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THE EXISTENCE OF FINITE ELEMENT MINIMIZERS

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ABSTRACT. We present a general theorem on the existence of finite element minimizers for the approximation of variational problems of multiple integrals. Our theorem applies to variational problems for which a minimum does not exist in infinite-dimensional spaces of functions. Such problems occur in models for microstructure in martensitic and ferromagnetic crystals.

1. INTRODUCTION

Let $n \geq 1$ be an integer and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Let $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a measurable function, where $m \geq 1$ is an integer and $\mathbb{R}^{m\times n}$ denotes the set of all real $m \times n$ matrices. For a suitably smooth function $u : \Omega \to \mathbb{R}^m$, we denote

$$I(u) := \int_\Omega F(x, u(x), \nabla u(x))\, dx, \tag{1.1}$$

where $\nabla u : \Omega \to \mathbb{R}^{m\times n}$ is the gradient of u. Suppose the function F is bounded below and satisfies a suitable growth condition at infinity with respect to its last two variables. Then, the existence of minimizers of the functional I on a set of admissible functions u in some Sobolev space is guaranteed if the functional I is sequentially lower semi-continuous with respect to the weak (or weak-star) topology of the underlying Sobolev space [14]. It turns out that the sequential weak lower semi-continuity of the functional I is, with some technical assumptions, equivalent to the *quasi-convexity* of $F(x, u, \cdot) : \mathbb{R}^{m\times n} \to \mathbb{R}$ for each $(x, u) \in \Omega \times \mathbb{R}^m$ [1, 5, 14, 29, 30, 31].

However, there are some kinds of variational integrals in which the function F are not quasi-convex. Consequently, there may not exist minimizers for the corresponding functionals over sets of admissible functions. An interesting example is the elastic energy functional modeling a crystalline solid capable

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of undergoing certain solid-to-solid phase transformation. Such an energy functional is defined by

$$\mathcal{E}(y) := \int_{\Omega} \phi(\nabla y(x))\, dx, \tag{1.2}$$

in which $m = n = 3$, $\Omega \subset \mathbb{R}^3$ represents a homogeneous reference configuration of the underlying crystal, $y : \Omega \longrightarrow \mathbb{R}^3$ is a deformation and $\nabla y : \Omega \longrightarrow \mathbb{R}^{3\times3}$ is the deformation gradient, and $\phi : \mathbb{R}^{3\times3} \longrightarrow \mathbb{R}$ is the elastic energy density. Typically, due to the rotational invariance the elastic energy density ϕ is minimized on several energy wells $\mathcal{U}_i := \mathrm{SO}(3)U_i, i = 1, \dots, N$, where $\mathrm{SO}(3) \subset \mathbb{R}^{3\times3}$ is the set of all proper rotations, each $U_i \in \mathbb{R}^{3\times3}$ is a symmetric, positive definite matrix representing a phase (or a variant of a phase), that is, a stable homogeneous transformation of the crystal, and where $N > 1$. Moreover, these energy wells are pairwise compatible in the sense that for any pair of indices i and j with $i \neq j$, there are rotations $R_i, R_j \in \mathrm{SO}(3)$ such that the difference $R_iU_i - R_jU_j$ is a rank-one matrix [3, 4, 10, 16, 17, 18, 20, 22, 26]. The energy density $\phi : \mathbb{R}^{3\times3} \longrightarrow \mathbb{R}$ is therefore not rank-one convex, hence not quasi-convex. We refer to [3, 4, 23] for the non-existence of energy minimizers for the energy functional $\mathcal{E}$ on a set of admissible deformations which satisfy a Dirichlet boundary condition.

When the finite element method is applied to approximate the variational integral (1.1), minimizers are sought in spaces of admissible finite element functions. If the underlying variational problem has a unique minimizer in an admissible set of functions, these finite element minimizers are approximations of such a minimizer. Otherwise, the finite element minimizers corresponding to a sequence of meshes with vanishing mesh sizes form a minimizing sequence which can serve as a "solution" to the original variational problem no matter whether it permits a minimizer or not [7, 8, 9, 13, 19, 23, 24, 25, 27, 28].

