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Nonlinear Stability of Time Dependent Differential Equations

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by

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1. Introduction

We consider partial differential equations of the form

$$u_t = P(x, \frac{\partial}{\partial x})u + \varepsilon f(x, t, u, u_x, \ldots) + F(x, t),$$
$$u(x, 0) = u_0(x). \tag{1.1}$$

We assume that F and its derivatives decay like $1/(t+1)^q$ for some suitable $q > 0$ and we want to investigate under what conditions

$$\lim_{t\to\infty} |u(\cdot, t)|_\infty = 0. \tag{1.2}$$

We shall use the following concepts.

Definition 1.1. *If (1.2) holds uniformly for* $0 \le \varepsilon \le \varepsilon_0,\ \varepsilon_0 > 0$, *then we call the problem nonlinearly stable.*

Definition 1.2. *The problem is called linearly stable if (1.2) holds for* $\varepsilon = 0$.

Usually, one discusses linear stability in the following way. For $F \equiv 0$, $\varepsilon = 0$, we construct special solutions of the form

$$u(x, t) = e^{\lambda t}\varphi(x) \tag{1.3}$$

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where λ, φ represent a solution of the eigenvalue problem

$$(\lambda I - P)\varphi = 0. \tag{1.4}$$

This leads to

Definition 1.3. *The system (1.3) satisfies the eigenvalue condition if the eigensolutions of (1.4) satisfy*

$$\text{Re } \lambda \leq -\delta < 0. \tag{1.5}$$

Here $\delta > 0$ is some fixed constant.

If one wants to prove nonlinear stability, one has to construct estimates for the solutions of the linear problem ($\varepsilon = 0$). We shall use the usual L_2-scalar product and norm and use the notations (u, v), $\|u\|^2$. There are two techniques available.

1) **The Liaponov technique.** Here one constructs a new scalar product

$$(u, Hv)$$

defined by a positive definite bounded Hermitian operator H such that, for $F \equiv 0$, the linear problem ($\varepsilon = 0$) becomes a contraction, i.e.,

$$\frac{\partial}{\partial t}(u, Hu) \leq -2\alpha(u, Hu), \quad \alpha = \text{ const. } > 0. \tag{1.6}$$

Then one extends the construction to the nonlinear problem.

2) **The resolvent technique.** In this case we consider (1.1) with homogeneous initial data $u_0(x) \equiv 0$ and solve the linear problem by Laplace transform, which leads to the resolvent equation

$$(sI - P(x, \frac{\partial}{\partial x}))\hat{u}(x, s) = \hat{F}(x, s). \tag{1.7}$$

If there is a constant K such that

$$\|\hat{u}(\cdot,s)\|^2 \le K^2 \|\hat{F}(\cdot,s)\|^2, \quad \text{for all } s \text{ with } \operatorname{Re} s \ge 0, \tag{1.8}$$

then Parseval's relation gives us

$$\int_0^\infty \|u(\cdot,t)\|^2 dt \le K^2 \int_0^\infty \|F(\cdot,t)\|^2 dt. \tag{1.9}$$

Mild smoothness properties of u imply convergence. For the nonlinear problem, one considers the nonlinear terms as part of the forcing. Complementing (1.9) with estimating the derivatives of u, we can estimate the solution of the nonlinear problem.

We shall use

Definition 1.4. *We say that the resolvent condition is satisfied if there is a constant K such that*

$$\|(P - sI)^{-1}\| \le K \quad \textit{for all } s \textit{ with } \operatorname{Re} s \ge 0.$$

In the next four sections we shall apply the above techniques to different classes of problems. In Section 2 we consider ordinary differential equations. In Sections 3 – 5 we discuss the Cauchy problem for hyperbolic, parabolic and general partial differential equations, respectively.

2. Ordinary differential equations

In this section we consider systems of ordinary differential equations

$$\begin{aligned} u_t &= Au + \varepsilon f(u,t) + F(t), \\ u(0) &= u_0. \end{aligned} \tag{2.1}$$

Here u, f, F are complex valued vector functions with n components and A is a constant $n \times n$ matrix. Also, $f(u,t) \in C^\infty$, $F(t) \in C^\infty$ are smooth functions of all variables and $\varepsilon > 0$ is a small parameter. We will denote the Euclidean scalar product and norm by $\langle u, v\rangle$, $|u| = \langle u, u\rangle^{1/2}$, respectively. For F, f, we make

Assumption 2.1

$$\int_0^\infty |F(t)|^2 dt < \infty, \quad \lim_{t\to\infty} F(t) = 0. \tag{2.2}$$

For every constant c_0 there is a constant C_0 such that

$$|f(u,t)| \le C_0 |u| \quad \textit{provided} \quad |u| \le c_0. \tag{2.3}$$

We want to discuss conditions such that the solutions of (2.1) converge to zero for $t \to \infty$, i.e., that the problem (2.1) is stable at $u = 0$.

The particular form of the system (2.1) is natural. Consider the apparently more general problem

$$y_t = F(y,t).$$

Assume that

$$\lim_{t\to\infty} F(y_0,t) = 0, \quad , \quad \lim_{t\to\infty} \frac{\partial F}{\partial y}(y_0,t) = A,$$

i.e., y_0 is in the limit a stationary point and the Jacobian converges to a constant matrix. Then $v = y - y_0$ solves

$$\begin{aligned} v_t &= F(y,t) - F(y_0,t) + F(y_0,t) \\ &= \frac{\partial F}{\partial y}(y_0,t)v + G(v,t) + F(y_0,t) \\ &= Av + (\frac{\partial F}{\partial y}(y_0,t) - A)v + G(v,t) + F(y_0,t). \end{aligned} \tag{2.4}$$

Here $G(v,t)$ represent the nonlinear terms. Consider (2.4) for $t \geq t_0$, t_0 sufficiently large, with initial data

$$v(t_0) = y(t) - y(t_0) = \varepsilon u_0, \quad |u_0| \leq 1.$$

Then we rescale the equations by introducing

$$u(t) = \frac{1}{\varepsilon} v(t)$$

as a new variable and obtain

$$\begin{aligned} u_t &= Au + \varepsilon \frac{\left(\frac{\partial F}{\partial y}(y_0,t) - A\right)}{\varepsilon} u + \varepsilon G_1(u,t) + \frac{1}{\varepsilon} F(y_0,t), \\ u(t_0) &= u_0. \end{aligned} \tag{2.5}$$

(2.5) is of the form (2.1) and Assumption 2.1 is satisfied, provided t_0 is sufficiently large.

We shall need the following well known results.

Lemma 2.1. *Let $\Phi = \Phi(y,t)$ be a C^1-function of y,t and let $y(t), y_0(t)$ denote nonnegative C^1-functions defined for $0 \leq t \leq T$. If*

$$y'(t) \leq \Phi\big(y(t),t\big), \quad y_0'(t) = \Phi\big(y_0(t),t\big),$$
$$y_0(0) \geq y(0),$$

then

$$y_0(t) \geq y(t) \quad \text{in} \quad 0 \leq t \leq T.$$

Proof. We have

$$\big(y_0(t) - y(t)\big)' \geq \Phi\big(y_0(t),t\big) - \Phi\big(y(t),t\big).$$

Since, for every fixed t,

$$\Phi(y_0,t)-\Phi(y,t)=\int_y^{y_0}\frac{\partial}{\partial y}\Phi(v,t)dv$$
$$=\Big\{\int_0^1\frac{\partial}{\partial y}\Phi\big(y+(y_0-y)s,t\big)ds\Big\}(y_0-y)=:c(t)(y_0-y)$$

we obtain

$$\frac{d}{dt}\Big(e^{-\int_0^t c(\tau)d\tau}(y_0-y)\Big)=e^{-\int_0^t c(\tau)d\tau}\Big\{(y_0-y)'-c(t)(y_0-y)\Big\}\geq 0$$

and the lemma follows.

Lemma 2.2. *Consider the differential equation*

$$y'=-\lambda(t)y+f(t),\quad y(0)=y_0.$$

Assume that $\lambda(t)>0$. *Then*

$$|y(t)|\leq e^{-\int_0^t\lambda(\xi)d\xi}|y(0)|+\int_0^t|f(\xi)|d\xi$$

or

$$|y(t)|\leq e^{-\int_0^t\lambda(\xi)d\xi}|y(0)|+\max_{0\leq\xi\leq t}|f(\xi)/\lambda(\xi)|.$$

Proof. By Duhamel's principle, we can write down the solution explicitly

$$y(t)=e^{-\int_0^t\lambda(\xi)d\xi}y(0)+\int_0^t e^{-\int_\xi^t\lambda(\eta)d\eta}f(\xi)d\xi.$$

Therefore, the first inequality follows directly. Since

$$|\int_0^t e^{-\int_\xi^t\lambda(\eta)d\eta}f(\xi)d\xi|\leq|\int_0^t\Big(\frac{d}{d\xi}e^{-\int_\xi^t\lambda(\eta)d\eta}\Big)\Big(f(\xi)/\lambda(\xi)\Big)d\xi|$$
$$\leq\int_0^t\frac{d}{d\xi}e^{-\int_\xi^t\lambda(\eta)d\eta}d\xi\ \max_{0\leq\xi\leq t}|f(\xi)/\lambda(\xi)|$$
$$\leq\max_{0\leq\xi\leq t}|f(\xi)/\lambda(\xi)|,$$

the lemma follows.

