(z_0,s)^2}}\ g_i^-(y,s;x,t) \\ &= O(1)\ e^{-\delta\ \bar{x}(s)}\ g_i^-(y,s;x,t), \\ g_i^-(y,s;x,t) &\equiv K\left(\frac{\bar{N}_i(y,s)}{\bar{N}_{iy}}, t-s\right); \end{aligned} \tag{4.54}$$

where the function $\bar{\lambda}_i^-$ is given by

$$\bar{\lambda}_i^-(y,s;x) \equiv \bar{\lambda}_i(y,s) - \bar{\lambda}_i(z_0,s) - \frac{\partial_s\ (m_i(z_0,s) - m_i(\bar{x}(s),s) - (t-s)\ )}{\partial_y m_i(z_0,s)}.$$

Then, by the equation (4.53) we have that

$$\begin{aligned} \frac{N(z_0,s)}{N_y(z_0,s)} &= -\bar{\lambda}_i^-(z_0,s)\ \frac{N(z_0,s)}{N_s(z_0,s)} \\ &= \frac{\partial_s\ (m_i(z_0,s) - m_i(\bar{x}(s),s) - (t-s)\ )}{\partial_y m_i(z_0,s)}\ \frac{N(z_0,s)}{N_s(z_0,s)} \\ &= \frac{m_i(z_0,s) - m_i(\bar{x},s) - (t-s)}{m_{iy}(z_0,s)}. \end{aligned}$$

From this we have match $g_i(y,s;x,t)$ at the point $y = z_0$.
Next, we turn to evaluate the approximate error of $g_i(y,s;x,t)$ for $y \le z_0$.
From the definition of $m_i(y,s)$ we have that

$$\|\partial_s^k m_i\| = O(\epsilon^k) \text{ for } k \in \mathbf{Z}.$$

Therefore, from this and the regularity of $\lambda_i(u^0)$ (2.16) there exists a function $\bar{N}_i(y,s)$ solves (4.53) and satisfies that for $s \le t \le O(1)\delta^3$

$$\sup_{y\le z_0} \partial_x^i \bar{\lambda}_i^-(y,s) = O(1)\epsilon^i \text{ for } i=1,2.$$

$$\bar{N}_{iyy}(y,s) = O(1)\frac{\epsilon}{\delta} \text{ for } y \le z_0.$$

This implies that

$$\begin{aligned} &\partial_s g_i^- + \partial_y \bar{\lambda}_i g_i^- + \partial_y^2 g_i^- \\ &\partial_s g_i^- + \partial_y \bar{\lambda}_i^- g_i^- + \partial_y^2 g_i^- + \partial_y \left(\bar{\lambda}_i - \bar{\lambda}_i^-\right) g_i^- \\ &= O(1)\left(\left(\frac{\epsilon}{\delta} + \epsilon|\log(\epsilon)|\right)\frac{1}{\sqrt{t-s}}\,\bar{g}_i^- + \frac{\epsilon}{\delta}\,\bar{g}_i^i\right), \end{aligned}$$

where

$$\bar{g}_i^-(y,s;x,t) \equiv K\left(\frac{\bar{N}_i}{2\bar{N}_{iy}}, t-s\right).$$

From this, (4.48b), and (4.49) we have Proposition 4.2.

Proposition 4.2. *There exists positive constant T_0 such that for $x \ge 0$ and $0 \le t \le T/\epsilon$*

$$(\partial_s + \partial_y\bar{\lambda}_i + \partial_y^2) g_i(y,s;x,t) = \begin{cases} O(1)\frac{\epsilon}{\delta}\,\bar{g}_i(y,s;x,t), & \textit{for } y > -z_0, \\ O(1)\left(\delta^2(\delta + \frac{1}{\sqrt{t-s}})\,e^{-\delta|y|/2} + \epsilon + \frac{\epsilon|\log\epsilon|}{\sqrt{t-s}}\right)\ \bar{g}_i(y,s;x,t) & \textit{for } |y| \le -z_0, \\ O(1)\left[\epsilon + e^{-\delta\,|\bar{x}|}\left(\frac{\epsilon|\log\epsilon|}{\sqrt{t-s}} + \epsilon\right)\right]\ \bar{g}_i^-(y,s;x,t) & \textit{for } y \le z_0. \end{cases}$$

4.3 Basic Properties on convolutions with Approximate Green's Functions

We continue to estimate the convolutions of the approximate Green's functions with element waves.

First, set $\Lambda_j(t)$ is j-characteristic curve starting at $x=0$,

$$\begin{aligned} \Lambda_j(t) &\equiv c_j(t,0) \text{ for } j \ne i, \\ \Lambda_i^{\pm}(t) &\equiv \bar{\lambda}_i(0\pm,0)t, \\ \Lambda_i(t) &\equiv 0, \end{aligned} \tag{4.55}$$

where the curve $\{(c_j(t,0),t) : t \in \mathbf{R}\}$ is given in (4.8).

Part 1. Dissipation of Diffusion Waves.
Diffusion waves of algebraic types.

$$\begin{aligned} \sigma^\alpha(x,t;\Lambda_j,D) &\equiv (t+1)^{-\alpha/2} e^{-\frac{(x-\Lambda_j(t+1))^2}{D(t+1)}}, \\ \zeta^\alpha(x,t;\Lambda_j) &\equiv [(x-\Lambda_j(t+1))^2 + t+1]^{-\alpha/2}, \\ \bar{\zeta}^\alpha(x,t;\Lambda_j) &\equiv [(|x-\Lambda_j(t+1)|)^3 + (t+1)^2]^{\alpha/3}. \end{aligned} \tag{4.56}$$

$$I^{\alpha,\beta}(x,t;k,\Lambda_j,D) \equiv \int_0^t \int_R (|t-s|+1)^{-\beta/2} \left(g_k(y,s;x,t)\cdot(t-s)^{\frac{1}{2}}\right)\cdot\sigma^\alpha(y,s;\Lambda_j,D)dyds;$$

$$J^{\alpha,\beta}(x,t;k,\Lambda_j,D) \equiv \int_0^t \int_R (|t-s|+1)^{-\beta/2} \left(g_k(y,s;x,t)\cdot(t-s)^{\frac{1}{2}}\right)\cdot\zeta^\alpha(y,s;\Lambda_j,D)dyds.$$

Set

$$\begin{aligned}\Gamma^\alpha(t) &\equiv \int_0^t (s+1)^{-\alpha/2}ds \\ &= O(1)\begin{cases} 1 \text{ for } \alpha > 2, \\ \log(t+1) \text{ for } \alpha = 2, \\ (t+1)^{(2-\alpha)/2} \text{ for } \alpha < 2. \end{cases}\end{aligned}$$

In the rest of this paper the constant D is assumed to be a positive constant such that

$$D > 8;$$

and the constants T and δ are also assumed small enough that for $t \le \frac{T}{\epsilon}$

$$|\ \|\partial_x m_k(\cdot,t)\|_\infty - 1\ | \le \frac{1}{100}. \tag{4.57}$$

Set

$$H_0 = 1 + \frac{1}{100}.$$

Lemma 4.1. *For $\alpha,\beta > 0$ and $k \ne i$ there exists a constant $T > 0$ such that for $t \le T/\epsilon$*

$$\begin{aligned}&I^{\alpha,\beta}(x,t;k,\Lambda_k,D) \\ &\quad = O(1)\left[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(t+1) + (t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(t+1)\right]\sigma(x,t;\Lambda_k,D).\end{aligned} \tag{4.58}$$

In particular,

$$I^{\alpha,\beta}(x,t;k,\Lambda_k;D) = \begin{cases} O(1)\ \sigma(x,t;\Lambda_k,D) \text{ for } \alpha \ge 3, \beta = 1, \\ O(1)\ \sigma^{3/2}(x,t;\Lambda_k,D), \text{ for } \alpha \ge 2.5,\ \beta = 2. \end{cases}$$

Proof: Similar to the estimates in (3.30) we need to compare the two functions $x - \Lambda_k(t)$ and $m_k(x,t)$. From (4.36) there exists a constant T such that for $t \le T/\epsilon$

$$\begin{aligned}(x - \Lambda_k(t)) &\le H_0\ m_k(x,t), \\ \|\partial_x m_k(x,t)\|_\infty &\le H_0.\end{aligned}$$

From this and (4.34), it follows that

$$\begin{aligned}&I^{\alpha,\beta}(x,t;k,\Lambda_k,D) \\ &= O(1)\int_0^t (t-s)^{-(\alpha-1)/2}s^{-(\beta-1)/2}ds\frac{e^{-\frac{m_k(x,t)^2}{D\ t}}}{\sqrt{t+1}} \\ &= O(1)\left(\int_0^{t/2} + \int_{t/2}^t\right)(t-s)^{-(\alpha-1)/2}s^{-(\beta-1)/2}ds\cdot\frac{e^{-\frac{(x-\Lambda_k(t))^2}{D\ t}}}{\sqrt{t+1}}.\end{aligned}$$

Q.E.D.

Lemma 4.2.A. *Suppose that ($\alpha, \beta \geq 1$), ($k < j$), and ($k, j \neq i$). Then, for any given constant $E > {H_0}^4\ D$ there exists a positive constant $T > 0$ such that for $t \in [0, \frac{T}{\epsilon}]$*

$$I^{\alpha,\beta}(x,t;k,\Lambda_j,D) \tag{4.59}$$
$$= O(1)(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\sigma(x,t;\Lambda_k,E) + O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\sigma(x,t;\Lambda_j,E)$$
$$+\begin{cases} 0 \text{ for } x < \Lambda_k(t+1)+\sqrt{t+1}, \text{ or } x > \Lambda_j(t+1)-\sqrt{t+1} \\ O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\Lambda_k(t))\sigma(x,t;\Lambda_k,E) + (\Lambda_j(t)-x)^{(-\beta+1)/2}(x-\Lambda_k(t))^{(-\alpha+1)/2} \\ \quad +(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\Lambda_j(t)-x)\sigma(x,t;\Lambda_j,E)] \\ \text{for } \Lambda_k(t+1)+\sqrt{t+1} < x < \Lambda_j(t+1)-\sqrt{t+1}. \end{cases}$$

In particular,

$$I^{\alpha,\beta}(x,t;k,\Lambda_j,D) = \begin{cases} O(1)\zeta^{1/2}(x,t;\Lambda_k), \text{ for } \alpha=2,\ \beta=1, \\ O(1)[\zeta^{3/2}(x,t;\Lambda_k)+\zeta^{3/2}(x,t;\Lambda_j)], \text{ for } \alpha=3,\ \beta=2. \end{cases} \tag{4.60}$$

Proof: We separate this problem into the following five cases.

Case 1. $(x \leq \Lambda_k(t+1)) \equiv (m_k(x,t) < 0)$.
Case 2. $(0 \leq \Lambda_k(t+1) \leq x \leq \Lambda_k(t+1)+\sqrt{t}) \equiv (|m_k(x,t)| \leq O(1)\sqrt{t}(1+O(\delta)))$.
Case 3. $(\Lambda_k(t+1)+\sqrt{t} \leq x \leq \Lambda_j(t+1)-\sqrt{t}) \equiv (m_j(x,t+1) \leq -O(1)\sqrt{t},\ O(1)\sqrt{t} \leq m_k(x,t))$.
Case 4. $(\Lambda_j(t+1)-\sqrt{t} \leq x \leq \Lambda_j(t+1)+\sqrt{t}) \equiv (|m_j(x,t)| \leq O(1)\sqrt{t})$.
Case 5. $(\Lambda_j(t+1)+\sqrt{t} \leq x) \equiv (O(1)\sqrt{t} \leq |m_j(x,t)|)$.

For Case 1 and Case 2 we need to compare $y-\Lambda_j(s)$ to $m_k(y,s)-m_k(\Lambda_j(s),s)$. From (4.57) there exist positive constants $T > 0$ and $D_0 > 0$ such that for $t \leq T/\epsilon$

$$\begin{aligned} &\|\partial_y m_j(\cdot,t)\|_\infty \leq\ H_0, \\ {H_0}^{-1}\ |m_k(y,s)-m_k(\Lambda_j(s),s)| \leq |y-\Lambda_j(s)|\ &\leq\ H_0\ |m_k(y,s)-m_k(\Lambda_j(s),s)|, \\ D_0\ s \leq\ {H_0}^{-1}\ (\Lambda_j(s)-\Lambda_k(s))\ &\leq m_k(\Lambda_j(s),s)-m_k(\Lambda_k(s),s) \\ &= m_k(\Lambda_j(s),s). \end{aligned} \tag{4.61}$$

From this we have that

$$\int_R \frac{e^{\frac{(m_k(y,s)-m_k(x,t))^2}{4m_y(y,s)^2\ (t-s)}}}{\sqrt{t-s}}\ \frac{e^{\frac{(y-\Lambda_j(s))^2}{D\ (t-s)}}}{\sqrt{s}}\ dyds\ \leq\ \int_R \frac{e^{\frac{(m_k(y,s)-m_k(x,t))^2}{4\ H_0\ (t-s)}}}{\sqrt{t-s}}\ \frac{e^{\frac{(m_k(y,s)-m_k(\Lambda_j(s),s))^2}{{H_0}^2\ D\ (t-s)}}}{\sqrt{s}}\ dyds$$
$$\leq\ \frac{e^{\frac{(m_k(x,t)-m_k(\Lambda_j(s),s))^2}{{H_0}^2 D\ t}}}{\sqrt{t}}.$$

From a straight calculation, (see [11]),

$$I^{\alpha,\beta}(x,t;k,\Lambda_j,{H_0}^4 D) \tag{4.62}$$
$$= \int_0^{t-1} O(1)(t-s)^{-(\beta-1)/2}(1+s)^{-(\alpha-1)/2} \cdot \frac{e^{-\frac{(m_k(x,t)-m_k(\Lambda_j(s),s))^2}{{H_0}^2\ D\ t}}}{\sqrt{t}} ds$$

$$= O(1)\int_0^t (t-s)^{(-\beta+1)/2} s^{-(\alpha-1)/2} e^{-\frac{m_k(x,t)^2}{H_0{}^2 Dt}} e^{-\frac{D_0^2\ s^2}{H_0{}^4\ D(t+1)}} ds$$

$$= O(1)\left\{(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1}) + (t+1)^{(-\alpha-2\beta+3)/4}\right\}\sigma(x,t;\Lambda_k,H_0{}^4\ D)$$

$$+O(1)\ (t+1)^{(-\alpha-\beta+4)/2} e^{-\frac{D_0^2\ t^2}{H_0^4 D(t+1)}}\ \sigma(x,t;\Lambda_k,H_0{}^4\ D).$$

The Lemma is true for Case 1.
Case 2. From the condition for Case 2 $\sigma(x,t;\Lambda_k) = O(1)\sqrt{t+1}$ we have that

$$I^{\alpha,\beta}(x,t;k,\Lambda_j,D) = O(1)\int_0^t (t-s)^{(-\beta+1)/2} s^{-(\alpha-1)/2} \frac{e^{-\frac{(O(\sqrt{t})-(D_1(s)\ s)^2}{H_0^2\ D\ t}}}{\sqrt{t+1}} ds \tag{4.63}$$

$$= O(1)\left\{(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1}) + (t+1)^{(-\alpha-2\beta+3)/4}\right\}\sigma(x,t;\Lambda_k,H_0{}^2 D) \tag{4.64}$$

$$+O(1)\ (t+1)^{(-\alpha-\beta+4)/2} e^{-\frac{D_2{}^2 t^2}{4H_0^2\ D\ (t+1)}}\ \sigma(x,t;\Lambda_k,H_0{}^2 D), \tag{4.65}$$

where $D_1(s)$ is a positive bounded function with a lower bound D_0 and upper bound D_2.
Thus, the Lemma is true for Case 2.

Case 3. In this case we need to separate the integration of the time domain $[0,t]$ into three regions, $[0, m_k(x,t)]$, $[m_k(x,t), t+m_j(x,t)]$ and $[t+m_j(x,t), t]$. For s in the first two domains, (that is, $0 \le s \le t+m_j(x,t)$), from (4.61) we can replace $|y-\Lambda_k(s)|$ by $O(1)|m_j(y,s)-m_j(\Lambda_k(s),s)|$. Similarly, for $s \in [t+m_j(x,t), t]$ we replace $m_k(y,s)$ by

$$m_k(y,s) = [m_k(y,s) - m_k(\Lambda_j(s),s)] + m_k(\Lambda_j(s),s). \tag{4.66}$$

Then, we replace $m_k(y,s)-m_k(\Lambda_j(s),s)$ and $m_k(\Lambda_j(s),s)$ by by $O(1)(y-\Lambda_j(s))$ and $D_2(s)\ s$ where $D_2(s)$ is the uniformly bounded positive used in Case 2.
When $s \in [t+m_j(x,t), t]$, we need to introduce a new variable

$$Y_j(y,s) \equiv m_k(y,s) - m_k(\Lambda_j(s),s). \tag{4.67}$$

Then, for $s \le t \le T_0/\epsilon$

$$\begin{aligned} m_k(x,t) - m_k(y,s) &= Y_j(x,t) - Y_j(y,s) + m_k(\Lambda_j(t),t) - m_k(\Lambda_j(s),s) \\ &= Y_j(x,t) - Y_j(y,s) + D_3(s)\ (t-s), \\ H_0{}^{-1}\ |y-\Lambda_j(s)| &\le |Y_j(y,s)| \le H_0\ |y-\Lambda_j(s)|, \end{aligned}$$

where $D_3(s)$ is a uniformly bounded positive function. Hence

$$\int_R \frac{e^{-\frac{(m_k(x,t)-m_k(y,s))^2}{4m_{ky}(y,s)^2\ (t-s)}}}{\sqrt{t-s}} \frac{e^{-\frac{(y-\Lambda_j(s))^2}{D\ s}}}{\sqrt{s}} dy = O(1)\int_R \frac{e^{-\frac{(Y_j(x,t)-Y_j(y,s)+D_2(s)\ (t-s)\ )^2}{4H_0^2\ (t-s)}}}{\sqrt{t-s}} \frac{e^{-\frac{Y_j(y,s)^2}{H_0{}^2\ D\ s}}}{\sqrt{s}} dy$$

$$= O(1)\ \frac{e^{-\frac{(Y_j(x,t)+D_3(s)\ (t-s)\ )^2}{H_0^2 D\ t}}}{\sqrt{t}} = O(1)\ \frac{e^{-\frac{(m_j(x,t)+D_4(s)\ (t-s)\ )^2}{H_0^4 D\ t}}}{\sqrt{t}}$$

the positive function $D_4(s)$ share the same property as $D_3(s)$. Then, we have that

$$
\begin{aligned}
&I^{\alpha,\beta}(x,t;k,\Lambda_j) \\
&= \int_1^{m_k(x,t)} (t-s)^{-(\beta-1)/2} s^{-(\alpha-1)/2} \frac{e^{-\frac{(m_k(x,t)-D_2(s)\ s)^2}{H_0{}^2\ D\ t}}}{\sqrt{t+1}} ds \\
&+ \int_{m_k(x,t)}^{t+m_j(x,t)} (t-s)^{-(\beta-1)/2} s^{-(\alpha-1)/2} \frac{e^{-\frac{(m_k(x,t)-D_2(s)\ s)^2}{H_0{}^2\ D\ t}}}{\sqrt{t+1}} ds \\
&+ \int_{t+m_j(x,t)}^{t} (t-s)^{-(\beta-1)/2} s^{-(\alpha-1)/2} \frac{e^{-\frac{m_j(x,t)+\ D_3\ (s)\ (t-s))^2}{H_0{}^4 D t}}}{\sqrt{t+1}} ds.
\end{aligned}
\tag{4.68}
$$

From a straight calculation one can evaluate the above three integrals and get

$$
\begin{aligned}
&I^{\alpha,\beta}(x,t;k,\Lambda_j) \\
&\quad = O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(m_k(x,t))\sigma(x,t;\Lambda_j,H_0^4\ D) + (1+|m_j(x,t)|)^{(-\beta+1)/2}(1+|m_j(x,t)|)^{(-\alpha+1)/2}] \\
&\quad +O(1)[(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(1+|m_j(x,t)|)\sigma_j(x,t;\Lambda_j,H_0^4\ D)].
\end{aligned}
$$

From this and (4.61) the Lemma is true for Case 3.
For Case 4 and Case 5 we need to use the variable $Y_j(y,s)$ again. Similarly to the integral for $s \in [t+m_j(x,t),t]$ in Case 3, by using this variable $Y_j(y,s)$ Case 4 becomes

$$
\begin{aligned}
&0 \le Y_j(x,t) \le O(1)\ \sqrt{t}, \\
&I^{\alpha,\beta}(x,t;k,\Lambda_j) \\
&\quad = O(1) \int_1^{t-1} (t-s)^{-(\beta-1)/2} s^{-(\alpha-1)/2} \frac{e^{-\frac{(Y_j(x,t)-m_k(\Lambda_j(t),t)+m_k(\Lambda_j(s),s))^2}{4(1+O(\delta))t}}}{\sqrt{t+1}} ds.
\end{aligned}
$$

By evaluate this integral we have obtained the Lemma for Case 4.
The same argument works for Case 5.
Q.E.D.

Remark: The evaluation of the integrals in (4.62), (4.63), and (4.68) is quite lengthy and straight, therefore we refer it Lemma 2 of [11].

Lemma 4.2.B. *For $\alpha,\beta \ge 1$ and $j \ne i$ there exist positive constants T_0 and η_0 such that for any given constant $E > H_0^4\ D$ for $\delta \le \eta_0$ and $t \le T_0/\epsilon$*
Case 1. $j > i$, $x > 0$

$$
\begin{aligned}
&I^{\alpha,\beta}(x,t;i,\Lambda_j,D) \\
&\quad = O(1)(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\sigma(x,t;\Lambda_i^+,E) + O(1)e^{-\delta x}(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\sigma(x,t;\Lambda_i^-,E) \\
&\qquad\qquad +O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\sigma(x,t;\Lambda_j,D) \\
&\quad + \begin{cases}
0 \\
\text{for } 0 \le x \le \max(\sqrt{t}+\Lambda_i(t),0); \\
O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\Lambda_i^+(t))\sigma(x,t;\Lambda_i^+,E) + (\Lambda_j(t)-x)^{(-\beta+1)/2}(x-\Lambda_i^+(t))^{(-\alpha+1)/2} \\
\quad +(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\Lambda_j(t)-x)\sigma(x,t;\Lambda_j,E)] \\
\text{for } \max(0,\Lambda_i^+(t)+\sqrt{t}) < x < \Lambda_j(t+1)-\sqrt{t+1}; \\
0 \\
\text{for } \Lambda_j(t+1)-\sqrt{t+1} < x;
\end{cases}
\end{aligned}
$$

$$+\begin{cases} 0 \text{ for } x < \delta t/E, \\ O(1)\ e^{-\delta|x|/E}\ (x-\Lambda_j(t))^{(-\beta+1)/2}(t+1)^{(-\alpha+1)/4}\ e^{-\delta^2 t/E} \text{ for } \delta t/E \le x \le \Lambda_j(t)/E, \\ 0 \text{ for } x \ge \Lambda_j(t)/E. \end{cases}$$

In particular

$$I^{\alpha,\beta}(x,t;i,\Lambda_j,D) = \begin{cases} O(1)\zeta^{1/2}(x,t;\Lambda_i^+), \text{ for } \alpha=2,\ \beta=1, \\ O(1)[\zeta^{3/2}(x,t;\Lambda_i^+)+\bar{\zeta}^{3/2}(x,t;\Lambda_j)], \text{ for } \alpha=3,\ \beta=2. \end{cases} \tag{4.69}$$

Case 2. $j < i,\ x > 0$.

$$\begin{aligned} &I^{\alpha,\beta}(x,t;i,\Lambda_j,D) \\ &= O(1)(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\sigma(x,t;\Lambda_i^+,E) + O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\sigma(x,t;\Lambda_j,E) \\ &+O(1)e^{-\delta x}(1+t)^{-(\beta-1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\sigma(x,t;\Lambda_i^-,E) \\ &+\begin{cases} 0 \text{ for } x > E\delta t, \\ e^{-\delta|x|/E}\ (x-\Lambda_j(t))^{-(\beta-1)/2}\ (t+1)^{(-\alpha+1)/4}e^{-\delta^2 t/E} \text{ for } 0 \le x \le E\delta t. \end{cases} \end{aligned}$$

Proof. For both Cases 1 and 2 we need to separate this problem into three situations.

Case A. $0 \le x \le \max(0, \sqrt{t}+\Lambda_i^+(t))$.

Case B. $\max(0, \sqrt{t}+\Lambda_i^+(t)) \le x \le \Lambda_j(t) - \sqrt{t}$.

Case C. $\Lambda_j(t) - \sqrt{t} \le x$.

For the Cases A, B, and C we need to separate the domain of the double integration into two components $y \le 0$ and $y \ge 0$, since our approximate green function is defined differently for $y \ge 0$ and $y \le z_0 \equiv 2\log(\epsilon)\ /\ \delta$, where z_0 is given in (4.16).

We consider the integral over $y \ge 0$ first.
According to (4.40) and (4.42a) for $y \ge 0$ the approximate green function for the compressive field satisfies that

$$g_i(y,s;x,t) \ = \ O(1)K\left(\frac{N(y,s)-N(x,t)}{N_y(y,s)}, t-s\right),$$

where the function $N(y,s)$ is defined on $y \ge z_0$ by (4.44) and (4.46).

