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Tony F. Chan
Pep Mulet

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90095-1555

ON THE CONVERGENCE OF THE LAGGED DIFFUSIVITY FIXED POINT METHOD IN TOTAL VARIATION IMAGE RESTORATION *

TONY F. CHAN† AND PEP MULET‡

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Abstract. In this paper we show that the *lagged diffusivity fixed point* algorithm introduced by Vogel and Oman in [10] to solve the problem of Total Variation denoising, proposed by Rudin, Osher and Fatemi in [9], is a particular instance of a class of algorithms introduced by Eckhardt and Voss in [11], whose origins can be traced back to Weiszfeld's original work for minimizing a sum of Euclidean lengths [12]. There have recently appeared several proofs for the convergence of this algorithm [2], [3], [6]. Here we present a proof of the global and linear convergence using the framework introduced in [11] and give a bound for the convergence rate of the fixed point iteration that agrees with our experimental results. These results are also valid for suitable generalizations of the fixed point algorithm.

1. Introduction. Recently, a new class of nonlinear PDE based techniques has emerged for image restoration problems, primarily because they preserve sharp edges better. A particularly popular technique is the *Total Variation* (TV) restoration method, introduced by Rudin, Osher and Fatemi in [9]. This can be posed as a variational problem, which gives a highly nonlinear Euler-Lagrange equation. In [9], the solution of this equation was originally obtained as the steady state solution of an auxiliary time dependent equation which was numerically integrated by an explicit method. Because of stability constraints, this algorithm can be slowly convergent. To improve the convergence behavior, Vogel and Oman proposed in [10] the use of a linearization technique for this nonlinear equation, resulting in the *lagged diffusivity fixed point algorithm* (FP). This idea is quite commonly used in other PDE applications, e.g. CFD and our theory may have applications there as well. The implementation of this iterative method amounts to solving a linear elliptic equation at each iteration. Experiments show that this algorithm is very robust and globally and linearly convergent.

Although faster convergent methods have been developed since the appearance of FP (e.g. [5]), it remains one of the most efficient and robust methods for Total Variation image restoration, and thus its convergence theory has attracted some attention in the literature. In particular, there have recently appeared several convergence proofs [2], [3], [6]. The main idea in these proofs is based on the introduction of an auxiliary variable v and alternatively minimizing an associated *half-quadratic functional* (see [8] for details) in the original variable u and the new variable v.

In this paper, we present a different proof of the global and linear convergence for the discrete version of this algorithm. We view our contributions as follows:

1. We point out that the fixed point algorithm is a generalized version of an algorithm proposed by Weiszfeld in 1937 for minimizing a sum of Euclidean

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†Department of Mathematics, University of California, Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90024-1555. E-mail address: chan@math.ucla.edu. URL: http://www.math.ucla.edu/~chan.

‡Department of Mathematics, University of California, Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90024-1555 and Departament de Matemàtica Aplicada, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, Spain. E-mail address: mulet@math.ucla.edu and mulet@uv.es. URL: http://www.math.ucla.edu/~mulet. Supported partially by DGICYT grants EX94 28990695 and PB94-0987.

lengths. Although this paper appeared 60 years ago it does not seem to be known in the image processing literature.

2. Our proof is based on concepts appearing in a paper by Eckhardt and Voss [11] and relies only on elementary facts of linear algebra and nonlinear equations, and not on the differential nature of the operators involved. This, we think, gives a new insight to the subject.
3. Our proof applies to a generalized version of the FP algorithm in which the regularization functional can be more general than TV. In particular it contains as a special case the regularization functional considered by Aubert et al. in [2].
4. The linear convergence of the fixed point algorithm is proven under no stringent assumptions on the data and parameters. Some experiments aiming to study the dependence of the convergence rate with respect to various parameters are shown as well. As far as we know, these constitute novel results not found in [2], [3], [6].

The outline of this paper is as follows. We introduce the image denoising problem using the Total Variation as regularization functional in section 2; in section 3 we define the denoising problem for a more general functional. The concept of generalized Weiszfeld's method is introduced in section 4. In section 5 we give a proof of the global convergence of these methods and in section 6 we obtain their linear convergence, as well as an estimate for the rate of convergence in terms of the spectral structure of a preconditioned Hessian. Finally, section 7 contains some experiments that validate the bounds obtained in the previous section, as well as a numerical study of this rate for varying parameters and data.

2. Total Variation Denoising. The Total variation denoising method that we are going to briefly survey in this section can be formulated for data of any dimensionality, but since it is normally utilized for 2-dimensional data (images), we will use its 2-dimensional formulation for the sake of clarity.

An image u can be interpreted as a 2-dimensional array $(u_{i,j})$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. For notational simplicity let us assume $m = n$, so that $N = n^2$. We use $||y||$ or $|y|$ to denote the 2-norm of a vector y.

Our interest is to restore an image which is contaminated with noise. The restoration process should recover the edges of the image. Let us denote by z the observed image and u the real image. The model of degradation we assume is $u + \tilde{n} = z$, where $\tilde{n}$ is a Gaussian white noise, with $||\tilde{n}||$ known.