Since each finite element space is finite-dimensional, the existence of finite element minimizers for the functional I defined by (1.1) does not depend on the quasi-convexity of the function $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ with respect to its last variable. Under less general conditions, such existence is proved in [7, 8, 9] for the approximation by the piecewise linear element of a non-convex variational problem of multiple integrals modeling oscillation. It is also proved in [13, 23, 24] and [25] for the conforming and non-conforming, respectively, finite element approximations of the energy functional (1.2) with a double-well or triple-well energy density $\phi : \mathbb{R}^{3\times3} \longrightarrow \mathbb{R}$ that satisfies an explicit quadratic growth condition at infinity.

In this paper, we prove a general theorem (Theorem 2.1) on the existence of finite element minimizers in the approximation of the variational integral (1.1) by any kind of finite elements for the function $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ that satisfies a lower boundedness condition and a growth property at infinity.

We remark that when the admissible set for the variational integral (1.1) consists of pointwise constrained functions such as those representing a harmonic mapping, a magnetization field, or an orientation-preserving deformation, the proof of the existence of finite element minimizers can be trivial in one case but nontrivial in another. See [28] for the finite element approximation in micromagnetics, and the recent work [13] for a proof of the existence of a minimizer for the functional (1.2) with a double-well density ϕ among a class of finite element deformations whose gradients have determinant bounded below by a small, positive constant.

2. The Existence Theorem

For simplicity of exposition, let us fix a function $g \in L^\infty(\partial\Omega; \mathbb{R}^m)$, and define the set of admissible functions

$$\mathcal{A} := \left\{ u \in W^{1,\infty}(\Omega; \mathbb{R}^m) : u = g \text{ on } \partial\Omega \right\}.$$

We consider the finite element approximation of the following variational problem:

$$\inf \{ I(u) : u \in \mathcal{A} \}, \tag{2.1}$$

where the functional I is defined by (1.1).

To this end, we let $\mathcal{T}_h$ be a finite element mesh with mesh size $h > 0$ of the underlying domain Ω so that the closure $\bar{\Omega}_h = \cup_{K \in \mathcal{T}_h} K$ is an approximation of the closure $\bar{\Omega}$, where Ω_h is the interior of $\cup_{K \in \mathcal{T}_h} K$. To cover general finite element partitions, we assume that [6, 11, 32]:

(a) $\mathcal{T}_h$ is a non-empty and finite collection of subsets of $\bar{\Omega}$;
(b) Each $K \in \mathcal{T}_h$ is the closure of a non-empty, open subset of Ω, the boundary of the subset being Lipschitz continuous;
(c) For any K_1 and K_2 in $\mathcal{T}_h$, either $K_1 = K_2$ or $\text{int}\,(K_1) \cap \text{int}\,(K_2) = \emptyset$, where $\text{int}\,(\omega)$ denotes the interior of a subset $\omega \subseteq \bar{\Omega}$.

By (b), we have $\Omega_h \subseteq \Omega$.

We denote by V_h a finite element space associated with the mesh $\mathcal{T}_h$, and assume that [6, 11, 32]:

(d) V_h is a finite-dimensional subspace of $L^\infty(\Omega_h)$;
(e) For each $K \in \mathcal{T}_h$, the restriction of V_h on K,

$$V_h(K) := \{ v_h|_K : v_h \in V_h \},$$

is a finite-dimensional subspace of $C^1(K)$, the space of continuously differentiable functions.

We will always identify any function $v_h \in V_h \subset L^\infty(\Omega_h)$ with a function whose restriction on each element $K \in \mathcal{T}_h$ is continuously differentiable. To treat

Dirichlet boundary conditions, we define

$$V_{h0} := \{v_h \in V_h : v_h = 0 \text{ on } \partial\Omega_h\},$$

where we recall $\Omega_h = \operatorname{int}(\cup_{K\in\tau_h} K)$ and where the identity $v_h = 0$ on $\partial\Omega_h$ means that $v_h|_K = 0$ on $K \cap \partial\Omega_h$, whenever the intersection is non-empty for $K \in \tau_h$. We denote for any $v_h \in V_h$

$$\|v_h\|_{1,\infty,h} := \max_{K\in\tau_h} \|v_h\|_{W^{1,\infty}(K)}, \qquad |v_h|_{1,\infty,h} := \max_{K\in\tau_h} \|\nabla v_h\|_{L^\infty(K;\mathbb{R}^n)}.$$

Obviously, $\|\cdot\|_{1,\infty,h}$ and $|\cdot|_{1,\infty,h}$ are a norm and a semi-norm, respectively, of both the spaces V_h and V_{h0}. We assume that the following Poincaré type inequality holds true:

(f) There exists a constant $C_h > 0$ such that

$$\|v_h\|_{L^\infty(\Omega_h)} \le C_h |v_h|_{1,\infty,h}, \qquad \forall v_h \in V_{h0}. \tag{2.2}$$

Now let us denote by $\mathcal{V}_h$ and $\mathcal{V}_{h0}$ the corresponding spaces of vector-valued finite element functions:

$$\mathcal{V}_h := \left\{v_h = \left(v_h^1, \ldots, v_h^m\right) : v_h^i \in V_h,\ i = 1, \ldots, m\right\},$$

and

$$\mathcal{V}_{h0} := \left\{v_h = \left(v_h^1, \ldots, v_h^m\right) : v_h^i \in V_{h0},\ i = 1, \ldots, m\right\}.$$

It is clear that $\mathcal{V}_{h0}$ is a subspace of $\mathcal{V}_h$ and that both $\mathcal{V}_h$ and $\mathcal{V}_{h0}$ are finite-dimensional subspaces of $L^\infty(\Omega_h;\mathbb{R}^m)$. If $v_h = (v_h^1, \ldots, v_h^m) \in \mathcal{V}_h$, we denote

$$\|v_h\|_{1,\infty,h} := \max_{1\le i\le m} \max_{K\in\tau_h} \left\|v_h^i\right\|_{W^{1,\infty}(K)}$$

and

$$|v_h|_{1,\infty,h} := \max_{1\le i\le m} \max_{K\in\tau_h} \left\|\nabla v_h^i\right\|_{L^\infty(K;\mathbb{R}^n)}.$$

These notation $\|\cdot\|_{1,\infty,h}$ and $|\cdot|_{1,\infty,h}$ are used for both the spaces V_h and $\mathcal{V}_h$. They are also a norm and a seminorm, respectively, of both the spaces $\mathcal{V}_h$ and $\mathcal{V}_{h0}$. Moreover, it follows from (2.2) that we have the Poincaré type inequality

$$\|v_h\|_{L^\infty(\Omega_h;\mathbb{R}^m)} \le C_h |v_h|_{1,\infty,h}, \qquad \forall v_h \in \mathcal{V}_{h0}, \tag{2.3}$$

where $C_h > 0$ is the same constant as in (2.2) and

$$\|v_h\|_{L^\infty(\Omega_h;\mathbb{R}^m)} := \max_{1\le i\le m} \left\|v_h^i\right\|_{L^\infty(\Omega_h)}.$$

We now let $g_h \in \mathcal{V}_h$ be a vector-valued finite element function which approximates $g \in L^\infty(\partial\Omega;\mathbb{R}^3)$. We then define the corresponding set of admissible, vector-valued finite element functions by

$$\mathcal{V}_{g_h} := \{v_h \in \mathcal{V}_h : v_h = g_h \text{ on } \partial\Omega_h\},$$

where again by the identity $v_h = g_h$ on $\partial\Omega_h$ it is understood that $v_h|_K = g_h|_K$ on $K \cap \partial\Omega_h$, whenever the intersection is non-empty for $K \in \tau_h$. Notice that $\mathcal{V}_{g_h} = g_h + \mathcal{V}_{h0}$. Finally, we define the approximate integral

$$(2.4) \qquad I_h(v_h) := \sum_{K \in \tau_h} \int_K F(x, v_h(x), \nabla v_h(x))\, dx, \qquad v_h \in \mathcal{V}_h.$$

The finite element approximation problem is now to minimize the integrals $I_h(v_h)$ among all $v_h \in \mathcal{V}_{g_h}$.

Several remarks are now in order.