For Case A we need to compare $m_j(y,s)$ with $\left((N(y,s)-N(\Lambda_j(s),s)\right)/N_y(y,s)$, where the function $N(y,s)$ is given in (4.48) for defining the approximate green's function $g_i(y,s;x,t)$. From the definition of the $N(y,s)$ (4.46) one can have that

$$\left|\sup_{y_1,y_2\ge z_0,\quad t\le\frac{T}{\epsilon}} \frac{\left|\frac{N(y,s)-N(\Lambda_j(s),s)}{N_y(y,s)}\right|}{|m_j(y,s)|} - 1\right| \le \frac{1}{100}, \tag{4.70}$$

$$\left|\sup_{y\ge z_0,\quad t\le\frac{T}{\epsilon}} \frac{|N_y(y_1,s)|}{|N_y(0,s)|} - 1\right| \le \frac{1}{100},$$

by assuming that T and δ are small enough. From this, (4.57) and (4.40) we have that for $t>1$

$$\int_{y>0}\frac{e^{-\frac{(N(y,s)-N(x,t))^2}{4\,N_y(y,s)^2(t-s)}}}{\sqrt{t-s}}\frac{e^{-\frac{(y-\Lambda_j(s))^2}{D\,s}}}{\sqrt{s}}dy = O(1)\int_{y>0}\frac{e^{-\frac{(N(x,t)-N(y,s))^2}{4\,H_0^2\,N_y(0,s)^2(t-s)}}}{\sqrt{t-s}}\frac{e^{-\frac{(N(y,s)-N(\Lambda_j(s),s))^2}{DH_0^2N_y(0,s)^2\,s}}}{\sqrt{s}}dy$$
$$= O(1)\,\frac{e^{-\frac{(N(x,t)-N(\Lambda_j(s),s))^2}{D\,H_0^2\,N_y(0,s)^2t}}}{\sqrt{t+1}} = O(1)\,\frac{e^{-\frac{(x-\Lambda_i^+(t)-D_3(s)\,s\,)^2}{D\,H_0^4t}}}{\sqrt{t+1}},$$

where $D_3(s)=\bar{\lambda}_j(0,0)-\bar{\lambda}_i(0+,0)+O(\delta+T)$ is a uniformly bounded positive real-valued function.
Similar to (4.62) we have that

$$\int_0^t\int_{y>0}(|t-s|+1)^{-(\beta-1)/2}g_i(y,s;x,t)e^{-\frac{(y-\Lambda_j(s))^2}{D(t-s)}}(1+s)^{-\alpha/2}dyds \tag{4.71}$$
$$=O(1)\int_0^t\frac{e^{-\frac{(x-\Lambda_i^+(t)-D_3(s)\,s\,)^2}{D\,H_0^4t}}}{\sqrt{t+1}}\,ds$$
$$=O(1)(|t-s|+1)^{-(\beta-1)/2}(|s|+1)^{-(\alpha-1)/2}\frac{e^{\frac{-(x-\Lambda_i^+(t)-D_3(s)\,s\,)^2}{H_0{}^2\,D\,(t+1)}}}{\sqrt{t+1}}.$$

For Case B we need to introduce different auxiliary variable for the space integral

$$\mathscr{I}_\pm\equiv\int_{\pm y>0}g_i(y,s;x,t)\;\sigma^\alpha(y,s;\Lambda_j,D)dy.$$

First, we need to separate the time domain into four regions,

I. $1<s\le\sqrt{t}$
II. $\sqrt{t}\le s\le t-\sqrt{t}$
III. $t-\sqrt{t}\le s\le t-1$
IV. $t-1\le s$.

When $s\in$I, the evaluation for $\mathscr{I}_+$ is identical to Case A.
When $s\in$ II $\cup$III$\cup$IV, we introduce a new auxiliary variable $Y_i(y,s)$ on domain $y>0$. This variable is similar to $Y_j(y,s)$ in (4.67), and it is given by

$$Y_i(y,s)\equiv N(y,s)-N(\Lambda_j(s),s);$$
$$N(y,s)-N(x,t)=Y_i(y,s)-Y_i(x,t)+N(\Lambda_j(s),s)-N(\Lambda_j(t),t).$$

Hence, by the restricted condition for (4.57) and the definition of N we have that

$$|m_j(y,s)|\;\le\;H_0\frac{|Y_i(y,s)\;|}{|\;\partial_yN(0,s)\;|}.$$

Hence there is a constant E such that

$$\int_0^\infty\frac{e^{-\frac{(\;N(x,t)-N(y,s)\;)^2}{4\;N_y(y,s)^2(t-s)}}}{\sqrt{t-s}}\frac{e^{-\frac{(\;m_j(y,s)\;)^2}{D\;s}}}{\sqrt{s}}dy\;\le\;\int_0^\infty\frac{e^{-\frac{(\;Y_i(x,t)-Y_i(y,s)+N(\Lambda_j(t),t)-N(\Lambda_j(s),s))^2\;)^2}{4H_0{}^2\;N_y(0,s)^2(t-s)}}}{\sqrt{t-s}}\frac{e^{-\frac{Y_i(y,s)^2}{DH_0{}^2\;N_y(0,t)^2s}}}{\sqrt{s}}dy$$
$$=O(1)\;\frac{e^{-\frac{(\;Y_i(x,t)+N(\Lambda_j(t),t)-N(\Lambda_j(s),s))^2\;)^2}{DH_0{}^2\;N_y(0,t)^2t}}}{\sqrt{t}}\;=\;O(1)\;\frac{e^{-\frac{(\;m_j(x,t)+D_5(s)\;(t-s)\;)^2\;)^2}{E\;t}}}{\sqrt{t}},$$

where $D_5(s)$ and $1/D_5(s)$ are uniformly bounded functions. Similarly Lemma 4.2.A, by this estimate we have that

$$\int_0^t \int_0^\infty (|t-s|+1)^{-(\beta-1)/2} g_i(y,s;x,t) e^{-\frac{(y-\Lambda_j(s))^2}{(s+1)D}} s^{-\alpha/2} dy ds \tag{4.72}$$
$$= O(1) \int_0^t (|t-s|+1)^{-(\beta-1)/2} s^{-\alpha/2} \frac{e^{-\frac{(\ m_j(x,t)+D_5(s)\ (t-s)\ )^2}{E\ t}}}{\sqrt{t}} ds.$$

The above integral can be handled the same way as Lemma 4.2.A.

$$\begin{aligned}\mathscr{I}_+ \ =\ & O(1)(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\sigma(x,t;\Lambda_i^+,E) + O(1)e^{-\delta x}(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\sigma(x,t;\Lambda_i^-,E) \\ & +O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\sigma(x,t;\Lambda_j,D) \\ & + \begin{cases} 0 \\ \text{for } 0 \le x \le \max(\sqrt{t}+\Lambda_i(t),0); \\ O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\Lambda_i^+(t))\sigma(x,t;\Lambda_i^+,E) + (\Lambda_j(t)-x)^{(-\beta+1)/2}(x-\Lambda_i^+(t))^{(-\alpha+1)/2} \\ \quad +(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\Lambda_j(t)-x)\sigma(x,t;\Lambda_j,E)] \\ \text{for } \max(0,\Lambda_i^+(t)+\sqrt{t}) < x < \Lambda_j(t+1)-\sqrt{t+1}; \\ 0 \\ \text{for } \Lambda_j(t+1)-\sqrt{t+1} < x; \end{cases}\end{aligned}$$

Next, we turn to the evaluation for $\mathscr{I}_-$.
According to (4.54) the green function is of the form $e^{-\delta|\bar{x}|}K(\bar{N}_i(y,s)/\bar{N}_{iy}(y,s),t-s)$ for $y \le z_0$. First, we need to extend to domain of the function $\bar{N}_i(y,s)$ into $\mathbf{R}$. For each given (x,t) with $x>0$ we can simply define $\bar{N}_i(y,s)$ as follows

$$\bar{N}_i(y,s) \equiv m_i(y,s) - m_i(\bar{x}(s),s) - (t-s) \quad \text{for } y \ge z_0, \tag{4.73}$$

where $m_i(y,s)$ is a globally defined function given in (4.41). Similar to the estimate (4.70), under the same hypothesis for (4.70) we have that

$$\left| \frac{\left|\frac{\bar{N}_i(y,s)-\bar{N}_i(\Lambda_j(s),s)}{\bar{N}_{iy}(y,s)}\right|}{|m_j(y,s)|} - 1 \right| \le \frac{1}{100}; \tag{4.74a}$$

$$\left| \frac{\bar{N}_i(\Lambda_j(s),s)}{\bar{N}_{iy}(y,s)} \right| \le H_0 \left|\ \Lambda_j(s) - \bar{x}(s) + \delta\ D_3(s)\ (t-s)\ \right|, \tag{4.74b}$$

where $D_3(s) = 1+O(\delta+T)$ is an uniformly bounded positive function. Substitute (4.74) into the following integral, then there exists a constant E_0 such that

$$\int_{-\infty}^0 g_i(y,s;x,t)\ e^{-\frac{(y-\Lambda_j(s))^2}{D(t-s)}}(1+s)^{-\alpha/2}dy$$
$$= O(1)\ e^{-\delta|\bar{x}|} \int_{-\infty}^0 K\left(\frac{\bar{N}_i(y,s)}{\bar{N}_{iy}(y,s)},t-s\right)\ K\left(\frac{\bar{N}_i(y,s)-\bar{N}_i(\Lambda_j(s),s)}{E_0\ \bar{N}_{iy}(0,s)},s\right)dy$$
$$= O(1)e^{-\delta|\bar{x}|}\ K\left(\frac{\bar{N}_i(\Lambda_j(s),s)}{E_0\ \bar{N}_{iy}(0,s)},t\right).$$

This yields that

$$\mathscr{I}_- = O(1)\, e^{-\delta \bar{x}} \int_0^t (|t-s|+1)^{-(\beta-1)/2} (|s|+1)^{-(\alpha-1)/2} \frac{e^{-\frac{(\bar{x}-\Lambda_j(s)-\delta\ D_3(s)\ (t-s))^2}{E_0\ (t+1)}}}{\sqrt{t+1}}\, ds. \tag{4.75}$$

From (4.51) and by modifying the proof of Lemma 2 of [11] it yield that there exist constant E such that

$$\begin{aligned} \mathscr{I}_- &\leq E\, \mathscr{I}_+ \\ &+ \begin{cases} 0 \text{ for } x < \delta t/E, \\ O(1)\, e^{-\delta |x|/E}\, (x-\Lambda_j(t))^{(-\beta+1)/2} (t+1)^{(-\alpha+1)/4}\, e^{-\delta^2 t/E} \text{ for } \delta t/E \leq x \leq \Lambda_j(t)/E, \\ 0 \text{ for } x \geq \Lambda_j(t)/E. \end{cases} \end{aligned}$$

Hence Case 1 is true.

Remark. One just need to make change of variables, then the integration in (4.75) can be bounded an integral of the standard form

$$e^{-\delta\ |X|/E} \int_0^T (T-\tau)^{\beta-1} \tau^{\alpha-1} \frac{e^{-\frac{(X+\tau)^2}{4T}}}{\sqrt{T}}\, d\tau.$$

The procedure for proving is Case 2 is identical to Case 1. Therefore, we omit it. **Q.E.D.**

From similar calculations as those in Lemma 4.2 one can conclude Lemma 4.3, Lemma 4.4. However, the calculation is quite lengthy and massive. One can also refer to the calculations for Lemma 3 and Lemma 4 in [11] with our modification given in Lemma 4.2.A and Lemma 4.2.B.

Lemma 4.3. *Suppose* $0 \leq \beta < 3$. *Then for any positive constant* $E > D$ *and* $k \neq i$

$$J^{\alpha,\beta}(x,t;k,\Lambda_k,D) = O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(t+1)\sigma(x,t;\Lambda_k,E) + \Gamma^{\beta-1}(t+1)\zeta^{\alpha}(x,t;\Lambda_k)]. \tag{4.76}$$

For $x > 0$

$$J^{\alpha,\beta}(x,t;i,\Lambda_i^+,D) = O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(t+1)\sigma(x,t;\Lambda_i^+,E) + \Gamma^{\beta-1}(t+1)\zeta^{\alpha}(x,t;\Lambda_i^+)].$$

In particular,

$$J^{\alpha,\beta}(x,t;k,\Lambda_k,D) = \begin{cases} O(1)\zeta^{1/2}(x,t;\Lambda_k), \text{ for } \alpha \geq 2.5,\ \beta = 1, \\ O(1)\zeta(x,t;\Lambda_k), \text{ for } \alpha > 3,\ \beta = 1, \\ O(1)\zeta^{3/2}(x,t;\Lambda_k), \text{ for } \alpha \geq 2.5,\ \beta = 2. \end{cases}$$

Lemma 4.4.A. *Suppose that* $\alpha > 1$, $3 > \beta \geq 1$ *and* $k < j$, k, $j \neq i$. *Then for any given constant* $E > D$

$$\begin{aligned} J^{\alpha,\beta}(x,t;k,\Lambda_j,D) &= O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1}) + (t+1)^{-\alpha}\Gamma^{\beta-1}(t+1)]\sigma(x,t;\Lambda_k,E) \\ &\quad + O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\sigma(x,t;\Lambda_j,E) \end{aligned} \tag{4.77}$$

$$
+O(1)\left\{\begin{array}{l}
[(t+1)^{(-\beta+1)/2}(x-\Lambda_k(t))^{(-\alpha+1)}[1+(1+\frac{t}{|x-\Lambda_k(t)|})^{-1/2}\Gamma^{2\alpha-2}](\frac{t}{|x-\Lambda_k(t)|}+1) \\
+\Gamma^{\beta-1}(t+1)|x-\Lambda_k(t)|^{-\alpha}+\{(t+1)^{(-2\alpha+1)/2}\Gamma^{2\alpha}(t+1) \\
+(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})+(t+1)^{(-2\alpha+1)/2}\Gamma^{\beta-1}(t+1)\}\sigma(x,t;\Lambda_k,E)] \\
\text{for } x<\Lambda_k(t)-\sqrt{t+1}; \\
(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\Lambda_k(t))\sigma(x,t;\Lambda_k,E) \\
+(\Lambda_j(t)-x)^{(-\beta+1)/2}(x-\Lambda_k(t))^{(-\alpha+1)/2}+(x-\Lambda_k(t))^{-\alpha}\Gamma^{\beta-1}(t+1) \\
\text{for } \Lambda_k(t)+\sqrt{t}<x<\Lambda_j(t)-\sqrt{t}, \\
(x-\Lambda_j(t))^{\alpha}\Gamma^{\beta-1}(\frac{(t+1)(x-\Lambda_j(t))}{x-\Lambda_k(t)}) \text{ for } x>\Lambda_j(t)+\sqrt{t}.
\end{array}\right.
$$

In particular,

$$
J^{\alpha,\beta}(x,t;k,\Lambda_j,D)=\left\{\begin{array}{l}
O(1)\zeta^{1/2}(x,t;\Lambda_k) \text{ for } \alpha\geq 2,\ \beta=1, \\
O(1)[\sigma(x,t;\Lambda_k,D)+\zeta^{3/2}(x,t;\Lambda_k)] \text{ for } \alpha\geq 4,\beta=1, \\
O(1)[\zeta^{3/2}(x,t;\Lambda_k)+\bar{\zeta}^{3/2}(x,t;\Lambda_j)], \text{ for } \alpha\geq 3,\ \beta=2.
\end{array}\right. \tag{4.78}
$$

Lemma 4.4.B. *Assume that $\alpha>1$, $3>\beta\geq 1$ and $x>0$ and $j\neq i$.*
Case 1. $j>i$.
For any given constant $E>D$

$$
\begin{aligned}
&J^{\alpha,\beta}(x,t;i,\Lambda_j,D)=O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})+(t+1)^{-\alpha}\Gamma^{\beta-1}(t+1)]\sigma(x,t;\Lambda_i^+,E) \\
&+O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\sigma(x,t;\Lambda_j,E) \qquad (4.79) \\
&+O(1)\left\{\begin{array}{l}
(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\Lambda_i^+(t))\sigma(x,t;\Lambda_i^+,E) \\
+(\Lambda_j^+(t)-x)^{(-\beta+1)/2}(x-\Lambda_i^+(t))^{(-\alpha+1)/2}+(x-\Lambda_i^+(t))^{-\alpha}\Gamma^{\beta-1}(t+1) \\
\text{for } \max(\Lambda_i^+(t)+\sqrt{t},0)<x<\Lambda_j(t)-\sqrt{t}, \\
(x-\Lambda_j(t))^{\alpha}\Gamma^{\beta-1}(\frac{(t+1)(x-\Lambda_j(t))}{x-\Lambda_i^+(t)}) \text{ for } x>\Lambda_j(t)+\sqrt{t},
\end{array}\right. \\
&+\{\ e^{-\delta|x|/E}\ (\Lambda_j(t)-x)^{(-\beta+1)/2}(1+t)^{(-\alpha+1)/4}\ e^{-\delta^2t/E} \text{ for } x\leq E\delta t,
\end{aligned}
$$

Case 2. $j<i$.

$$
\begin{aligned}
&J^{\alpha,\beta}(x,t;i,\Lambda_j,D) \\
&=O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})+(t+1)^{-\alpha}\Gamma^{\beta-1}(t+1)]\sigma(x,t;\Lambda_i^+,E) \\
&+O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\sigma(x,t;\Lambda_j,E) \\
&+O(e^{-\delta|x|})\left\{\begin{array}{l}
t^{-(\beta-1)/2} \text{ for } \alpha=2, \\
(t+1)^{-\beta/2} \text{ for } \alpha\geq 3.
\end{array}\right.
\end{aligned} \tag{4.80}
$$

Part 2 Dissipation of Damping Waves
In this part, we introduce the propagation of damping waves:

$$
\mathbb{K}^{\alpha,\beta}(x,t;\Lambda_k,D)\equiv\int_0^t\int_R(t-s+1)^{-\beta/2}e^{-\frac{[y-x+\Lambda_k(t)-\Lambda_k(s)]^2}{D(t-s)}}(s+1)^{-\alpha/2}e^{-\frac{4\delta|y|}{D}}dyds.
$$

Lemma 4.5. *Suppose that $\alpha,\beta \geq 0$ and Λ_k is positive constant. Then there exists a positive constant C such that for any fixed constant $E > D + O(1)\delta$ and a bound $O(1)$ independent of δ*

$$\mathbb{K}^{\alpha,\beta}(x,t;\Lambda_k,D) \tag{4.81}$$

$$= O(1)\begin{cases} [(t+1)^{(-\beta+1)/2}\Gamma^{\alpha}(t+1)e^{-C\delta t} + (t+1)^{-\alpha/2}\Gamma^{\beta-1}(\delta^{-1})]e^{-4\delta|x|/D} \text{ for } x \leq 0, \\[2mm] (x+1)^{(-\beta+2)/2}(\Lambda_k(t)-x)^{-\alpha/2}(1+\delta\sqrt{1+x})^{-1} + \Gamma^{\beta-1}(\delta^{-1})(t+1)^{-\alpha/2}e^{-C\delta x} \\ +\Gamma^{\alpha}(\delta^{-1})(t+1)^{(-\beta+1)/2}e^{-C\delta|\Lambda_k(t)-x|}, \text{ for } 0 \leq x \leq \Lambda_k(t+1) - \sqrt{t+1}, \\[2mm] (t+1)^{(-\beta+1)/2}[\Gamma^{\alpha}(t+1)e^{-C\delta t} + \Gamma^{\alpha}(\sqrt{t+1})(1+\delta\sqrt{t})^{-1}] + (t+1)^{-\alpha/2}\Gamma^{\beta-1}(t+1)e^{-C\delta t}, \\ \text{for } |x-\Lambda_k(t+1)| \leq \sqrt{t+1}, \\[2mm] (t+1)^{(-\beta+1)/2}[\Gamma^{\alpha}(\sqrt{t+1})(1+\epsilon\sqrt{t})^{-1}e^{-\frac{(x-\Lambda_k(t))^2}{D(t+1)}} + (t+1)^{-\alpha/2}e^{-\frac{4\delta(x-\Lambda_k(t))}{D}}] \\ +(x-\Lambda_k(t))^{(-\alpha-2\beta+3)/2}e^{-C\delta(x-\Lambda_k(t))} \\ +(t+1)^{-\alpha/2}[\Gamma^{\beta-1}(\delta^{-1}) + \Gamma^{\beta-1}(t+1)]e^{-C\delta t}e^{-C\delta t}e^{-\frac{4\delta(x-\Lambda_k(t))}{D}}, \\ \text{for } x > \Lambda_k(t+1) + \sqrt{t+1}. \end{cases}$$

Proof. We use $\Lambda_j(s)$ to replace the time coordinate s. It follows

$$\begin{aligned} &\mathbb{K}^{\alpha,\beta}(x,t;\Lambda_k,D) \\ &= O(1)\int_0^{\Lambda_k(t)}\int_R (\Lambda_k(t)-\Lambda_k(s)+1)^{-\beta/2}e^{-\frac{[y-x+\Lambda_k(t)-\Lambda_k(s)]^2}{D(\Lambda_k(t)-\Lambda_k(s))}}(\Lambda_k(s)+1)^{-\alpha/2}e^{-\frac{4\delta|y|}{D}}dyd\Lambda_k(s). \end{aligned}$$

By a straight calculation given in Lemma 5 in [11], one can evaluate the above integral.
Hence, Lemma 4.5 follows. **Q.E.D.**

Set

$$\begin{aligned} \mathbb{L}^{\alpha,\beta}(x,t;\delta,D) &\equiv e^{-\frac{4\delta|x|}{D}}\int_0^t\int_{-\infty}^0 (t-s+1)^{-\beta/2}(s+1)^{-\alpha/2}e^{-\frac{[x-y-\delta(t-s)]^2}{D(t-s)}}e^{\frac{4\delta y}{D}}dyds, \\ \mathbb{M}^{\alpha,\beta}(x,t;\delta,D) &\equiv \int_0^t\int_0^{\infty}(t-s+1)^{-\beta/2}(s+1)^{-\alpha/2}e^{-\frac{[x-y+\delta(t-s)]^2}{D(t-s)}}e^{\frac{-4\delta y}{D}}dyds, \\ \mathbb{N}^{\alpha,\beta,\gamma}(x,t;\delta,D) &\equiv \int_0^t\int_{-\infty}^0 e^{-\frac{\delta|y|}{D}}(t-s+1)^{-\beta/2}(s+1)^{-\gamma/2}[y+\delta(s+1)]^{-\alpha/2}e^{-\frac{[x-y-\delta(t-s)]^2}{D(t-s)}}dyds \\ &\quad+\int_0^t\int_0^{\infty} e^{-\frac{\delta|y|}{D}}(t-s+1)^{-\beta/2}(s+1)^{-\gamma/2}[y+\delta(s+1)]^{-\alpha/2}e^{-\frac{[x-y+\delta(t-s)]^2}{D(t-s)}}dyds. \end{aligned}$$

By a similar transformation of the time variables from a similar calculation for Lemma 4.5, we concludes Lemma 4.6, Lemma 4.7 and Lemma 4.8.

Lemma 4.6. *Suppose that $\alpha,\beta \geq 0$ and $x \geq 0$. Then*

$$\begin{aligned} \mathbb{L}^{\alpha,\beta}(x,t;\delta,D) = O(1)[&(t+1)^{(-\beta+1)/2}\Gamma^{\alpha}(t+1)e^{-\frac{4\delta x}{D}}e^{-\frac{\delta^2 t}{4D}}e^{-\frac{x^2}{D(t+1)}} \\ &+(t+1)^{-\alpha/2}\delta^{-3+\beta}e^{-\frac{4\delta x}{D}}(1-e^{-\frac{\delta^2 t}{2D}})e^{-\frac{2x^2}{D(t+1)}}]. \end{aligned} \tag{4.82}$$

Lemma 4.7. *Suppose that $\alpha, \beta \geq 0$ and $x \geq 0$. Then*

$$\mathbb{M}^{\alpha,\beta}(x,t;\delta,D) = O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha}(t+1)e^{-C\delta(x+\delta t)} + (t+1)^{-\alpha/2}\Gamma^{\beta-1}(\delta^{-2})e^{-C\delta x}]. \tag{4.83}$$

Lemma 4.8. *Suppose that $x > 0$, $2 > \alpha > 0$, $3 > \beta > 0$. Then there exists a positive constant C such that*

$$\mathbb{N}^{\alpha,\beta,\gamma}(x,t;D,\delta) = O(1)(t+1)^{-\gamma/2}(x+\delta(t+1))^{-\alpha/2}\delta^{-\beta+3}e^{-C\delta x}, \tag{4.84}$$

where the term $\epsilon^{-\beta+3}$ can be replaced by $(t+1)^{(-\beta+3)/2}$.

4.4 Construction of the Reaction Wave System

We continue to establish the reaction wave system $\Theta_2(x,t)$ for the system (4.7) due to the source term $\sum_{j=1}^{n} E_j^2 \ \bar{r}_j$, which is estimated in (4.5).

Proposition 4.3. *For the of solution (4.7) $\Theta_2(x,t)$ there exists a positive constant $T > 0$ such that for $t \in [0, T/\epsilon]$*

$$\sup_{t \leq \delta^2/\epsilon} \|\Theta_2(\cdot,t)\|_\infty = O(1)\delta\epsilon.$$

Proof. We need to rewrite the differential equation for $\Theta_2(x,t)$ as integral equations in terms our approximate green's functions. Then, we introduce a sequence of iterated functions $\{{}^k\Theta_2\}_{k\geq 0}$ to construct the reaction wave system $\Theta_2(x,t)$. The integral equations for $\Theta_2(x,t)$ as follows:
For $j \neq i$

$$\begin{aligned}\Theta_2^j(x,t) = &\int_0^t \int_R -(\partial_s + \partial_y \bar{\lambda}_j + \partial_y^2) g_j(y,s;x,t) \cdot \Theta_2(y,s)^j \\ &+O(1)\int_0^t \int_R g_j(y,s;x,t)\left(\delta^2 e^{-\delta|y|} + \epsilon\right) \|\Theta_2(y,s)\| dyds \\ &-\int_0^t \int_R g_j(y,s;x,t) \ E_2^j(y,s) \ dyds.\end{aligned}$$

For $j = i$

$$\begin{aligned}\Theta_2^i(x,t) \ = \ &\int_0^t \int_R -\left(\partial_s + \bar{\lambda}_i \partial_y + \partial_y^2\right) g_i(y,s;x,t) \ \Theta_2^i(y,s) dyds \\ &+ \ O(1)\int_0^t \int_R g_i(y,s;x,t)\left(\delta^2 e^{-\delta|y|} + \epsilon\right) |\bar{\lambda}_i(y,s)| \cdot \Theta_2^i(y,s) dyds \\ &+ \ O(1)\int_0^t \int_R g_i(y,s;x,t)\left(\delta^2 e^{-\delta|y|} + \epsilon\right) \sum_{j\neq i}{}^{k-1}\Theta_2^j(y,s) dyds \\ &-\int_0^t \int_R g_i(y,s;x,t) \ E_2^i(y,s) \ dyds.\end{aligned}$$

By modifying this integral representation we introduce an sequence of iterated functions $\{{}^k\Theta_2(x,t)\}$ by the following recursive relationship.