We want to prove

Theorem 2.1. *The problem (2.1) is nonlinearly stable if and only if the eigenvalue condition holds.*

By (1.3) and (1.4), the eigenvalue condition is clearly necessary and, therefore, we assume that (1.5) holds. To begin with, we assume that

$$A = A^*.$$

Then

$$\langle u, Au \rangle \le -\delta |u|^2,$$

and we obtain, for the solution of the linear problem ($\varepsilon = 0$),

$$\begin{aligned} |u|_t^2 &= \langle u, u_t \rangle + \langle u_t, u \rangle = 2\mathrm{Re}\langle u, u_t \rangle \\ &= 2\mathrm{Re}\langle u, Au \rangle + 2\mathrm{Re}\langle u, F \rangle \\ &\le -2\delta |u|^2 + 2|u|\,|F|, \end{aligned}$$

i.e.,

$$|u|_t \le -\delta |u| + |F|, \quad |u(0)| = |u_0|. \tag{2.6}$$

Lemmata 2.1 and 2.2 give us the estimate

$$\max_{0 \le t \le T} |u| \le \frac{1}{\delta} \max_{0 \le t \le T} |F| + |u_0| =: K_1. \tag{2.7}$$

Now we consider the nonlinear equation. For every ε, there exists an interval $0 \le t \le T_\varepsilon$, $T_\varepsilon > 0$ such that, instead of (2.7)

$$|u(t)| \le 2K_1. \tag{2.8}$$

We choose T_ε as large as possible. There are two possibilities.

1) $T_\varepsilon = \infty$. In this case the inequality holds for all t.

2) $T_\varepsilon = T_0 < \infty$. Now

$$|u(T_0)| = 2K_1. \tag{2.9}$$

We want to show that (2.9) is not possible if ε is sufficiently small. By Assumption 2.1, there is a constant K_2 which depends only on the bound (2.8) such that

$$|f(u,t)| \le K_2|u|, \quad 0 \le t \le T_\varepsilon. \tag{2.10}$$

We consider εf as part of the forcing and obtain from (2.7)

$$\begin{aligned}\max_{0\le t\le T_\varepsilon} |u| &\le \frac{1}{\delta} \max_{0\le t\le T_\varepsilon} |F + \varepsilon f| + |u_0| \\ &\le K_1 + \frac{\varepsilon K_2}{\delta} \max_{0\le t\le T_\varepsilon} |u|.\end{aligned}$$

Therefore,

$$\max_{0\le t\le T_\varepsilon} |u| < 2K_1 \quad \text{if} \quad \varepsilon < \frac{\delta}{2K_2}$$

which contradicts (2.9). Thus, (2.8) and (2.10) hold for all times if ε satisfies the above inequality.

We can integrate the differential equation starting at any time $t = t_0$. Replacing F by $F + \varepsilon f$, we obtain, from (2.6) and (2.10),

$$|u|_t \le -\delta|u| + |F| + \varepsilon K_2|u| \le -\frac{\delta}{2}|u| + |F|.$$

By (2.8) and Lemma 2.2,

$$|u(t)| \le \frac{2}{\delta} \max_{t_0\le \xi\le t} |F(\xi)| + e^{-\frac{\delta}{2}(t-t_0)}|u(t_0)|, \quad |u(t_0)| \le 2K_1,$$

i.e.

$$\overline{\lim_{t\to\infty}}|u(t)| \le \frac{2}{\delta} \max_{t_0\le\xi<\infty} |F(\xi)|.$$

Since t_0 is arbitrary, the convergence follows.

We now consider the general case. We shall construct a new norm such that an estimate of type (2.8) holds. We need

Lemma 2.3. *Assume that the eigenvalues of A satisfy (1.5). Then there is a positive definite Hermitian matrix H such that*

$$HA + A^*H \le -\delta H. \tag{2.11}$$

(Note that for Hermitian matrices $A \ge B$ means $\langle x, Ax\rangle \ge \langle x, Bx\rangle$.)

Proof. By Schur's lemma, there is a unitary transformation U such that

$$U^*AU = \begin{pmatrix} \lambda_1 & \tilde a_{12} & \cdots & \tilde a_{1n} \\ & \lambda_2 & & \vdots \\ & & \ddots & \tilde a_{n-1\,n} \\ 0 & & & \lambda_n \end{pmatrix}.$$

Let

$$D = \begin{pmatrix} 1 & & & 0 \\ & d & & \\ & & d^2 & \\ 0 & & & d^{n-1} \end{pmatrix}.$$

Then

$$D^{-1}U^*AUD = \begin{pmatrix} \lambda_1 & d\tilde a_{12} & \cdots & \cdots & d^{n-1}\tilde a_{1n} \\ & \lambda_2 & d\tilde a_{23} & \cdots & d^{n-2}\tilde a_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & d\tilde a_{n-1\,n} \\ 0 & & & & \lambda_n \end{pmatrix} =: \Lambda + \tilde A.$$

We can choose d so small that

$$\Lambda + \tilde A + \Lambda^* + \tilde A^* \le -\delta I.$$

Let $T = UD$ and define

$$H = (T^*)^{-1}T^{-1}.$$

Then

$$\begin{aligned} HA + A^*H &= (T^*)^{-1}T^{-1}A + A^*(T^*)^{-1}T^{-1} \\ &= (T^*)^{-1}\big(T^{-1}AT + T^*A^*(T^*)^{-1}\big)T^{-1} \\ &\le -\delta(T^*)^{-1}T^{-1} = -\delta H. \end{aligned}$$

This proves the lemma.

We now define a new scalar product and norm by

$$\langle u, v\rangle_H = \langle u, Hv\rangle, \quad |u|_H^2 = \langle u, Hu\rangle. \tag{2.12a}$$

Since

$$|H^{-1}|^{-1}|u|^2 = |H^{-1}|^{-1}\langle u, HH^{-1}u\rangle \le |u|_H^2 \le |H|\,|u|^2, \tag{2.12b}$$

the new norm is equivalent with the original Euclidean norm. By Lemma 2.3,

$$\text{Re}\langle u, Au\rangle_H = \frac{1}{2}\langle u, (HA + A^*H)u\rangle \le -\frac{\delta}{2}|u|_H^2$$

and, therefore, (2.6) becomes

$$\begin{aligned} \frac{\partial}{\partial t}|u|_H &\le -\frac{\delta}{2}|u|_H + |F|_H, \\ |u(0)|_H &= |u_0|_H. \end{aligned} \tag{2.13}$$

Now we can repeat the same argument as before. Thus, we have proved the theorem.

The above construction has been known for a long time and H is called a Liaponov function.

We will discuss another technique which is based on Laplace transform. We start with the linear problem

$$u_t = Au + F(t),$$
$$u(0) = 0. \tag{2.14}$$

The assumption that $u(0) = 0$ is no restriction. If it is not true, we would introduce a new variable $v = u - e^{-\delta t}u(0)$. To begin with we also assume that F has compact support, i.e.,

$$F(t) = 0 \quad \text{for} \quad t \geq T. \tag{2.15}$$

Now we Laplace transform (2.14). Let

$$\hat{u} = \int_0^\infty e^{-st}u(t)dt, \quad \hat{F} = \int_0^\infty e^{-st}F(t)dt, \quad s = \eta + i\xi.$$

By (2.15), $\hat{F}$ is a well defined analytic function of s. In particular,

$$|\hat{F}(s)| \leq \int_0^\infty e^{-\eta t}|F(t)|dt = \int_0^T e^{-\eta t}|F(t)|dt \text{ for } \operatorname{Re} s \geq 0.$$

(2.14) becomes the so called *resolvent* equation

$$(sI - A)\hat{u} = \hat{F}. \tag{2.16}$$

We need

Lemma 2.4. *$(sI - A)^{-1}$ exists for $\operatorname{Re} s \geq 0$ and*

$$\sup_{\operatorname{Re} s \geq 0} |(sI - A)^{-1}| =: R < \infty \tag{2.17}$$

if and only if there is a constant $\delta > 0$ such that, for all eigenvalues λ of A,

$$\operatorname{Re} \lambda \leq -\delta \tag{2.18}$$

holds. Also, for $|s| > |A|$,

$$|(sI - A)^{-1}| \le \frac{1}{|s| - |A|}. \tag{2.19}$$

Proof. We start with (2.19). (2.16) gives us

$$|s|\ |\hat{u}| = |A\hat{u} + \hat{F}| \le |A|\ |\hat{u}| + |\hat{F}|.$$

Therefore, by Fredholm's alternative, (2.16) has a unique solution for $|s| > |A|$ and

$$|\hat{u}| \le \frac{1}{|s| - |A|}|\hat{F}|$$

shows that (2.19) holds. Thus, we need only consider a compact domain Ω, say $|s| \le |A| + 1$, $\quad \text{Re } s \ge 0$. Again using Fredholm's alternative, we know that the resolvent $(sI - A)^{-1}$ exists if and only if A has no eigenvalue $\lambda \in \Omega$. In this case the resolvent is a continuous function of s and therefore uniformly bounded. This proves the lemma.