Of course, the problem $u + \tilde{n} = z$, $||\tilde{n}|| = \sigma$ has many solutions, but if we impose a certain *regularity* condition on the solution u, then it may become well-posed. In [9], it is proposed to use as regularization functional the so-called *Total Variation* (TV) functional. If we denote $h = \frac{1}{n}$, then a standard discretization of this functional is

$$TV(u) = \sum_{i,j=1}^{n} h^2 \sqrt{(\frac{u_{i+1,j} - u_{i,j}}{h})^2 + (\frac{u_{i,j+1} - u_{i,j}}{h})^2} = \frac{1}{N} \sum_{i,j=1}^{n} |\nabla_{i,j} u|, \tag{2.1}$$

where we have taken into account that $N = n^2$ and used the notation

$$\nabla_{i,j} u = (\frac{u_{i+1,j} - u_{i,j}}{h}, \frac{u_{i,j+1} - u_{i,j}}{h})$$

for the linear operator $\nabla_{i,j}$ from $\mathbb{R}^N$ to $\mathbb{R}^2$ (which is a discrete gradient operator).

The TV functional does not penalize discontinuities in u, and thus allows us to recover the edges of the original image. The restoration problem can now be written as

$$\min_u \frac{1}{N}\alpha \sum_{i,j=1}^{n} |\nabla_{i,j}u| + \frac{1}{N}\sum_{i,j=1}^{n} \frac{1}{2}(u_{ij} - z_{ij})^2, \tag{2.2}$$

or simply as

$$\min_u \sum_{i,j=1}^{n} \left[\alpha|\nabla_{i,j}u| + \frac{1}{2}(u_{ij} - z_{ij})^2\right], \tag{2.3}$$

where $\alpha > 0$. Since this functional is not differentiable, a commonly used technique to overcome this difficulty is to slightly perturb the TV norm to:

$$\frac{1}{N}\sum_{i,j=1}^{n} \sqrt{|\nabla_{i,j}u|^2 + \beta},$$

where β is a small positive parameter. The problem is stable with respect to this perturbation, in the sense that we can obtain the solution as the limit of the solutions of the perturbed problems when $\beta \to 0$, see [1]. We will denote $\sqrt{|\nabla_{i,j}u|^2 + \beta}$ by $|\nabla_{i,j}u|_\beta$ or simply $|\nabla_{i,j}u|$ if there is no possible confusion.

The first order condition for problem (2.3) is:

$$\alpha \sum_{i,j=1}^{n} \nabla_{i,j}^T \left(\frac{\nabla_{i,j}u}{|\nabla_{i,j}u|}\right) + u - z = 0, \tag{2.4}$$

where $\nabla_{i,j}^T$ is the transpose of the linear operator $\nabla_{i,j}$.

Note that (2.4) can be viewed as a nonlinear algebraic system of discretized partial differential equations. The presence of the diffusivity coefficient $\frac{1}{|\nabla_{i,j}u|}$ in (2.4) makes it highly nonlinear and Newton's method does not work satisfactorily on it (see [10], [4], [5]).

C. Vogel and M. Oman [10] proposed the following fixed point iteration to linearize equation (2.4) by solving

$$\alpha \sum_{i,j=1}^{n} \nabla_{ij}^T \left(\frac{\nabla_{i,j}u_{k+1}}{|\nabla_{i,j}u_k|}\right) + u_{k+1} - z = 0 \tag{2.5}$$

for u_{k+1}, given u_k. The rest of the paper is about a convergence proof for the iteration given in (2.5).

Let us briefly review the *half-quadratic regularization* approach of [8] that has been used in [2] to obtain a proof of the global convergence of the lagged diffusivity fixed point iteration.

Using the fact that $t = \min_v vt^2 + \frac{1}{4v}$, attained for $v = \frac{1}{2t}$, the minimization problem (2.3) can be written as

$$\begin{aligned}
&\min_u \sum_{i,j=1}^{n} \left[\alpha|\nabla_{i,j}u|_\beta + \frac{1}{2}(u_{ij}-z_{ij})^2\right] \\
&= \min_u \sum_{i,j=1}^{n} \left[\alpha \min_{v_{ij}}(v_{ij}|\nabla_{i,j}u|_\beta{}^2 + \frac{1}{4v_{ij}}) + \frac{1}{2}(u_{ij}-z_{ij})^2\right] \\
&= \min_u \min_v \sum_{i,j=1}^{n} \left[\alpha(v_{ij}|\nabla_{i,j}u|_\beta{}^2 + \frac{1}{4v_{ij}}) + \frac{1}{2}(u_{ij}-z_{ij})^2\right] \\
&= \min_u \min_v \Theta(u,v),
\end{aligned}$$

where $\frac{1}{2|\nabla_{i,j}u|_\beta} = \operatorname{argmin}_{v_{ij}}(v_{ij}|\nabla_{i,j}u|_\beta{}^2 + \frac{1}{4v_{ij}})$, $\forall i,j$. Note that $\Theta(u,v)$ is quadratic in u, thus the name *half-quadratic*.

The algorithm ARTUR used in [2] consists in the following alternate minimization procedure:

$$\begin{aligned}
v^{n+1} &= \operatorname*{argmin}_v \Theta(u^n, v), \\
u^{n+1} &= \operatorname*{argmin}_u \Theta(u, v^{n+1}).
\end{aligned} \tag{2.6}$$

The first equation is trivially solved by $v_{ij}^{n+1} = \frac{1}{2|\nabla_{i,j}u^n|_\beta}$ $\forall i,j$. Since $\Theta(u,v^{n+1})$ is a convex quadratic function of u it then follows that u^{n+1} is given as the solution of the equation

$$\begin{aligned}
0 = \nabla_u\Theta(u^{n+1},v^{n+1}) &= \sum_{i,j=1}^{n}\left[\alpha 2v_{ij}^{n+1}\nabla_{ij}^T\nabla_{ij}u^{n+1} + u_{ij}^{n+1} - z_{ij}\right] \\
&= \sum_{i,j=1}^{n}\left[\alpha\frac{\nabla_{ij}^T\nabla_{ij}u^{n+1}}{|\nabla_{i,j}u^n|_\beta} + u_{ij}^{n+1} - z_{ij}\right],
\end{aligned} \tag{2.7}$$

which is precisely the fixed point iteration (2.5).