For $k=1$ the initial recursive condition is

$$
\begin{aligned}
{}^{1}\Theta_2^j(x,t) &= -\int_0^t \int_R g_j(y,s;x,t)\ E_2^j(y,t)\ dy \text{ for } j=1,2,\cdots,n, \\
{}^{1}\Theta_2(x,t) &\equiv \sum_{k=1}^{n} {}^{1}\Theta_2^k(x,t)\ \bar{r}_k(x,t).
\end{aligned}
\tag{4.85a}
$$

For $k \geq 2$ and $j \neq i$

$$
\begin{aligned}
{}^{k}\Theta_2^j(x,t) = &\int_0^t \int_R -(\partial_s + \partial_y \bar{\lambda}_j + \partial_y^2) g_j(y,s;x,t) \cdot {}^{k-1}\Theta_2^j(y,s) \\
&+O(1) \int_0^t \int_R g_j(y,s;x,t) \left(\delta^2 e^{-\delta|y|} + \epsilon\right) \|{}^{k-1}\Theta_2(y,s)\| dyds \\
&-\int_0^t \int_R g_j(y,s;x,t)\ E_2^j(y,s)\ dyds,
\end{aligned}
\tag{4.85b}
$$

For $k \geq 2$ and $j = i$

$$
\begin{aligned}
{}^{k}\Theta_2^i(x,t) &= \int_0^t \int_R -\left(\partial_s + \bar{\lambda}_i \partial_y + \partial_y^2\right) g_i(y,s;x,t)\ {}^{k-1}\Theta_2^i(y,s) dyds \\
&+\ O(1) \int_0^t \int_R g_i(y,s;x,t) \left(\delta^2 e^{-\delta|y|} + \epsilon\right) |\bar{\lambda}_i(y,s)| \cdot {}^{k-1}\Theta_2^i(y,s) dyds \\
&+\ O(1) \int_0^t \int_R g_i(y,s;x,t) \left(\delta^2 e^{-\delta|y|} + \epsilon\right) \sum_{j \neq i} {}^{k-1}\Theta_2^j(y,s) dyds \\
&-\int_0^t \int_R g_i(y,s;x,t)\ E_2^i(y,s)\ dyds, \\
{}^{k}\Theta_2(x,t) &\equiv \sum_{j=1}^{n} {}^{k}\Theta_2^j(x,t)\ \bar{r}^j(x,t).
\end{aligned}
\tag{4.85c}
$$

Let's briefly state how this iterated functions converge. The function obtained from the convolutions (4.85a) yields a sharper estimate for the solution ${}^{k}\Theta(x,t)$ for $k \in \mathbf{N}$. One can use this function to define a priori bound for all the iterated functions and to shows the convergence of the sequence of solutions $\{{}^{k}\Theta_2\}_{k \in \mathbf{N}}$. However, we need to point out that if one does not add $-\bar{\lambda}_{ix}\Theta_2^i\ \bar{r}_i$ to (4.6) then the iterated function $\{{}^{k}\Theta_2\}_k$ may not converge.

Substituting (4.5b) and (4.38) into (4.85a) we have that for $j \neq i$

$$
\begin{aligned}
&\int_0^t \int_R g_j(y,s;x,t)\ E_2^j(y,s) dyds \\
&= \sum_{k \neq j,i} O(1)\ \epsilon\ \delta \int_0^t \int_R K\left(\frac{m_j(x,t) - m_j(y,s)}{2B_+^j}, t-s\right) K\left(\frac{M_k(y,s)}{2}, s\right) dyds, \\
&+O(1)\ \epsilon\ \delta^2 \int_0^t \int_R K\left(\frac{m_j(x,t) - m_j(y,s)}{2B_+^j}, t-s\right) \min(1, e^{-\delta \operatorname{sgn}(j-i)x})\ K\left(\frac{M_j(y,s)}{2}, s\right) dyds
\end{aligned}
\tag{4.86a}
$$

$$\equiv I_1 \; + \; I_2,$$

$$\int_0^t \int_R g_i(y,s;x,t) E_2^i(y,s) dyds \tag{4.86b}$$

$$= \sum_{k \neq i} O(1) \; \epsilon \; \delta \; \int_0^t \int_R g_i(y,s;x,t) K\left(\frac{M_k(y,s)}{2}, s\right) \; dyds \tag{4.86c}$$

In fact we can replace $|M_k(y,s)|$ by $H_0 \; |(y - \Lambda_k(s))|$ and Apply Lemma 4.2.A with $\beta = 1$ and $\alpha = 1$ to I_1 given in (4.86a). Then, we have that

$$I_1 \; = \; O(1) \epsilon \; \delta. \tag{4.87a}$$

One also has the estimate

$$\delta \; K((y - \Lambda_j(s))/(2H_0), s) \min(1, e^{-\delta \; \mathrm{sgn}(j-i) \; y}) = O(1)(1+s)^{-1} K((y - \Lambda_j(s))/(4H_0), s),$$

and then substitute this into I_2 given in (4.86a). Thus, applying Lemma 4.1 with $\beta = 1$ and $\alpha = 3$ to I_2 we have that

$$I_2 \; = \; O(1) \; \epsilon \; \delta.$$

Applying Lemma 4.2.B to (4.86b) it yields that

$$\int_0^t \int_R g_i(y,s;x,t) E_i(x,t) dyds \; = \; O(1) \; \epsilon \; \delta. \tag{4.87b}$$

From the estimates in (4.87) we conclude that

$$\| {}^1\Theta_2(\cdot,t) \|_\infty = O(1) \; \epsilon \; \delta. \tag{4.88}$$

Now we proceed to show the convergence of the sequence $\{{}^k\Theta_2\}_k$. Set

$$\Delta^k \Theta_2(x,t) \equiv {}^{k+1}\Theta_2(x,t) - {}^k\Theta_2(x,t).$$

First, we make a priori assumption on $\Delta^k \Theta_2(x,t)$ for $t \leq \delta^2/\epsilon$ and $k \in \mathbf{N}$

$$\sup_{t \leq \delta^2/\epsilon} \| \Delta^k \Theta_2(\cdot,t) \|_\infty = O(1) \; \epsilon \; \delta^{k+1}. \tag{4.89}$$

For $k = 1$ from (4.85) and (4.88) we have that for $j \neq i$

$$\Delta^1 \Theta_2(x,t)^j(x,t) = -\int_0^t \int_R \left[(\partial_s + \partial_y \bar{\lambda}_j + \partial_y^2) g_j(y,s;x,t)\right] \; \epsilon\delta \; dyds \tag{4.90a}$$

$$+O(1) \int_0^t \int_R g_j(y,s;x,t) \left(\delta^2 e^{-\delta|y|} + \epsilon\right) \; \epsilon \; \delta \; dyds$$

$$\equiv \mathfrak{I}_1 \; + \; \mathfrak{I}_2;$$

$$\Delta^1 \Theta_2(x,t)^i(x,t) = \int_0^t \int_R -(\partial_s + \partial_y \bar{\lambda}_j + \partial_y^2) g_i(y,s;x,t) \; \epsilon \; \delta \; dyds \tag{4.90b}$$

$$+O(1) \int_0^t \int_R g_i(y,s;x,t) \left(|\bar{\lambda}_i(y,s)| \delta^2 e^{-\delta \; |y|} + \epsilon\right) \cdot \epsilon\delta \; dyds$$

$$\equiv \mathfrak{B}_1 \; + \; \mathfrak{B}_2. \tag{4.90c}$$

From (4.39), Lemma 4.5 with $\alpha = 0$ and $\beta = 2$, and the estimates

$$\int_R |g_j(y,s;x,t)| \, dy \;=\; O(1) \tag{4.91}$$

we have that for $t \leq \delta^2/\epsilon$

$$|\ \mathfrak{I}_1(x,t)\ | \;=\; O(1)\ \epsilon\ \delta^2.$$

Similar to this and from Lemma 4.5 with $\alpha = 0$ and $\beta = 1$ we have that we have that for $t \leq \delta^2/\epsilon$

$$|\ \mathfrak{I}_2(x,t)\ | \;=\; O(1)\ \epsilon\ \delta^2.$$

From Proposition 4.2, Lemma 4.6 and 4.7 with the cases $(\alpha,\beta) = (0,1)$ and $(\alpha,\beta) = (0,2)$, and (4.91) for $t \leq \delta^2/\epsilon$

$$\mathfrak{B}_1 \;=\; O(1)\delta^2\ \epsilon.$$

Similarly from Lemma 4.7 and 4.7 with $(\alpha,\beta) = (0,1)$, (4.91), and condition

$$\bar{\lambda}_i(x,t) \;=\; O(\delta) \;\; \text{for } |x| \leq -z_0$$

we have that for $t \leq \delta^2/\epsilon$

$$\mathfrak{B}_2 \;=\; O(1)\epsilon\ \delta^2.$$

From the above estimates about $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ we have that for $t \leq \delta^2/\epsilon$

$$\|\Delta^1\Theta_2(x,t)\| \;=\; O(1)\ \delta^2\epsilon.$$

Therefore, (4.89) is true for $k = 1$.
Suppose that (4.89) is true for some $k \in \mathbf{N}$.
The representation of the function $\Delta^{k+1}\Theta_2(x,t)$ is the following.
Case $j \neq i$.

$$\begin{aligned}
\Delta^{k+1}\Theta_2(x,t)^j \;&=\; -\int_0^t \int_R [(\partial_s + \partial_y + \partial_y^2) g_j(y,s;x,t)]\ \Delta^k\Theta_2(y,t)^j\ dyds \\
&\quad +O(1)\int_0^t \int_R [(\ \delta^2\ e^{-\delta|y|/2}\ + \epsilon\ ) g_j(y,s;x,t)]\ \|\Delta^k\Theta_2(y,t)\|\ dyds \\
&= \; O(1)\int_0^t \int_R [(\partial_s + \partial_y + \partial_y^2) g_j(y,s;x,t)]\ \delta^{k+1}\ \epsilon\ dyds \\
&\quad +O(1)\int_0^t \int_R [(\ \delta^2\ e^{-\delta|y|/2}\ + \epsilon\ ) g_j(y,s;x,t)]\ \delta^{k+1}\ \epsilon\ dyds
\end{aligned}$$

Case $j = i$.

$$\begin{aligned}
\Delta^{k+1}\Theta_2(x,t)^i \;&=\; -\int_0^t \int_R [(\partial_s + \partial_y + \partial_y^2) g_i(y,s;x,t)]\ \Delta^k\Theta_2(y,t)^i\ dyds \\
&\quad +O(1)\int_0^t \int_R [\delta\ (\ \delta^2\ e^{-\delta|y|/2}\ + \epsilon\ ) g_i(y,s;x,t)]\ \|\Delta^k\Theta_2(y,t)^i\|\ dyds \\
&\quad +O(1)\sum_{m\neq i}\int_0^t \int_R [(\ \delta^2\ e^{-\delta|y|/2}\ + \epsilon\ ) g_i(y,s;x,t)]\ \|\Delta^k\Theta_2(y,t)^m\|\ dyds
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\int_0^t\int_R [(\partial_s+\partial_y+\partial_y^2)g_i(y,s;x,t)]\ \delta^{k+1}\ \epsilon\ dyds \\
&\quad +O(1)\int_0^t\int_R [\delta\ (\ \delta^2\ e^{-\delta|y|/2}\ +\epsilon\ )g_i(y,s;x,t)]\ \delta^{k+1}\ \epsilon\ \ dyds \\
&\quad +O(1)\sum_{m\neq i}\int_0^t\int_R [(\ \delta^2\ e^{-\delta|y|/2}\ +\epsilon\ )g_i(y,s;x,t)]\ \delta^{k+1}\ \epsilon\ dyds.
\end{aligned}
$$

By applying the procedure for obtaining the estimates for $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ we have that for $t \le \delta^2/\epsilon$

$$
\|{}^{k+1}\Theta_2(x,t)\| \;=\; O(1)\delta^{k+2}\ \epsilon.
$$

Hence, (4.89) is true for $k+1$. Therefore, the sequence $\{\Delta^k\Theta_2(x,t)\}$ is a geometric sequence for $t \le \delta^2/\epsilon$. The limit $\lim_{k\to\infty} {}^k\Theta_2(x,t)$ exists for $t \in [0,\delta^2/\epsilon]$ and it satisfies that

$$
\sup_{t\in[0,\epsilon/\delta^2]} \|{}^k\Theta_2(\cdot,t)\|_\infty = O(1) \sup_{t\in[0,\epsilon/\delta^2]} \|{}^1\Theta_2(\cdot,t)\|_\infty = O(1)\delta\epsilon \text{ for } k\in \mathbf{N}.
$$

Thus, this limit solves $\Theta_2(x,t)$ for $t \in [0,\delta^2/\epsilon]$ and

$$
\sup_{t\in[0,\epsilon/\delta^2]} \|\Theta_2(\cdot,t)\|_\infty \;=\; O(1)\ \delta\ \epsilon.
$$

Q.E.D.

5 Shock Fronts and Higher Order Diffusion Waves, II

In Section 4 we have constructed a reaction wave system $\Theta_2(x,t)$ to correct the approximation error $E_2(x,t)$. This correction is still not good enough for our nonlinear analysis, because it does not reveal the interaction between nonlinear diffusion waves and shock locations. It is still necessary to explore the interactions between the nonlinear diffusion waves and shock fronts of in order to find a good correction to the approximation error $E_2(x,t)$.

Since the reaction wave system $\Theta_2(x,t)$ is obtained from a modified linear system in stead of the original linearized system around $\bar u^1$. Hence, when we substitute $\Theta_2(x,t)$ into the original system, it will produce an extra error $\bar\lambda_{ix}\ \Theta_2^i\ \bar r_i$,

$$
\Theta_{2t} - s(t)\ \Theta_{2x} + \left(f'(\bar u^1)\Theta_2\right)_x - \Theta_{2xx} + E_2 = \bar\lambda_{ix}\ \Theta_2^i\ \bar r_i.
$$

Next we will introduce a sequence of functions $\{E_m(x,t)\}_{m\ge 3}$ to correct the approximation $\bar\lambda_{ix}\ \Theta_2^i\ \bar r_i$ and E_1, which is the approximation error due to $\bar u^1$ given in (2.37).
Set

$$
\begin{aligned}
E_3 \;&\equiv\; \bar\lambda_{ix}\ \Theta_2^i\ \bar r_i + E_1 \\
&=\; \left[\Theta_{2t} - s(t)\ \Theta_{2x} + \left(f'(\bar u^1)\Theta_2\right)_x - \Theta_{2xx} + E_2\right] + E_1.
\end{aligned}
\tag{5.1}
$$

From Proposition 4.3 we have that for $t \le \delta^2/\epsilon$

$$
\|\bar\lambda_{ix}\ \Theta_2\ \bar r_i\|(x,t) \;=\; O(1)\ \delta\ \epsilon\ (\epsilon + \delta^2 e^{-\delta|x|}).
$$

From this and Lemma 2.1 we have that

$$E_3(x,t) = -E_1(x,t) + O(1)\ \epsilon\delta\ \left(\epsilon + \delta^2 e^{-\delta|x|}\right) \tag{5.2a}$$

$$= \begin{cases} O(1)\epsilon^2, & \text{for } |x| \geq |\log\epsilon|/4\delta; \\ O(1)(\epsilon^2|\log\epsilon| + \delta^3\epsilon e^{-\delta|x|}), & \text{for } 1 \leq |x| \leq |\log\epsilon|/4\delta; \\ O(1)\epsilon & \text{for } |x| \leq 1. \end{cases}$$

For $m \geq 3$ the function $E_m(x,t)$ will be constructed by an iteration scheme such that for $t \leq \delta^2/\epsilon$

$$E_m(x,t) = O(1)\ \delta^{m-3}\ \epsilon(\ \epsilon\ +\ \ e^{-\delta|x|}\ ). \tag{5.2b}$$

We will construct the iteration scheme for $\{E_m\}_{m\geq 3}$ and verify (5.2b) in the next section.

5.1 The Iteration Scheme for Correcting Approximation Error, I

Let $E_m(x,t)$ be a function satisfying (5.2b). We decompose $E_m(x,t)$ into $E_m^a + E_m^b$, where E_m^a is wave around the shock front and $E_m^b(x,t)$ is wave away from shock

$$\begin{aligned} E_m^a(x,t) &\equiv E_m(x,t) \text{ for } |x| \leq 3|\log\epsilon|/\delta, \\ E_m^a(x,t) &\equiv 0 \text{ for } |x| \geq 3|\log\epsilon|/\delta, \\ E_m^b(x,t) &\equiv E_m(x,t) - E_m^a(x,t). \end{aligned}$$

Decompose the mass in the wave $E_m^a(x,t)$ with the same rule as (3.5).

$$\int_R E_m^a(x,t)dx = e_m^i(t)(\bar{u}_+(t) - \bar{u}_-(t)) + \sum_{j<i} e_m^j(t) r_j(\bar{u}_-(t)) + \sum_{j>i} e_m^j(t) r_j(\bar{u}_+(t)); \tag{5.3a}$$

$$\text{where} \quad \bar{u}_-(t) \equiv u^0(0-,t), \quad \bar{u}_+(t) \equiv u^0(0+,t).$$

From (5.2b) we have that

$$e_m^j(t) = O(1)\delta^{m-4}\epsilon \text{ for } j \neq i \text{ and } e_m^i(t) = O(1)\delta^{m-5}\ \epsilon. \tag{5.3b}$$

Similar to the construction of $\theta_j(x,t)$ we define the reaction wave system $\Psi_m(x,t)$ generated by the source $E_m^a(x,t)$,

$$l_{j,j} \cdot \left(\partial_t(\psi_m^j\ r_{j,j}) + \partial_x(\lambda_{j,j}\ \psi_m^j\ r_{j,j}) - \partial_x^2(\psi_m^j\ r_{j,j})\right) = -e_m^j(t)\ K_0(x) \text{ for } j \neq i, \tag{5.4a}$$

$$\psi_m^j(x,0) = 0,$$

$$\Psi_m \equiv \textstyle\sum_{j\neq i} \psi_m^j\ r_{j,j} \tag{5.4b}$$

where $K_0(x)$ is a non-negative C^∞ function satisfying that

$$K_0 : \mathbf{R} \longmapsto [0,2], \quad supp(K_0) \subset [-2,2],$$

$$\int_R K_0(y)\ dy = 1.$$

By Proposition 5.1, which is given in the end of this subsection, we can obtain the solutions for (5.4a). Hence it follows

$$\sup_{t\leq\delta/\epsilon} \|\Psi_m(\cdot,t)\|_\infty = O(\delta^{m-4}\epsilon). \tag{5.5}$$

Set

$$Mass_m(x,t) \equiv \left(\sum_{j<i} e_m^j(t) r_j(0-,t) + \sum_{j>i} e_m^j(t) r_j(0+,t)\right) K_0(x);$$

This function $Mass_m(x,t)$ will take the masses, which not in the compressive field, from $\Psi_m(x,t)$. Let $\mathscr{D}_m(x,t)$ be the resulting function,

$$\mathscr{D}_m \equiv \partial_t\Psi_m + \partial_x\left(\sum_{j\neq i}\lambda_{j,j}\ \psi_m^j r_{j,j}\right) - \partial_x^2\Psi_m + Mass_m,$$

$$\int_R E_m^a(x,t) - Mass_m(x,t)\ dx = e_m^i(t)\cdot(u(0+,t)-u(0-,t)). \tag{5.6}$$

From Proposition 5.1 the function $\mathscr{D}_m$ satisfies that for $t\leq[0,\delta/\epsilon]$

$$\mathscr{D}_m(x,t) = O(1)\ \delta^{m-4}\epsilon^2. \tag{5.7}$$

Next, we consider another function $\bar{\Psi}_m$ to correct $\mathscr{D}_m(x,t)$ as well as $E_m^b(x,t)$ subject to the approximation around $\bar{u}^1$,

$$\begin{cases} \partial_t\bar{\Psi}_m - s(t)\ \partial_x\bar{\Psi}_m + \partial_x f'(\bar{u}^1)\bar{\Psi}_m - \partial_x^2\bar{\Psi}_m - \bar{\lambda}_{ix}\ \bar{\Psi}_m^i\ \bar{r}_i = -\mathscr{D}_m\ -\ E_m^b, \\ \bar{\Psi}_m(x,0)\equiv 0. \end{cases} \tag{5.8}$$

Similar to solving the system (4.7), one can decompose this system into n scalar equations with source terms of the order $O(\delta^{m-4}\epsilon^2)$. Then, we convolute the source terms with the approximate Green's function in domain $\mathbf{R}\times[0,\delta^2/\epsilon]$. Since a convolution with the approximate Green's function in space is a bounded operator in $\|\cdot\|_\infty$. One can have that

$$\sup_{t\leq\delta^2/\epsilon}\|\bar{\Psi}_m(\cdot,t)\|_\infty = O(1)\int_0^{\delta^2/\epsilon}\ \|\mathscr{D}_m(\cdot,s)\|_\infty + \|E_m^b(\cdot,s)\|_\infty\ ds = O(1)\ \delta^{m-2}\epsilon. \tag{5.9}$$

By using this function $\bar{\Psi}_m$ we can define the function $E_{m+1}(x,t)$ as follows

$$\begin{aligned} E_{m+1} &\equiv \partial_t\bar{\Psi}_m - s(t)\partial_x\bar{\Psi}_m + \partial_x f'(\bar{u}^1)\bar{\Psi}_m - \partial_x^2\bar{\Psi}_m + \mathscr{D}_m + E_m^b & (5.10)\\ &= \bar{\lambda}_{ix}\ \bar{\Psi}_m^i. & (5.11)\end{aligned}$$

Hence, we have that for $t\in[0,\delta^2/\epsilon]$

$$E_{m+1} = O(1)\ |\bar{\lambda}_{ix}|\ \cdot\|\bar{\Psi}_m\|_\infty = O(1)\ \delta^{m-2}\ \epsilon\ \left(\epsilon+\delta^2 e^{-\delta|x|}\right). \tag{5.12}$$

Thus, we have defined an recursive relationship between E_m and E_{m+1}, and the sequence $\{E_m\}_{m\in\mathbf{N}}$ satisfying (5.2b) for $m\geq 3$. In the meantime our procedure also shows that $\{\Psi_m\}_m$, $\{\bar{\Psi}_m\}_m$, $\{Mass_m\}_m$ and $\{\mathscr{D}_m\}_m$ are all geometric sequences.

From $\bar{\Psi}_m$ and Ψ_m we define a reaction wave $\Theta_m(x,t)$,

$$\Theta_m(x,t)\equiv\Psi_m(x,t)+\bar{\Psi}_m(x,t).$$

Next, we need to compare the function $\Theta_m(x,t)$ to a system of conservation laws with inhomogeneous term $E_{m+1}-E_m$,
for $m \geq 3$

$$\begin{aligned}
&\partial_t\,\Theta_m - s(t)\,\partial_x\Theta_m + \partial_x f'(\bar{u}^1)\,\Theta_m - \partial_x^2\Theta_m = -E_m + E_{m+1} \qquad (5.13)\\
&\quad +(\,E_m^a \;-\; Mass_m\,) \;+\; \partial_x\left\{\sum_{j\neq i}\psi_m^j\cdot(\,f'(\bar{u}^1)\;-\;f'(\bar{u}_j^0)\,)\;r_{j,j}\right\},\\
&\int_R(\,E_m^a - Mass_m\,)\;(x,t)\;dx = e_m^i(t)\;\cdot(\,u^0(0+,t)\;-\;u^0(0-,t)\,).
\end{aligned}$$

By combining $\Theta_m(x,t)$ for $m \geq 1$ we define a modified diffusion wave $\Pi_1(x,t)$ for our nonlinear system (3.2) as follows:

$$\begin{aligned}
\Pi_1(x,t) &\equiv \sum_{m\geq 1}\Theta_m(x,t), \qquad (5.14)\\
\mathscr{G}(x,t) &\equiv \sum_{m\geq 3}\Theta_m(x,t);\\
Net(x,t) &\equiv \sum_{m\geq 3}(E_m^a - Mass_m)\,(x,t),
\end{aligned}$$

where $\mathscr{G}(x,t)$ a higher correction function to the nonlinear diffusion waves and $Net(x,t)$ a shock location correction function. From (4.4), (5.1) and (5.13) this correction diffusion wave $\Pi_1(x,t)$ satisfies the following equation

$$\begin{aligned}
&\partial_t(\Pi_1+\bar{u}^1) - s(t)\,\partial_x(\Pi_1+\bar{u}^1) + \partial_x\,(f'(\bar{u}^1)\Pi_1\;+\;f(\bar{u}^1)\,) - \;\partial_x^2(\Pi_1+\bar{u}^1) \qquad (5.15)\\
&= -\frac{1}{2}\partial_x\sum_{j\neq i}C_{j,jj}^j\theta_j^2\;r_{j,j}\;+\;Net[E_3]\;+\;\partial_x\left(\sum_{m=3}^{\infty}\sum_{j\neq i}\psi_m^j\;\cdot(f'(\bar{u}^1)-f'(\bar{u}_j^0))\;r_{j,j}\right)\\
&\quad +\partial_x\left\{\sum_{j\neq i}\theta_j\;(f'(u^0)-f'(\bar{u}_j^0))\;r_{j,j}\right\}.
\end{aligned}$$

Furthermore $\int_R Net(y,t)\;dy$ and $u^0(0-,t)-u^0(0+,t)$ are parallel. Therefore, we can define a new function $net(t)$,

$$net(t) \equiv \int_R Net(x,t)\;dx\;/\;\;(\,u^0(0+,t)-u^0(0-,t)\,). \qquad (5.16)$$

Since for $t \leq \delta^2/\epsilon$ the sequence $\{e_m(t)\}_m$ is a geometric sequence with ratio $O(1)\delta$, from (5.3b) for $t \leq \delta^2/\epsilon$

$$net[E_3](t) \;=\; O(1)\;e_3^i(t) \;=\; O(1)\delta^{-2}\,\epsilon. \qquad (5.17)$$

Remark I: From Proposition 5.1 there is a directional effect on the solution $\psi_j^m(x,t)$

$$\psi_j^m(x,t) = O(1)\delta^{m-4}\;\epsilon\;e^{-\frac{|\bar{\lambda}_j(0,0)\;x|}{4H_0}}\;\text{ for } x\cdot(j-i)<0.$$

By combining this and $|u^0-\bar{u}_j^0| = O(1)\delta\;\min(1, e^{-\mathrm{sgn}(j-i)\;x})$ it yields

$$\psi_m^j(x,t)\cdot(u^0-\bar{u}_j^0) \;=\; O(1)\delta^{m-3}\;\epsilon\;e^{-\delta\;|x|}. \qquad (5.18)$$

Remark II: The functions $\mathscr{G}(x,t)$, $Net(x,t)$, $net(t)$ and $\sum_{m=3}^{\infty}\sum_{j\neq i}\psi_m^j\cdot\left(f'(\bar{u}^1)-f'(\bar{u}_j^0)\right)r_{j,j}(x,t)$ are uniquely determined by the function $E_3(x,t)$. Thus, we can write $\mathscr{G}(x,t)$ and $Net(x,t)$ as

$$\begin{aligned}
\mathscr{G}[E_3](x,t) &\equiv \mathscr{G}(x,t) && (5.19)\\
Net[E_3](x,t) &\equiv Net(x,t), \\
net[E_3](t) &\equiv net(t), && (5.20)\\
\Psi[E_3](x,t) &\equiv \sum_{m=3}^{\infty}\sum_{j\neq i}\psi_m^j\cdot\left(f'(\bar{u}^1)-f'(\bar{u}_j^0)\right)r_{j,j}(x,t)
\end{aligned}$$

to keep their dependence on $E_3(x,t)$. The function $net[E_3]$ will measure the influence of the nonlinear diffusion $\Theta_1(x,t)$ on the shock location.