We now assume that (2.18) holds. By (2.16),

$$\hat{u} = (sI - A)^{-1}\hat{F} \tag{2.20}$$

and, by Lemma 2.4,

$$|\hat{u}(s)| \le R|\hat{F}(s)|.$$

By (2.15), the solution of (2.14) decays, for $t > T$, exponentially like $(t - T)^p e^{-\delta(t-T)}$. Therefore, the Laplace transform is well defined for $\eta > -\delta$ and we can invert it. By (2.17) and Parseval's relation, we obtain, for $\eta = 0$,

$$\int_0^\infty |u(t)|^2 dt = \int_{-\infty}^\infty |\hat{u}(i\xi + 0)|^2 d\xi \le R^2 \int_{-\infty}^\infty |\hat{F}(i\xi + 0)|^2 d\xi = R^2 \int_0^T |F(t)|^2 dt. \tag{2.21}$$

Using the differential equation we can sharpen the estimate. We have

$$\int_0^T |u_t|^2 dt = \int_0^T |Au + F|^2 dt \leq 2|A|^2 \int_0^T |u|^2 dt + 2 \int_0^T |F(t)|^2 dt$$
$$\leq 2(|A|^2 R^2 + 1) \int_0^T |F(t)|^2 dt,$$

and (2.21) becomes

$$\int_0^T (|u|^2 + |u_t|^2) dt \leq K_1 \int_0^T |F(t)|^2 dt, \quad K_1 = 2(|A|^2 R^2 + 1) + R^2. \tag{2.22}$$

The reason why we want to estimate also u_t is that we can use the Sobolev inequality

$$\max_{0 \leq t \leq T} |u|^2 \leq \frac{1}{2} \int_0^T (|u|^2 + |u_t|^2) dt \tag{2.23}$$

to estimate the maximum norm of u in terms of F.

Up to now we have assumed that $F(t) = 0$ for $t \geq T$. We now consider arbitrary $F(t)$ with

$$\int_0^\infty |F(t)|^2 dt < \infty \tag{2.24}$$

and show that (2.22) still holds. This follows from the observation that the solution of (2.14) for $0 \leq t \leq T$ does not depend on values of $F(t)$ with $t > T$. Therefore, we can replace $F(t)$ for $t > T$ by zero for general F satisfying (2.24). Thus, we have the estimate

$$\int_0^\infty |u|^2 + |u_t|^2 dt \leq K_1 \int_0^\infty |F(t)|^2 dt. \tag{2.25}$$

Now we consider the general nonlinear problem (2.1) with $u_0 = 0$. We assume that (2.24) holds. For $\varepsilon = 0$ the estimate (2.25) holds. Therefore, for every $\varepsilon > 0$, there is an interval $0 \leq t \leq T_\varepsilon$ such that

$$\int_0^{T_\varepsilon} (|u|^2 + |u_t|^2) dt \leq 4K_1 \int_0^\infty |F(t)|^2 dt.$$

As before, we choose T_ε as large as possible. If T_ε is finite, then we have equality. By (2.23),

$$\max_{0\le t\le T_\varepsilon} |u|^2 \le 2K_1 \int_0^\infty |F(t)|^2 dt =: c_0,$$

and, by assumption,

$$|f(u,t)| \le C_0|u|.$$

We now consider $\varepsilon f(u,t) + F(t)$ as a forcing function. Then (2.22) becomes

$$\begin{aligned}\int_0^{T_\varepsilon} (|u|^2 + |u_t|^2)dt &\le 2K_1 \int_0^{T_\varepsilon} |F(t)|^2 dt + 2\varepsilon^2 K_1 \int_0^{T_\varepsilon} |f(u,t)|^2 dt \\ &\le 2K_1 \int_0^{T_\varepsilon} |F(t)|^2 dt + 2\varepsilon^2 K_1 C_0^2 \int_0^{T_\varepsilon} |u|^2 dt.\end{aligned}$$

Thus, for $2\varepsilon^2 C_0^2 K_1 < 1/3$, we obtain, from (2.23),

$$\max_{0\le t\le T_\varepsilon} |u|^2 \le \frac{1}{2}\int_0^{T_\varepsilon} (|u|^2 + |u_t|^2)dt \le \frac{3}{2}K_1 \int_0^{T_\varepsilon} |F(t)|^2 dt.$$

Therefore, by the same argument as before, $T_\varepsilon = \infty$, for all ε with $2\varepsilon^2 C_0^2 K_1 < 1/3$. Since

$$\int_0^\infty |u|^2 + |u_t|^2 dt < \infty \quad \text{implies} \quad \lim_{t\to\infty} |u(t)| = 0,$$

we have again proved Theorem 2.1.

The estimates for the size of the admissible perturbations resulting from (2.6) or (2.13) are rather satisfactory, because they are proportional to δ. This dependence cannot be improved. For example, the solutions of

$$y' = -\delta u + \varepsilon u$$

grow exponentially for $\varepsilon > \delta$.

The corresponding estimates for the Laplace transform technique are proportional to R. Let λ, y with $|y| = 1$ be an eigenvalue and eigenvector, respectively, of A, i.e.,

$$Ay = \lambda y$$

and let

$$F(t) = e^{\alpha t} y, \quad \alpha = \mathrm{Im}\,\lambda - 1.$$

Then (2.16) becomes

$$(sI - A)\hat{u} = \frac{1}{\alpha - s} y,$$

i.e.,

$$\hat{u} = \frac{1}{s - \lambda} \cdot \frac{1}{\alpha - s} y.$$

Thus, for $s = \mathrm{Im}\,\lambda$,

$$|\hat{u}| = |\frac{1}{\mathrm{Re}\,\lambda}|.$$

Therefore,

$$R^{-1} \leq \min |\mathrm{Re}\,\lambda| = \delta.$$

If A is symmetric, then $R^{-1} = \delta$ and the admissible perturbations are again proportional to δ. If A is very "skew", then the admissible perturbations can be much smaller. For example, if

$$A = \begin{pmatrix} -1 & 10^p \\ 0 & -1 \end{pmatrix},$$

then

$$\sup_{\mathrm{Re}\ s \geq 0} |(sI - A)^{-1}| \geq 10^p.$$

However, in the H-norm we obtain, from (2.16),

$$\mathrm{Re}\langle u, (sI - A)u\rangle_H = \mathrm{Re}\langle u, \hat{F}\rangle_H$$

and, therefore,

$$(\ \mathrm{Re}\ s + \frac{\delta}{2})|u|_H^2 \leq |u|_H|F|_H.$$

Thus, in the H-norm,

$$R_H = 2\delta^{-1}.$$

The H-norm provides a scaling of the dependent variables such that the problem behaves like a symmetric one.