1. The space $W^{1,\infty}(\Omega; \mathbb{R}^m)$ in the definition of the set of admissible functions $\mathcal{A}$ can be replaced by other Sobolev spaces. For instance, if the function F satisfies certain growth condition with respect to its last two variables, one can define $\mathcal{A}$ to be the set of all functions $u \in L^p(\Omega; \mathbb{R}^m)$ with $\nabla u \in L^q(\Omega; \mathbb{R}^{m\times n})$ that satisfy the Dirichlet boundary condition $u = g$ on $\partial\Omega$ with $g \in L^r(\partial\Omega; \mathbb{R})$ for some suitable $p, q, r \in [1, \infty]$, see [1, 2, 5, 14, 30, 31]. But the choice of the set of admissible functions $\mathcal{A}$ for the continuum problem (2.1) will essentially not affect the finite element approximation.
2. With regards to the existence of finite element minimizers, one can easily treat a boundary condition which is more general than the Dirichlet condition in the definition of the set of admissible functions $\mathcal{A}$ [2, 12].
3. Our definition of a finite element mesh and that of a finite element space are different from but more general than the standard ones [6, 11, 32]. We have covered all kinds of practically possible meshes and finite element (conforming and non-conforming, Lagrange type and Hermite type) spaces.
4. The Poincaré type inequality (2.2) is obviously true for a conforming finite element space V_h, since by definition $V_h \subset W^{1,\infty}(\Omega_h)$ in this case [11]. For some non-conforming finite element spaces, such an inequality can also be established [21].
5. The approximate boundary condition in the definition of the set of admissible finite element functions $\mathcal{V}_{g_h}$ can be replaced by a different one when we apply a finite element whose degrees of freedom are not nodal values of functions [21].

We recall that a function $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is called a Carathéodory function if $F(x, \cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous for almost all $x \in \Omega$ and $F(\cdot, u, \xi) : \Omega \to \mathbb{R}$ is measurable for each $(u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{m\times n}$ [15]. It is obvious that any continuous function $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function.

We are now ready to state the existence theorem for the finite element approximation of the variational problem (2.1).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, and let all of the τ_h, V_h, V_{h0}, $\mathcal{V}_h$, $\mathcal{V}_{h0}$, g_h, and $\mathcal{V}_{g_h}$ be defined as above. Assume that conditions (a) – (f) are satisfied.*

Suppose that $F : \Omega \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function satisfying the following two conditions:

1. *Lower boundedness.*

$$f \in L^1(\Omega), \qquad \text{where} \quad f(x) := \inf_{(u,\xi)\in\mathbb{R}^m\times\mathbb{R}^{m\times n}} F(x,u,\xi), \quad \forall x \in \Omega; \tag{2.5}$$

2. *Growth at infinity.*

$$\lim_{|\xi|\to\infty} \inf_{u\in\mathbb{R}^m} F(x,u,\xi) = +\infty, \qquad a.e.\ x \in \Omega. \tag{2.6}$$

Then there exists $u_h \in \mathcal{V}_{g_h}$ such that

$$I_h(u_h) = \inf_{v_h\in\mathcal{V}_{g_h}} I_h(v_h),$$

where I_h is defined by (2.4).

We remark that our conditions on the lower boundedness and the growth at infinity for the function $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ are weaker than those in the existence theorems in the calculus of variations [1, 2, 5, 14, 30, 31]. If these conditions are not satisfied, then the corresponding functional I_h may take the value $-\infty$ or I_h may not attain its minimum in a set of admissible finite element functions. This can be seen from the following two examples.

1. Let $m = n = 1$, $\Omega = (0,1)$, and $F(x,u,\xi) := x^{-4}\left(u^2 + \xi^2 - 1\right)$. Then, $F : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies the growth condition (2.6). But it is easy to see that the corresponding integral functional I defined by (1.1) and its finite element approximation I_h defined by (2.4) can be $-\infty$.
2. Let $m = n = 1$ and $\Omega = (-1,1)$. Define $F(x,u,\xi) := \exp(-|\xi|)$. Thus, the function $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the boundedness condition (2.5). Let τ_h consist of only two elements $(-1,0)$ and $(0,1)$. Let V_{h0} be the corresponding set of piecewise linear polynomials which vanish at $x = -1$ and $x = 1$. Such a polynomial v_h is completely determined by the value $v_0 := v_h(0)$. A simple calculation shows that $I_h(v_h) = 2\exp(-|v_0|)$ which is always positive. Moreover, the infimum of I_h over V_{h0} is 0. But it can never be attained in V_{h0}.

As a direct consequence of Theorem 2.1, we have the following corollary for the existence of finite element minimizers in the approximation of the energy functional $\mathcal{E}$ defined by (1.2) [13, 23, 24, 25, 27].