Following the procedure for constructing Θ_m, $Mass_m$ and $e_m^i(t)$ from E_m we have the corollary.

Corollary 5.1. *The functions $\mathscr{G}[e^{-\delta|x|}](x,t)$, $Net_1[e^{-\delta|x|}](x,t)$, and $net[e^{-\delta|x|}](x,t)$ satisfy that for $t\le\delta^2/\epsilon$*

$$\begin{aligned}
\mathscr{G}[\ e^{-\delta|x|}\ ](x,t) &= O(1)\ \frac{1}{\delta},\\
Net[\ e^{-\delta|x|}\ ](x,t) &= O(1)\left(e^{-\delta|x|}+\frac{K_0(x)}{\delta}\right),\\
net[\ e^{-\delta|x|}\ ](t) &= O(1)\ \frac{1}{\delta^2}.
\end{aligned}$$

Next we prepare for Proposition 5.1. Introduce a model equation for solving $\psi_m^j(x,t)$,

$$\begin{gathered}
l_{j,j}(x,t)\cdot\left(\partial_t(\rho_j(x,t)r_{j,j})+\partial_x(\lambda_{j,j}\rho_j r_{j,j})-\partial_x^2(\rho_j r_{j,j})\right)=-S_j(x,t)\ \text{for}\ j\neq i, \qquad (5.21)\\
\psi(x,0)=0,\ \text{where}\ S_j(x,t)=O(1)K_0(x).
\end{gathered}$$

Proposition 5.1. *The solution $\rho_j(x,t)$ of this model equation satisfies that for $t\le\delta/\epsilon$*

$$\sup\nolimits_{t\le\delta/\epsilon}\|\rho_j(\cdot,t)\|_\infty=O(1)\ \delta, \qquad (5.22)$$

$$\left\|\sum\nolimits_{j\neq i}\left(\partial_t\ \rho_j\ r_{j,j}+\partial_x\ \lambda_{j,j}\ \rho_j\ r_{j,j}-\partial_x^2\ \rho_j r_{j,j}+\sum\nolimits_{j\neq i}\ S_j\ r_{j,j}\right)\right\|_\infty=O(1)\ \epsilon, \qquad (5.23)$$

$$\rho_j(x)=O(1)\min(1,e^{-\mathrm{sgn}(j-i)Q_0\ x/4}), \qquad (5.24)$$

where the positive constant $Q_0=\min_{j\neq i}\ |\bar{\lambda}_j(0,0)\ |/H_0$.

Proof: Use the approximate Green's function given in (3.14), then we have the following representation for $\rho_j(x,t)$.

$$\begin{aligned}
\rho_j(x,t)&=O(\epsilon)\int_0^t\int_R\left(1+\frac{1}{\sqrt{t-s}}\right)K\left(\frac{M_j(x,t)-M_j(y,s)}{2A_+},t-s\right)\rho_j(y,s)dyds \qquad (5.25)\\
&+\int_0^t\int_R K\left(\frac{M_j(x,t)-M_j(y,s)}{2A_+},t-s\right)K_0(y)dyds\\
&\equiv \mathfrak{B}_3+\mathfrak{B}_4.
\end{aligned}$$

For the term $\mathfrak{B}_4$ we may assume the case $j>i$. Then, there exist positive constant T such that for $t\le T/\epsilon$

$$\mathfrak{B}_4 \ = \ \int_0^t K\left(\frac{x-y-D_3(s)\ (t-s+1)}{4H_0^2},t-s+1\right)\ ds,$$

where $D_3(s) = \bar{\lambda}_j(0,0) + O(\delta)$ for $t \in [0,T]$. From this we have that there exists constant M_0 such that

$$\mathfrak{B}_4 \ \le \ \begin{cases} M_0 \text{ for } x > 0, \\ \\ M_0 \ e^{\frac{\bar{\lambda}_j(0,0) \ x}{4H_0^2}} \text{ for } x \le 0. \end{cases}$$

From this estimate we may assume that for $t \in [0, \delta/\epsilon]$

$$\rho_j(x,t) \ \le 2 \begin{cases} M_0 \text{ for } x > 0, \\ \\ M_0 \ e^{\frac{\bar{\lambda}_j(0,0) \ x}{8H_0^2}} \text{ for } x \le 0. \end{cases} \tag{5.26}$$

From this estimate We can interchange the order of the integration for $\mathfrak{B}_3$ and substitute the assumption for ρ_j into it. Then, we have that for $t \in \delta/\epsilon$ and $x \le 0$

$$\begin{aligned} \mathfrak{B}_4 \ &= \ O(1) \ \epsilon M_0 \int_0^{\infty} \int_0^t K\left(\frac{x-y-D_3(s) \ (t-s+1)}{4H_0^2}, t-s+1\right) \ dsdy \\ &\quad + O(1) \ \epsilon M_0 \int_{-\infty}^{0} \int_0^t K\left(\frac{x-y-D_3(s) \ (t-s+1)}{4H_0^2}, t-s+1\right) \ e^{\frac{\bar{\lambda}_j(0,0) \ y}{8H_0^2}} \ dsdy \\ &= \ O(1) \ \epsilon \ M_0 \left(\int_0^{\infty} e^{\frac{\bar{\lambda}_j(0,0) \ (x-y)}{4H_0^2}} dy + \left[\int_x^0 e^{\frac{\bar{\lambda}_j(0,0) \ (x-y)}{4H_0^2}} e^{\frac{\bar{\lambda}_j(0,0) \ y}{8H_0^2}} dy \ + \int_{-\infty}^{x} e^{\frac{\bar{\lambda}_j(0,0) \ y}{4H_0^2}} dy\right]\right) \\ &= \ O(1) \ \epsilon \ M e^{\frac{\bar{\lambda}_j(0,0) \ x}{8H_0^2}}. \end{aligned}$$

On the other hand for $t \in [0, \delta/\epsilon]$ and $x > 0$ we need not to interchange the order of integration for $\mathfrak{B}_3$,

$$\mathfrak{B}_3 \ = \ O(1)\epsilon M \int_0^t \|\rho_j(\cdot, s)\|_\infty ds \ = \ O(1)\delta\epsilon M.$$

From the estimates of $\mathfrak{B}_3$ and $\mathfrak{B}_4$ we have that for $t \in [0, \delta/\epsilon]$

$$\rho_j(x,t) \ = \ (\ 1 \ + \ O(1) \ \delta \ ) \begin{cases} M_0 \text{ for } x > 0, \\ \\ M_0 \ e^{\frac{\bar{\lambda}_j(0,0) \ x}{4H_0^2}} \text{ for } x \le 0. \end{cases}$$

Thus, (5.26) is valid. Therefore, (5.22) and (5.24) follows. From (5.22) and (3.7) we have (5.23). **Q.E.D.**

5.2 Correction to the Shock Fronts

The equation (5.15) is almost in a conservative form expect the term $Net[E_3](x,t)$. By change the shock location one may reduce the influence from $Net[E_3]$. In this subsection we will proceed to construct a sequence of updating the shock location to make (5.15) in a conservative form.

For the approximate solution $\bar{u}^1(x,t)$ the viscous shock front remains at $x = 0$. Let's denote $S_0(t) \equiv 0$ the shock front for $\bar{u}^1(x,t)$.
Set

$$S_1(t) - S_0(t) \equiv \int_0^t net[E_3](\rho) \ d\rho. \tag{5.27}$$

From (5.17) we have that for $t \le \delta^2/\epsilon$

$$S_1(t) \;=\; O(1)\,\frac{\epsilon\, t}{\delta^2}. \tag{5.28}$$

Now, we update the location of the viscous profile $\phi(x - S_0(t), t)$ in $\bar{u}^1(x,t)$ to $\phi(x - S_1(t), t)$. Then, we use $\bar{u}^2(x,t)$ to approximate (2.15),

$$\mathfrak{E}_1^2(x,t) \equiv \partial_t \bar{u}^2 - s(t)\ \partial_x \bar{u}^2 + \partial_x f(\bar{u}^2) - \partial_x^2 \bar{u}^2. \tag{5.29}$$

where

$$\bar{u}^2(x,t) \equiv \bar{u}^1(x,t) + \phi(x - S_1(t), t) - \phi(x - S_0(t), t).$$

Let's rewrite (5.29),

$$\begin{aligned}
\mathfrak{E}_1^2 &= \partial_t \bar{u}^2 - s(t)\ \partial_x \bar{u}^2 + \partial_x f(\bar{u}^2) - \partial_x^2 \bar{u}^2 \\
&= \ E_0 - S_1'(t)\ \phi_x(x - S_1(t), t) + S_0'(t)\ \phi_x(x - S_0(t)) \\
&\quad + \phi_t(x - S_1(t), t) - \phi_t(x - S_0(t), t) \\
&\quad + \partial_x \left\{ -s(t)\ (\bar{u}^2 - \bar{u}^1) + [f(\bar{u}^2) - f(\bar{u}^1)] - \partial_x(\bar{u}^2 - \bar{u}^1) \right\}.
\end{aligned}$$

From this, (5.15) and (5.27), it yields

$$\begin{aligned}
&\int_R \left(\partial_t\ \Pi_1 - s(t)\ \partial_x \Pi_1 + \partial_x\ f'(\bar{u}^1)\ \Pi_1 - \partial_x^2\ \Pi_1\right) + \mathfrak{E}_1^2(x,t) dx \\
&\quad = \left\{\ net[E_3](t) - \ (S_1(t) - S_0(t))'\ \right\}\ \left(\ u^0(0+, t) - u^0(0-, t)\ \right) \\
&\quad - \int_R (\phi_t(x - S_1(t), t) - \phi_t(x - S_0(t), t))\ dx \\
&\quad = O(\epsilon) |\ S_1(t) - S_0(t)\ |.
\end{aligned} \tag{5.30}$$

We will proceed to construct a sequence of updating the shock front such that we can modify (5.30) to a system of conservation laws.
Now we define sequences of iterations $\{\bar{u}^k\}_{k\ge1}$, $\{\mathfrak{E}_1^k\}_{k\ge1}$, $\{\mathfrak{E}_3^k\}_{k\ge1}$ and $\{S_k\}_{k\ge0}$. The recursive relationship is given as follows.

For $k \ge 1$ suppose that $S_k(t)$ and $\bar{u}^k(x,t)$ have been constructed. Then, we set

$$\begin{aligned}
\mathfrak{E}_1^1 &\equiv E_1, \\
\bar{u}^{k+1} &\equiv \bar{u}^1 + \phi(x - S_k(t)) - \phi(x - S_0(t)), \\
\mathfrak{E}_1^k &\equiv \phi_t(x - S_k(t), t) - \phi_t(x - S_0(t), t) + E_1, \\
\mathfrak{E}_3^k &\equiv -E_2 - \mathfrak{E}_1^k, \\
\Pi_{k+1} &\equiv \Theta_1 + \Theta_2 + \mathscr{G}[\mathfrak{E}_3^k], \\
net_k &\equiv net[\mathfrak{E}_3^k], \\
S_{k+1}(t) &\equiv \int_0^t net[\mathfrak{E}_3^k](\rho)\ d\rho.
\end{aligned}$$

Corollary 5.2. *For $t \in [0, \delta^3/\epsilon]$ the sequences of functions $\{\bar{u}^k\}_{k\ge1}$, $\{\mathfrak{E}_1^k\}_{k\ge1}$, $\{\mathfrak{E}_3^k\}_{k\ge1}$, $\{\Pi_k\}_{k\ge1}$, $\{\Psi[\mathfrak{E}_3^k]\}_k$ and $\{S_k\}_{k\ge0}$ converge uniformly.*

Proof. It is sufficient to show that the sequence $\{S_k - S_{k-1}\}_{k\ge1}$ is a geometric sequence. Then, the convergence of the other sequences follows. Since the functionals *net* and $\mathscr{G}$ are linear functionals, from the above definition of sequences we have that

$$S_{k+1}(t) - S_k(t) = \int_0^t \left(net[\mathfrak{e}_3^k] - net[\mathfrak{e}_3^{k-1}]\right)(\rho)\, d\rho = \int_0^t net[\mathfrak{e}_3^k - \mathfrak{e}_3^{k-1}](\rho)\, d\rho \tag{5.31}$$

$$= \int_0^t net[\phi_t(x - S_k(\rho), \rho) - \phi_t(x - S_{k-1}(\rho), \rho)](\rho)\, d\rho$$

$$= \int_0^t net\left[\, O(1)\, \epsilon\, |S_k(\rho) - S_{k-1}(\rho)|\, e^{-\delta\, |y|}\, \right](\rho)\, d\rho.$$

Let's introduce a norm $|||\cdot|||_T$ for $S_k - S_{k-1}$,

$$|||S_k - S_{k-1}|||_T \equiv \sup_{\rho\in[0,T]} |\, S_k(\rho) - S_{k-1}(\rho)\, |.$$

From Corollary 5.1 and (5.31) we have that

$$|||S_{k+1} - S_k|||_{\delta^3} = O(1)\, \delta\, |||S_k - S_{k-1}|||_{\delta^3}.$$

This concludes that the sequence $\{|||\, S_k - S_{k-1}\, |||_{\delta^3}\}_k$ is a geometric sequence. Therefore, for $t \in [0, \delta^3/\epsilon]$ the sequence $\{S_k\}_k$ converges, so do $\{\Pi_k\}_k$, $\{\bar{u}^k\}_k$, $\{net[\mathfrak{e}_3^k]\}_k$ and $\{\Psi[\mathfrak{e}_3^k]\}_k$.
Q.E.D.

Since the sequences converge, for $t \le \delta^3/\epsilon$ we set

$$\bar{u}^\infty(x,t) \equiv \lim_{k\to\infty} \bar{u}^k(x,t),$$
$$\Pi_\infty(x,t) \equiv \lim_{k\to\infty} \Pi_k(x,t),$$
$$\mathfrak{e}_3^\infty(x,t) \equiv \lim_{k\to\infty} \mathfrak{e}_3^k(x,t)$$
$$\Psi^\infty(x,t) \equiv \lim_{k\to\infty} \Psi[\mathfrak{e}_3^k](x,t).$$

The above functions satisfy that

$$\int_R \left(\partial_t\, (\bar{u}^\infty + \Pi_\infty) - s(t)\, \partial_x\, (\, f(\bar{u}^\infty) + f'(\bar{u}^1)\, \Pi_\infty\,) + \partial_x^2\, (\bar{u}^\infty + \Pi_\infty)\right)\, dx = 0.$$

Hence, we can rewrite the equation of $\bar{u}^\infty + \Pi_\infty$ in a conservative form.
For $t \le \delta^3/\epsilon$

$$\partial_t\, (\bar{u}^\infty + \Pi_\infty) - s(t)\, \partial_x\, (\Pi_\infty + \bar{u}^\infty) + \partial_x\, [f(\bar{u}^\infty) + f'(\bar{u}^\infty)\Pi_\infty] - \partial_x^2\, (\bar{u}^\infty + \Pi_\infty) \tag{5.32}$$

$$= -\partial_x\, \frac{1}{2} \sum_{j\neq i} C^j_{j,jj}\, {\theta_j}^2\, r_{j,j} + \partial_x\, \mathfrak{Error},$$

$$\mathfrak{Error} \equiv \Psi^\infty + \sum_{j\neq i} \theta_j\, (f'(u^0) - f'(u_j^0))\, r_{j,j} + [f'(\bar{u}^\infty) - f'(\bar{u}^1)]\, \Pi_\infty + Error;$$

where

$$|\bar{u}^1(x,t) - \bar{u}^\infty| = O(1)\, \delta^2\, |S_\infty(t) - S_0(t)|\, e^{-\delta|x|} = O(1)\, \delta^3\, e^{-\delta\, |x|},$$

$$Error(x,t) \equiv \int_{-\infty}^x \phi_t(\rho - S_\infty(\rho), t) - \phi_t(\rho - S_0(\rho), t)\, d\rho = O(1)\, \frac{\epsilon^2\, t}{\delta^3}\, e^{-\delta\, |x|} = O(1)\, \epsilon\, e^{-\delta\, |x|},$$

$$\|\Pi_\infty(x,t)\| = O(1)\left(\epsilon + \sum_{j\neq i} \theta_j(x,t)\right),$$

$$\|\, \Psi^\infty(x,t)\, \| = O(1)\, \epsilon\, e^{-\delta\, |x|}, \tag{5.33}$$

and $\theta_j(x,t)$ is a wave along the characteristic curve with mass $O(\delta)$,

$$\theta_j(x,t) = O(\delta)\ K\left(\frac{x-\mathscr{S}_j(0,t)}{2A_+^j},t\right).$$

6 Shock Waves in Initial Data and Asymptotic Stability

In (5.32) we have established a well approximate function to the solution $u(x,t)$ provided that $u(x,t)$ and $u^\infty(x,t)$ are sufficiently small at some given time $t=t_0$. However, the initial data $u(x,0)$ with a shock wave seems not close enough to our approximate function at $t=0$. Hence, one needs to show that $u(x,t)$ will approach $\bar{u}^\infty$ at some given time, (that is, the solution $u(x,t)$ will evolve from a shock wave data toward a viscous shock layer in finite time).

6.1 Initial Layer

Now, we consider the perturbation for $\bar{u}^\infty+\Pi_\infty$. First we introduce the following functions:

$$\begin{cases} \mathbf{v}(x,t) & \equiv & u(x,t)-\bar{u}^\infty(x,t)-\Pi_\infty(x,t),\\ v^j(x,t) & \equiv & \mathbf{l}_j(x,t)\cdot\mathbf{v}(x,t),\\ \mathbf{w}(x,t) & \equiv & \int_{-\infty}^x \mathbf{v}(\rho,t)\ d\rho\\ w^j(x,t) & \equiv & \mathbf{l}_j(x,t)\cdot\mathbf{w}(x,t),\\ \bar{\lambda}_j(x,t) & \equiv & \lambda_j(\bar{u}^\infty(x,t)) \ - \ s(t),\\ \bar{v}^j(x,t) & \equiv & \partial_x w^j(x,t),\\ \mathbf{l}_j(x,t) & \equiv & l_j(\bar{u}^\infty(x,t)),\\ \mathbf{r}_j(x,t) & \equiv & r_j(\bar{u}^\infty(x,t)),\\ C_{jk}^q & \equiv & \mathbf{l}_q\cdot f''(\bar{u}^\infty)\ (\mathbf{r}_j,\mathbf{r}_k). \end{cases}$$

From the definition of $u(x,t)$ and $\bar{u}^\infty(x,t)$ we have the equation for $\mathbf{v}(x,t)$

$$\partial_t\mathbf{v}-s(t)\ \partial_x\ \mathbf{v}\ +\partial_x f'(\bar{u}^\infty)\ \mathbf{v}-\partial_x^2\mathbf{v}\ =\ -\frac{1}{2}\partial_x\ C_{ii}^i\ v^iv^i\ \mathbf{r}_i\ +\ \partial_x\sum_{1\le j\le n}\mathbf{F}[\ \mathbf{v}\ ]^j\ \mathbf{r}_j, \tag{6.1}$$

$$\begin{aligned} \sum_{1\le j\le n}\mathbf{F}[\mathbf{v}]^j\ \mathbf{r}_j\ &\equiv -\ \frac{1}{2}\sum_{\substack{1\le j,k,l\le n\\(j,k,l)\ne(i,i,i)}} C_{kj}^l\ v^k\ v^j\ \mathbf{r}_l\ +\ O(1)\ \|\mathbf{v}\|^3\\ &+O(1)\ \mathfrak{Error}\ +\ O(1)\sum_{\substack{j,k,p\ne i\\(k,p)\ne(j,j)}}\theta_k\ \theta_p\ \mathbf{r}_j\ +\ O(1)\ \|\mathbf{v}\|\cdot\|\Theta_1\|\ +\ O(1)\ \|\Theta_1\|^3\\ &+\mathfrak{Error}_{\ 1},\end{aligned}$$

$$\mathfrak{Error}_{\ 1}\ \equiv\ O(1)\ \delta^3\sum_{j\ne i}K\left(\frac{x-\Lambda_j(t)}{2A_+^j},t\right)\ \min\left(1,e^{-\delta\ \mathrm{sgn}(j-i)\ x}\right); \tag{6.2}$$

and the initial data for $\mathbf{v}(x,0)$ satisfies that

$$\int_R \mathbf{v}(x,0)\ dx \ = \ 0, \tag{6.3a}$$

$$|v^j(x,0)| \ = \ O(1)\ \delta^2\ e^{-\delta|x|}\ \text{for}\ j\ne i, \tag{6.3b}$$

$$|v^i(x,0)| \ = \ O(1)\ \delta\ e^{-\delta|x|}. \tag{6.3c}$$

From (6.3c) the initial data in the compressive field is in the same order as $(1+|x|)^{-1}$ without any small parameter in front of it. This causes difficulty in analyzing the formation of a shock layer. We will use the nonlinearity of the Burgers' equation to study the formation of the shock layer.

Burgers' Equation.

Let $\lambda_{tw}(x)$ be a stationary wave solution of Burgers' equation

$$\begin{aligned} \partial_t \lambda_{tw} \ + \ \partial_x \left(\frac{C^i_{jj}(0,0)\ {\lambda_{tw}}^2}{2} \right) &= \partial_x^2 \lambda_{tw}, \\ \lim_{x\to\infty} \lambda_{tw}(x) &= \lambda_i(u^0(0+,0)) - s(0), \\ \lim_{x\to-\infty} \lambda_{tw}(x) &= -\lambda_i(u^0(0+,0)) + s(0), \\ \lambda_{tw}(0) &= 0, \text{ (normalization condition for travelling waves)}. \end{aligned} \tag{6.4}$$

Due to the genuinely nonlinearity, $(\mathbf{l}_i f''(u)(\mathbf{r}_i, \mathbf{r}_i) \neq 0)$, we may assume

$$C^i_{ii}(0,0) = 1.$$

Let $\lambda_B(x,t)$ be the solution of Burgers' equation,

$$\partial_t \lambda_B + \frac{1}{2}\partial_x {\lambda_B}^2 - \partial_x^2 \lambda_B = 0,$$

with a given initial value obtained by modifying the compressive field

$$\lambda_B(x,0) = \lambda_{tw}(x) + v^i(x,0).$$

One can apply the first estimate in the proof of Theorem 2.2 in [14] to show that

$$|\ \lambda_{tw}(x) - \lambda_B(x,t)\ | = O(1)\ \delta\ e^{-\delta|x|} \cdot \begin{cases} \frac{\exp[-(\delta t-|x|)^2/\ 4t]}{[\delta t-|x|]/\sqrt{t}} \text{ for } |x| < \delta t, \\ 1 \text{ else.} \end{cases} \tag{6.5a}$$

As a consequence of (6.5a) it follows

$$\begin{aligned} |\ \lambda_{tw}(x) - \lambda_B(x,t)\ | &= O(1)\ \delta\ e^{-\delta|x|} \cdot \begin{cases} \exp[-\delta^2 t/\ 32] \text{ for } |x| < \delta t/2 \ - 1, \\ 1 \text{ else,} \end{cases} \\ &= O(1)\ \delta\ e^{-\delta|x|/2\ -\ \delta^2 t/32}. \end{aligned} \tag{6.5b}$$

Remark I. The estimates in the proof of Theorem 2.2 in [14] are obtained from a straight Cole-Hopf transformation.

Remark II. By a suitable rescaling one can have that

$$|\partial_x \lambda_B(x,t)| = O(1)\ \delta\ \frac{e^{-\delta^2 t/2\ -\ \delta|x|/2}}{\sqrt{t+1}}. \tag{6.5c}$$

We make the following proposition about the viscous profile $\phi(x,0)$ and the Burgers' shock profile $\lambda_{tw}(x)$ without giving the proof.