Finally, we consider an equation of convolution type

$$\begin{aligned} u_t &= \int_0^t K(t-\xi)u(\xi)d\xi + \varepsilon f(u,t) + F(t), \\ u(x,0) &= 0. \end{aligned} \tag{2.26}$$

Here $K(\tau)$ is a smooth kernel with the property

$$|K(\tau)| \leq \frac{K_0}{(1+\tau)^{1+\beta}}, \quad \beta > 0,\ \tau \geq 0. \tag{2.27}$$

We again start with the linear problem

$$\begin{aligned} u_t &= \int_0^t K(t-\xi)u(\xi)d\xi + F(t) \\ u(x,0) &= 0, \end{aligned} \tag{2.28}$$

and assume that $F(t)$ has compact support. We need

Lemma 2.5. *There is a constant $\alpha > 0$ such that the solutions of the homogeneous equation*

$$v_t = \int_0^t K(t-\xi)v(\xi)d\xi, \quad v(0) = v_0, \tag{2.29}$$

satisfies the estimate

$$|v(t)| \le e^{\alpha t}|v(0)|. \tag{2.30}$$

Proof. Introducing into (2.29) a new variable $v = e^{\alpha t} w$ we obtain

$$w_t = \int_0^t K_1(t-\xi)w(\xi)d\xi \; - \alpha w, \quad K_1(\tau) = e^{-\alpha\tau}K(\tau). \tag{2.31}$$

Consider a fixed interval $0 \le t \le T$ and assume that

$$\max_{0\le t\le T} |w|^2 = |w(T_1)|^2 > 0.$$

If $T_1 \ne 0$, we have $d|w(T_1)|^2/dt \ge 0$. Also, (2.31) gives us, for $t = T_1$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w(T_1)|^2 &\le |\langle w(T_1), \int_0^{T_1} K_1(t-\xi)w(\xi)d\xi\rangle| - \alpha|w(T_1)|^2 \\ &\le \left(\int_0^{T_1} |K_1(t-\xi)|\, d\xi - \alpha\right)\cdot|w(T_1)|^2.\end{aligned}$$

Therefore, for $\alpha > \int_0^\infty |K_1(t-\xi)|d\xi$, we arrive at a contradiction. Thus, $T_1 = 0$ and $|w(t)|^2$ is nonincreasing. Therefore, v does not grow faster than $e^{\alpha t}$.

We solve (2.28) by Laplace transform. Let $\text{Re}\, s > \alpha$. Then $\hat{u}(s)$ is well defined. Also, using the definition

$$\tilde{K}(\tau) = \begin{cases} K(\tau) & \text{for } \tau \ge 0 \\ 0 & \text{for } \tau < 0 \end{cases},$$

we have

$$\begin{aligned}\int_0^\infty e^{-st}\int_0^t K(t-\xi)u(\xi)d\xi\; dt &= \int_0^\infty\int_0^t e^{-s(t-\xi)}K(t-\xi)e^{-\xi t}u(\xi)d\xi\; dt \\ &= \int_0^\infty\int_0^\infty e^{-s(t-\xi)}\tilde{K}(t-\xi)dt\; e^{-\xi t}u(\xi)d\xi \\ &= \hat{K}(s)\hat{u}(s).\end{aligned}$$

Here

$$\hat{K}(s) = \int_0^\infty e^{-s\tau} K(\tau) d\tau. \tag{2.32}$$

Therefore, the Laplace transformed equation (2.28) is given by

$$(sI - \hat{K}(s))\hat{u} = \hat{F}. \tag{2.33}$$

By (2.27),

$$\sup_{\text{Re } s \geq 0} |\hat{K}(s)| \leq \int_0^\infty |K(\tau)| d\tau \leq \frac{K_0}{\beta}. \tag{2.34}$$

Also, for Re $s < 0$, we cannot expect $\hat{K}(s)$ to exist. By (2.34), we can solve (2.33) for $|s| > K_0/\beta$, Re $s \geq 0$, and obtain the estimate

$$|\hat{u}| \leq \frac{|\hat{F}|}{|s| - \frac{K_0}{\beta}}. \tag{2.35}$$

We now make

Assumption 2.2. *The resolvent equation (2.33) has a unique solution for* Re $s \geq 0$.

Since $\hat{K}(s)$ is an analytic function for Re $s > 0$ and continuous for Re $s \geq 0$, it follows that $(sI - \hat{K}(s))^{-1}$ has the same property. Therefore, we obtain

Lemma 2.6. *There exists a constant R such that*

$$|\hat{u}(s)| \leq R|\hat{F}|$$

for Re $s \geq 0$. *Therefore, we can choose $\eta = 0$ in the inversion formula and obtain the estimate*

$$\int_0^T |u(t)|^2 dt \leq R^2 \int_0^T |F(t)|^2 dt. \tag{2.36}$$

The last inequality holds for all $F(t)$ with $\int_0^\infty |F(t)|^2 dt < \infty$ and all T.

We shall now estimate $\int_0^T |u_t|^2 dt$. (2.28) gives us

$$|u_t|^2 \le 2|\int_0^t K(t-\xi)u(\xi)d\xi|^2 + 2|F(t)|^2.$$

Therefore,

$$\int_0^T |u_t|^2 dt \le 2\int_0^T |\int_0^t K(t-\xi)u(\xi)d\xi|^2 dt + 2\int_0^T |F(t)|^2 dt.$$

Since

$$\begin{aligned}\int_0^T |\int_0^t K(t-\xi)u(\xi)d\xi|^2 dt &\le \int_0^T \int_0^t |K(t-\xi)|d\xi \cdot \int_0^t |K(t-\xi)|\,|u(\xi)|^2 d\xi\,dt \\ &\le \int_0^\infty |K(\tau)|d\tau \cdot \int_0^T \int_0^\infty |\tilde{K}(t-\xi)|\,|u(\xi)|^2 d\xi\,dt \\ &\le \frac{K_0^2}{\beta^2}\int_0^T |u(\xi)|^2 d\xi,\end{aligned}$$

we obtain the desired estimate from (2.36). Now we can proceed as before and obtain

Theorem 2.2. *Consider the convolution equation (2.27) and assume that Assumptions 2.1 and 2.2 hold. Then the solution of the nonlinear problem converges to zero, provided ε is sufficiently small.*

3. Hyperbolic first order systems

In this section we consider semilinear hyperbolic systems

$$\begin{aligned}u_t &= P_0(\frac{\partial}{\partial x})u + Bu + \varepsilon f(x,t,u) + F(x,t), \quad t \ge 0,\\ u(x,0) &= u_0(x).\end{aligned} \tag{3.1}$$

Here

$$P_0(\frac{\partial}{\partial x}) = \sum_{\nu=1}^{d} A_\nu \frac{\partial}{\partial x_\nu}$$

where A_ν, B are constant $n \times n$ matrices. f, F and u_0 are smooth functions of all variables which are 2π-periodic in all space variables. We are interested in solutions with the same properties.

We shall use the following notations. $p = (p_1, \ldots, p_d)$, $q = (q_1, \ldots, q_n)$, $|p| = \sum p_i$, $|q| = \sum q_i$ denote multi- indices and

$$D_x^p = \frac{\partial^p}{\partial x_1^{p_1} \cdots \partial x_d^{p_d}}, \quad D_u^q = \frac{\partial^q}{\partial u_1^{q_1} \cdots \partial u_n^{q_n}},$$

denote derivatives with respect to the x and u variables, respectively. Also,

$$(u, v) = \int_0^{2\pi} \cdots \int_0^{2\pi} \langle u, v \rangle dx_1 \cdots dx_d, \quad \|u\|^2 = (u, u), \quad \|u\|_{H^p}^2 = \sum_{|j| \le p} \|D_x^j u\|^2,$$

denote the usual L_2-scalar product,-norm and H^p-norm, respectively.

Corresponding to Assumption 2.1, we make

Assumption 3.1. *For every c_0, there are constants $C_{|p|,|q|}$ such that*

$$|D_x^p f(x, t, u)| \le C_{|p|,0} |u|, \quad |D_x^p D_u^q f(x, t, u)| \le C_{|p|,|q|}, \; |q| \ge 1,$$

provided $|u| \le c_0$.

We start with the symmetric case, i.e.,

$$A_j = A_j^*, \quad B = B^*, \tag{3.2a}$$

and assume that

$$B + B^* \le -2\delta I < 0. \tag{3.2b}$$

As for the ordinary differential equation, we derive an estimate for the linear problem. Since

$$\mathrm{Re}(u, \partial u / \partial x_j) = 0, \quad \mathrm{Re}(u, Bu) \le -\delta \|u\|^2,$$

integration by parts gives, corresponding to (2.6),

$$\frac{\partial}{\partial t}\|u\| \le -\delta\|u\| + \|F\|.$$

Differentiating the differential equation we obtain

$$\frac{\partial}{\partial t}\|D_x^p u(\cdot,t)\| \le -\delta\|D_x^p u(\cdot,t)\| + \|D_x^p F(\cdot,t)\|,$$

i.e.,

$$\frac{\partial}{\partial t}\|u(\cdot,t)\|_{H^p} \le -\delta\|u(\cdot,t)\|_{H^p} + \|F(\cdot,t)\|_{H^p}. \tag{3.3}$$

Therefore, by Lemmata 2.1 and 2.2,

$$\max_{0\le t\le T}\|u(\cdot,t)\|_{H^p} \le \frac{1}{\delta}\max_{0\le t\le T}\|F(\cdot,t)\|_{H^p} + \|u(\cdot,0)\|_{H^p} =: K_1. \tag{3.4a}$$

Using Sobolev inequalities there is, for every $p > [d/2]+1$, a universal constant E_p such that

$$|D_x^\alpha u(\cdot,t)|_\infty \le E_p\|u(\cdot,t)\|_{H^p} \quad \text{for all } \alpha \text{ with } p \ge [d/2]+|\alpha|+1. \tag{3.4b}$$

($[x]$ largest integer $\le x$).