3. A generalized problem. In this section we introduce a two-fold generalization of problem (2.3). Let us consider the problem

$$\min_u F(u) \equiv \min_u \alpha\sum_{i=1}^{m}\phi(|A_i^T u|_\beta) + \frac{1}{2}||u-z||^2. \tag{3.1}$$

where A_i, $i = 1,\ldots,m$ is a $n\times 2$ matrix and ϕ is a real function, which respectively generalize $\nabla_{i,j}^T$ and $\phi(x) = x$. We assume that ϕ verifies the following

HYPOTHESIS 3.1.

1. *ϕ is C^2.*
2. *$\phi(0) = 0$.*
3. *ϕ is increasing for $x \geq 0$, i.e., $\phi'(x) > 0$ $x \geq 0$.*
4. *ϕ is convex, i.e., $\phi''(x) \geq 0$.*
5. *$\psi(x) = \phi(\sqrt{x})$ is concave for $x \geq 0$.*

Note that by 2 and 3 in Hypothesis 3.1 $\phi(x) \geq 0$ $\forall x \geq 0$. We can cite as examples of such functions ϕ the functions x^p, $1 \leq p \leq 2$. Note that problem (2.3) corresponds to the case $A_i^T = \nabla_{i,j}$ and $p = 1$ and that $p = 2$ corresponds to the H^1-norm restoration. The exponents p such that $1 < p < 2$ might be potentially useful for image restoration applications.

It is easy to show that

$$F'(u) = \alpha \sum_i A_i \left(\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} A_i^T u \right) + u - z, \tag{3.2}$$

$$F''(u) = \alpha \sum_i A_i \Big(\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} (I_2 - \frac{A_i^T u \otimes A_i^T u}{|A_i^T u|_\beta^2}) + \phi''(|A_i^T u|_\beta) \frac{A_i^T u \otimes A_i^T u}{|A_i^T u|_\beta^2} \Big) A_i^T + I_n, \tag{3.3}$$

where $\otimes$ denotes the Kronecker product.

LEMMA 3.2. *Under hypothesis 3.1 on ϕ, we have:*

1. *F is twice continuously differentiable and strictly convex.*
2. *F is coercive, i.e., $\lim_{||x|| \to \infty} F(x) = \infty$.*
3. *F is bounded below.*

Therefore problem (3.1) has a unique solution.

Proof. Since $F(u) \geq \frac{1}{2}||u - z||^2$, F is coercive and bounded below by 0. By (3.2) and the fact that $|A_i^T u|_\beta \neq 0$, F is twice continuously differentiable. By Cauchy-Schwartz, we have $|A_i^T v|^2 - \frac{(A_i^T u, A_i^T v)^2}{|A_i^T u|_\beta^2} \geq 0$. Moreover, since $\phi'(x) \geq 0$ if $x > 0$ and $\phi''(x) \geq 0$, we have for $v \neq 0$

$$F''(u)(v,v) = \alpha \sum_i \Big[\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} (|A_i^T v|^2 - \frac{(A_i^T u, A_i^T v)^2}{|A_i^T u|_\beta^2}) + \phi''(|A_i^T u|_\beta) \frac{(A_i^T u, A_i^T v)^2}{|A_i^T u|_\beta^2} \Big] + ||v||^2 > 0,$$

hence F is a strictly convex functional. By standard arguments in convex analysis (cf. [7] for example), problem (3.1) has a unique solution. □

We consider the more general fixed point algorithm given by solving for u_{k+1}, given u_k, in:

$$\alpha \sum_i A_i \left(\frac{\phi'(|A_i^T u_k|_\beta)}{|A_i^T u_k|_\beta} A_i^T u_{k+1} \right) + u_{k+1} - z = 0. \tag{3.4}$$

Note the connection with (3.2); also, if $\phi(x) = x$ and $A_i^T \equiv \nabla_{i,j}$, i.e., the original functional is the Total Variation functional, then (3.4) is just (2.5).

4. Generalized Weiszfeld's method. Let us consider now a general minimization problem

$$\min_u F(u), \tag{4.1}$$

for a functional F defined on $\mathbb{R}^n$, and let us assume the usual hypothesis that ensure that problem (4.1) has a unique solution

HYPOTHESIS 4.1.

1. *F is a twice continuously differentiable and strictly convex.*

2. *F is coercive, i.e.,* $\lim_{\|x\|\to\infty} F(x) = \infty$.

3. *F is bounded below.*

Note that (3.1) verifies these conditions by Lemma 3.2.

E. Weiszfeld proposed in [12] a linearization technique for a class of problems similar to (3.1) that differs from Newton's method in that the original functional is approximated from above. This *generalized Weiszfeld's algorithm* for (4.1) consists, first of all, in the choice of a *uniformly strictly convex* quadratic functional $G(v,u)$ that is an approximation from above for $F(u)$, i.e.,

HYPOTHESIS 4.2.