Proposition 6.1. *Suppose that $\|u^0(0-,0) - u^0(0+,0)\|$ is sufficiently small. Then the normalized profile $\phi(x,0)$ satisfies that*

$$
\begin{aligned}
|\lambda_i(\phi(0,0)) - s(0)| \;&=\; O(1)\|u^0(0-,0) - u^0(0+,0)\|^2 \;=\; O(1)\,\delta^2, \\
\sup_{x\in R} |\;(\lambda_i(\phi(x,0)) - s(0)) \;-\; \lambda_{tw}(x)\;| \;&=\; O(1)\;\|u^0(0-,0) - u^0(0+,0)\|^2 \;=\; O(1)\,\delta^2.
\end{aligned}
\tag{6.6}
$$

Set

$$\bar{\lambda}_B(x,t) \;\equiv\; \lambda_B(x,t) \;-\; \lambda_{tw}(x).$$

Consider a linear problem

$$\mathbf{u}_t \;+\; \lambda_B\;\mathbf{u}_x \;=\; \mathbf{u}_{xx}.$$

By a linear Cole-Hopf transformation

$$
\begin{aligned}
\bar{\mathbf{u}}(x,t) \;&\equiv\; e^{-\frac{1}{2}\,\tilde{V}(x,t)}\;\mathbf{u}(x,t), \\
\tilde{V}(x,t) \;&\equiv\; \int_0^x \lambda_B(\rho,t)\;d\rho \;=\; \int_0^x (\;\lambda_{tw}(\rho) \;+\; \bar{\lambda}_B(\rho,s)\;)\;d\rho,
\end{aligned}
$$

one can transform this equation into a heat equation

$$\bar{\mathbf{u}}_t \;+\; \frac{1}{4}{\lambda_{tw,+}}^2\;\bar{\mathbf{u}} = {\partial_x}^2\;\bar{\mathbf{u}}, \text{ where } \lambda_{tw,\pm} \;\equiv\; \lim_{x\to\pm\infty}\lambda_{tw}(x).$$

Therefore, we can have the representation

$$
\begin{aligned}
\bar{\mathbf{u}}(x,t) \;&=\; \int_R e^{-\frac{{\lambda_{tw,+}}^2\;t}{4}}\;k(x-y,0)\;\bar{\mathbf{u}}(y,0)\;dy, \\
\mathbf{u}(x,0) \;&=\; \int_R e^{\frac{1}{2}\;(\tilde{V}(x,t)-\tilde{V}(y,0)\;)-\frac{{\lambda_{tw,+}}^2\;t}{4}}\;k(x-y,t)\;\mathbf{u}(y,0)\;dy.
\end{aligned}
$$

Hence, we have the exact Green's function $\mathbb{G}_j(y,s;x,t)$ for this linear problem.

$$\mathbb{G}_i(y,s;x,t) \;\equiv\; \frac{e^{-\frac{1}{2}\tilde{V}(y,s)-\frac{{\lambda_{tw,+}}^2}{4}\;(t-s)}}{e^{-\frac{1}{2}\tilde{V}(x,t)}}K(x-y,t-s). \tag{6.7}$$

This Green's function $\mathbb{G}_i(y,s;x,t)$ has the same form as (4.40).
We continue to analyze $\mathbb{G}_j(y,s;x,t)$.
Case $x>0,\ y>0.\ (x<0,\ y<0)$.

$$\tilde{V}(x,t) - \tilde{V}(y,s) = \int_y^x \lambda_{tw}(\rho)d\rho + \int_y^x \bar{\lambda}_B(\rho,s) - \bar{\lambda}_B(\rho,s)\;d\rho + \int_0^x \bar{\lambda}_B(\rho,t) - \bar{\lambda}_B(\rho,s)\;d\rho.$$

Using (6.5) for the last two integrals, it yields that

$$
\begin{aligned}
&\tilde{V}(x,t) - \tilde{V}(y,s) = \int_y^x \lambda_{tw}(\rho)\;d\rho + O(1) \;=\; (x-y)\;\lambda_{tw,+} + O(1)\int_y^x (\;\lambda_{tw}(\rho) - \lambda_{tw,+}\;)\;d\rho + O(1) \\
&= \lambda_{tw,+}\;(x-y) + O(1).
\end{aligned}
$$

By substituting this into the definition of $\mathbb{G}(y,s;x,t)$ it follows

$$\mathbb{G}_i(y,s;x,t) = O(1)\ \frac{1}{\sqrt{t-s}}\ \exp\left[\frac{1}{2}\lambda_{tw,+}\ (x-y) - \frac{\lambda_{tw,+}{}^2(t-s)}{4} - \frac{(x-y)^2}{4(t-s)}\right]$$

$$= O(1)\ \frac{1}{\sqrt{t-s}} \exp\left[-\frac{(x-y-\lambda_{tw,+}\ (t-s))^2}{4(t-s)}\right].$$

Similarly for $x<0$ and $y<0$,

$$\mathbb{G}_i(y,s;x,t) = O(1)\ \frac{1}{\sqrt{t-s}}\ \exp\left[-\frac{(x-y-\lambda_{tw,-}\ (t-s))^2}{4(t-s)}\right].$$

Case $x>0$, $y<0$. ($x<0$, $y>0$)
Similar to the above we have that

$$\tilde{V}(x,t)-\tilde{V}(y,s) = \left(\int_y^0 + \int_0^x\right) \lambda_{tw}(\rho)\ d\rho + O(1)\ =\ -y\ \lambda_{tw,-} +\ x\ \lambda_{tw,+} + O(1)$$

$$= x\ \lambda_{tw,+} + y\ \lambda_{tw,+} + O(1)\ =\ 2x\ \lambda_{tw,+} + \lambda_{tw,+}\ (-x+y) + O(1).$$

By substituting this into the definition of $\mathbb{G}(y,s;x,t)$ we have that

$$\begin{aligned}\mathbb{G}_i(y,s;x,t) &= O(1)\ \frac{1}{\sqrt{t-s}}\ \exp\left[-\frac{(x-y+\lambda_{tw,+}\ (t-s))^2}{4(t-s)}\right]\\ &= O(1)\ \frac{e^{\lambda_{tw,+}\ x}}{\sqrt{t-s}}\ \exp\left[-\frac{(x-y-\lambda_{tw,-}\ (t-s))^2}{4(t-s)}\right].\end{aligned}$$

Similarly for $x<0$ and $y>0$ it follows

$$\mathbb{G}_i(y,s;x,t) = O(1)\ \frac{e^{\lambda_{tw,-}\ x}}{\sqrt{t-s}}\ \exp\left[-\frac{(x-y-\lambda_{tw,+}\ (t-s))^2}{4(t-s)}\right].$$

Set

$$\mathbf{v}_B(x,t)\ \equiv\ \bar{\lambda}_B(x,t)\ \mathbf{a},$$

where $\mathbf{a}$ is a vector of the form

$$\mathbf{a} \equiv \frac{u^0(0+,0)-u^0(0-,0)}{a_0\cdot \|u^0(0+,0)-u^0(0-,0)\|},$$

such that

$$\left\|\ \mathbf{r}_i(x,0)\ -\ \frac{u^0(0+,0)-u^0(0-,0)}{a_0\cdot \|u^0(0+,0)-u^0(0-,0)\|}\ \right\|_\infty\ =\ O(1)\ \delta.$$

From (6.5b) we have the following estimate for $\mathbf{v}_B$

$$\|\ \mathbf{v}_B\ \|\ =\ O(1)\ \delta\ e^{-\frac{\delta\ |x|}{2}\ -\ \frac{\delta^2\ t}{32}}. \tag{6.8}$$

We rewrite the equation for $\mathbf{v}_B(x,t)$ as follows

$$\begin{aligned} 0\ &=\ \partial_t\ \bar{\lambda}_B\ \mathbf{a} + \partial_x\ (\ \lambda_{tw}\ \bar{\lambda}_B + \bar{\lambda}_B{}^2\ )\ \mathbf{a}\ -\ \partial_x^2\ \bar{\lambda}_B\ \mathbf{a}\\ &=\ \partial_t \mathbf{v}_B\ +\ \partial_x\ (\lambda_{tw}+\bar{\lambda}_B)\ \mathbf{v}_B\ -\partial_x{}^2\ \mathbf{v}_B.\end{aligned}$$

From this identity we consider the equation

$$\partial_t \mathbf{v}_B + \partial_x \left((f'(\bar{u}^\infty) - s(0)) \mathbf{v}_B + \frac{1}{2}\bar{\lambda}_B{}^2 \mathbf{a} \right) - \partial_x{}^2 \mathbf{v}_B \tag{6.9}$$
$$= \partial_x \left((f'(\bar{u}^\infty) - s(0)) - \lambda_{tw} \right) \mathbf{v}_B.$$

The term

$$\begin{aligned} &(f'(\bar{u}^\infty) - s(0)) \mathbf{v}_B \\ &= \bar{\lambda}_B \left\{ (f'(\bar{u}^\infty) - s(0)) (\mathbf{a} - \mathbf{r}_i) + \bar{\lambda}_i (\mathbf{r}_i - \mathbf{a}) + \bar{\lambda}_i \mathbf{a} \right\} \\ &= O(1) e^{-\delta |x|} \left(\delta^3 \mathbf{r}_i + \delta^2 \sum_{j\neq i} \mathbf{r}_j \right) + \bar{\lambda}_i \mathbf{v}_B . \end{aligned}$$

Consider the following functions:

$$\begin{aligned} \mathbf{v}_I(x,t) &\equiv \mathbf{v}(x,t) - \mathbf{v}_B(x,t), \\ \sum_{1\le j\le n} v_I^j(x,t) \, \mathbf{r}_j(x,t) &\equiv \mathbf{v}_I(x,t), \\ \mathbf{w}_I(x,t) &\equiv \int_{-\infty}^{x} \mathbf{v}_I(r,t) \, dr, \\ \sum_{1\le j\le n} w_I^j(x,t) \, \mathbf{r}_j(x,t) &\equiv \mathbf{w}_I(x,t), \\ \bar{v}_I^j(x,t) &\equiv \partial_x w_I^j(x,t) \text{ for } j = 1,\cdots,n, \\ v_I^j(x,t) &= \bar{v}_I^j + O(1) \, \|\mathbf{w}_I(\cdot,t)\|_\infty \, (\delta^2 e^{-\delta|x|} + \epsilon). \end{aligned}$$

From (6.3) we also have that

$$\bar{v}_I^j(x,0) = O(1) \, \delta^2 e^{-\delta|x|}, \quad w_I^j(x,0) = O(1) \, \delta \, e^{-\delta|x|}, \text{ for } j = 1,2,\cdots,n. \tag{6.10}$$

Subtracting (6.9) from (6.1) we have that The equation for $\mathbf{v}_I(x,t)$ is

$$\partial_t \mathbf{v}_I + \partial_x ((\bar{\lambda}_i - \lambda_{tw}) + \lambda_B) v_I^i \mathbf{r}_i + \partial_x \sum_{j\neq i} \bar{\lambda}_j (\mathbf{v}_I + \mathbf{v}_B)^j \mathbf{r}_j - \partial_x^2 \mathbf{v}_I \tag{6.11}$$
$$\begin{aligned} &= \partial_t \mathbf{v}_I + \partial_x (\bar{\lambda}_i + \bar{\lambda}_B) v_I^i \mathbf{r}_i + \partial_x \sum_{j\neq i} \bar{\lambda}_j (\mathbf{v}_I + \mathbf{v}_B)^j \mathbf{r}_j - \partial_x^2 \mathbf{v}_I \\ &= -\frac{1}{2}\partial_x \left[C_{ii}^i(0,0) \cdot (v_I^i)^2 + O(1) \, (C_{ii}^i(x,t) - C_{ii}^i(0,0)) \, (v^i)^2 + O(1) \, \delta^3 e^{-\delta |x|} \right] \mathbf{r}_i \\ &\quad + \partial_x \left[\sum_{(k,l)\neq(i,i)} v^k v^l \mathbf{r}_i + \sum_{\substack{1\le k,l\le n \\ m\neq i}} C_{kl}^m v^k v^l \mathbf{r}_m + O(1) \sum_{j\neq i} \delta^2 e^{-\delta |x|} \mathbf{r}_j \right] \\ &\quad + \partial_x \left(O(1) \, \|\mathbf{v}\| \, \|\Theta_1\| + O(1) \, \|\Theta_1\|^3 + \mathfrak{Error} + \mathfrak{Error}_1 + O(1) \sum_{\substack{(k,p)\neq(j,j) \\ j,k,p\neq i}} \theta_k \, \theta_p \, \mathbf{r}_j \right), \end{aligned}$$

with that

$$\begin{aligned} &C_{ii}^i(x,t) - C_{ii}^i(0,0) = O(1) \, (\delta + \epsilon (|x| + t)), \\ &v^i = v_I^i + O(1) \, \bar{\lambda}_B, \\ &v^j = v_I^j + O(1) \, \delta \, \bar{\lambda}_B \text{ for } j \neq i. \end{aligned}$$

By integrating (6.11) we have the following system

$$\partial_t w_I^i + \lambda_B\ \partial_x w_I^i - \partial_{xx} w_I^i \ = \ \mathfrak{F}^i[\ \mathbf{w}_I\ ],$$

$$\mathfrak{F}^i[\ \mathbf{w}_I\ ] \ \equiv \ O(1)\ (\delta\ + \epsilon\ |x|\ )\ (\ \bar{v}_I^i\ +\ \bar{\lambda}_B{}^2\ )$$

$$+O(1)\ \|\bar{u}_x^\infty\|\ w_I^i\ \bar{\lambda}_B\ +\ O(1)\ \sum_{1\le k\le n}\ (\ w_{Ix}^k\ \|\ \bar{u}_x^\infty\ \|\ +\ w_I^k\ \|\ \bar{u}_{xx}^\infty\ \|\ )\ +\ O(1)\ (v_I^i)^2$$

$$+O(1)(v_I^i\ \|\ \mathbf{v}_I + \mathbf{v}_B\ \|\ +\ \sum_{(j,k)\neq(i,i)} v^j v^k\ +\ \|\ \mathbf{v}_I + \mathbf{v}_B\ \|\ \|\ \Theta_1\ \|\ +\ \sum_{j,k\neq i} \theta_j\ \theta_k\ )$$

$$+O(1)\ (\ \|\ \mathbf{v}_I + \mathbf{v}_B\ \|^3\ +\ \|\Theta_1\|^3\ ) + \mathfrak{Error}^{\ i} + \mathfrak{Error}^{\ i}_{\ 1}; \tag{6.12a}$$

$$\partial_t w_I^j + \bar{\lambda}_j\ \partial_x w_I^j - \partial_{xx} w_I^j \ = \ \mathfrak{F}^j[\mathbf{w}_I],$$

$$\mathfrak{F}^j[\mathbf{w}_I] \ \equiv \ O(1)\ \|\ \bar{u}_x^\infty\ \|\ w_I^j\ \bar{\lambda}_j\ +\ O(1)\ \sum_{1\le k\le n}\ (\ w_{Ix}^k\ \|\ \bar{u}_x^\infty\ \|\ +\ w_I^k\ \|\ \bar{u}_{xx}^\infty\ \|\ )$$

$$+O(1)\ (\ \bar{\lambda}_B{}^2\ +\ \sum_{(m,k)\neq(i,i)} v^m\ v^k\ +\ O(1)\ \|\ \mathbf{v}_I + \mathbf{v}_B\ \|\ \|\ \Theta_1\ \|\ +\ \sum_{\substack{m,k\neq i\\ (m,k)\neq(j,j)}} \theta_m\theta_k\ \ )$$

$$+O(1)\ (\ \|\ \mathbf{v}_I + \mathbf{v}_B\ \|^3\ +\ \|\Theta\|^3\ )\ +\ \mathfrak{Error}^{\ j}\ +\ \mathfrak{Error}^{\ j}_{\ 1}; \tag{6.12b}$$

Next, we substitute that

$$\begin{aligned} \partial_x{}^q\ \mathbf{r}_k &= O(1)\ (\ \delta^{1+q}\ e^{-\delta|x|}\ +\ \epsilon^q\ )\ \text{for}\ q\ge 1, \\ \bar{\lambda}_i &= O(1)\ (\ \delta\ +\ \epsilon\ |x|\ ), \\ \|\bar{u}_x^\infty(x,0)\| &= O(1)\ (\delta^2\ e^{-\delta|x|}\ +\ \epsilon\ ) \end{aligned}$$

into (6.12a), and by Duhamel's principle we have the representation for $w_I^j(x,t)$,

$$w_I^i(x,t) \tag{6.13}$$

$$= O(1) \int_0^t \int_R \mathbb{G}_i(y,s;x,t) \sum_{1\le k\le n} (\ w_{Iy}^k\ +\ \delta\ \ w_I^k\ )\ (\ \delta^2\ e^{-\delta|y|}\ +\ \epsilon\ )\ \ dyds$$

$$+O(1) \int_0^t \int_R \mathbb{G}_i(y,s;x,t)\ (\ \delta\ +\ \epsilon\ (\ |y|\ +\ s\ )\ )\ \ (\ v_I^{i\ 2}\ +\ \bar{\lambda}_B{}^2\ )$$

$$+O(1) \int_0^t \int_R \mathbb{G}_i(y,s;x,t) \left(v_I^{i2} + \|\mathbf{v}_I + \mathbf{v}_B\| \cdot \|\Theta_1\| + \sum_{j\neq i} (\ v^j\ )^2 \right) dyds$$

$$+O(1) \int_0^t \int_R \mathbb{G}_i(y,s;x,t) \left(\|\mathbf{v}_I + \mathbf{v}_B\|^3 + \|\Theta_1\|^2\ +\ \mathfrak{Error}^{\ i}\ + \mathfrak{Error}^{\ i}_{\ 1}\right) dyds;$$

for $j \neq i$,

$$w_I^j(x,t) = O(1) \int_R \bar{g}_j(y,s;x,t)\ w_I^j(y,0)\ dyds \tag{6.14}$$

$$+O(1) \int_0^t \int_R \frac{\bar{g}_j(y,s;x,t)}{\sqrt{t-s}}\ (\ \delta^2 e^{-\delta|y|}\ +\ \epsilon\ )\ w_I^j\ dyds$$

$$+O(1) \int_0^t \int_R \bar{g}_j(y,s;x,t)\ \left(\ w_I^j\ +\ \sum_{1\le k\le n}\ \ w_{Iy}^k\ +\ \delta\ w_I^k \right)\ (\ \delta^2\ e^{-\delta|y|} + \epsilon\ )\ dyds$$

$$+O(1)\int_0^t\int_R \bar{g}_j(y,s;x,t)\,(\;\bar{\lambda}_B^2 \;+\; \|\mathbf{v}_I\|\cdot\|\Theta_1\| \;+\; \|\mathbf{v}_B\|\cdot\|\Theta_1\| \;+\; \|\mathbf{v}_I\|^2 \;+\; \|\Theta_1\|^3\;)\,dyds$$

$$+O(1)\int_0^t\int_R \bar{g}_j(y,s;x,t)\left(\|\mathbf{v}_I+\mathbf{v}_B\|^3 + \sum_{\substack{(m,k)\neq(j,j);\\ m,k\neq i}} \theta_m\,\theta_k \;+\; \mathfrak{Error}^{\,j} \;+\; \mathfrak{Error}^{\,j}_{\,1}\right)dyds,$$

where $\bar{g}_j(y,s;x,t)$ is given in (4.37).

To obtain optimal estimates for $w_I^j(x,t)$ and $v_I(x,t)$ we will need to estimate another quantity:

$$z_I^j(x,t) \;\equiv\; \partial_t w_I^j(x,t) \text{ for } j=1,\cdots,n. \tag{6.15}$$

From (6.12) and (6.10),

$$z_I^j(x,0) \;=\; O(1)\;\delta^2\;e^{-\delta|x|} \text{ for } j=1,\cdots,n. \tag{6.16}$$

By differentiating (6.12) with respect to t, we have the equation for $z_I^j(x,t)$.

Case: $j\neq i$.

$$\begin{aligned}
&\partial_t z_I^j+\lambda_j\partial_x z_I^j=\partial_x^2 z_I^j+O(1)\;(\;\delta^2\;e^{-\delta|x|}\;+\;\epsilon\;)\;(\;z_{Ix}^j\;+\;z_I^j\;) \qquad (6.17a)\\
&\quad+O(1)\left(\bar{\lambda}_B\;\partial_t\bar{\lambda}_B\;+\sum_{1\le m,k\le n}\eta_{mk}\;+\sum_{\substack{l\le l\le n\\ l\neq i}}\partial_t\bar{\lambda}_B\;\theta_l\;+\;\bar{\lambda}_B\;\partial_t\theta_l\;)\right)\\
&\quad+O(1)\sum_{\substack{1\le l\le n\\ k\neq i}}\Big[\;(\;\delta^2\;e^{-\delta|x|}\;+\;\epsilon\;)\;\|\mathbf{w}_I\|\;\;+\;\bar{v}_I^l\;\Big]\;\partial_t\theta_k\\
&\quad+O(1)\sum_{\substack{1\le l\le n\\ k\neq i}}\Big[\;z_I^l\;\theta_k\;(\;\delta^2\;e^{-\delta|x|}+\epsilon\;)\;+\;z_I^l\partial_x\theta_k\;+\;(\;z_I^l\;\theta_k\;)_x\;\Big]\\
&\quad+O(1)\left(\sum_{1\le m\le n}(\;z_I^m\;\bar{\lambda}_B\;)_x\;+\;z_I^m\;\partial_x\bar{\lambda}_B\right)\\
&\quad+O(1)\left(\sum_{\substack{m,k\neq i\\ (m,k)\neq(j,j)}}\theta_m\theta_k\;+\;\|\mathbf{v}_I+\mathbf{v}_B\|^3\;+\;\|\Theta\|^3\;+\;\mathfrak{Error}^{\,j}\;+\mathfrak{Error}^{\,j}_{\,1}\right)_t,
\end{aligned}$$

where

$$\eta_{kl}\;\equiv\;(\;\bar{v}_I^k\;\bar{v}_I^l\;)_t\;=\;(z_I^k\bar{v}_I^l\;+\;z_I^l\bar{v}_I^k\;)_x\;-\;(\;z_I^k\partial_x\bar{v}_I^l\;+\;z_I^l\partial_x\bar{v}_I^k\;). \tag{6.17b}$$

Case $j=i$.

$$\begin{aligned}
&\partial_t z_I^i+\lambda_j\partial_x z_I^i=\partial_x^2 z_I^i+O(1)\;(\;\delta^2\;e^{-\delta|x|}\;+\;\epsilon\;(|x|+|t|)\;)\,(\;z_{Ix}^i\;+\;\delta\;z_I^i\;) \qquad (6.17c)\\
&\quad+O(1)\left(\bar{\lambda}_B{}^2\;\partial_t\bar{\lambda}_B\;+\sum_{1\le m,k\le n}\eta_{mk}\;+\sum_{l\le l\le n}\partial_t\bar{\lambda}_B\;\theta_l\;+\;\bar{\lambda}_B\;\partial_t\theta_l\;)\right)\\
&\quad+O(1)\sum_{\substack{1\le l\le n\\ k\neq i}}\Big[\;(\;\delta^2\;\;e^{-\delta|x|}+\;\epsilon\;)\;\|\mathbf{w}_I\|+\;\bar{v}_I^l\;\Big]\;\partial_t\theta_k
\end{aligned}$$

$$+ O(1) \sum_{\substack{1\le l\le n\\ k\neq i}} \left[z_I^l \, \theta_k \, (\delta^2 \; e^{-\delta|x|} + \epsilon) + z_I^l \partial_x \theta_k + (z_I^l \, \theta_k)_x \right]$$

$$+ O(1) \left(\sum_{1\le m\le n} (z_I^m \, \bar\lambda_B)_x + z_I^m \, \partial_x \bar\lambda_B \right)$$

$$+ O(1) \left(\sum_{m,k\neq i} \theta_m \theta_k + \|\mathbf{v}_I + \mathbf{v}_B\|^3 + \mathfrak{Error}^{\,i} + \mathfrak{Error}_1^{\,i} \right)_t .$$

Ansatz

We make a priori assumption on the solutions.