Now we consider the nonlinear problem (3.1) and proceed in the same way as for ordinary differential equations. Local existence causes no difficulty, i.e., if we have estimates of u and its derivatives, the solution exists and changes continuously with time. Let p with

$$p - [p/2] > [d/2] + 1 \tag{3.5}$$

be fixed. For every $\varepsilon > 0$, there exists an interval $0 \le t \le T_\varepsilon$, $T_\varepsilon > 0$, such that (3.4a) holds with K_1 replaced by $2K_1$. We choose T_ε as large as possible. Either $T_\varepsilon = \infty$ or if

$T_\varepsilon < \infty$, then we have equality at $t = T_\varepsilon$. We want to show that, for sufficiently small ε, we have $T_\varepsilon = \infty$.

Let $|\alpha| \le p$. By Leibniz rule (see for example [1 ,Sec.6.4.2]),

$$D_x^\alpha f(x,t,u(x,t)) = \sum_{r=1}^{|\alpha|} \sum_{|\sigma_1|+\cdots+|\sigma_r|\le|\alpha|} C_\sigma D_x^{\sigma_1} u(x,t) \cdots D_x^{\sigma_r} u(x,t).$$

Here C_σ consists of partial derivatives of f with respect to x and u of order $\tau \le |\alpha|$. By Assumption 3.1, we have uniform bounds for these derivatives. If $|\sigma_j| \le [p/2]$, it follows from (3.5) and (3.4b) that

$$|D_x^{\sigma_j} u(\cdot,t)|_\infty \le E_p \|u(\cdot,t)\|_{H^p}.$$

Since $|\alpha| \le p$, there is at most one σ_j with $|\sigma_j| > [p/2]$. Therefore,

$$\|D_x^\alpha f(\cdot,t,u(\cdot,t))\| \le \text{ const. } \|u(\cdot,t)\|_{H^p}^{|\alpha|}.$$

Thus, there is a constant K_2 which depends only on $2K_1$ such that

$$\|f(\cdot,t,u(\cdot,t))\|_{H^p} \le K_2\Big(1 + \|u(\cdot,t)\|_{H^p}^{p-1}\Big)\|u(\cdot,t)\|_{H^p}, \quad 0 \le t \le T_\varepsilon. \tag{3.6}$$

We consider f as part of the forcing and obtain instead of (3.4a), using (3.6),

$$\begin{aligned}\max_{0\le t\le T_\varepsilon} \|u(\cdot,t)\|_{H^p} &\le \frac{1}{\delta}\Big(\max_{0\le t\le T_\varepsilon} \|F(\cdot,t)\|_{H^p} + \varepsilon \max_{0\le t\le T_\varepsilon} \|f(\cdot,t,u(\cdot,t))\|_{H^p}\Big) + \|u(\cdot,0)\|_{H^p} \\ &\le K_1\Big(1 + 2\varepsilon K_2(1 + (2K_1)^{p-1})\Big).\end{aligned}$$

Therefore, the inequality (3.4a) with K_1 replaced by $2K_1$ and the inequality (3.6) hold for all times for the solution of the nonlinear problem if

$$\varepsilon < \frac{1}{2K_2(1 + (2K_1)^{p-1})}.$$

Considering again f as part of the forcing, we obtain, from (3.3) and (3.6),

$$\begin{aligned}\frac{\partial}{\partial t}\|u(\cdot,t)\|_{H^p} &\leq -\delta\|u(\cdot,t)\|_{H^p} + \varepsilon K_3\|u(\cdot,t)\|_{H^p} + \|F(\cdot,t)\|_{H^p} \\ &\leq -\frac{\delta}{2}\|u(\cdot,t)\|_{H^p} + \|F(\cdot,t)\|_{H^p}.\end{aligned}$$

Here $K_3 = K_2\big(1 + (1 + (2K_1)^{p-1})\big)$. Using the same argument as for ordinary differential equations, we obtain

Theorem 3.1. *Suppose p satisfies (3.5). For symmetric systems satisfying (3.2), we have, for sufficiently small* ε,

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{H^p} = 0 \quad \text{if} \quad \lim_{t\to\infty}\|F(\cdot,t)\|_{H^p} = 0.$$

Now we consider strongly hyperbolic systems, i.e., the symbol

$$P_0(i\omega) = i\sum_{\nu=1}^{d} A_\nu\omega_\nu = |\omega|\hat{P}_0(i\omega'), \quad \omega' = \omega/|\omega| \text{ real}$$

has the following property: There is a constant K and, for every frequency $\omega = (\omega_1, \ldots, \omega_d)$ $\neq 0$, ω_j integer, a transformation T such that

$$\begin{aligned}T^{-1}(\omega)\hat{P}_0(i\omega)T(\omega) &= |\omega|T^{-1}(\omega')\hat{P}_0(i\omega')T(\omega') \\ &= |\omega|\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix} =: |\omega|\Lambda, \quad \operatorname{Re}\lambda_j = 0,\end{aligned} \tag{3.8}$$

where

$$\max\{|T|, |T^{-1}|\} \leq K^{1/2}, \quad K \geq 1.$$

We now construct a new norm. Let

$$\hat{H}(0) = I, \quad \hat{H}(\omega) = (T^{-1}(\omega'))^* T^{-1}(\omega') \quad \text{for } \omega \neq 0.$$

$\hat{H}$ is Hermitian and

$$K^{-1}I \le \hat{H}(\omega) \le KI.$$

This follows from the relation

$$K^{-1}|\hat{u}|^2 = K^{-1}|TT^{-1}\hat{u}|^2 \le |T^{-1}u|^2 = (\hat{u}, \hat{H}\hat{u})$$
$$= |T^{-1}\hat{u}|^2 \le |T^{-1}|^2|\hat{u}|^2 \le K|\hat{u}|^2.$$

Also,

$$\hat{H}(0)\hat{P}_0(0) + \hat{P}_0^*(0)H(0) = 0$$

and, for $\omega \neq 0$,

$$\begin{aligned}\hat{H}\hat{P}_0 + \hat{P}_0^*\hat{H} &= (T^{-1})^*\big(T^{-1}\hat{P}_0T + T^*\hat{P}_0^*(T^{-1})^*\big)T^{-1} \\ &= (T^{-1})^*(\Lambda + \Lambda^*)T^{-1} = 0.\end{aligned} \tag{3.9}$$

We define a new norm in the following way:

Let $f \in L_2$. It can be represented by its Fourier series

$$f = \sum_\omega e^{i\langle\omega,x\rangle}\hat{f}(\omega).$$

We define the operator H by

$$Hf = \sum_\omega e^{i\langle\omega,x\rangle}\hat{H}(\omega)\hat{f}(\omega).$$

By Parseval's relation,

$$\|Hf\|^2 = \sum_\omega |\hat{H}(\omega)\hat{f}(\omega)|^2 \le K^2\sum_\omega |\hat{f}(\omega)|^2 \le K^2\|f\|^2,$$
$$(g, Hf) = \sum_\omega \langle\hat{g}, \hat{H}\hat{f}\rangle = \sum_\omega \langle\hat{H}\hat{g}, \hat{f}\rangle = (Hg, f).$$

Also, H^{-1} exists and

$$H^{-1}f = \sum_\omega e^{i\langle\omega,x\rangle}\hat{H}^{-1}(\omega)\hat{f}(\omega),$$

i.e.,

$$\|H^{-1}f\|^2 \le K^{-2}\|f\|^2.$$

Thus, H is a selfadjoint positive definite bounded operator.

We now define a new scalar product by

$$(u,v)_H = (u, Hv), \quad \|u\|_H^2 = (u, Hu).$$

Since,

$$\langle \hat{u}, \hat{H}^{-1}\hat{H}\hat{u}\rangle \le |\hat{H}^{-1}|\langle \hat{u}, \hat{H}\hat{u}\rangle \le K\langle \hat{u}, \hat{H}\hat{u}\rangle,$$

we obtain, by Parseval's relation,

$$K^{-1}\|u\|^2 = K^{-1}\sum_{\omega}\langle \hat{u}, \hat{H}^{-1}\hat{H}\hat{u}\rangle \le \sum_{\omega}\langle \hat{u}, \hat{H}\hat{u}\rangle = (u, Hu) \le K\|u\|^2.$$

Thus, the H-scalar product defines a norm which is equivalent with the L_2-norm.