1. $G(v,u) = F(u) + (v-u)^T F'(u) + \frac{1}{2}(v-u)^T C(u)(v-u)$.
2. *C is continuous.*
3. $\lambda_{\min}(C(u)) \geq \mu > 0$, $\forall u$.
4. $F(v) \leq G(v,u)$ $\forall v$.

Then one defines the iteration for this generalized Weiszfeld's method by

$$u_{n+1} = \operatorname*{argmin}_v G(v, u_n), \tag{4.2}$$

see Figure 4.1. Since, for fixed u, G is $\mathcal{C}^2$, coercive, bounded below and strictly convex,

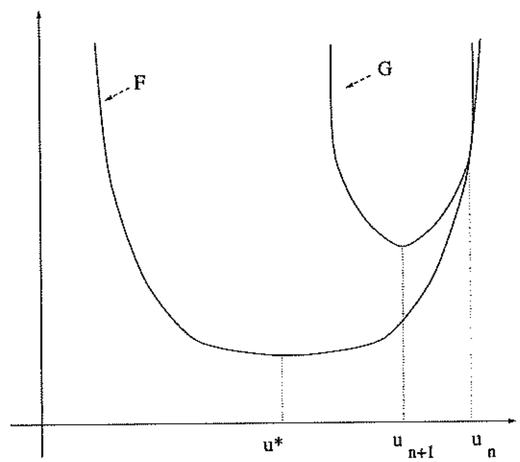

FIG. 4.1. *Weiszfeld's algorithm. G is a quadratic osculating super approximation to F.*

this minimum exists and it is given by the following equation:

$$0 = G_v(u_{n+1}, u_n) = F'(u_n) + C(u_n)(u_{n+1} - u_n). \tag{4.3}$$

We show next that the fixed point algorithm (2.5) is an instance of a generalized Weiszfeld's method.

LEMMA 4.3. *If* $C(u)$ *is given by*

$$C(u) = A\ diag\Big(\frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta}\Big) A^T + I_n, \qquad A = [A_1 \dots A_m] \tag{4.4}$$

then the functional

$$G(v,u) = F(u) + (v-u)^T F'(u) + \frac{1}{2}(v-u)^T C(u)(v-u)$$

defines a generalized Weiszfeld's method for problem (3.1) and (3.4) corresponds to its iteration.

Proof. The smallest eigenvalue of $C(u)$ is characterized as

$$\lambda_{\min} = \min_{||v||_2=1} v^T C(u) v.$$

Since $v^T C(u) v \geq ||v||^2$, we deduce that $\lambda_{\min}(C(u)) \geq 1$, so we can take $\mu = 1$ in 3 of Hypothesis 4.2.

The continuity of $C(u)$ can be deduced easily from the fact that $|A_i^T u|_\beta \neq 0$.

Let us now see that $F(v) \leq G(v,u)$, $\forall v$. By using 1 in Definition 4.2 and the definition of F in (3.1) we have

$$G(v,u) - F(v) = F(u) - F(v) + (v-u)^T F'(u) + \frac{1}{2}(v-u)^T C(u)(v-u) =$$

$$\alpha \sum_i \left[\phi(|A_i^T u|_\beta) - \phi(|A_i^T v|_\beta) + \frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta}((A_i^T u, A_i^T v - A_i^T u) + \frac{1}{2}|A_i^T v - A_i^T u|^2) \right] =$$

$$\alpha \sum_i \left[\phi(|A_i^T u|_\beta) - \phi(|A_i^T v|_\beta) + \frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} \frac{1}{2}(|A_i^T v|^2 - |A_i^T u|^2) \right] =$$

$$\alpha \sum_i \left[\phi(|A_i^T u|_\beta) - \phi(|A_i^T v|_\beta) + \frac{\phi'(|A_i^T u|_\beta)}{|A_i^T u|_\beta} \frac{1}{2}(|A_i^T v|_\beta^2 - |A_i^T u|_\beta^2) \right]. \quad (4.5)$$

In the equations above we have used the fact that

$$|A_i^T v|_\beta^2 - |A_i^T u|_\beta^2 = |A_i^T v|^2 + \beta - (|A_i^T u|^2 + \beta) = |A_i^T v|^2 - |A_i^T u|^2.$$

Each of the terms in this sum is of the form

$$\phi(a) - \phi(b) + \frac{\phi'(a)}{2a}(b^2 - a^2),$$

and, by the definition of ψ in Hypothesis 3.1, it can be written as

$$\psi(a^2) - \psi(b^2) + \psi'(a^2)(b^2 - a^2). \quad (4.6)$$

Since ψ is concave by hypothesis,

$$\psi(c) + \psi'(c)(d-c) \geq \psi(d) \quad \forall c, d \geq 0. \quad (4.7)$$

From (4.5), (4.6) and (4.7), we deduce $F(v) \leq G(v,u)$.