There exists a constant M such that for $t \in [0, \delta^{-2-\alpha_0}]$ with $\alpha_0 \in (0, 1/8)$

$$w_I{}^j(x,t) \;\le\; M \left(\delta^{\,\alpha_0{}^2/4} \, \zeta_j(x,t)^{1/2\,(\,1-\alpha_0/2\,)} + \frac{\epsilon}{\delta^2} \right), \tag{6.18a}$$

$$\bar v_I^j(x,t) \;\le\; M \left(\delta^{\,\alpha_0{}^2/4} \sum_{1\le k\le n} \zeta_k(x,t)^{3/2\,-\,\alpha_0/4} + \frac{\epsilon}{\delta^2} \right) \text{ for } j \neq i,$$

$$\bar v_I^i(x,t) \;\le\; M \left(\delta^{\,\alpha_0{}^2/4} \sum_{1\le k\le n} \zeta_k(x,t)^{3/2\,-\,\alpha_0/4} + \frac{\epsilon}{\delta^{3/2}} \right),$$

$$\bar v_{Ix}^j(x,t) \;\le\; M \left(\delta^{\,\alpha_0{}^2/4} \, \zeta_j(x,t)^{2\,-\,\alpha_0/4} + \frac{\epsilon}{\delta^{3/2}} \right), \tag{6.18b}$$

$$z_I^j(x,t) \;\le\; M \, \delta^{\,\alpha_0{}^2/4} \left(\sum_{1\le k\le n} \zeta_k(x,t)^{3/2\,-\,\alpha_0/4} + \sigma(x,t;\Lambda_j,E) + \frac{\epsilon}{\delta^{3/2}} \right) \tag{6.18c}$$

$$+ \;\; M \, \delta \; ch(x,t) \min((\,1 + |x - \Lambda_j(t)\,|\,)^{-1}, \delta\,) \text{ for } j \neq i,$$

$$z_I^i(x,t) \;\le\; M \, \delta^{\,\alpha_0{}^2/4} \left(\sum_{1\le k\le n} \zeta_k(x,t)^{3/2\,-\,\alpha_0/4} + \chi_i(x,t) + \frac{\epsilon}{\delta^{3/2}} \right). \tag{6.18d}$$

where algebraic-decaying function are defined by

$$\begin{aligned}
\zeta_j(x,t) &\equiv [(x-\Lambda_j(t))^2 + t + 1]^{-1/2} \text{ for } j \neq i,\\
\zeta_i(x,t) &\equiv [(|x| + \delta t)^2 + t + 1]^{-1/2};\\
\bar\zeta_j(x,t) &\equiv (\, t^2 + |x - \Lambda_j(t)|^3 \,)^{-1/3}\\
ch(x,t) &\equiv \begin{cases} 1 & \text{for } |x| < \max_j\{|\Lambda_j(t)| + \sqrt{t}\},\\ 0 & \text{else.} \end{cases}\\
\chi_i(x,t) &\equiv (t+1)^{-1/2}(1 + \delta(|x| + \delta t))^{-1}\\
&\quad\cdot \begin{cases} 1, & \text{for } |x| < C(t+1),\\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Proposition 6.2. *The function $w_I{}^j(x,0)$ in (6.10) satisfies that*

$$\partial_x^k \int_R g_j(y,0;x,t) w_I{}^j(y,0)dy \;=\; O(1)\,\sqrt{\delta}\;\zeta_j(x,t)^{(1+2k)/2} \textit{ for } k = 0,1,\; j \neq i.$$

Moreover,

$$\partial_x^k \int_R g_i(x,t;y,0)\;\delta\; e^{-\delta|y|}\;dy \;=\; O(1)\sqrt{\delta}\;\zeta_i(x,t)^{(1+2k)/2} \textit{ for } |x| + \delta t \le \sqrt{t} \textit{ for } k = 0,1.$$

Proof: We separate this proof into two cases.
Case 1. $k = 0$.
From (6.10) we have that

$$\left| \int_R g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy \right| = O(1) \int_R g_j(y,0;x,t)\ \delta\ dy = O(1)\ \delta. \tag{6.19}$$

For $|m_j(x,t)| \le \sqrt{t}$

$$\left| \int_R g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy \right| = O(1) \int_R \frac{1}{\sqrt{t}}\ \delta\ e^{-\delta|y|}\ dy = O(1)\ \frac{1}{\sqrt{t}}. \tag{6.20}$$

This and (6.19) yield that for $|m_j(x,t)| \le \sqrt{t}$

$$\left| \int_R g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy \right| = O(1)\ \min(\ \delta, \sqrt{t}\ ) = O(1)\ \sqrt{\delta}\ /\ t^{1/4}. \tag{6.21}$$

For $m_j(x,t) \ge \sqrt{t}$,

$$\int_R g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy = \left(\int_{|y| \le \frac{1}{2}|m_j(x,t)|} + \int_{|y| \ge \frac{1}{2}|m_j(x,t)|} \right) g_j(y,0;x,t) w_I{}^j(y,0)\ dy \tag{6.22}$$
$$= O(1) \left(\frac{e^{-\frac{m_j(x,t)^2}{16t}}}{\sqrt{t}} + \delta e^{-\delta|m_j(x,t)|/8} \right) = O(1)\ \frac{1}{|m_j(x,t)|}.$$

By combining (6.19) and (6.22) it yields that for $m_j(x,t) \ge \sqrt{t}$

$$\left| \int_R g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy \right| = O(1)\ \sqrt{\delta}\ /\ \sqrt{|m_j(x,t)|}. \tag{6.23}$$

From this and (6.20) we have that

$$\left| \int_R g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy \right| = O(1)\ \sqrt{\delta}\ \zeta_j(x,t)^{1/2}.$$

Case 2. $k = 1$.

$$\left| \int_R \partial_x g_j(y,0;x,t) w_I{}^j(y,0) dy \right| = O(1) \int_R \frac{1}{\sqrt{t}} K\left(\frac{m_j(x,t) - m_j(y,0)}{2}, t \right)\ \delta\ dy = O(1)\ \frac{\delta}{\sqrt{t}}. \tag{6.24}$$

On the other hand

$$\left| \int_R \partial_x g_j(y,0;x,t) w_I{}^j(y,0) dy \right| = O(1) \int_R \frac{1}{t}\ \delta\ e^{-\delta|y|}\ dy = \frac{O(1)}{t}.$$

This and (6.24) yield that

$$\left| \int_R \partial_x g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy \right| = O(1)\ \min\left(\frac{1}{t}, \frac{\delta}{\sqrt{t}} \right) = O(1)\ \frac{\sqrt{\delta}}{t^{3/4}}. \tag{6.25}$$

For $|m_j(x,t)| > \sqrt{t}$,

$$\int_R \partial_x g_j(y,0;x,t)\ w_I{}^j(y,0)\ dy = \int_R \partial_x K\left(\frac{m_j(y,0)-m_j(x,t)}{m_{jy}(y,0)},t\right) w_I{}^j(y,0)\ dy \tag{6.26}$$

$$= -\int_R \partial_y K\left(\frac{m_j(y,0)-m_j(x,t)}{m_{jy}(y,0)},t\right)\ w_I{}^j(y,0)\ dy$$

$$-\int_R K_X\left(\frac{m_j(y,0)-m_j(x,t)}{m_{jy}(y,0)},t\right)\ \frac{(m_j(y,0)-m_j(x,t))m_{jyy}(y,0)}{m_{jy}(y,0)^2}\ w_I{}^j(y,0)\ dy$$

$$= \int_R K\left(\frac{m_j(y,0)-m_j(x,t)}{m_{jy}(y,0)},t\right) w_{Iy}^j(y,0)dy$$

$$+\int_R K_X\left(\frac{m_j(y,0)-m_j(x,t)}{m_{jy}(y,0)},t\right)\ \frac{(m_j(y,0)-m_j(x,t))m_{jyy}(y,0)}{m_{jy}(y,0)^2}\ w_I{}^j(y,0)\ dy$$

$$= O(1)\ \int_R K\left(\frac{m_j(y,0)-m_j(x,t)}{2m_{jy}(y,0)},t\right)\ \left(\delta^2\ e^{-\delta|y|}\ +\ (\ \epsilon\ +\delta^2\ e^{-\delta|y|}\ )\ \delta\ e^{-\delta|y|}\ \right)\ dy$$

$$= O(1)\ \left(\int_{|y|\le|m_j(x,t)|/2} + \int_{|y|\ge|m_j(x,t)|/2}\right) K\left(\frac{m_j(y,0)-m_j(x,t)}{2m_{jy}(y,0)},t\right)\ (\ \delta^2\ +\delta\ \epsilon\ )\ e^{-\delta|y|}\ dy$$

$$= O(1)\ \delta\ \left(K\left(\frac{m_j(x,t)}{4m_{jy}(y,0)},t\right) + \delta e^{-\delta|m_j(x,t)|/2}\right) = O(1)\ \delta\ (1+|m_j(x,t)|)^{-1}.$$

Similar to this derivation one can obtain that

$$\int_{|y|\le|m_j(x,t)|/2} \partial_x g_j(y,0;x,t) w_I{}^j(y,0)\ dy\ =\ O(1)\ \delta^2\ e^{-\delta|x|}\ =\ O(1)\ (1+|m_j(x,t)|)^{-2}. \tag{6.27}$$

On the other hand,

$$\int_{|y|\le|m_j(x,t)|/2} \partial_x g_j(y,0;x,t)\ w_I{}^j(y,s)\ dy = O(1)\ \frac{e^{-|m_j(x,t)|^2/\ 16t}}{t} \int_{|y|\le|m_j(x,t)|/2} |w_I{}^j(y,s)|\ dy$$

$$= O(1)\ \frac{e^{-|m_j(x,t)|^2/\ 16t}}{t}\ = O(1)\ (\ 1\ +\ |m_j(x,t)|\ )^{-2}. \tag{6.28}$$

From (6.27) and (6.28), we have that

$$\left|\int_R \partial_x g_j(y,0;x,t)\ w_I{}^j(y,s)\ dy\right| = O(1)\ (1+|m_j(x,t)|)^{-2}. \tag{6.29}$$

Combine (6.26) and (6.29) to obtain that for $m_j(x,t) \ge \sqrt{t}$

$$\left|\int_R \partial_x g_j(y,0;x,t)\ w_I{}^j(y,s)\ dy\right| = O(1)\ \sqrt{\delta}\ (1+\ m_j(x,t)\ )^{-3/2}. \tag{6.30}$$

From (6.30) and (6.25), we have that

$$\left|\int_R \partial_x g_j(y,0;x,t)\ w_I{}^j(y,s)\ dy\right| = O(1)\ \sqrt{\delta}\ \zeta^{3/2}(x,t).$$

By the same derivation for (6.21), we have that

$$\int_R \partial_x^k g_i(x,t;y,0)\delta e^{-\delta|y|}dy = O(1)\ \sqrt{\delta}\ \zeta_i(x,t)^{(1+2k)/2} \text{ for } |x|+\delta t \le \sqrt{t} \text{ and }\ k=0,1.$$

Thus, the proposition follows. **Q.E.D.**

Verification of Ansatz
By substituting the ansatz (6.18) into the integrals in the R.H.S. of (6.13), (6.14) and (6.17) we can verify the ansatz (6.18) is valid for $t \le \delta^{-(2+\alpha_0)}$.

We begin with the estimate of the integrals in the R.H.S. of (6.14).
From Proposition 6.2 the first integral satisfies that

$$\begin{aligned} \int_R \bar{g}_j(y,0;x,t)\ w_I^j(y,0)\ dy &= \int_R \bar{g}_j(y,0;x,t)\ \delta\ e^{-\delta|y|}\ dy \\ &= O(1)\ \sqrt{\delta}\ \zeta_j(x,t)^{1/2}. \end{aligned}$$

Before estimating the second and the third integrals, we need to estimate the function $\delta e^{-\delta|y|}\zeta_j^\rho$ with $\rho \ge 0$ as follows

$$\begin{aligned} \delta e^{-\delta|y|}\ \zeta_j(y,s)^\rho &= e^{-\delta|y|/2}\ \left(\delta\ e^{-\delta|y|/2}\zeta_j(y,s)^\rho\right) \\ &= O(1)\ e^{-\delta|y|/2}\left(\delta\ (1+s)^{-\rho}\ + \sqrt{\delta}\ (1+s)^{-(1+\rho)/2}\ \right). \end{aligned}$$

From this with $\rho = \frac{2-\alpha_0}{4}$ and (6.18) that

$$\begin{aligned} \delta\ e^{-\delta|y|}\ w_I^j(y,s) &= O(1)\ M\ \left(\ \delta^{\frac{\alpha_0^2}{4}} e^{-\delta|y|/2}\left(\delta\ (1+s)^{-\ (2\ -\ \alpha_0\ )/4}\ +\ \sqrt{\delta}\ (1+s)\ ^{-\ (6\ -\ \alpha_0\ )/8}\ \right)\ +\ \epsilon\ \right); \\ \delta\ e^{-\delta|y|}\bar{v}_I^j(y,s) &= O(1)\ M\ \left(\ \delta^{\frac{\alpha_0^2}{4}}\ e^{-\delta|y|/2}\left(\delta\ (1+s)^{-\ (6\ -\ \alpha_0\ )/4}\ +\ \sqrt{\delta}\ (1+s)\ ^{-\ (10\ -\ \alpha_0\ )/8}\ \right)\ +\ \epsilon\ \right); \end{aligned}$$

Consider the integral

$$\int_0^t \int_R \frac{\bar{g}_j(y,s;x,t)}{\sqrt{t-s}}\ \epsilon\ w_j(y,s)\ dyds\ \equiv \mathfrak{T}.$$

Since α_0 satisfies that $\alpha_0 \in [0,1/8]$, we have that for $|x| > \delta^3$ and $t \in [0, \delta^{-(2+\alpha_0)}]$

$$\begin{aligned} \mathfrak{T} &= O(1)\ M \int_0^t \int_{|y| \le \frac{\delta^3}{2}} \frac{K(x/2,t-s)}{\sqrt{t-s}}\ \epsilon\ \delta^{\frac{\alpha_0^2}{4}}\ dyds \\ &\quad +\ O(1)\ M \int_0^t \int_{|y| \ge \frac{\delta^3}{2}} \frac{\bar{g}_j(y,s;x,t)}{\sqrt{t-s}}\ \epsilon\ (\ \delta^{\frac{\alpha_0^2}{4}}\ \delta^{\frac{3\ (2-\alpha_0)}{4}}\ +\ \frac{\epsilon}{\delta^2}\ )\ dyds \\ &= O(1)\ M\ \frac{e^{-\frac{x^2}{8t}}}{t}\ (\ \log(t)\ t\ \epsilon\ )\ \delta^{\frac{\alpha_0^2}{4}}\ +\ O(1)\ M\ \epsilon\ \sqrt{t}\ \delta^{\frac{\alpha_0^2}{4}\ +\ \frac{3\ (2-\alpha_0)}{4}} \\ &= O(1)\ M\ \left(\ \epsilon\ t \log(t)\ \delta^{\frac{\alpha_0^2}{4}}\ \ (\ 1\ +\ |x|\ )^{-2}\ +\ \delta^{\frac{\alpha_0^2}{4}}\ \delta^{\frac{2-5\alpha_0}{4}}\ \epsilon\ \right). \end{aligned} \tag{6.31}$$

Since ϵ satisfies that $\epsilon \ll \delta^6$, (6.31) becomes

$$\mathfrak{T}\ \le\ O(1)\ \delta^{1/8}\ M\ (\delta^{\frac{\alpha_0^2}{4}}\ \ (\ 1\ +\ |x|\ )^{-2}\ +\ \epsilon\ ). \tag{6.32}$$

On the other hand for $|x| < \delta^3$ and for $t \in [0, \delta^{-(2+\alpha_0)}]$ we have that

$$\begin{aligned}
\mathfrak{T} &\equiv O(1)\ M \int_0^t \frac{1}{\sqrt{t-s}}\ \epsilon \left(\delta^{\frac{\alpha_0{}^2}{4}}\ s^{-(2-\alpha_0)/8} + \frac{\epsilon}{\delta^2} \right) ds \\
&= O(1)\ M\ (\delta^{\frac{\alpha_0{}^2}{4}}\ t^{(2+\alpha_0)/8}\ \epsilon + \epsilon^{\ 2} \frac{\sqrt{t}}{\delta^2}) \\
&= O(1)\ \delta^{1/2}\ M\ \delta^{\frac{\alpha_0{}^2}{4}}\ \zeta_j^{5/3}(x,t).
\end{aligned}$$

From this and (6.32) we have that

$$\mathfrak{T} = O(1)\delta^{\frac{1}{8}}\ M\ (\delta^{\frac{\alpha_0{}^2}{4}}\ \zeta_j^{5/3}(x,t) + \epsilon). \tag{6.33}$$

Then, by applying Lemma 4.5 with $\alpha = (2-\alpha_0)/2,\ 3/2 - \alpha_0/2$ and $\beta = 2$ we have that

$$\begin{aligned}
&\int_0^t \int_R \frac{\bar{g}_j(y,s;x,t)}{\sqrt{t-s}}\ \delta^2 e^{-\delta|y|} w_I^j\ dyds \\
&= O(1)\ M\ \delta^{\ 1\ +\ \alpha_0^2/4} \left(\delta\ (\ 1 + |x-\Lambda_j(t)|\ )^{-(1-\alpha_0/2)/2} + \sqrt{\delta}\ (\ 1 + |x-\Lambda_j(t)|\ )^{-3/4-\alpha_0/4} \right);
\end{aligned}$$

and with $\alpha = 3/2 - \alpha_0/2,\ 5/2 - \alpha_0/4$ and $\beta = 1$,

$$\begin{aligned}
&\int_0^t \int_R \delta^2\ \bar{g}_j(y,s;x,t) \sum_{k\neq i} (w_{Iy}^k + \delta\ w_I^k)\ e^{-\delta|y|}\ dyds \qquad (6.34) \\
&= O(1)\ M\ \delta^{\ \alpha_0^2/4} \left(\delta\ (\ 1 + |x-\Lambda_j(t)|\ )^{-(1-\alpha_0/2)/2} + \sqrt{\delta}\ (\ 1 + |x-\Lambda_j(t)|\ )^{-3/4-\alpha_0/4} \right).
\end{aligned}$$

From the ansatz (6.18) we have that $\|w_I^i(\cdot,t)\|_\infty = O(1)\ M\ \delta^{-(2-\alpha_0)/4\ +\ \alpha_0^2/4}\ (\ 1 + t\ )^{\ -(2-\alpha_0)/4}$. This, the procedure for obtaining (6.33), and Lemma 4.5 with $\alpha = (2-\alpha_0)/2$ and $\beta = 1$ yield that

$$\int_0^t \int_R \delta^2\ \bar{g}_j(y,s;x,t)\ (\ w_{Iy}^i + w_I^i\ )\ e^{-\delta|y|}\ dyds = O(1)\ M\ \delta^{\frac{(\alpha_0^2+2+\alpha_0)}{4}}\ \zeta_j(x,t)^{(2-\alpha_0)/4}.$$

By Lemma 4.5 with $\alpha = 0$ and $\beta = 1$ the first integrand in the fourth integral satisfies that

$$\int_0^t \int_R \bar{g}_j(y,s;x,t)\ \bar{\lambda}_B(y,s)^2\ dyds = O(1)\ \delta^{\alpha_0^2/4}\ \zeta_j(x,t)^{(2-\alpha_0)/4} \text{ for } 1 \le t \le \delta^{-2-\alpha_0}. \tag{6.35}$$

The second integrand satisfies that for $t \in [0, \delta^{-(2+\alpha_0)}]$

$$\|\ \mathbf{v}_I\ \| \cdot \|\ \Theta_1\ \| = O(1)\ M \left(\delta^{\ \alpha_0^2/4} \frac{1}{(1\ +\ s\ )^{\frac{3}{4}-\frac{\alpha_0}{8}}} + \epsilon \right) \|\ \Theta_1\ \|.$$

By this estimate and by using Lemma 4.1, 4.2.A, and 4.2.B with $\alpha = 2.5 - \alpha_0/4$ and $\beta = 1$ we have that

$$\begin{aligned}
&\int_0^t \int_R \bar{g}_j(y,s;x,t)\ (\|\Theta_1\|\ \|\mathbf{v}_I\|\ )\ (y,s)\ dyds \\
&= O(1)\ M\ \delta^{\alpha_0^2/4\ +\ 1} \left((t+1)^{1/4\ +\ \alpha_0/8}\ \sigma(x,t;\Lambda_j,E) + \sum_{k\neq i,j} \zeta_k(x,t)^{1/2} \right) \\
&= O(1)\ M\ \delta^{\alpha_0^2/2}\ \zeta_j(x,t)^{(2-\alpha_0)/4}.
\end{aligned}$$

For the third integrand from (6.8) there exists a constant $E > 1$ such that

$$\begin{aligned} \|\mathbf{v}_B(y,s)\| \cdot \|\Theta_1(y,s)\| &= O(1)\ \delta\ e^{-\delta s/E} \sum_{k\neq i} \sigma(y,s;\Lambda_k,E) \\ &= O(1)\ \delta^{\ 1/4}(1+s)^{-3/4} \sum_{k\neq i} \sigma(y,s;\Lambda_k,E). \end{aligned}$$

Then, from this estimate and from Lemma 4.1, 4.2.A and 4.2.B with $\alpha = 5/2$ and $\beta = 1$ we have that for $t \in [0, \delta^{\ -(2+\alpha_0)}]$

$$\begin{aligned} &\int_0^t \int_R \bar{g}_j(y,s;x,t) \|\Theta_1\| \cdot \|\mathbf{v}_B\|\ dyds \\ &= O(1)\ \delta^{\alpha_0^2/4\ +\ 1/4}\ \left((t+1)^{1/4} \sigma(x,t;\Lambda_j,E)\ +\ \zeta_j(x,t)^{1/2} \right) \\ &= O(1)\ \delta^{\alpha_0^2/4+\ 1/4}\ \zeta_j(x,t)^{1/2}. \end{aligned} \tag{6.36}$$

The fourth integrand satisfies that

$$\|\mathbf{v}_I(y,s)\|^2 = O(1)\ M^2\ (\ \delta^{\alpha_0^2/2} \sum_{1\leq k\leq n} \zeta_k(y,s)^{3-\frac{\alpha_0}{2}}\ +\ \epsilon^2\ ).$$

Then, by Lemma 4.3, 4.4.A and 4.4.B with $\alpha = 2.5$ and $\beta = 1$ as well as by the above estimate it yields that

$$\int_0^t \int_R \bar{g}_j(y,s;x,t) \|\mathbf{v}_I\|^2\ dyds = O(1)\ M^2\ (\ \delta^{\ \alpha_0^2/2}\ \zeta_j(x,t)^{1/2}\ +\ \epsilon^2\ t\ ).$$

By Lemma 4.1, 4.2.A and 4.2.B with $\alpha = 3$ and $\beta = 1$ the integral for fifth integrand satisfies that

$$\begin{aligned} \int_0^t \int_R \bar{g}_j(y,s;x,t)\ \|\Theta_1(y,s)\|^3\ dyds &= O(1)\ \delta^3\ \left(\ \zeta_j(x,t)^{1/2} + \sigma(x,t;\Lambda_j,E)\ \right) \\ &= O(1)\ \delta^3\ \zeta_j(x,t)^{1/2}. \end{aligned} \tag{6.37}$$

The first integrand in the fifth integral satisfies that

$$\|\mathbf{v}_B + \mathbf{v}_I\|^3 = O(1)\ \left(\delta^3\ e^{-3\ \delta\ |y|}\ + \epsilon^3\ +\ M^3\ \delta^{\ 3\ \alpha_0{}^2/4} \sum_{1\leq l\leq n} \zeta_l{}^3 \right).$$

By Lemma 4.5 with $\alpha = 0$ and $\beta = 1$ for $t \in\ [0, \delta^{\ -(\ 2\ +\ \alpha_0\ )}]$

$$\begin{aligned} &\int_0^t \int_R \bar{g}_j(y,s;x,t) \delta^3\ e^{-\delta|y|}\ dyds \\ &=\ O(1)\ \ \delta^2\ ch(x,t)\ =\ \delta^{\ 1+\alpha_0^2/4}\ \zeta_j(x,t), \end{aligned} \tag{6.38}$$

and by Lemma 4.3, 4.4.A and 4.4.B with $\alpha = 3$ and $\beta = 1$

$$\begin{aligned} &\int_0^t \int_R \bar{g}_j(y,s;x,t)\ \cdot \|\mathbf{v}_I\|\ (y,s)^3\ dyds \\ &= O(1)\ M^3\ \delta^{\frac{3\alpha_0^2}{4}}\ \zeta_j(x,t)^{1/2}. \end{aligned} \tag{6.39}$$

There exists a constant E such that the second integrand satisfies that

$$\sum_{\substack{(m,k)\neq(j,j)\\ m,k\neq i}} (\ \theta_m\ \theta_k\ )(x,t) \ = \ O(1)\left(\sum_{l\neq j,i}\theta_l^{\,2}\ +\ \theta_j\sum_{l\neq j,i}\theta_l\right)(x,t) \qquad (6.40)$$

$$= \ O(1)\ \delta^2\ \left(\sum_{l\neq j,i}\sigma(x,t;\Lambda_l,E)^2\ +\ e^{-t/E}\sum_{l\neq j,i}\sigma(x,t;\Lambda_l,E)\right).$$

From this and by using Lemma 4.2.A and 4.2.B with $\alpha=2$ and $\beta=1$,

$$\int_0^t\int_R \bar{g}_j(y,s;x,t)\ \sum_{\substack{(m,k)\neq(j,j)\\ m,k\neq i}}(\ \theta_m\ \theta_k\ )(y,s)\ \ dyds\ =\ O(1)\ \delta^2\ \zeta_j(x,t)^{1/2}. \qquad (6.41)$$

From (6.2) and (5.32) the last integrand satisfies that

$$|\mathfrak{Error}^{\ j}+\mathfrak{Error}^{\ j}_{\ 1}|\ (y,s)\ =\ O(1)\ \epsilon\ e^{-\delta|y|}$$
$$+O(1)\ \delta^{\ (2+\alpha_0{}^2)/4}\ (\ 1\ +\ s\ )^{-(\ 6\ -\ \alpha_0{}^2\ )/4}\ e^{\ -\delta|y|/4}\ K\left(\ \frac{y-\lambda_j s}{E},\ s\ \right).$$

By Lemma 4.1, 4.2.A and 4.2.B with $\alpha=2.5-\alpha_0/4$ and $\beta=1$ as well as by Lemma 4.5 with $\beta=1$ and $\alpha=0$ we have that

$$\int_0^t\int_R \bar{g}_j(y,s;x,t)\ \ (\mathfrak{Error}^{\ j}+\mathfrak{Error}^{\ j}_{\ 1})\ (y,s)\ \ dyds \qquad (6.42)$$
$$=\ O(1)\ \left(\ \delta^{\ (2+\alpha_0{}^2)/4}\ \zeta_j(x,t)^{1/2}\ +\ \frac{\epsilon}{\delta}\ \right).$$

The estimates for (6.35), (6.37), (6.38), (6.39), (6.40) and (6.42) do not contain the factor M. The others contain both the factor M and$o(\delta^{\alpha_0{}^2}4)$. Thus, the above estimates for integrals in the R.H.S. of (6.14) yields that there is a constant M_0 such that for $t\in[0,\delta^{-2-\alpha_0}]$ and for $j\neq i$

$$w_I^j(x,t)\ \leq\left(M_0\ +O(1)\ \ (M+M^2+M^3)\ \delta^{\frac{\alpha_0{}^2}{4}}\right)\ \left(\ \delta^{\frac{\alpha_0{}^2}{4}}\ \zeta_j(x,t)^{(2-\alpha_0)/4}\ +\ \frac{\epsilon}{\delta^2}\ \right). \qquad (6.43)$$

Now we continue to verify the ansatz for $w_I^i(x,t)$.
We also begin with the estimate the double integral $\int_0^t\int_R \mathbb{G}_i(y,s;x,t)\ \delta^3\ e^{-\delta|y|}\ w_I^k\ dyds$ in the first integral in the R.H.S. of (6.13). One need to treat this integral in two cases, ($k\neq i$ or $k=i$).