Corollary 2.1. *Let $\Omega, \tau_h, V_h, V_{h0}, \mathcal{V}_h, \mathcal{V}_{h0}, g_h$, and $\mathcal{V}_{g_h}$ satisfy the conditions of Theorem 2.1. Suppose that $\phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a continuous function such that*

$$\lim_{|\xi|\to\infty} \phi(\xi) = +\infty.$$

Then there exists $y_h \in \mathcal{V}_{g_h}$ such that

$$\mathcal{E}_h(y_h) = \inf_{z_h\in\mathcal{V}_{g_h}} \mathcal{E}_h(z_h),$$

where

$$\mathcal{E}_h(z_h) := \sum_{K\in\tau_h} \int_K \phi(\nabla z_h(x))\,dx, \qquad \forall z_h \in \mathcal{V}_h.$$

3. Proof of the Existence Theorem

To prove Theorem 2.1, we need several lemmas. The first one is elementary but we give a proof for completeness.

Lemma 3.1. *Let X be a real, finite-dimensional vector space and $J : X \to (-\infty, \infty]$ a lower semi-continuous functional* [14, 15]. *Suppose*

$$\lim_{\|x\|\to\infty} J(x) = +\infty. \tag{3.1}$$

Then there exists $x_0 \in X$ such that

$$J(x_0) = \inf_{x\in X} J(x). \tag{3.2}$$

Proof. Let $\alpha := \inf_{x\in X} J(x)$. If $\alpha = +\infty$, then (3.2) is true for any $x_0 \in X$. Suppose $\alpha < +\infty$. Let $\{x_k\}$ be a sequence of points in X such that

$$\lim_{k\to\infty} J(x_k) = \alpha.$$

By (3.1), $\{x_k\}$ must be bounded. Since X is finite-dimensional, there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that

$$\lim_{j\to\infty} x_{k_j} = x_0$$

for some $x_0 \in X$. It then follows from the lower semi-continuity of $J : X \to (-\infty, \infty]$ that

$$\alpha = \liminf_{j\to\infty} J(x_{k_j}) \ge J(x_0) \ge \alpha,$$

implying (3.2). □

The following lemma follows from the uniform continuity of a continuous function defined on a compact metric space and the equivalence of norms on finite-dimensional vector spaces.

Lemma 3.2. *Given any $K \in \tau_h$ and any $\epsilon > 0$, there exists $\delta = \delta(K, \epsilon) > 0$ such that if x, $y \in K$ and $|x - y| < \delta$, then*

$$|\nabla v_h(x) - \nabla v_h(y)| \le \epsilon \|\nabla v_h\|_{L^\infty(K;\mathbb{R}^n)}, \qquad \forall v_h \in V_h(K). \tag{3.3}$$

Proof. Let $\{w_1, \dots, w_M\}$ be a basis for the finite-dimensional space

$$\nabla V_h(K) := \{\nabla v_h \in C(K; \mathbb{R}^n) : v_h \in V_h(K)\}.$$

We define the function $W : K \times [-1, 1]^M \to \mathbb{R}^n$ by

$$W(x, \mu) = \sum_{i=1}^{M} \mu_i w_i(x), \qquad x \in K, \ \mu = (\mu_1, \dots, \mu_M) \in [-1, 1]^M.$$

Obviously, W is a continuous function defined on a compact metric space. Hence, we have for any $\hat{\epsilon} > 0$ that there exists $\delta = \delta(K, \hat{\epsilon}) > 0$ such that if

$$|x - y| + |\mu - \nu| < \delta, \qquad x, \, y \in K \text{ and } \mu, \, \nu \in [-1, 1]^M,$$

then

$$|W(x, \mu) - W(y, \nu)| \le \hat{\epsilon}. \tag{3.4}$$

We obtain from (3.4) with $\mu = \nu$ and from scaling that

$$\left| \sum_{i=1}^{M} \mu_i w_i(x) - \sum_{i=1}^{M} \mu_i w_i(y) \right| \le \hat{\epsilon} \max_{1 \le i \le M} |\mu_i|, \qquad \forall \mu \in \mathbb{R}^M, \tag{3.5}$$

for any $x, y \in K$ such that $|x - y| < \delta$.