Let us now finally see that (3.4) corresponds to the iteration of the generalized Weiszfeld's method. From equations (3.2) and (4.4) we deduce that equation (4.3) can be written as

$$0 = \alpha \sum_i \left[A_i (\frac{\phi'(|A_i^T u_k|_\beta)}{|A_i^T u_k|_\beta} A_i^T u_k) \right] + u_k - z +$$

$$\alpha \sum_i \left[A_i (\frac{\phi'(|A_i^T u_k|_\beta)}{|A_i^T u_k|_\beta} A_i^T (u_{k+1} - u_k)) \right] + u_{k+1} - u_k =$$

$$\alpha \sum_i \left[A_i (\frac{\phi'(|A_i^T u_k|_\beta)}{|A_i^T u_k|_\beta} A_i^T u_{k+1}) \right] + u_{k+1} - z. \quad (4.8)$$

Therefore, equation (4.3) is equivalent to the generalized fixed point iteration given in equation (3.4). □

5. Global convergence theory. In this section, we establish the linear convergence of the generalized Weiszfeld's algorithm. H. Voss and U. Eckhardt have shown in [11] that the generalized Weiszfeld's method has linear and monotonic convergence. In our proof, we follow essentially the same approach.

First, we establish global convergence.

LEMMA 5.1.

1. $F(u_{n+1}) \leq F(u_n)$, $\forall n$.

2. $\lim_{n\to\infty} ||u_{n+1} - u_n|| = 0$.

3. The sequence (u_n) *converges to the unique global minimum of (3.1).*

Proof. 1. By 4 in Hypothesis 4.2, $F(u_{n+1}) \leq G(u_{n+1}, u_n)$. Now, $u_{n+1} = \operatorname{argmin}_v G(v, u_n)$ implies $G(u_{n+1}, u_n) \leq G(u_n, u_n) = F(u_n)$.

2. By 4 in Definition 4.2 and (4.3),

$$F(u_{n+1}) \leq G(u_{n+1}, u_n) = F(u_n) + (u_{n+1} - u_n)^T F'(u_n) + \frac{1}{2}(u_{n+1} - u_n)^T C(u_n)(u_{n+1} - u_n) = F(u_n) - \frac{1}{2}(u_{n+1} - u_n)^T C(u_n)(u_{n+1} - u_n).$$

From this, taking into account that $\mu \leq \lambda_{\min}(C(u))$, $\forall u$, by 3 in Definition 4.2:

$$\frac{1}{2}\mu||u_{n+1} - u_n||^2 \leq \frac{1}{2}(u_{n+1} - u_n)^T C(u_n)(u_{n+1} - u_n) \leq F(u_n) - F(u_{n+1}),$$

which implies

$$0 \leq ||u_{n+1} - u_n|| \leq \sqrt{\frac{2}{\mu}(F(u_n) - F(u_{n+1}))}. \tag{5.1}$$

On the other hand, since F is bounded below and $(F(u_n))$ is monotonically decreasing, then $F(u_n)$ converges and, in particular, $\lim_{n\to\infty} F(u_n) - F(u_{n+1}) = 0$. This and (5.1) gives us that $\lim_{n\to\infty} ||u_{n+1} - u_n|| = 0$.

Let us finally see that u_n converges to the unique minimum of (3.1). Since $(F(u_n))$ is monotonically decreasing, it is bounded above. The fact that F is coercive then implies that u_n is bounded. By compactness, it suffices to show that the limit of every convergent subsequence of (u_n) is the global minimum of (3.1). By the strict convexity of F, it will then suffice to show that those limits are stationary points of F.

Let (u_{n_k}) be a subsequence of (u_n) that converges to $\overline{u}$. Since $G_v(v, u) = F'(u) + C(u)(v - u)$ and by the hypothesis on F and C, we deduce that G_v is continuous. Since $\lim_{k\to\infty} ||u_{n_k+1} - u_{n_k}|| = 0$, as we have just seen, we have that $\lim_{k\to\infty} u_{n_k+1} = \lim_{k\to\infty} u_{n_k} = \overline{u}$. By (4.3)

$$0 = G_v(u_{n_k+1}, u_{n_k}) = F'(u_{n_k}) + C(u_{n_k})(u_{n_k+1} - u_{n_k}).$$

Taking limits, we deduce from the facts stated above that

$$0 = \lim_{k\to\infty} G_v(u_{n_k+1}, u_{n_k}) = G_v(\lim_{k\to\infty} u_{n_k+1}, \lim_{k\to\infty} u_{n_k}) = G_v(\overline{u}, \overline{u}) = F'(\overline{u}) + C(\overline{u})(\overline{u} - \overline{u}) = F'(\overline{u}).$$

□

6. Linear convergence analysis. We derive in this section the linear convergence of the generalized Weiszfeld's method and give a bound of its convergence rate in terms of spectral structure of a preconditioned Hessian.

Let us denote by u^* the unique minimum of (3.1) and

$$\begin{aligned} \lambda_n &= \frac{G(u^*, u_n) - F(u^*)}{\frac{1}{2}(u^* - u_n)^T C(u_n)(u^* - u_n)}, \\ \Lambda &= 1 - \lambda_{\min}(C(u^*)^{-1} F''(u^*)). \end{aligned} \tag{6.1}$$

See Figure 6.1 and equation (6.3) below for a geometrical interpretation of λ_n.