For $k\neq i$ this integral can be treated by the same procedure as that for $w^j(x,t)$ with $j\neq i$ in the above.
For $k=i$ substitute the ansatz (6.18) into this integral to yield that

$$\int_0^t\int_R \mathbb{G}_i(y,s;x,t)\delta^3\ w_I^i\ e^{-\delta|y|}\ dyds$$
$$=\ \ O(1)\ M\ \delta^3\int_0^t\int_R \mathbb{G}_i(y,s;x,t)(\ (|y|+\delta s)^2\ +\ s\ )^{-(2-\alpha_0)/8}\ e^{-\delta|y|}\ dyds.$$

From this and Lemma 4.8 with $\alpha=(2-\alpha_0)/2$, $\beta=1$ and $\gamma=0$ there is a positive constant C such that

$$\int_0^t\int_R \mathbb{G}_i(y,s;x,t)\delta^3\ w_I^i\ e^{-\delta|y|}\ dyds\ =\ O(1)\ \delta^{1-\alpha_0^2/4}\ M\ \delta^{\alpha_0^2/4}\ (\ |x|+\delta t\ ))^{-(2-\alpha_0)/4}e^{-C|x|}. \qquad (6.44)$$

For $x \le \sqrt{t} \le \delta^{-1}$ we need a sharper estimate for this integral,

$$\begin{aligned}
&\int_0^t \int_R \mathrm{G}_i(y,s;x,t)\delta^3\ w_I^i\ e^{-\delta|y|}\ dyds \\
&= \ O(1)\ M\delta^3 \int_0^t \left(\int_{|y|\le\sqrt{s}} + \int_{|y|\ge\sqrt{s}} \right)\ \mathrm{G}_i(y,s;x,t)(\ (|y|+\delta s)^2\ +\ s\ )^{-(2-\alpha_0)/8}\ e^{-\delta|y|}\ dyds \\
&= \ O(1)\ M\delta^2 \int_0^t (t-s)^{-1/2} s^{-(2-\alpha_0)/8} ds \\
&= \ O(1)\ \delta^{1\ -\alpha_0^2/4}\ M\delta^{\alpha_0^2/4}\ \zeta_i(x,t).
\end{aligned} \tag{6.45}$$

Hence from (6.44) and (6.45)

$$\int_0^t \int_R \mathrm{G}_i(y,s;x,t)\delta^3\ w_I^i\ e^{-\delta|y|}\ dyds\ =\ O(1)\ \delta^{1-\alpha_0^2/4} M\ \zeta_i(x,t).$$

We decompose the integrand $\delta^2 w_{Ix}^i$ as

$$\begin{aligned}
\delta^2 w_{Iy}^i(y,s)\ &=\ O(1)\ M\ \left(\ \delta^{2+\alpha_0^2/4}\ (\ (|y|+\delta s)^2\ +\ s\ )^{(6-\alpha_0)/4}\ +\ \delta^{1/2}\ \epsilon\ \right) \\
&=\ O(1)\ M\ \left(\ \delta^{1+\alpha_0^2/4}\ (1+s)^{-1}(\ (|y|+\delta s)^2\ +\ s\ )^{(2-\alpha_0)/4}\ +\ \delta^{1/2}\ \epsilon\ \right).
\end{aligned}$$

By Lemma 4.8 with $\alpha = (2-\alpha_0)/2$, $\gamma = 2$, $\beta = 1$ and replacing $\delta^{(3-\beta)/2}$ by $(s+1)^{(3-\beta)/2}$,

$$\int_0^t \int_R \mathrm{G}_i(y,s;x,t)\delta^2\ w_{Iy}^i\ e^{-\delta|y|}\ dyds\ =\ O(\delta M)\ (t+\delta|x|)^{-(2-\alpha_0)/2}. \tag{6.46}$$

For $|x| \le \sqrt{t} \le \delta^{-1}$ by the same argument as (6.45) we have that

$$\begin{aligned}
&\int_0^t \int_R \mathrm{G}_i(y,s;x,t)\delta^2\ w_{Iy}^i\ e^{-\delta|y|}\ dyds \\
&= \ O(M\delta^2) \int_0^t \left(\int_{|y|\le\sqrt{s}} + \int_{|y|\ge\sqrt{s}} \right)\ \mathrm{G}_i(y,s;x,t)(\ (|y|+\delta s)^2\ +\ s\ )^{-(6-\alpha_0)/8}\ e^{-\delta|y|}\ dyds \\
&= \ O(1)\ M\ \delta \int_0^t (t-s)^{-1/2} s^{-(6-\alpha_0)/8} ds = \delta O(1)\ M\ t^{-(2-\alpha_0)/8} \\
&= \ O(1)\ \delta^{1-\alpha_0^2/4}\ M\delta^{\alpha_0^2/4}\ \zeta_i(x,t).
\end{aligned} \tag{6.47}$$

From (6.46) and (6.47) it yields that

$$\int_0^t \int_R \mathrm{G}_i(y,s;x,t)\ \delta^2\ w_{Iy}^i\ e^{-\delta|y|}\ dyds\ =\ O(1)\ \delta^{1-\alpha_0^2/4}\ M\delta^{\alpha_0^2/4}\ \zeta_i(x,t). \tag{6.48}$$

For $t \in [0, \delta^{-(2+\alpha_0)}]$ the integral for $(w_{Iy}^i\ +\ \delta w_I^i)\ \epsilon$ is

$$\int_0^t \int_R \mathrm{G}_j(y,s;x,t)\ (w_{Iy}^i\ +\ \delta w_I^i)\ \epsilon\ dyds\ =\ O(1)\ M\ \epsilon\ \left(\delta^{-\frac{(4+\alpha_0{}^2)}{8}}\ +\ \delta^{1-\frac{(2+\alpha_0)^2}{8}} \right).$$

The integral for $\epsilon\ |y|\ \mathbb{G}_i(y,s;x,t)\ \bar{v}_I^i(y,s)^2$ we have that

$$\begin{aligned}
&\int_0^t \int_R \epsilon\ |y|\ \mathbb{G}_i(y,s;x,t)\ \bar{v}_I^i(y,s)^2\ dyds \\
&= O(1)\ M^2 \int_0^t \int_R \mathbb{G}_i(y,s;x,t)\ \epsilon\ |y|\ \delta^{\frac{\alpha_0{}^2}{2}}(|y|^2+s)^{-\frac{6-\alpha_0}{4}}\ dyds \\
&= O(1)\ M^2\ \epsilon\ \delta^{\frac{\alpha_0{}^2}{2}} \int_0^t \int_R \mathbb{G}_i(y,s;x,t)\ s^{-\frac{4-\alpha_0}{4}}\ dyds = O(1)\ M^2\ \epsilon\ \delta^{\frac{\alpha_0{}^2}{2}}\ t^{\frac{\alpha_0}{4}} \\
&= O(1)\ M^2\ \epsilon\ \delta^{\frac{\alpha_0{}^2}{2}-\frac{\alpha_0\ (\ 2\ +\ \alpha_0\ )}{4}}.
\end{aligned}$$

By Lemma 4.6 with $\alpha = 0$ and $\beta = 1$ it yields that

$$\begin{aligned}
&\int_0^t \int_R \mathbb{G}_j(y,s;x,t)\ \delta\ \bar{\lambda}_B(y,s)^2\ dyds \\
&= O(1)\ \delta^3 \int_0^t \int_R \mathbb{G}_j(y,s;x,t)\ e^{-\delta|y|}\ dyds\ =\ O(1)\ \delta\ e^{-\delta|x|} \\
&= O(1)\ \delta^{\frac{2+\alpha_0}{4}}\ \zeta_i^{\frac{2-\alpha_0}{4}}(x,t).
\end{aligned}$$

By Lemma 4.3 with $\alpha = 3 - \alpha_0/2$ and $\beta = 1$ the double integral for $v_I^{i\,2}$ is

$$\int_0^t \int_R \mathbb{G}_i(y,s;x,t)\ v_I^i(y,s)^2 dyds\ =\ O(1)\ M^2\ \delta^{\alpha_0^2/4}\ \zeta_i(x,t)^{1/2}.$$

The integrals for the higher order terms in (6.13) can be handled by the same procedure for obtaining (6.43). It yields that for $t \in [\ 0,\ \delta^{-2-\alpha_0}\ ]$ there exists a constant M_0 such that

$$w_I^i(x,t)\ \leq\ \left(M_0\ +\ O(1)\ \delta^{\alpha_0{}^2/4}\ \left(M+M^2+M^3\right)\right)\left(\delta^{\frac{\alpha_0{}^2}{4}}\ \zeta_i(x,t)^{(2-\alpha_0)/4}\ +\ \frac{\epsilon}{\delta^2}\right). \tag{6.49}$$

Hence, it justifies the ansatz $\mathbf{w}_I$ for $t \in [0, \delta^{-(2+\alpha_0)}]$ if δ is sufficiently small and if M is chosen such that $|M/M_0|$ is sufficiently large.

Next, we verify the ansatz $z_I^j(x,t)$ in (6.18c). The procedures for verifying ansatz in compressive field and non-compressive fields are different just as we did in the verification for $\mathbf{w}_I$.

From (6.16), (6.17) and (6.18) we have the representation for $z_I^j(x,t)$ with $j \neq i$

$$\begin{aligned}
z_I^j(x,t)\ &=\ O(1)\ \delta^2 \int_R \bar{g}_j(y,0;x,t)e^{-\delta|y|}\ dy \qquad \text{(6.50a)}\\
&+O(1) \int_0^t \int_R \left(\frac{\bar{g}_j(y,s;x,t)}{\sqrt{t-s}} + \bar{g}_j(y,s;x,t)\right)\ (\ \delta^2\ e^{-\delta|y|}\ +\ \epsilon\ )\ z_I^j\ dyds \\
&+O(1) \int_0^t \int_R \bar{g}_j(y,s;x,t) \left(\bar{\lambda}_B \partial_s \bar{\lambda}_B + \sum_{l\neq i} \partial_s\left(\bar{\lambda}_B \theta_l\right) + \sum_{\substack{m,k\neq i\\ (m,k)\neq(j,j)}} (\ \theta_m \theta_k\ )_s\right)\ dyds
\end{aligned}$$

$$+O(1)\int_0^t\int_R \bar{g}_j(y,s;x,t)\sum_{1\le m,k\le n}\eta_{mk}\,dyds$$
$$+O(1)\int_0^t\int_R \bar{g}_j(y,s;x,t)F(y,s)dyds,$$

where the function $F(x,t)$ is

$$F(x,t)\equiv$$
$$O(1)\sum_{\substack{1\le l\le n\\ k\ne i}}\Big[\,(\,\delta^2\;e^{-\delta|x|}\;+\;\epsilon\,)\;\|\mathbf{w}_I\|\;+\;\bar{v}_I^l\,\Big]\;\partial_t\theta_k$$
$$+\;O(1)\sum_{\substack{1\le l\le n\\ k\ne i}}\Big[\,(\,\delta^2\;e^{-\delta|x|}\;+\;\epsilon\,)\;z_I^l\;\theta_k\;+\;z_I^l\partial_x\theta_k\;+\;(\;z_I^l\;\theta_k\;)_x\,\Big]$$
$$+\;O(1)\;(\sum_{1\le m\le n}\;(\;z_I^m\;\bar{\lambda}_B\;)_x\;+\;z_I^m\;\partial_x\bar{\lambda}_B\;)$$
$$+\;O(1)\;(\sum_{\substack{m,k\ne i\\(m,k)\ne(j,j)}}\theta_m\theta_k\;+\;\|\mathbf{v}_I+\mathbf{v}_B\|^3\;+\;\|\Theta\|^3\;+\;\mathfrak{Error}^{\,j}+\mathfrak{Error}_1^{\,j}\;)_t.$$

$$z_I^i(x,t)\;=\;O(\delta^2)\int_R \mathbb{G}_i(y,0;x,t)e^{-\delta|y|}\,dy \tag{6.50b}$$
$$+O(1)\int_0^t\int_R \mathbb{G}_i(y,s;x,t)\;(\;\delta^2e^{-\delta|y|}\;+\;\epsilon\;(\;|x|\;+\;t\;))\;(\;\delta\;z_I^i\;+\;\partial_y z_I^i\;)\;dyds$$
$$+O(1)\int_0^t\int_R \mathbb{G}_i(y,s;x,t)\left(\bar{\lambda}_B^2\partial_s\bar{\lambda}_B+\sum_{l\ne i}\partial_s\left(\bar{\lambda}_B\theta_l\right)+\sum_{\substack{m,k\ne i\\(m,k)\ne(j,j)}}\theta_m\theta_k\right)dyds$$
$$+O(1)\int_0^t\int_R \mathbb{G}_i(y,s;x,t)\;\sum_{1\le m,k\le n}\eta_{mk}\;dyds$$
$$+O(1)\int_0^t\int_R \mathbb{G}_i(y,s;x,t)F(y,s)\;dyds.$$

The first integral in (6.50a) and (6.50b) can be obtained by Proposition 6.2. For the integrand in the second integral in R.H.S. of (6.50a) it can be estimated as follows

$$O(1)\;\delta^2\int_0^t\int_R \bar{g}_j(y,s;x,t)\;e^{-\delta|y|}\;z_I^j(y,s)\;dyds \tag{6.51}$$
$$=\;O(1)\;M\delta^{2+\frac{\alpha_0^2}{4}}\int_0^t\int_R \bar{g}_j(y,s;x,t)\;e^{-\delta|y|}\sum_{1\le k\le n}\zeta_k(y,s)^{(6-\alpha_0)/4}\;dyds$$
$$+\;O(1)\;M\;\delta^{3+\frac{\alpha_0^2}{4}}\int_0^t\int_R \bar{g}_j(y,s;x,t)\;e^{-\delta|y|}\;ch(y,s)\;\;(\;1+|y-\Lambda_j(s)|\;)^{-1}dyds.$$

The first integral in R.H.S. of (6.51) can be estimated by the same way for obtaining (6.34). The integrand in the second integral in the R.H.S. of (6.51) satisfies that

$$\delta^3\;ch(y,s)\;(\;1+|y-\Lambda_j(s)|\;)^{-1}e^{-\delta|y|}$$

$$\leq C\begin{cases} \delta^3\ (1+s)^{-1}\ e^{-\delta|y|} \text{ for } |y| \leq \frac{1}{2}|\Lambda_j(s)|, \\ \delta^{(6+\alpha_0)/4}\ (1+s)^{-(6-\alpha_0)/4}\ e^{-\delta|y|/2} \text{ for } |y| \geq \frac{1}{2}|\Lambda_j(s)|. \end{cases}$$

By Lemma 4.5 with $\beta = 2$ and $\alpha = 2$ the function $(1+s)^{-(6-\alpha_0)/4}\ e^{-\delta|y|/2}$ satisfies that for $t \leq \delta^{-2-\alpha_0}$

$$\begin{aligned} &\delta^3\ M \int_0^t \int_R \bar{g}_j(y,s;x,t)(1+s)^{-1}\ e^{-\delta|y|}\ dyds \\ &= O(1)\ M\delta^{(4-\alpha_0)/2} \int_0^t \int_R \frac{1}{\sqrt{|t-s|+1}} \bar{g}_j(y,s;x,t)(1+s)^{-1}\ e^{-\delta|y|}\ dyds \\ &= O(1)\ M\ \delta^{(2-\alpha_0)/2} \cdot |\ln\delta|\ \zeta_j^{3/2}(x,t). \end{aligned}$$

By Lemma 4.5 with $\beta = 1$ and $\alpha = (6-\alpha_0)/2$ the function $(1+s)^{-(6-\alpha_0)/4}\ e^{-\delta|y|/2}$ satisfies that

$$\begin{aligned} &M\ \delta^{(6+\alpha_0)/4} \int_0^t \int_R \bar{g}_j(y,s;x,t)\ \ (1+s)^{-(6-\alpha_0)/4}\ e^{-\delta|y|/2}\ dyds \\ &= O(1)\ M\ \delta^{\alpha_0/2}\ (\ \zeta_j(x,t)^{(6-\alpha_0)/4}\ +\ \sqrt{\delta}\sigma_j(x,t;E)\ ). \end{aligned}$$

By Lemma 4.5 with $\alpha = 2$ and $\beta = 1$ we the integral in (6.50a) satisfies that for $t \in [\ 0,\ \delta^{-(2+\alpha_0)}\ ]$

$$\begin{aligned} &\int_0^t \int_R \bar{g}_j(y,s;x,t)\ \partial_s\ \bar{\lambda}_B(y,s)^2 dyds \\ &= O(\delta^2) \int_0^t \int_R \bar{g}_j(y,s;x,t)\ \frac{e^{-2\delta|y|}}{1+s} dyds \\ &= O(\delta)\ \left[ch(x,t)\ (\ 1+|x-\Lambda_j(t)|\ )^{-1}\right] \\ &+\ O(1)\ \delta^{(2-\alpha_0)/4}|\log\delta|\ \left(\zeta_i(x,t)^{3/2}+\zeta_j(x,t)^{3/2}\right). \end{aligned}$$

The integrand $\partial_s(\bar{\lambda}_B\ \theta_l)$ next to $\bar{\lambda}_B\partial_s\bar{\lambda}_B$ in (6.50a) satisfies that

$$\begin{aligned} \partial_s(\bar{\lambda}_B\ \theta_l) &= O(1)\ \delta^2\ \sigma(y,s;\Lambda_l,E)^2\ e^{-\delta|y|} \\ &= O(1)\delta^{3/2}\ \sigma(y,s;\Lambda_l,E')^3\ e^{-\delta|y|/2} \end{aligned}$$

for some positive constants $E > E'$. Therefore,

$$\begin{aligned} &\int_0^t \int_R \bar{g}_j(y,s;x,t)\partial_s(\bar{\lambda}_B\ \theta_l)\ dyds \\ &= O(1)\ \delta^{3/2} \int_0^t \int_R \bar{g}_j(y,s;x,t)\ \sigma(y,s;\Lambda_l,E')^3\ e^{-\delta|y|/2}\ dyds. \end{aligned}$$

By a similar method for obtaining (6.36) we have that

$$\begin{aligned} &\delta^2 \int_0^t \int_R \bar{g}_j(y,s;x,t)\ \sigma(y,s;\Lambda_l,E)^2\ e^{-\delta|y|}\ dyds \\ &= \begin{cases} O(1)\ \delta\ \sigma(x,t;\Lambda_j,E') \text{ for } l=j, \\ O(1)\ \delta^{3/2}\ (\sigma(x,t;\Lambda_j,E')+\sigma(x,t;\Lambda_k,E')+ch(x,t)(1+|x-\Lambda_l(t)|)^{-1}) \text{ for } l \neq j. \end{cases} \end{aligned}$$

To obtain a sharper estimate for the integral of the third integrand $(\theta_l\theta_k)_s$ is not similar to (6.41). We consider the integral $\int_0^t \int_R (\theta_m\theta_k)_s dyds$ in two cases, $k \neq m$ and $k = m$.

Case $k < m$.

We can separate the two waves $\sigma(x,t;\Lambda_k,E)$ and $\sigma(x,t;\Lambda_m,E)$ by a curve $x = \frac{1}{2}(\Lambda_k(t)+\Lambda_m(t))$. There exist positive constants D_1 and D_2 such that

$$\theta_m\theta_k(y,s) \;=\; O(1)\;\delta^2 \left\{ \begin{array}{l} e^{-D_1|s|}\sigma(y,s;\Lambda_m,D_2) \text{ for } x \le \frac{1}{2}(\Lambda_k(t)+\Lambda_m(t)), \\ e^{-D_1|s|}\sigma(y,s;\Lambda_k,D_2) \text{ for } x \ge \frac{1}{2}(\Lambda_k(t)+\Lambda_m(t)). \end{array} \right.$$

The product of two diffusion waves decays exponential fast, hence this function can be compared with any function with algebraic decaying rate. It yields that

$$\theta_m\theta_k(y,s) \;=\; O(\delta^2)\;(1+s)^{-R}\;\left(\sigma(y,s;\Lambda_m,E)+\sigma(y,s;\Lambda_k,E)\right) \text{ for any } R>0.$$

Set $R \equiv 5$, then by Lemma 4.1, 4.2.A and 4.2.B with $\alpha = 5$ and $\beta = 1$ we have that

$$\begin{aligned}&\int_0^t \int_R \bar{g}_j(y,s;x,t)(\theta_k\theta_m)dyds \\ &\quad = \; O(1)\;\delta^2\;\left(\sigma(x,t;\Lambda_j,E)+\zeta_j(x,t)^{3/2}\right).\end{aligned}$$

Case $k = m$, $m \neq i$

From (3.8) the nonlinear diffusion wave $\theta_k(x,t)$ satisfies that

$$\partial_s\theta_k^2 + \lambda_{k,k}\partial_y\theta_k^2 - \partial_y^2\theta_k^2 \;=\; O(\delta^2)\;\sigma(x,t,\Lambda_k,E)^4$$

for some positive constant E. Rewrite this equation as follows

$$\partial_s\theta_k^2 \;=\; \frac{\lambda_{k,k}}{\lambda_{,kk}-\lambda_{j,j}}\left(\partial_s\theta_k^2 + \lambda_{j,j}\partial_y\theta_k^2 - \partial_y^2\theta_k^2\right) + O(\delta^2)\;\sigma(x,t,\Lambda_k,E)^4.$$

From this and by Lemma 4.2 with $\beta = 1$, $\alpha = 4$ and $\beta = 2$, $\alpha = 6$ we can have the following estimates

$$\begin{aligned}&\int_0^t \int_R \bar{g}_j(y,s;x,t)\partial_s\theta_k(y,s)^2 dyds \\ &= \; O(\delta^2)\int_0^t \int_R \;\bar{g}_j(y,s;x,t)\sigma(y,s;\Lambda_k,E)^4 - \left((\partial_s+\lambda_{j,j}+\partial_y^2)\bar{g}_j(y,s;x,t)\right)\;\sigma(y,s;\Lambda_k,E)^2 dyds \\ &= \; O(\delta^2)\int_0^t \int_R \;\bar{g}_j(y,s;x,t)\sigma(y,s;\Lambda_k,E)^4 - \frac{\delta^2 e^{-\delta|y|}}{(1+|t-s|)^{1/2}}\bar{g}_j(y,s;x,t)\sigma(y,s;\Lambda_k,E)^2 dyds \\ &= \; O(\delta^2)\int_0^t \int_R \;\bar{g}_j(y,s;x,t)\sigma(y,s;\Lambda_k,E)^4 - (1+|t-s|)^{-1/2}(1+s)^{-2}\bar{g}_j(y,s;x,t)\sigma(y,s;\Lambda_k,E')^2 dyds \\ &= \; O(\delta^2)\left(\sigma(x,t;\Lambda_j,E')+\zeta_j(x,t)^{3/2}\right).\end{aligned}$$

Let's return to the fourth integral $\int_0^t \int_R \eta_m\eta_k dyds$ in R.H.S. of (6.50a). From (6.17b) we have the following estimates

$$\int_0^t \int_R \bar{g}_j(y,s;x,t)\eta_{kl}(y,s)dyds \tag{6.52}$$

$$= \left(\int_0^{t-1} + \int_{t-1}^{t}\right)\int_R \bar{g}_j(y,s;x,t)\partial_s(\bar{v}_I^k\bar{v}_I^l)dyds$$

$$= \int_R \bar{g}_j(y,t-1;x,t)\ (\bar{v}_I^k\bar{v}_I^l)(y,t-1)\ -\ \bar{g}_j(y,0;x,t)\ (\bar{v}_I^k\bar{v}_I^l)(y,0)\ dyds$$

$$-\int_0^{t-1}\int_R \partial_s\bar{g}_j(y,s;x,t)\ (\bar{v}_I^k\bar{v}_I^l)(y,s)\ dy$$

$$+\int_{t-1}^{t}\int_R \partial_y\bar{g}_j(y,s;x,t)\ (z_I^k\bar{v}_I^l + z_I^l\bar{v}_I^k)\ +\ \bar{g}_j(y,s;x,t)(z_I^k\partial_y\bar{v}_I^l + z_I^l\partial_y\bar{v}_I^k)\ dyds.$$

The estimate of the first integral in the R.H.S. of (6.52) is

$$\int_R \bar{g}_j(y,t-1;x,t)\ (\bar{v}_I^k\bar{v}_I^l)(y,t-1)\ -\ \bar{g}_j(y,0;x,t)\ (\bar{v}_I^k\bar{v}_I^l)(y,0)\ dy$$

$$= O(1)\ M^2\int_R \bar{g}_j(y,t-1;x,t)\ \left(\delta^{\alpha_0^2/2}(\ \zeta_k(y,t-1)^{3-\frac{\alpha_0}{2}}\ +\ \zeta_l(y,t-1)^{3-\frac{\alpha_0}{2}}\ )\ +\frac{\epsilon^2}{\delta^2}\right)\ dy$$

$$+O(1)\ M^2\ \delta^{\alpha_0^2/2}\int_R \bar{g}_j(y,0;x,t)\ (\ \zeta_k(y,0)^{3-\frac{\alpha_0}{2}}\ +\ \zeta_l(y,0)^{3-\frac{\alpha_0}{2}}\ )\ dy$$

$$=\ O(1)\ M^2\left(\delta^{\alpha_0^2/2}\ \left(\zeta_k(x,t)^{3-\frac{\alpha_0}{2}}\ +\ \zeta_k(x,t)^{3-\frac{\alpha_0}{2}}\ +\ \sigma(x,t;\Lambda_j,E') + \zeta_j(x,t)^{3-\frac{\alpha_0}{2}}\right) + \frac{\epsilon^2}{\delta^2}\ \right).$$

By Lemma 4.3, 4.4.A and 4.4.B with $\alpha = 3-\alpha_0/2$ and $\beta = 2$ estimate for second one in the R.H.S. of (6.52) is

$$\int_0^{t-1}\int_R \partial_s\bar{g}_j(y,s;x,t)\ (\bar{v}_I^k\bar{v}_I^l)(y,t-1)\ dyds$$

$$= O(1)\ M^2\int_0^{t-1}\int_R \partial_s\bar{g}_j(y,s;x,t)\ \left(\ \delta^{\alpha_0^2/2}(\ \zeta_k(y,s)^{\frac{6-\alpha_0}{2}}\ +\ \zeta_l(y,s)^{\frac{6-\alpha_0}{2}}\ )\ +\ \frac{\epsilon^2}{\delta^3}\ \right)\ dyds$$

$$=\ O(1)\ M^2\left(\ \delta^{\alpha_0^2/2}\ \left(\ \zeta_j(x,t)^{\frac{6-\alpha_0}{4}}\ +\ \zeta_k(x,t)^{\frac{6-\alpha_0}{4}}\ +\ \zeta_l(x,t)^{\frac{6-\alpha_0}{4}}\ \right)\ +\ \frac{t\ \epsilon^2}{\delta^3}\ \right).$$