Now we can prove

Lemma 3.1. *Assume that the system is strongly hyperbolic. Then we can construct a new norm which is equivalent with the L_2-norm such that*

$$\text{Re}(u, P_0 u)_H = 0.$$

Proof. By (3.9) and Parseval's relation,

$$\begin{aligned} 2\text{Re}(u, P_0 u)_H &= \big(u, (HP_0 + P_0^* H)u\big) \\ &= \sum_{\omega}\langle \hat{u}(\omega), \big(\hat{H}(\omega)\hat{P}_0(i\omega) + \hat{P}_0^*(i\omega)\hat{H}(\omega)\big)\hat{u}(\omega)\rangle = 0. \end{aligned}$$

This proves the lemma.

The next step is to consider the system

$$u_t = \Big(P_0(\frac{\partial}{\partial x}) + B\Big)u, \quad B \text{ constant matrix.} \tag{3.10}$$

We want to prove

Lemma 3.2. *Assume that (3.10) is strongly hyperbolic and that the eigenvalues μ of $\hat{P}_0(i\omega) + B$ satisfy*

$$\operatorname{Re}\mu \le -\delta < 0. \tag{3.11}$$

Then we can construct a new norm such that

$$\operatorname{Re}\Big(u, \big(P_0(\frac{\partial}{\partial x}) + B\big)u\Big)_H \le -\delta(u,u)_H.$$

Proof. We consider the symbol

$$\hat{P}_0(i\omega) + B.$$

By (3.8), we obtain, for $\omega \neq 0$,

$$T^{-1}\hat{P}_0 T + T^{-1}BT =: |\omega|\Lambda + \tilde{B}.$$

Now we can find a unitary matrix $U(\omega)$ such that

$$U^*(|\omega|\Lambda + \tilde{B})U = \begin{pmatrix} \mu_1 & \tilde{\tilde{b}}_{12} & \tilde{\tilde{b}}_{13} & \cdots & \tilde{\tilde{b}}_{1n} \\ & \mu_2 & \tilde{\tilde{b}}_{23} & \cdots & \tilde{\tilde{b}}_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \tilde{\tilde{b}}_{n-1,n} \\ 0 & & & & \mu_n \end{pmatrix}.$$

We want to show that the $\tilde{\tilde{b}}_{ij}$ are bounded independently of $|\omega|$.

Clearly, the entries of $U^*\tilde{B}U$ are bounded independently of $|\omega|$. Since the sum of the entries of $|\omega|U^*\Lambda U$ and $U^*\tilde{B}U$ below the diagonal add up to zero, these entries of $|\omega|U^*\Lambda U$ are bounded independently of $|\omega|$. Since $U^*\Lambda U$ is Hermitian, the entries of $|\omega|U^*\Lambda U$ above the diagonal are also bounded independently of $|\omega|$. Therefore, the $\tilde{\tilde{b}}_{ij}$ are bounded independently of $|\omega|$. By (3.11), we can construct a diagonal matrix D,

$$D = \begin{pmatrix} 1 & & & 0 \\ & d_1 & & \\ & & \ddots & \\ 0 & & & d_{n-1} \end{pmatrix},$$

$d > 0$ sufficiently small but independent of ω,

such that

$$2\mathrm{Re}\Big(D^{-1}U^*(|\omega|\Lambda + \tilde{B})UD\Big) \leq -\delta I.$$

For $\omega = 0$,

$$\hat{P}_0(i\omega) + B = B$$

and, by (3.11), there is a transformation $T_1(0)$ such that

$$\mathrm{Re}\Big(T_1^{-1}(0)BT_1(0)\Big) \leq -\delta I.$$

Now we define $\hat{H}(\omega)$ by

$$\hat{H}(\omega) = \Big(T_1^{-1}(\omega)\Big)^* T_1^{-1}(\omega), \quad T_1(\omega) = T(\omega')U(\omega)D \text{ for } \omega \neq 0$$

and the lemma follows.

We can now prove

Theorem 3.2. *Assume that the system (3.1) is strongly hyperbolic and that the eigenvalue condition (3.11) is satisfied. Then the results of Theorem 3.1 hold.*

Proof. We follow the proof of Theorem 3.1 in the H-norm, constructed in Lemma 3.2.

In applications one is interested in quasilinear systems

$$u_t = P_0(\frac{\partial}{\partial x})u + Bu + \varepsilon\Big(P_1(x,t,u,\frac{\partial}{\partial x})u + f(x,t,u)\Big) + F(x,t). \tag{3.12}$$

Here

$$P_1 = \sum \tilde{A}_\nu(x,t,u)\frac{\partial}{\partial x_\nu}$$

is also a first order operator whose coefficients depend on u. One could be tempted to consider

$$P_1(x,t,u,\frac{\partial}{\partial x})u + f(x,t,u) = \tilde{f}(x,t,u,Du)$$

as part of the forcing and use the same proof as above. However, this process does not work because, instead of (3.5), one can only prove that

$$\|\tilde{f}(\cdot,t,u(\cdot,t),Du(\cdot,t)\|_{H^p} \le K_2\|u(\cdot,t)\|_{H^{p+1}},$$

which cannot be dominated by the left hand side of (3.4a). To treat systems (3.12), one has to construct a new norm which depends on u such that

$$\text{Re}\Big(u,(P+\varepsilon P_1+B)u\Big)_H \le -\delta(u,u)_H.$$

For details, see [3] where Theorem 3.1 is proven for systems (3.12).

Instead of the Liaponov technique, we could also use the resolvent method to prove Theorem 3.1. However, this technique cannot be used for the more general systems (3.12).

4. Second order parabolic systems

In this section we consider parabolic systems

$$u_t = \Delta u + \Big(P_0(\frac{\partial}{\partial x}) + B\Big)u + \varepsilon f(x,t,u,\frac{\partial u}{\partial x_i},\frac{\partial^2 u}{\partial x_i \partial x_j}) + F(x,t),$$
$$u(x,0) = 0. \tag{4.1}$$

Here

$$\Delta = \sum_{j=1}^{s} \frac{\partial^2}{\partial x_j^2}$$

denotes the Laplacian. As in Section 3, $P_0 + B$ is a first order operator with constant coefficients. f, F are smooth functions of all variables which are 2π-periodic with respect to the space variables. We are interested in solutions with the same properties.

Corresponding to Assumption 3.1, we make

Assumption 4.1. *For every c_0, there are constants $C_{|p|}$ such that*

$$|D_x^p f| \le C_{|p|}\Big(|u| + \sum_{j=1}^{d} |\frac{\partial u}{\partial x_j}| + \sum_{i,j=1}^{d} |\frac{\partial^2 u}{\partial x_i \partial x_j}|\Big),$$
$$|D^p f| \le C_{|p|},$$

provided

$$|u| + \sum_{j=1}^{d} |\frac{\partial u}{\partial x_j}| + \sum_{i,j=1}^{d} |\frac{\partial^2 u}{\partial x_i \partial x_j}| \le c_0.$$

(As before, D_x^p stands for space derivatives, D^p for derivatives with respect to all variables.)

We shall use the resolvent technique. We start with the linear problem ($\varepsilon = 0$).

$$u_t = \Delta u + \Big(P_0(\frac{\partial}{\partial x}) + B\Big)u + F(x,t),$$
$$u(x,0) = 0. \tag{4.2}$$

Fourier transform with respect to the space variables and Laplace transform with respect to t gives us

$$\Big((s + |\omega|^2)I - \Big(P_0(i\omega) + B\Big)\hat{u}(\omega, s) = \hat{F}(\omega, s). \tag{4.3}$$

Corresponding to Lemma 2.4 we have

Lemma 4.1. *(4.3) has a unique solution for* Re $s \geq 0$ *and there is a constant R such that*

$$\sup_{\text{Re } s \geq 0} \left| \Big((s + |\omega|^2)I - (P_0(i\omega) + B) \Big)^{-1} \right| \leq R \tag{4.4}$$

if and only if there is a constant $\delta > 0$ such that, for all real ω and all complex s with Re $s \geq 0$, *the eigenvalues λ of $P_0(i\omega) + B - (s + |\omega|^2)I$ satisfy*

$$\text{Re}\, \lambda \leq -\delta. \tag{4.5}$$

Proof. The condition is certainly necessary. We shall show that it is also sufficient. Since P_0 is a first order operator, $(s+|\omega|^2)I$ dominates $P_0(i\omega)+B$ for sufficiently large $|s|+|\omega|^2$, and we obtain the estimate

$$(|s| + |\omega|^2)|\hat{u}(\omega, s)| \leq 2|\hat{F}(\omega, s)|, \quad |s| + |\omega|^2 >> 1. \tag{4.6}$$

For bounded $|s| + |\omega|^2$, the estimate follows in the same way as in Lemma 2.4 by a compactness argument.