It next follows from the equivalence of norms on finite-dimensional vector spaces that there exists $\zeta > 0$ such that

$$\max_{1 \le i \le M} |\mu_i| \le \zeta \left\| \sum_{i=1}^{M} \mu_i w_i \right\|_{L^\infty(K;\mathbb{R}^n)}, \qquad \forall \mu \in \mathbb{R}^M. \tag{3.6}$$

The result (3.3) now follows from (3.5) and (3.6) with $\hat{\epsilon}$ chosen to be ϵ/ζ. $\square$

For any $K \in \tau_h$, we denote by $\mathcal{V}_h(K)$ the restriction of $\mathcal{V}_h$ on K. If $\beta \in [0, 1]$ and $v_h \in \mathcal{V}_h(K)$, we also denote

$$K(v_h; \beta) := \{x \in K : |\nabla v_h(x)| \ge \beta |v_h|_{1,\infty,K}\}.$$

The following lemma is a consequence of Lemma 3.2.

Lemma 3.3. *Given any $K \in \tau_h$ and any $\beta \in (0, 1)$, there exists $\gamma = \gamma(K, \beta) > 0$ such that the subset $K(v_h; \beta)$ contains a ball in K of radius γ for any $v_h \in \mathcal{V}_h(K)$.*

Proof. Let $\epsilon = 1 - \beta > 0$. By Lemma 3.2, there exists $\delta = \delta(K, \beta) > 0$ such that for any $x, y \in K$ with $|x - y| < \delta$ and any $v_h = (v_h^1, \ldots, v_h^m) \in \mathcal{V}_h(K)$,

$$\max_{1\le i\le m} \left|\nabla v_h^i(x) - \nabla v_h^i(y)\right| \le (1-\beta) \left|v_h\right|_{1,\infty,K}, \tag{3.7}$$

where $|v_h|_{1,\infty,K} := \|\nabla v_h\|_{L^\infty(K;\mathbb{R}^{m\times n})}$.

Suppose $w_h = (w_h^1, \ldots, w_h^m) \in \mathcal{V}_h(K)$ is such that

$$\max_{1\le i\le m} \left|\nabla w_h^i(x)\right| = |w_h|_{1,\infty,K}$$

for some point $x \in K$. Then, it follows from (3.7) that for any $y \in Q_{x,\delta} := \{z \in K : |x - z| < \delta\}$, we have

$$\begin{aligned}\max_{1\le i\le m} \left|\nabla w_h^i(y)\right| &\ge \max_{1\le i\le m} \left\{\left|\nabla w_h^i(x)\right| - \left|\nabla w_h^i(x) - \nabla w_h^i(y)\right|\right\} \\ &\ge \left|w_h\right|_{1,\infty,K} - (1-\beta)\left|w_h\right|_{1,\infty,K} \\ &= \beta \left|w_h\right|_{1,\infty,K}.\end{aligned}$$

Consequently, $Q_{x,\delta} \subseteq K(w_h; \beta)$. Since the boundary of K is Lipschitz continuous, there exists a constant $\gamma = \gamma(K, \beta) > 0$ such that for any $x \in K$, $Q_{x,\delta}$ contains a ball of radius γ. This proves the assertion of the lemma. □

The last lemma is a consequence of the previous lemma and the boundedness of the underlying domain Ω.

Lemma 3.4. *Let $K \in \tau_h$ and $\beta \in (0, 1)$. Suppose $\{u_k\}$ is a sequence of functions in $\mathcal{V}_h(K)$. Then there exist a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a fixed cube $Q \subseteq K$ such that $Q \subseteq K(u_{k_j}; \beta)$ for all $j \ge 1$.*

Proof. By Lemma 3.3, there exists $\gamma = \gamma(K, \beta) > 0$ such that each set $K(u_k; \beta)$ contains a ball of radius γ. The boundedness of K implies that $K \subseteq [-a, a]^n$ for some $a > 0$. Divide the cube $[-a, a]^n$ into 2^{ns} small cubes with the size of each side $2a/2^s$, where $s \ge 1$ is a fixed integer. If we choose s large enough, then for each $k \ge 1$ the set $K(u_k; \beta)$ will contain at least one of those small cubes. Therefore, there exist a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a fixed small cube $Q \subseteq K$ such that $Q \subseteq K\left(u_{k_j}; \beta\right)$ for all $j \ge 1$. □

We now give the proof of the main theorem.