For $t \le \delta^{-(2+\alpha_0)}$ the estimate for the last integral in R.H.S. of (6.52) satisfies that

$$\int_{t-1}^{t}\int_R \partial_y\bar{g}_j(y,s;x,t)(z_I^k\bar{v}_I^l + z_I^l\bar{v}_I^k)\ +\ \bar{g}_j(y,s;x,t)(z_I^k\partial_y\bar{v}_I^l + z_I^l\partial_y\bar{v}_I^k)\ dyds$$

$$=\ O(1)\ M^2\int_{t-1}^{t}\int_R \bar{g}_j(y,s;x,t)\ \left(\delta^{\alpha_0^2/2}\ (\ \zeta_k(y,s)^{3-\frac{\alpha_0}{2}}\ +\ \zeta_l(y,s)^{3-\frac{\alpha_0}{2}}\ ) + \frac{\epsilon^2}{\delta^2}\right)\ dyds$$

$$+\ \ O(1)\ \delta^2 M^2\int_{t-1}^{t}\int_R \bar{g}_j(y,s;x,t)\ ch(y,s)\ (\ (1+|y-\Lambda_k(s)|\ )^{-2}\ +\ (1+|y-\Lambda_l(s)|\ )^{-2}\ )\ dyds$$

$$=\ O(1)\ M^2\ \left(\delta^{\alpha_0^2/2}\ \left(\ \zeta_j(x,t)^{3-\frac{\alpha_0}{2}}\ +\ \zeta_k(x,t)^{3-\frac{\alpha_0}{2}}\ +\ \zeta_l(x,t)^{3-\frac{\alpha_0}{2}}\ \right)\ +\ \frac{\epsilon^2}{\delta^3}\ \right)$$

$$+\ \ O(1)\ M^2\delta^2\ \ (\ (1+|y-\Lambda_k(t)|\ )^{-2}\ +\ (1+|y-\Lambda_l(t)|\ )^{-2}\ )$$

$$=\ O(1)\ M^2\left(\delta^{\alpha_0^2/2}\ \left(\ \zeta_j(x,t)^{3/2}\ +\ \zeta_k(x,t)^{3/2}\ +\ \zeta_l(x,t)^{3/2}\ \right)\ +\frac{\epsilon^2}{\delta^3}\right).$$

Similar to the procedures for obtaining the estimates for w_I^j we have that

$$\int_0^t\int_R \bar{g}_j(y,s;x,t)F(y,s)dyds\ =\ O(1)\ (M+M^2+M^3)\ \delta^{\alpha_0^2/2}\ \sum_{1\le k\le n}\ \zeta_k(x,t)^{3/2}.$$

Thus, from all above estimates we have that for $t \in [0, \delta^{-(2+\alpha_0)}]$ there exists a constant M_0 such that

$$z_I^j(x,t) = (\ M_0 + O(1)\ \delta^{\alpha_0^2/4}(M + M^2 + M^3)\ )\ \delta^{\alpha_0^2/4} \left(\sum_{1\le l\le n} \zeta_l(x,t)^{3/2\ -\ \alpha_0/4} + \sigma(x,t;\Lambda_j,E) \right) \quad (6.53)$$
$$+(\ M_0 + O(1)\ \delta^{\alpha_0^2/4}(M + M^2 + M^3)\ )\ \left(\ \delta\ ch(x,t)\ (1 + |x - \Lambda_j(t)|)^{-1}\ +\ \frac{\epsilon}{\delta^{3/2}}\ \right).$$

Similarly, we can have the estimates for $z_I^i(x,t)$

$$z_I^i(x,t) \ = \ (\ M_0 + O(1)\ \delta^{\alpha_0^2/4}(M + M^2 + M^3))\ \delta^{\alpha_0{}^2/4} \left(\sum_{1\le k\le n} \zeta_k(x,t)^{3/2\ -\ \alpha_0/4}\ +\ \frac{e^{-\frac{x^2}{E\ t}}}{\sqrt{t+1}} \right) \quad (6.54)$$
$$+\ \ (\ M_0 + O(1)\ \delta^{\alpha_0^2/4}(M + M^2 + M^3))\ \frac{\epsilon}{\delta^{3/2}}.$$

By differentiating (6.13) and (6.14) with respect to x we will obtain the estimate for $\bar{v}_I^j$ and $\bar{v}_{Ix}^l$ for $j \neq i$ and $l = 1, \cdots, n$.
For $t \in [0, \delta^{-(2+\alpha_0)}]$ and $j \neq i$

$$|\bar{v}_I^j(x,t)| = (M_0 + O(1)\ \delta^{\alpha_0^2/4}(M + M^2 + M^3))(\delta^{\alpha_0^2/4}\zeta_j(x,t)^{3/2\ -\ \alpha_0/4} + \frac{\epsilon}{\delta^{3/2}}) \text{ for } |x - \Lambda_j(t)| \le \sqrt{t}. \quad (6.55)$$
$$|\bar{v}_{Ix}^l(x,t)| = (M_0 + O(1)\ \delta^{\alpha_0^2/4}(M + M^2 + M^3))(\delta^{\alpha_0^2/4}\zeta_l(x,t)^{3/2\ -\ \alpha_0/4} + \frac{\epsilon}{\delta^{3/2}}) \text{ for } x \in \mathbf{R}. \quad (6.56)$$

From the definition of $z_I^j(x,t)$ (6.15) and (6.12b) we have that for $j \neq i$

$$v_I^j \ = \ \frac{1}{\bar{\lambda}_j}\ \left(-z_I^j + \bar{v}_{Ix}^j + \mathfrak{F}^j[\mathbf{w}_I]\right).$$

From this we can use the estimate of z_I^j in the region $|x - \Lambda_j(t)| > \sqrt{t}$. Therefore, substituting (6.18) into $\mathfrak{F}[\mathbf{v}_I]^j$ and using the estimate (6.53) as well as (6.56), then we combine the result with (6.55) to obtain that for $t \in [0, \delta^{-(2+\alpha_0)}]$

$$\bar{v}^j \ = \ O(1)\ (M_0\ \ + \delta^{\frac{\alpha_0{}^2}{4}}(M + M^2 + M^3))\ \left(\ \delta^{\frac{\alpha_0{}^2}{4}}\ \zeta_j(x,t)^{\frac{6-\alpha_0}{4}}\ +\ \frac{\epsilon}{\delta^{\frac{3}{2}}}\ \right). \quad (6.57)$$

Thus, if δ, $M^3\delta^{\frac{\alpha_0{}^2}{4}}$, and M_0/M are sufficiently small, then (6.56) and (6.57) justify the ansatz for $\|\partial_x \mathbf{v}_I\|$ and $\bar{v}_I^j$ for $j \neq i$, respectively.

The estimate of $v_I^i(x,t)$ can be obtained by considering $(6.13)_x$. However, we just carry out the integration $\partial_x \mathbb{G}_i(y,s;x,t)\theta_j(y,s)^2$ for which the estimate procedure is different from the other.
From the definition of $\mathbb{G}_i(y,s;x,t)$ (6.7) we have that for $t \in [0, \delta^{-(2+\alpha_0)}]$

$$\begin{aligned} \partial_x \mathbb{G}_i(y,s;x,t) \ &= \ -\partial_y \mathbb{G}_i(y,s;x,t)\ + O(1)\ (\ \partial_x \tilde{V}(x,t)\ + \partial_y \tilde{V}(y,s)\ ) \\ &= \ -\partial_y \mathbb{G}_i(y,s;x,t) +\ O(1)\ (e^{-\delta|y|}\ +\ e^{-\delta|x|}\ ). \end{aligned}$$

From this we have that for $t \in [0, \delta^{-(2+\alpha_0)}$

$$
\begin{aligned}
&\int_0^t \int_R \partial_x \mathrm{G}_i(y,s;x,t)\ \theta_j(y,s)^2\ dyds \\
&\quad = O(\delta) \int_0^t \int_R \mathrm{G}_i(y,s;x,t)\ e^{-\delta|y|}\ \theta_j(y,s)^2 dyds\ +\ O(\delta) e^{-\delta|x|} \int_0^t \int_R \mathrm{G}_i(y,s;x,t)\ \ \theta_j(y,s)^2 dyds \\
&\quad + \int_0^t \int_R \mathrm{G}_i(y,s;x,t)\ \partial_y \theta_j(y,s)^2\ dyds \\
&= \ i\ +\ ii\ +iii.
\end{aligned}
$$

Estimate $\delta e^{-\delta|y|}\theta_j(y,s)^2$ as follows

$$
\begin{aligned}
&\delta e^{-\delta|y|}\theta_j(y,s)^2 = O(1)\ \delta^3 e^{-\delta|y|}\sigma(y,s;\Lambda_j,D) = O(1)\delta^{\frac{\alpha_0^2}{2}}\ s^{3-\frac{\alpha_0^2}{2}}\ \sigma(y,s;\Lambda_j,E'') \\
&= O(1)\ \delta^{\frac{\alpha_0^2}{2}}\ \sigma(y,s;\Lambda_j,E')^{6-\alpha_0^2}
\end{aligned}
$$

for some positive constants E' and E''. Therefore,

$$
\begin{aligned}
i \ &= \ O(\delta^{\alpha_0^2/2}) \int_0^t \int_R \mathrm{G}_i(y,s;x,t)\ \ \sigma(y,s;\Lambda_j,E')^{6-\alpha_0^2}\ dyds \\
&= \ O(\delta^{\alpha_0^2/2})\ (\zeta_i(x,t)^{3/2} + \zeta_j(x,t)^{3/2} + \sigma(x,t,\Lambda_i^+,E'))\ \text{for } x \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
ii \ &= \ O(\delta^3) e^{-\delta|x|} \ \int_0^t \int_R \mathrm{G}_i(y,s;x,t)\sigma(x,t;\Lambda_j,E')^2\ dyds \\
&= \ O(\delta^3)\ e^{-\delta|x|}\zeta_i(x,t)^{1/2} \ = \ O(\delta^{3/2})(\ 1\ +\ |x|\ )^{-1}(1\ +\ |x|\ +\ t\ )^{-1/2}.
\end{aligned}
$$

By using the following identity

$$
\begin{aligned}
\partial_y \theta_j(y,s)^2 \ &= \ \frac{1}{\lambda_{j,j} - \lambda_B}\ (\ [\partial_s + \lambda_{j,j}\partial_y - \partial_y^2]\ \theta_j(y,s)^2\ -\ [\partial_s + \lambda_B \partial_y - \partial_y^2]\ \theta_j(y,s)^2\ ) \\
&= \ O(\delta^2)\ \sigma(y,s;\Lambda_j,E')^4\ -\ \frac{1}{\lambda_{j,j} - \lambda_B}\ [\partial_s + \lambda_B \partial_s - \partial_s]\ \theta_j(y,s)^2
\end{aligned}
$$

we have that

$$
\begin{aligned}
iii \ &= \ O(1)\ \delta^2 \left(\frac{e^{-\frac{(\ |x|+\delta t\ )^2}{4E't}}}{\sqrt{t+1}} \ +\ \zeta_i(x,t)^{3/2}\ +\ \zeta_j(x,t)^{3/2} \right)\ + O(1)\ \sqrt{t}\epsilon\delta^2 \\
&= \ O(1)\ \delta^2 \left(\frac{e^{-\frac{x^2}{E't}\ -\ \frac{2\delta|x|}{E'}\ -\ \frac{\delta^2 t}{E'}}}{\sqrt{t+1}} \ +\ \zeta_i(x,t)^{3/2}\ +\ \zeta_j(x,t)^{3/2} \right)\ +\ O(1)\ \epsilon\ \delta^{1-\frac{\alpha_0}{2}} \\
&= \ \ O(\delta^{\alpha_0/2})\ (\ |x|\ +\ t)^{-(6-\alpha_0)/4}\ +\ O(\delta^2)\ \left(\ \zeta_i(x,t)^{3/2}\ +\ \zeta_j(x,t)^{3/2} \right) +\ O(1)\ \epsilon\ \delta^{1-\frac{\alpha_0}{2}}.
\end{aligned}
$$

Then, from the procedure for obtaining estimates for z_I^j we have that

$$
\bar{v}_I^i(x,t) \ = \ O(1)\ (M_0 + \delta^{\alpha_0^2/4}\ (M + M^2 + M^3)\ ) \left(\ \delta^{\alpha_0^2/4} \left(\ \sum_{1 \leq k \leq n} \zeta_k(x,t)^{3/2}\ +\ (1+|x|)^{-1}(t+|x|)^{-1/2} \right) + \frac{\epsilon}{\delta^{3/2}} \ \right).
$$

This justifies the ansatz for $\bar{v}_I^i(x,t)$ for $t \in [0, \delta^{-(2+\alpha_0)}$ if δ, $\delta^{\alpha_0^2/4} M^3$, and M_0/M are sufficiently small.

The ansatz (6.18) shows that $\mathbf{v}_I(x,t)$ becomes small at time $t = \delta^{-(2+\alpha_0)}$. This shows that the $u(x,t)$ will approach $\bar{u}^\infty + \Pi_\infty + \mathbf{v}_B$. The function $\mathbf{v}_B$ measures the formation of the shock layer for the Burgers' equation. From (6.8) and (6.18) $\mathbf{v}_B(x,t)$ becomes so small $t = \delta^{-(2+\alpha_0)}$ that

$$\|\mathbf{v}_B\|(x,t) \ \ll \ \|\mathbf{v}_I\|(x,t)$$

if δ is small enough. From this we have the estimate

$$\|u - \bar{u}^\infty - \Pi_\infty\|(x, \delta^{-(2+\alpha_0)}) \ \leq \ 2\|\mathbf{v}_I\|(x,t) \tag{6.58}$$

if δ is small enough.
Thus, $u(x,t)$ will approach the approximate solution $\bar{u}^\infty + \Pi_\infty$ at the time $t = \delta^{-(2+\alpha_0)}$. This establishes the formation of the shock layer.

6.2 Asymptotic Stability

Since we have shown that $u(x,t)$ and $\bar{u}^\infty + \Pi_\infty$ at time $t = \delta^{-(2+\alpha_0)}$, we can freeze $u(x,t)$ at time $t = \delta^{-(2+\alpha_0)}$. Then, take this as an initial data, which becomes a small perturbation of the approximate function $\bar{u}^\infty + \Pi_\infty$. So, we need to consider the asymptotic stability of the system (6.1)

$$\partial_t \mathbf{v} - s(t)\ \partial_x\ \mathbf{v}\ + \partial_x f'(\bar{u}^\infty)\ \mathbf{v} - \partial_x^2 \mathbf{v} \ = \ -\frac{1}{2}\partial_x\ C^i_{ii}\ v^i v^i\ \mathbf{r}_i \ + \ \partial_x \sum_{1\leq j\leq n} \mathbf{F}[\ \mathbf{v}\ ]^j\ \mathbf{r}_j,$$

with an initial data satisfying that

$$\begin{aligned}
w^j(x,\delta^{-(2+\alpha_0)}) &= O(1)\ M\ (\delta^{\alpha_0{}^2/4}\ \zeta_j(x,\delta^{-(2+\alpha_0)})^{1/2\ (\ 1-\alpha_0/2\ )} + \frac{\epsilon}{\delta^2}\ ), \\
\bar{v}^j(x,\delta^{-(2+\alpha_0)}) &= O(1)\ M\ \left(\delta^{\alpha_0{}^2/4}\ (\sum_{1\leq k\leq n} \zeta_k(x,\delta^{-(2+\alpha_0)})^{3/2\ -\ \alpha_0/4} + (|x|+1)^{-1}(|x|+t+1)^{-1/2} + \chi_j\right) \\
&\quad + O(1)\ M\ \frac{\epsilon}{\delta^{3/2}}\ \text{for}\ j \neq i, \\
\bar{v}^i(x,\delta^{-(2+\alpha_0)}) &= O(1)\ M\ \left(\delta^{\alpha_0{}^2/4}\ \sum_{1\leq k\leq n} \zeta_k(x,\delta^{-(2+\alpha_0)})^{3/2\ -\ \alpha_0/4} + (|x|+1)^{-1}(|x|+t+1)^{-1/2}\ \right) \\
&\quad + O(1)\ M\ \frac{\epsilon}{\delta^{3/2}}, \\
\bar{v}^j_x(x,\delta^{-(2+\alpha_0)}) &= O(1)\ M\ \left(\ \delta^{\alpha_0{}^2/4}\ \sum_{k=1}^{n} \zeta_k(x,\delta^{-(2+\alpha_0)})^{2\ -\ \alpha_0/4}\ +\ \frac{\epsilon}{\delta^{3/2}}\right).
\end{aligned}$$

Similar to the variable $z_I^i(x,t)$, we introduce the variable

$$\begin{aligned}
z^j(x,t) &\equiv w^j_t(x,t) \\
&= \bar{\lambda}_j \bar{v}^j(x,t) + \partial_x \bar{v}^j(x,t) + \mathfrak{G}[\mathbf{v}]^j \\
\mathfrak{G}[\mathbf{v}]^j &\equiv \mathbf{F}[\mathbf{v}]^j(x,t) + O(1)\sum_{k=1}^{n} \mathbf{l}_j \partial_x \mathbf{r}_k(\bar{u}^\infty) w^k.
\end{aligned}$$

We make the following a priori assumptions for $t \in [\delta^{-(2+\alpha_0)}, \delta^3/\epsilon]$:
There exists a constant M_1 such that

$$
\begin{aligned}
w^j(x,t) &\leq M_1 \left(\delta^{\alpha_0^2/4}\, \zeta_j(x,t)^{1/2\,(\,1-\alpha_0/2\,)} + \frac{\epsilon}{\delta^2}\right), \\
\bar{v}^j(x,t) &\leq M_1\, \delta^{\alpha_0^2/4}\Big(\chi_j(x,t) + (\,|x| + 1\,)^{-1}(\,1+t+|x|\,)^{-1/2}\Big) \\
&\quad + M_1 \left(\delta^{\alpha_0^2/4}\sum_{k=1}^{n}\zeta_k(x,t)^{3/2-\alpha_0/4} + \frac{\epsilon}{\delta^{3/2}}\right) \text{ for } j \neq i, \\
\bar{v}^i(x,t) &\leq M_1\, \delta^{\alpha_0^2/4}\left((\,|x| + 1\,)^{-1}(\,1+t+|x|\,)^{-1/2} + \sum_{k=1}^{n}\zeta_k(x,t)^{3/2-\alpha_0/4}\right) + M_1\frac{\epsilon}{\delta^{3/2}}, \\
\bar{v}^j_x(x,t) &\leq M_1\left(\sum_{k=1}^{n}\delta^{\alpha_0^2/4}\,\zeta_k(x,t)^{3/2-\alpha_0/4} + \frac{\epsilon}{\delta^{3/2}}\right) \\
z^j(x,t) &\leq M_1\, \delta^{\alpha_0^2/4}\left(\sum_{k=1}^{n}\zeta_k(x,t)^{3/2-\alpha_0/4} + \sigma(x,t;\Lambda_j,E) + \bar{\chi}_j(x,t)\right) \\
&\quad + M_1\frac{\epsilon}{\delta^{3/2}} \text{ for } j \neq i, \\
z^i(x,t) &= M_1\left(\delta^{\alpha_0^2/4}\left(\bar{\zeta}_i(x,t)^{3/2-\alpha_0/4} + \sum_{k\neq i}\zeta_k(x,t)^{3/2-\alpha_0/4} + \chi_i(x,t)\right) + \frac{\epsilon}{\delta^{3/2}}\right).
\end{aligned}
$$

where algebraic-decaying functions $\chi_j(x,t)$ and $\bar{\chi}_j(x,t)$ for $j \neq i$ are defined by

$$
\begin{aligned}
\chi_j(x,t) &\equiv \min(\bar{\chi}_i(x,t), (t+1)^{-1/2}(|x|+1)^{-1/2}), \\
\bar{\chi}_j(x,t) &\equiv |x-\Lambda_j(t)|^{-1}(1+\delta^2|x-\Lambda_j(t)|)^{-1/2} \\
&\quad \cdot \begin{cases} 1 & \text{for } 0 \leq x \leq \Lambda_j(t) - \sqrt{t}, j > i \\ 1 & \text{for } 0 \geq x \geq \Lambda_j(t) - \sqrt{t}, j < i \\ 0 & \text{else.} \end{cases}
\end{aligned}
$$

By Duhamel's principle we have the following representation

$$
\begin{aligned}
w^j(x,t) &= \int_R g_j(y,\delta^{-2-\alpha_0};x,t)w^j(y,\delta^{-2-\alpha_0})dy \\
&\quad + \int_{\delta^{-2-\alpha_0}}^{t}\int_R\left(\frac{1}{\sqrt{t-s}}+1\right)\delta^2 e^{-\delta|y|}w^j(y,s)dyds \\
&\quad + \int_{\delta^{-2-\alpha_0}}^{t}\int_R \bar{g}_j(y,s;x,t)\mathfrak{G}[\mathrm{v}]^j(y,s)dyds \text{ for } j \neq i, \\
w^i(x,t) &= \int_R g_i(y,\delta^{-2-\alpha_0};x,t)w^i(y,\delta^{-2-\alpha_0})dy \\
&\quad + \int_{\delta^{-2-\alpha_0}}^{t}\int_R \delta^2(\delta + \frac{1}{\sqrt{t-s}})\bar{g}_i(y,s;x,t)e^{-\delta|y|}w^j(y,s)dyds \\
&\quad + \int_{\delta^{-2-\alpha_0}}^{t}\int_R \bar{g}_i(y,s;x,t)\mathfrak{G}[\mathrm{v}]^i(y,s)dyds,
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}^j(x,t) = &\int_R \partial_x g_j(y,\delta^{-2-\alpha_0};x,t)w^j(y,\delta^{-2-\alpha_0})dy \\
&+\int_{\delta^{-2-\alpha_0}}^t \int_R (1+\frac{1}{\sqrt{t-s}})\bar{g}_j(y,s;x,t)\delta^2(v^j(y,s)+\delta^2 w^j(y,s))e^{-\delta|y|}dyds \\
&+\int_{\delta^{-2-\alpha_0}}^t \int_R \frac{\bar{g}_j(y,s;x,t)}{\sqrt{t-s}}\mathfrak{G}[\mathrm{v}]^j(y,s)dyds \text{ for } j\neq i, \\
\bar{v}^i(x,t) \;=\; &\int_R g_i(y,\delta^{-2-\alpha_0};x,t)w^i(y,\delta^{-2-\alpha_0})dy \\
&+\int_{\delta^{-2-\alpha_0}}^t \int_R \delta^2(\delta+\frac{1}{\sqrt{t-s}})\bar{g}_i(y,s;x,t)e^{-\delta|y|}w^j(y,s)dyds \\
&+\int_{\delta^{-2-\alpha_0}}^t \int_R \frac{\bar{g}_i(y,s;x,t)}{\sqrt{t-s}}\mathfrak{G}[\mathrm{v}]^i(y,s)dyds,
\end{aligned}
$$

The verification of the above a priori assumption is almost identical to the verification of the ansatz for the initial layer. It is even easier because the slower decaying term $\mathbf{v}_B$ does not show up in the above representation. Therefore, we omit the lengthy calculations and conclude that the above a priori assumption is true. So, the Main Theorem follows.
Q.E.D.

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References

[1] G., Chen, *The theory of compensated compactness and the system of isentropic gas dynamics,* preprint, MSRI-00527-91, Mathematical Science Research Institute, Berkeley.

[2] R. J. DiPerna, *Convergence of the viscosity method for isentropic gas dynamics,* Comm. Math. Phys. **91** (1983) 1-30.

[3] Rosa Donat & Stanley Osher, *Propagation of Error into Regions of Smoothness for Non-linear Approximations to Hyperbolic Equations,* Computer Methods in Applied Mechanics and Engineering 80 (1990) 59-64

[4] B. Engquist & S.-H. Yu, *Convergence of a finite difference scheme.*

[5] J. Goodman & Z. Xin, Viscous Limits for Piecewise Smooth Solutions to Systems of Conservation Laws, Arch. Rational Mech. Anal. 121 (1992) 235-265

[6] Gunilla Kreiss, & Heinz-Otto Kreiss, *Stability of steady solutions of Burgers' equation,* submitted to SIAM J. Numer. Anal.,

[7] Gunilla Kreiss, & Heinz-Otto Kreiss, and Peterson N. A., *On the convergence to steady state of solutions of nonlinear hyperbolic-parabolic systems,* SIAM J. Numer. Anal., Vol 31, No.6., pp 1577-1064, 1994

[8] P. Lax, *Hyperbolic Systems of Conservation Laws II,* Comm. Pure & Appl. Vol 10, pp 537-566, 1957

[9] Harten, & Hyman, &P. Lax On finite-difference approximations and entropy conditions for shocks, Comm. Pure Appl. Math. 29, 1976, pp 297-322

[10] T.-T. Li, W-C. Yu, *Boundary Value Problems for the First Order Quasi-linear Hyperbolic Systems and Their Applications, J. Differ. Equations* **41** (1981), 1-26.

[11] T.-P. Liu, *Pointwise Convergence to Shock Waves for Viscous Conservation Laws,* to appear in Comm. and Pure Appl. Math.

[12] A. Majda, & S.J. Osher, *Propagation of error into regions of smoothness for accurate difference approximation to hyperbolic equations.* Comm. Pure Appl. Math., **30**, 1977, 671-705

[13] A. Majda, & S.J. Osher, *The Fourier method for non-smooth initial data.* Math. Comp., **32**, 1978, 1041-1081

[14] D. Hoff & T.-P. Liu, *The inviscid limit for the Navier-Stokes equations of compressible isentropic flow with shock data,* Indiana Univ. Math. J. 38 (1989) 861-915

[15] R. Pego. *Stable Viscosities and Shock Profiles for Systems of Conservation Laws,* Transactions of the American Mathematical Society, Vol 282, No 2, April 1984

[16] J. Smoller, *Shock Waves and Reaction Diffusion Equations,* Springer-Verlag, New York, Berlin (1983).

[17] A. Volpert, *The space BV and quasi-linear equations,* Mat, Sb. **73** (1967) 255-302; English transl. in Math. USSR. Sb. 2 (1967) 255-267.