We can now invert the Laplace and Fourier transform and obtain by Parseval's relation

$$\int_0^T \|u(\cdot, t)\|^2 dt \leq R^2 \int_0^T \|F(\cdot, t)\|^2 dt. \tag{4.7}$$

(4.6) tells us that we can also estimate the second space derivatives in terms of F. Therefore, we can sharpen (4.7) to

$$\int_0^T \|u(\cdot, t)\|_{H^2}^2 dt \leq R^2 \int_0^T \|F(\cdot, t)\|^2 dt. \tag{4.8}$$

Thus, we gain two derivatives.

Differentiating the differential equations gives us

$$\int_0^T \|u(\cdot,t)\|_{H^{p+2}}^2 dt \leq R_p^2 \int_0^T \|F(\cdot,t)\|_{H^p}^2 dt. \tag{4.9}$$

Using the differential equation and (4.9), we can also estimate u_t. We obtain

$$\begin{aligned}\int_0^T \|u_t(\cdot,t)\|_{H^p}^2 dt &\leq 2\int_0^T \|\Delta u(\cdot,t) + \Big(P_0(\frac{\partial}{\partial x}) + B\Big)u(\cdot,t)\|_{H^p}^2 dt \\ &\quad + 2\int_0^T \|F(\cdot,t)\|_{H^p}^2 dt \\ &\leq \text{ const.} \int_0^T \|u(\cdot,t)\|_{H^{p+2}}^2 dt + 2\int_0^T \|F(\cdot,t)\|_{H^p}^2 dt \\ &\leq \text{ const.} \int_0^T \|F(\cdot,t)\|_{H^p}^2 dt.\end{aligned}$$

Thus, there are constants $\tilde{R}_p$ such that

$$\int_0^T \|u(\cdot,t)\|_{H^{p+2}}^2 dt + \int_0^T \|u_t(\cdot,t)\|_{H^p}^2 dt \leq \tilde{R}_p^2 \int_0^T \|F(\cdot,t)\|_{H^p}^2 dt. \tag{4.10}$$

By Sobolev inequalities, there are universal constants $\tilde{E}_p$ such that, for all T (observe that $u(x,0) \equiv 0$)

$$\begin{aligned}\max_{0\leq t\leq T} |D_x^\alpha u(\cdot,t)|^2 &\leq \tilde{E}_p^2\Big(\int_0^T \|u(\cdot,t)\|_{H^{p+2}}^2 dt + \int_0^T \|u_t(\cdot,t)\|_{H^p}^2 dt\Big) \\ &\leq \tilde{E}_p^2\tilde{R}_p^2 \int_0^T \|F(\cdot,t)\|_{H^p}^2 dt \quad \text{for all } \alpha \text{ with } p \geq [d/2] + |\alpha| + 1.\end{aligned} \tag{4.11}$$

Now we consider the nonlinear problem and we proceed as before. We choose p satisfying (3.5). There is an interval $0 \leq t \leq T_\varepsilon$, $T_\varepsilon > 0$, where u satisfies (4.10) with $\tilde{R}_p^2$ replaced by $2\tilde{R}_p^2$, i.e.,

$$\int_0^T \|u(\cdot,t)\|_{H^{p+2}}^2 dt + \int_0^T \|u_t(\cdot,t)\|_{H^p}^2 dt \leq 2\tilde{R}_p^2 \int_0^T \|F(\cdot,t)\|_{H^p}^2 dt, \quad 0 \leq T \leq T_\varepsilon. \tag{4.12}$$

We consider f as a part of the forcing and obtain, from (4.10),

$$\int_0^T \|u(\cdot,t)\|^2_{H^{p+2}}dt + \int_0^T \|u_t(\cdot,t)\|^2_{H^p}dt$$
$$\leq \tilde{R}_p^2\Big(\frac{3}{2}\int_0^T \|F(\cdot,t)\|^2_{H^p}dt + 3\varepsilon^2\int_0^T \|f\|^2_{H^p}dt\Big), \quad T \leq T_\varepsilon.$$

As in the previous section, we can estimate the nonlinear term by

$$\int_0^T \|f\|^2_{H^p}dt \leq \text{ const. } (1+\tilde{R}_p^{2p-2})\int_0^T \|F\|^2_{H^p}dt.$$

Therefore, for sufficiently small ε, we never achieve equality in (4.12), and $T_\varepsilon = \infty$. We have proved

Theorem 4.1. *The problem (4.1) is nonlinearly stable if and only if the eigenvalue condition (4.5) holds.*

In [2] the above results have been proven for very general mixed hyperbolic-parabolic systems.

5. General partial differential equations

In this section we consider perturbations of general systems of partial differential equations with constant coefficients

$$\begin{aligned} u_t &= P_0(\frac{\partial}{\partial x})u + \varepsilon P_1(x,t,u,\frac{\partial}{\partial x})u + F(x,t), \\ u(x,0) &= u_0(x). \end{aligned} \tag{5.1}$$

Here $x = (x_1,\ldots,x_d),\ u = (u^{(1)},\ldots,u^{(n)})^T$ and

$$P_0(\frac{\partial}{\partial x}) = \sum_{|\nu|\leq m} A_\nu \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}\cdots\partial x_d^{\nu_d}}, \quad \nu = (\nu_1,\ldots,\nu_s),\ |\nu| = \sum \nu_i,$$

denotes a general differential operator with constant matrix coefficients. Assume $\lim_{t\to\infty}$ $\|F(x,t)=0\|$. We are interested in solutions which are 2π-periodic in all space directions.

Therefore, we assume that all data and coefficients have the same property. P_1 is a nonlinear differential operator whose special properties we discuss later.

To begin with, we discuss well posed linear problems

$$
\begin{aligned}
v_t &= P_0(\frac{\partial}{\partial x})v, \\
v(x,0) &= v_0(x).
\end{aligned}
\tag{5.2}
$$

Definition 5.1. *The problem (5.2) is well posed if, for smooth data, there is a solution which satisfies the estimate*

$$\|v(\cdot,t)\| \leq Ke^{\alpha t}\|v(\cdot,0)\|. \tag{5.3}$$

Here K, α are universal constants which do not depend on the particular initial data.

We can solve (5.2) by Fourier expansion. Let

$$v(x,t) = \sum_{\omega} e^{i\langle\omega,x\rangle}\hat{v}(\omega,t).$$

Then $\hat{v}(\omega,t)$ solves

$$
\begin{aligned}
&\frac{d\hat{v}}{dt} = \hat{P}(i\omega)\hat{v}, \quad \hat{P} = \sum A_\nu(i\omega_1)^{\nu_1}\cdots(i\omega_d)^{\nu_d}, \\
&\hat{v}(\omega,0) = \hat{v}_0(\omega).
\end{aligned}
\tag{5.4}
$$

We have

Lemma 5.1. *A necessary condition for wellposedness is that the eigenvalues λ of $\hat{P}(i\omega)$ satisfy the Petrovskii condition*

$$\operatorname{Re}\lambda \leq \alpha. \tag{5.5}$$

Proof. For every fixed ω, the solution of (5.4) is of the form

$$\hat{v}(\omega,t) = e^{\hat{P}(i\omega)t}\hat{v}_0(\omega),$$

which gives us the solution

$$v(x,t) = e^{\hat{P}(i\omega)t} e^{i\langle\omega,x\rangle} \hat{v}_0(\omega).$$

Therefore,

$$\|v(\cdot,t)\|^2 = |e^{\hat{P}(i\omega)t}\hat{v}_0(\omega)|^2. \tag{5.6}$$

Since

$$|e^{\hat{P}(i\omega)t}| \geq e^{(\text{Re }\lambda)t} \quad \text{for any eigenvalue } \lambda,$$

the inequality (5.3) can only be satisfied if (5.5) holds.

Theorem 5.1. *The problem (5.2) is well posed if and only if, for all frequencies,*

$$|e^{\hat{P}(i\omega)t}| \leq Ke^{\alpha t}. \tag{5.7}$$

Proof. That (5.7) is necessary follows from (5.6). (5.7) is also sufficient. Let v be a solution. It can be represented as the Fourier series

$$v(x,t) = \sum_{\omega} e^{\hat{P}(i\omega)t} e^{i\langle\omega,x\rangle} \hat{v}_0(\omega).$$

By Parseval's relation and (5.7),

$$\|v(\cdot,t)\|^2 = \sum_{\omega} |e^{\hat{P}(i\omega)t}\hat{v}_0(\omega)|^2 \leq K^2 e^{2\alpha t} \cdot \|v(\cdot,0)\|^2. \tag{5.8}$$

Therefore, (5.3) is satisfied. The representation (5.8) also shows that, for smooth data, there is a smooth solution. This proves the theorem.