Proof of Theorem 2.1. Since $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function, for any $v_h \in \mathcal{V}_h$ and any $K \in \tau_h$, the function $x \to F(x, v_h(x), \nabla v_h(x))$ is measurable on K (cf. [15], pp. 232–234). Recall that $g_h \in \mathcal{V}_h$. By the definition of I_h (2.4) and the lower boundedness (2.5), $J_h(v_h) := I_h(g_h + v_h)$ defines a functional $J_h : \mathcal{V}_{h0} \to (-\infty, \infty]$. It follows from the fact that $\mathcal{V}_{g_h} = g_h + \mathcal{V}_{h0}$ that we need only to show that the functional $J_h : \mathcal{V}_{h0} \to (-\infty, \infty]$ attains its minimum in $\mathcal{V}_{h0}$.

Let $\{v_k\}$ be a sequence of functions in $\mathcal{V}_{h0}$ that converge to $v \in \mathcal{V}_{h0}$ in the norm $\|\cdot\|_{1,\infty,h}$. Since $F(x,\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous for a.e. $x \in \Omega$, we have

$$\lim_{k\to\infty} F(x, g_h(x) + v_k(x), \nabla g_h(x) + \nabla v_k(x)) \\ = F(x, g_h(x) + v(x), \nabla g_h(x) + \nabla v(x)), \qquad \text{a.e. } x \in \Omega_h.$$

Due to the lower boundedness (2.5), for each $K \in \tau_h$, all the measurable functions

$$F(x, g_h(x) + v_k(x), \nabla g_h(x) + \nabla v_k(x)), \qquad x \in K,$$

for $k \geq 1$ are bounded below by an integrable function. Applying Fatou's Lemma [33] we can conclude by the definition of J_h that

$$\liminf_{k\to\infty} J_h(v_k) \geq J_h(v).$$

This proves that the functional $J_h : \mathcal{V}_{h0} \to (-\infty, \infty]$ is lower semi-continuous.

Assume now that $w_k \in \mathcal{V}_{h0}$ for $k = 1, \ldots$ satisfies the condition

$$\lim_{k\to\infty} \|w_k\|_{1,\infty,h} = \infty.$$

By the Poincaré type inequality (2.3), this implies that

$$\lim_{k\to\infty} |w_k|_{1,\infty,h} = \infty. \tag{3.8}$$

We want to show that

$$\lim_{k\to\infty} J_h(w_k) = +\infty. \tag{3.9}$$

To this end, we let $\{w_{k_j}\}$ be any subsequence of $\{w_k\}$. Since $\bar{\Omega}_h = \cup_{K\in\tau_h} K$ and τ_h is a finite collection of subsets of $\bar{\Omega}_h$, there exist a fixed $K \in \tau_h$ and a further subsequence of $\{w_{k_j}\}$, still denoted by $\{w_{k_j}\}$, such that

$$\left|w_{k_j}\right|_{1,\infty,h} = \left|w_{k_j}\right|_{1,\infty,K}, \qquad \forall j \geq 1. \tag{3.10}$$

By Lemma 3.4, there exists a fixed cube $Q \subseteq K$ such that $Q \subseteq K\left(w_{k_j}; \frac{1}{2}\right)$ for all $j \geq 1$. Consequently, we have by the definition of $K\left(w_{k_j}; \frac{1}{2}\right)$, (3.8), and (3.10) that

$$\lim_{j\to\infty} \left|\nabla w_{k_j}(x)\right| = \infty, \qquad x \in Q.$$

This, together with the growth condition (2.6), implies that

$$\lim_{j\to\infty} F(x, g_h(x) + w_{k_j}(x), \nabla g_h(x) + \nabla w_{k_j}(x)) = +\infty, \qquad \text{a.e. } x \in Q. \tag{3.11}$$

By the fact that meas$\,(Q) > 0$, we infer from the lower boundedness (2.5) and (3.11) that

$$J_h(w_{k_j}) \geq -\int_{\Omega\setminus Q} |f(x)|\, dx + \int_Q F(x, g_h(x) + w_{k_j}(x), \nabla g_h(x) + \nabla w_{k_j}(x))\, dx$$

$$\to \infty, \qquad \text{as } j \to \infty.$$

We have in fact proved (3.9), since any subsequence of $\{w_k\}$ has a further subsequence $\{w_{k_j}\}$ such that $J_h(w_{k_j}) \to +\infty$ as $j \to \infty$.

Now the conclusion of Theorem 2.1 directly follows from Lemma 3.1 with $X = \mathcal{V}_{h0}$ equipped with the norm $\|\cdot\|_{1,\infty,h}$ and $J = J_h$. □

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