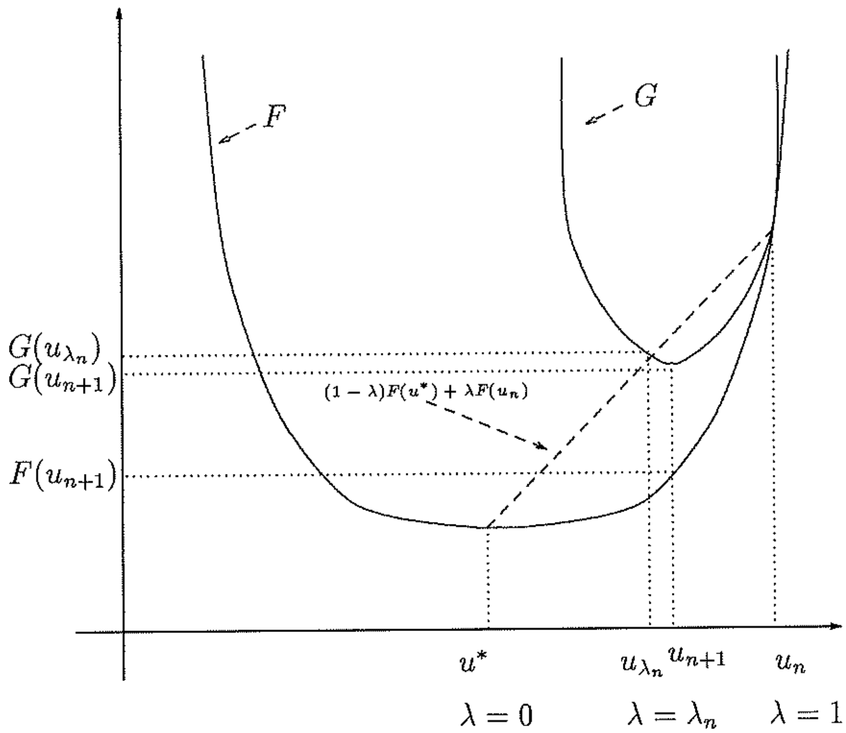

FIG. 6.1. *Convergence of Weiszfeld's algorithm.* $u_{\lambda_n} = \lambda_n u_n + (1 - \lambda_n) u^*$.

THEOREM 6.1.

1. $F(u_{n+1}) - F(u^*) \le \lambda_n (F(u_n) - F(u^*))$.
2. $\Lambda < 1$ *and* $0 \le \lambda_n \le \Lambda$, *for* n *sufficiently large. In particular,* $F(u_n)$ *has a linear convergent rate of at most* Λ.
3. u_n *is r-linearly convergent with a convergence rate of at most* $\sqrt{\Lambda}$.

Proof. By 4 in Definition 4.2 and (4.2)

$$F(u_{n+1}) \le G(u_{n+1}, u_n) \le G(\lambda_n u_n + (1 - \lambda_n) u^*, u_n). \tag{6.2}$$

A direct computation using (6.1) shows that

$$
\begin{aligned}
&G(\lambda_n u_n + (1-\lambda_n)u^*, u_n) = \\
&\quad F(u_n) + (1-\lambda_n)(u^*-u_n)^T F'(u_n) + \frac{(1-\lambda_n)^2}{2}(u^*-u_n)^T C(u_n)(u^*-u_n) = \\
&\qquad \lambda_n F(u_n) + (1-\lambda_n)\Big(G(u^*,u_n) - \frac{\lambda_n}{2}(u^*-u_n)^T C(u_n)(u^*-u_n)\Big) = \\
&\qquad\qquad \lambda_n F(u_n) + (1-\lambda_n)F(u^*). \quad (6.3)
\end{aligned}
$$

Substituting (6.3) into (6.2), subtracting $F(u^*)$ from both sides and dividing by $1-\lambda_n$ (which is nonzero as can be seen easily) gives the first statement (see Fig 6.1 for an illustration).

Recall that $F''(u)$ is the Hessian of F at u. By using the definition of G and Taylor expansions at u_n,

$$G(u^*,u_n) = F(u_n) + (u^*-u_n)^T F'(u_n) + \frac{1}{2}(u^*-u_n)^T C(u_n)(u^*-u_n), \quad (6.4)$$

$$F(u^*) = F(u_n) + (u^*-u_n)^T F'(u_n) + \frac{1}{2}(u^*-u_n)^T F''(u_n)(u^*-u_n) + \mathcal{O}(||u^*-u_n||^3). \quad (6.5)$$

If we subtract (6.5) from (6.4) and divide by $\frac{1}{2}(u^*-u_n)^T C(u_n)(u^*-u_n)$ we get

$$
\begin{aligned}
&\lambda_n = \frac{G(u^*,u_n)-F(u^*)}{\frac{1}{2}(u^*-u_n)^T C(u_n)(u^*-u_n)} = \\
&\frac{\frac{1}{2}(u^*-u_n)^T C(u_n)(u^*-u_n) - \frac{1}{2}(u^*-u_n)^T F''(u_n)(u^*-u_n) + \mathcal{O}(||u^*-u_n||^3)}{\frac{1}{2}(u^*-u_n)^T C(u_n)(u^*-u_n)} = \\
&\qquad 1 - \frac{\frac{1}{2}(u^*-u_n)^T F''(u_n)(u^*-u_n) + \mathcal{O}(||u^*-u_n||^3)}{\frac{1}{2}(u^*-u_n)^T C(u_n)(u^*-u_n)} = \\
&\qquad 1 - \frac{\frac{1}{2}(u^*-u_n)^T F''(u_n)(u^*-u_n)}{\frac{1}{2}(u^*-u_n)^T C(u_n)(u^*-u_n)} + \mathcal{O}(||u^*-u_n||) \le \\
&\qquad\qquad 1 - \lambda_{\min}(C(u_n)^{-1}F''(u_n)) + \mathcal{O}(||u^*-u_n||). \quad (6.6)
\end{aligned}
$$