We can give necessary and sufficient algebraic conditions such that (5.7) holds. The Kreiss matrix theorem (see, for example,[1]) gives us

Theorem 5.2. *The following conditions are equivalent.*

1) There are constants K_1, α *such that, for all frequencies* ω,

$$|e^{\hat{P}(i\omega)t}| \le K_1 e^{\alpha t}, \quad \text{Re}\,\lambda \le \alpha. \tag{5.9}$$

2) There are constants K_2, α *and, for every* $\hat{P}(i\omega)$, *a positive transformation* $S(\omega)$ *with*

$$|S(\omega)| + |S^{-1}(\omega)| \le K_2$$

such that

$$S^{-1}(\omega)\hat{P}(i\omega)S(\omega) = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & \cdots & b_{1n} \\ 0 & \lambda_2 & b_{23} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \lambda_n \end{pmatrix}$$

is upperdiagonal and the eigenvalues are ordered

$$0 \ge \text{Re}(\lambda_1 - \alpha) \ge \text{Re}(\lambda_2 - \alpha) \ge \cdots \ge \text{Re}(\lambda_n - \alpha).$$

The upper diagonal elements satisfy the estimate

$$|b_{ij}| \le K_2 |\text{Re}(\lambda_i - \alpha)|, \quad 1 \le i < j \le n.$$

3) There are constants K_3, α *such that, for all complex* s *with* $\text{Re}\, s > \alpha$,

$$|\left(sI - \hat{P}(i\omega)\right)^{-1}| \le \frac{K_3}{\text{Re}\, s - \alpha}. \tag{5.10}$$

4) There are constants K_4, α *and, for every* $\hat{P}(i\omega)$, *a positive definite Hermitian matrix* $\hat{H} = \hat{H}(i\omega)$ *with*

$$K_4^{-1} I \le \hat{H} \le I K_4 \tag{5.11a}$$

such that

$$\hat{H}\hat{P}^* + \hat{P}^*\hat{H} \le 2\alpha\,\hat{H}. \tag{5.11b}$$

If we know an α for any of the above conditions, then this α can be chosen for all conditions. This is not true for the K_i. However, if one of them is known, then the other can be estimated.

We shall now connect these results with the stability results for ordinary differential equations in Section 2. Immediately, we have

Theorem 5.3. *The initial value problem (5.1) is linearly stable if it is well posed and $\alpha < 0$ in (5.3).*

Proof. (5.7) and Parseval's relation (5.5) show that the solution of the linear problem (5.1) with $\varepsilon = 0$ anf $F \equiv 0$ converges exponentially to zero. If $F \not\equiv 0$ and $\lim_{t\to\infty} \|F(\cdot, t)\| = 0$, then we also obtain convergence by Duhamel's principle.

We shall now connect α with the eigenvalue condition. Introducing into the homogeneous linear equation (5.1) the ansatz

$$u = e^{\lambda t}\varphi(x)$$

gives us

$$\lambda\varphi = P_0(\frac{\partial}{\partial x})\varphi. \tag{5.12}$$

Fourier transform shows that the eigenfunctions are

$$\varphi = e^{i\langle\omega,x\rangle}\varphi_0$$

and the eigenvalues satisfy

$$(\lambda I - \hat{P}(i\omega))\varphi_0 = 0, \tag{5.13}$$

i.e., the eigenvalues of the differential operator are the eigenvalues of the symbol $\hat{P}(i\omega)$.

We have

Theorem 5.4. *Assume that the linear problem (5.2) is well posed and that the eigenvalue condition is satisfied, i.e., there is a constant* $\delta > 0$ *such that, for all* ω, *the eigenvalues of (5.13) satisfy*

$$\text{Re}\,\lambda \leq -\delta. \tag{5.14}$$

Therefore, the problem (5.1) is linearly stable.

Proof. We use the the second condition of Theorem 5.2. Wellposedness tells us that there is a fixed constant α such that

$$|b_{ij}| \leq K_2 |\text{Re}(\lambda_i - \alpha)|$$

holds for all frequencies ω. Since

$$\begin{aligned} |\text{Re}(\lambda_i - \alpha)| = |\text{Re}\,\lambda_i| \Big(1 + \frac{\alpha}{|\text{Re}\,\lambda_i|}\Big) &\leq 2(1 + \frac{\alpha}{\delta}) |\text{Re}\,\frac{\lambda_i}{2}| \\ &\leq 2(1 + \frac{\alpha}{\delta}) |\text{Re}\,\lambda_i - \frac{\delta}{2}|, \end{aligned}$$

we also have

$$|b_{ij}| \leq \tilde{K}_2 |\text{Re}(\lambda_i - \frac{\delta}{2})|, \quad \tilde{K}_2 = 2(1 + \frac{\alpha}{\delta}) K_2.$$

Thus, we can choose $\alpha = -\frac{1}{2}\delta$ and apply Theorem 5.3.

Now consider (5.1) with $\varepsilon = 0$ and $u_0(x) \equiv 0$ and Fourier transform it with respect to x and Laplace transform it with respect to time. The transformed system has the form

$$\Big(sI - \hat{P}(i\omega)\Big)\hat{u} = \hat{F}.$$

If the problem is well posed and $\alpha < 0$, it follows from (5.10) that, for $\text{Re}\,s = 0$,

$$|\hat{u}| \leq \frac{K_1}{|\alpha|} |\hat{F}|. \tag{5.15}$$

Therefore, by Parseval's relation,

$$\int_0^T \|u(\cdot,t)\|^2 dt \le \frac{K_1^2}{|\alpha|^2}\int_0^T \|F(\cdot,t)\|^2 dt.$$

Differentiating (5.1) with respect to the space variable we obtain

$$\int_0^T \|D^\nu u(\cdot,t)\|^2 dt \le \frac{K_1^2}{|\alpha|^2}\int_0^T \|D^\nu F(\cdot,t)\|^2 dt.$$

Here D^ν stands for any space derivative. As in the previous sections, we can use these relations to obtain estimates in the maximum norm and discuss nonlinear perturbations. However, in general, we do not "gain" any derivatives. Therefore, the nonlinear perturbations which we can handle are of zero order, i.e.,

$$P_1(x,t,u,\frac{\partial}{\partial x})u = B(x,t,u)\,u.$$

As we have seen in the last section, we can do better for parabolic partial differential equations. In this case m is even and, for linearly stable problems, there is a constant $\delta > 0$ such that

$$\text{Re}\,\lambda \le -\delta(|\omega|^m + 1).$$

In this case we obtain, instead of (5.15),

$$\delta(|\omega|^m + 1)|\hat{u}| \le K_1|\hat{F}|$$

which leads to

$$\int_0^T \sum_{|\nu|\le m} \|D^\nu u(\cdot,t)\|^2 dt \le \tilde{K}_1 \int_0^T \|F\|^2 dt.$$

Thus, we gain m derivatives and, therefore, the admissible perturbations can be any nonlinear differential operator of order m.

For hyperbolic partial differential equations, the estimate (5.15) is sharp and only zero order perturbations are allowed when using the resolvent technique.

We consider the fourth condition of Theorem 5.2. The matrices $\hat{H}$ define an operator

$$Hu =: \sum e^{i(\omega,x)} \hat{H}(i\omega)\hat{u}(\omega).$$

By (5.11a) and Parseval's relation,

$$(u, Hu) = \sum_{\omega} \langle \hat{u}(\omega), \hat{H}(i\omega)\hat{u}(\omega)\rangle = \sum_{\omega} |\hat{H}^{1/2}\hat{u}(\omega)|^2.$$

Since

$$K_4^{-1}|\hat{u}(\omega)|^2 \le |\hat{H}^{-1}|^{-1}|\hat{u}(\omega)|^2 \le |\hat{H}^{1/2}\hat{u}(\omega)|^2 \le |\hat{H}|\;|\hat{u}(\omega)|^2 \le K_4|\hat{u}(\omega)|^2,$$

we have, by Parseval's relation,

$$K_4^{-1}\|u\|^2 \le (u, Hu) \le K_4\|u\|^2.$$

Therefore, (u, Hv) defines a scalar product which is equivalent with the L_2 scalar product and norm. Also, as for the ordinary differential equations, the linear problem (5.2) becomes a contraction in the new norm if $\alpha < 0$. We have

$$\begin{aligned}(u, Hu)_t &= (u, HP_0u) + (P_0u, Hu) = \sum_{\omega} \langle \hat{u}(\omega), (\hat{H}\hat{P}_0 + \hat{P}_0^*\hat{H})\hat{u}(\omega)\rangle \\ &\le -2|\alpha|(u, Hu).\end{aligned} \tag{5.16}$$

In general, one can only treat zero order nonlinear perturbations. However, for parabolic systems, one can again treat perturbations of order m. As we have already mentioned in the last section, first order perturbations of hyperbolic systems can be treated if one uses a H-norm which depends on the perturbation.

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