Since F'' and C are continuous

$$1 - \lambda_{\min}(C(u_n)^{-1}F''(u_n)) \to 1 - \lambda_{\min}(C(u^*)^{-1}F''(u^*)) \quad (6.7)$$

Since F is strictly convex, its Hessian at u^*, $F''(u^*)$, is symmetric positive definite. By assumption, $C(u^*)$ is also symmetric positive definite, therefore

$$C(u^*)^{-1}F''(u^*) = C(u^*)^{-\frac{1}{2}}C(u^*)^{-\frac{1}{2}}F''(u^*)C(u^*)^{-\frac{1}{2}}C(u^*)^{\frac{1}{2}},$$

and $C(u^*)^{-1}F''(u^*)$ is the similar to $C(u^*)^{-\frac{1}{2}}F''(u^*)C(u^*)^{-\frac{1}{2}}$, which is positive definite, thus

$$\lambda_{\min}(C(u^*)^{-1}F''(u^*)) = \lambda_{\min}(C(u^*)^{-\frac{1}{2}}F''(u^*)C(u^*)^{-\frac{1}{2}}) > 0. \quad (6.8)$$

By (6.7) and (6.8):

$$\Lambda = 1 - \lambda_{\min}(C(u^*)^{-1}F''(u^*)) < 1, \quad (6.9)$$

and, together with (6.6) we get that $\lambda_n \leq \Lambda < 1$, for n sufficiently big. To conclude the second statement, note that by 4 in Hypothesis 4.2, we have $F(u^*) \leq G(u^*, u_n)$, which, by the definition of λ_n, in turn implies that $\lambda_n \geq 0$.

Since $F'(u^*) = 0$, by using the Taylor expansion at u^* we get

$$F(u_n) - F(u^*) = \frac{1}{2}(u_n - u^*)^T F''(u^*)(u_n - u^*) + \mathcal{O}(||u_n - u^*||^3) \geq \frac{1}{2}\lambda_{\min}(F''(u^*))\,||u_n - u^*||^2 + \mathcal{O}(||u_n - u^*||^3)$$

and, from here, there exists $K > 2$ such that

$$||u_n - u^*|| \leq \sqrt{\frac{K}{\lambda_{\min}(F''(u^*))}(F(u_n) - F(u^*))} \stackrel{\text{def}}{=} y_n, \tag{6.10}$$

for n sufficiently big. Now, by 1 in Theorem 6.1,

$$y_{n+1}^2 = \frac{K}{\lambda_{\min}(F''(u^*))}(F(u_{n+1}) - F(u^*)) \leq \frac{K}{\lambda_{\min}(F''(u^*))}\Lambda(F(u_n) - F(u^*)) = \Lambda y_n^2,$$

that is, $y_{n+1} \leq \sqrt{\Lambda} y_n$, hence (u_n) is r-linearly convergent with at most $\sqrt{\Lambda}$ as the r-convergence rate by (6.10). □

7. Numerical results. For typical image restoration problems it has been observed that the rate of convergence of the fixed point algorithm dramatically deteriorates when the regularizing parameter β takes the small values that are necessary for obtaining solutions with sharp edges. The purpose of this section is to present a numerical study of the dependence of the linear convergence rate of the FP algorithm on the parameter β. The result that we obtain in this section is that, although the convergence of the fixed point method can be slow, the convergence rate does not get asymptotically to 1 when $\beta \to 0$.

We have used one-dimensional signals u, to which we have added Gaussian white noise with some variance σ^2 to obtain z. In the following experiments, the parameter α is selected such that the solution of the problem with that parameter and $\beta = 10^{-4}$ matches the noise constraints $||u - z||^2 = \sigma^2$.

We estimate the linear convergence rate of the sequence $z_n(\beta) = F_\beta(u_n) - F_\beta(u^*)$ appearing in Theorem 6.1 for the functional of problem (2.3) and for several values of the parameter β forming a geometric sequence between 1 and 10^{-16}. The convergence rate for $z_n(\beta)$, $r(\beta) = \lim_n z_{n+1}/z_n$, is related to that for $u_n(\beta)$ by (6.10), thus it is enough to consider the convergence rate for $z_n(\beta)$.

The computations have been carried in high precision (33 digits) arithmetic to ensure the accuracy of the estimates. The solution of each of the problems, which is needed to estimate its convergence rate, is computed to full precision by the primal-dual method introduced in [5].

The signal in the first experiment has a size of 64 and the *signal to noise ratio* (SNR$= \frac{||u-\text{mean}(u)||_{\mathcal{L}^2}}{||u-z||_{\mathcal{L}^2}}$) is approximately 1. We show in the second column of Table 7.1 $r(\beta)$ for the different β's displayed in the first column. In the third column we display $\overline{r}(\beta) = \lim_{n\to\infty} \lambda_n(\beta)$, for $\lambda_n = \lambda_n(\beta)$ defined in (6.1), since this limit is precisely what we have used in (6.6), (6.7) and (6.9) to estimate $\Lambda = 1 - \lambda_{\min}(C(u^*)^{-1}F''(u^*))$, which is shown in the fourth column of this table. We plot in Figure 7.1 columns 2, 3 and 4 versus column 1 of this table. The conclusions we can draw from this experiment are the following:

- The convergence rate of the fixed point algorithm seems to approach asymptotically a value strictly smaller than 1, when $\beta \to 0$, i.e., $\lim_{\beta \to 0} r(\beta) < 1$. To the best of the author's knowledge, this rcsult seems not to have noticed before.
- The estimate of convergence rate of the fixed point algorithm $\overline{r}(\beta) = \lim_n \lambda_n$, obtained in Theorem 6.1 (1), seems to approach asymptotically a value strictly smaller than 1, when $\beta \to 0$, i.e., $\lim_{\beta \to 0} \overline{r}(\beta) < 1$.
- $\lim_{\beta \to 0} r(\beta) < \lim_{\beta \to 0} \overline{r}(\beta) < 1$
- The estimate of (6.6) seems to be asymptotically sharp.

The significance of the first point is that the convergence rate of the fixed point iteration does not get significantly worse once the parameter β gets below some threshold. The other points imply that a study of the convergence rate of the fixed point iteration can be based upon an analysis of the spectrum of the preconditioned matrix $C(u^*)^{-1}F''(u^*)$ for the typical piecewise constant solutions u^* of the total variation image restoration problem.

β	$r(\beta)$	$\overline{r}(\beta)$	Λ
$1.0e+00$	$8.52e-01$	$9.232202e-01$	$9.241033e-01$
$7.8e-02$	$8.85e-01$	$9.412004e-01$	$9.414071e-01$
$6.2e-03$	$9.09e-01$	$9.536306e-01$	$9.536308e-01$
$4.8e-04$	$9.08e-01$	$9.533806e-01$	$9.533810e-01$
$3.8e-05$	$9.06e-01$	$9.518686e-01$	$9.518715e-01$
$3.0e-06$	$9.05e-01$	$9.513539e-01$	$9.513572e-01$
$2.3e-07$	$9.04e-01$	$9.512019e-01$	$9.512063e-01$
$1.8e-08$	$9.04e-01$	$9.511583e-01$	$9.511635e-01$
$1.4e-09$	$9.04e-01$	$9.511459e-01$	$9.511514e-01$
$1.1e-10$	$9.04e-01$	$9.511433e-01$	$9.511480e-01$
$9.2e-12$	$9.04e-01$	$9.511406e-01$	$9.511471e-01$
$7.2e-13$	$9.04e-01$	$9.511422e-01$	$9.511468e-01$
$5.7e-14$	$9.04e-01$	$9.511388e-01$	$9.511467e-01$
$4.5e-15$	$9.04e-01$	$9.511407e-01$	$9.511467e-01$
$3.5e-16$	$9.04e-01$	$9.511417e-01$	$9.511467e-01$
$1.0e-16$	$9.04e-01$	$9.511420e-01$	$9.511467e-01$

TABLE 7.1

Estimates of the convergence rate $r(\beta)$ and predicted convergence rate $\overline{r}(\beta)$, see Section 7 for an explanation.

In the second experiment we obtain the convergence rate of the fixed point algorithm applied to signals of sizes 64, 128 and 256. These signals are obtained by adding Gaussian white noise to some original signals, resulting in noisy signals with a SNR$\approx$ 1. The signals of sizes 64 and 128 have been obtained by repeated subsampling of the one of size 256. The values of β are the same as in the previous experiment and the results are shown in Figure7.2. This experiment seems to suggest that the gap between 1 and the asymptotic convergence rate of the fixed point algorithm tends to shrink, i.e., the convergence gets slower, when the computational grid size grows, but the dependence of this gap on several parameters, such as the original image, α, and noise level, is still an open research question.

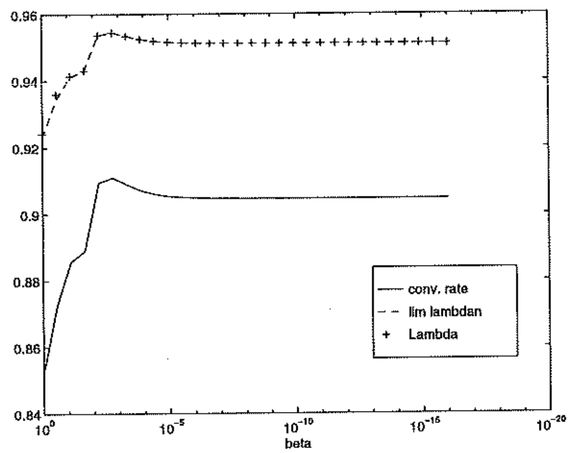

FIG. 7.1. *Comparison of convergence rate and estimates for signal of size 64 and SNR≈ 1.*

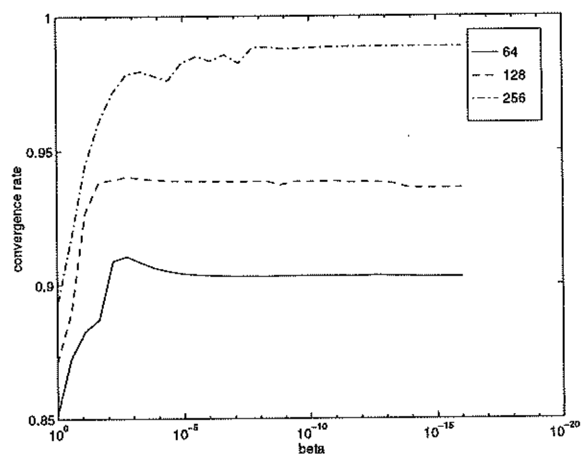

FIG. 7.2. *Comparison of convergence rate for signals of sizes 64, 128, 256 and SNR≈ 1.*

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