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Stability of Systems of Viscous Conservation Laws

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1 Introduction

Consider the Cauchy problem for a system of real conservation laws

$$\begin{aligned} v_t + f_x(v) &= \nu v_{xx}, \quad t \geq 0, \\ v(x,0) &= v_0(x), \quad -\infty < x < \infty. \end{aligned}$$

We assume that there is a steady state solution with

$$\lim_{x\to\infty} U(x) = U_R, \quad \lim_{x\to-\infty} U(x) = U_L, \quad U_r, U_L \text{ const.},$$

and want to investigate its stability. We assume that the initial data are of the form

$$v_0(x) = U(x) + \varepsilon\Big(\tilde{v}_0(x)\Big)_x, \quad |\varepsilon| << 1.$$

The change of variables $v = U(x) + \varepsilon\tilde{v}$ leads to a system of the form

$$\begin{aligned} \tilde{v}_t \quad + \quad & \Big(A(x)\tilde{v}\Big)_x + \varepsilon g_x(x,\tilde{v}) = \tilde{v}_{xx}, \\ & \tilde{v}(x,0) = \Big(\tilde{v}_0(x)\Big)_x. \end{aligned}$$

For our purpose, it is convenient to have homogeneous initial data. Therefore, we introduce

$$u = \tilde{v} - e^{-t}\Big(\tilde{v}_0(x)\Big)_x$$

as a new variable and obtain a system of type

$$\begin{aligned} u_t \quad &+ \quad \Big(A(x)u\Big)_x + \varepsilon_1\Big(B(x,t)u\Big)_x + \varepsilon_2\Big(g(x,t,u)\Big)_x \\ &= \quad u_{xx} - \Big(F(x,t)\Big)_x, \quad t \geq 0, \end{aligned} \tag{1}$$

with homogeneous initial data

$$u(x,0) = 0, \quad -\infty < x < \infty. \tag{2}$$

Here $u = u(x,t)$ takes values in R^n and $A(x) \in R^{n\times n}$, $B(x,t) \in R^{n\times n}$ are C^∞-smooth functions of x and x,t, respectively. We shall use the notations

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}.$$

More specifically, we shall assume:

1. A converges for $x \to \pm\infty$ exponentially fast to constant matrices A_R, A_L, respectively, i.e., there are constants $\alpha > 0$, $K_q > 0$ such that

$$\begin{aligned} |A(x) - A_R| &\leq K_0 e^{-\alpha x} \text{ for } x > 0, \\ |A(x) - A_L| &\leq K_0 e^{\alpha x} \text{ for } x < 0, \\ |D_x^q A(x)| &\leq K_q e^{-\alpha|x|}, \quad D_x = \partial/\partial x. \end{aligned} \tag{3}$$

   Also, A_L, A_R are nonsingular and strictly hyperbolic, i.e., their eigenvalues are real, nonzero and distinct.

2. B, F and their derivatives belong to $L_2(x,t)$, i.e., for every q there are constants K_q such that

$$\int_0^\infty \int_{-\infty}^\infty \left(|D_x^{q_1} D_t^{q_2} B|^2 + |D_x^{q_1} D_t^{q_2} F|^2\right) dx dt \leq K_q, \quad q = (q_1, q_2). \tag{4}$$

3. F and its derivatives belong to $L_1(x,t)$, i.e.,

$$\int_0^\infty \int_{-\infty}^\infty |D_x^{q_1} D_t^{q_2} F| dx dt \leq K_q. \tag{5}$$

4. For every $c > 0$ there are constants $C_q = C_q(c), C_{\tilde{q}}(c)$ such that, for $|u| \leq c$, the nonlinear function g satisfies the estimate

$$\begin{aligned} |D_x^{q_1} D_t^{q_2} g| &\leq C_q |u|^2, \\ |D_x^{q_1} D_t^{q_2} D_u g| &\leq C_q |u|, \\ |D_x^{q_1} D_t^{q_2} D_u^{q_3} g| &\leq C_{\tilde{q}}, \quad \tilde{q} = (q_1, q_2, q_3). \end{aligned} \tag{6}$$

We call (1),(2) nonlinearly stable if the solution $u(x,t)$ remains smooth for all $t \geq 0$ and the maximum norm $|u(\cdot,t)|_\infty$ tends to zero for $t \to \infty$, provided that $\varepsilon_1^2 + \varepsilon_2^2$ is sufficiently small. In particular, we call the problem linearly stable if the convergence takes place for $\varepsilon_1 = \varepsilon_2 = 0$. The aim of this paper is to supplement the above smoothness assumptions for the coefficients and data by structural conditions which imply nonlinear stability.

In the next section we assume that $A = A_R = A_L$ is a constant matrix. We shall prove

Theorem 1 *If $A = A_R = A_L$ is constant and the estimates (3)– (6) hold, then the problem (1), (2) is nonlinearly stable.*

In [6] we have generalized this result to general systems in multiple space dimensions.

In Section 3 we consider variable A. By assumption, there are transformations S_R and S_L such that

$$S_R^{-1} A_R S_R = \Lambda_R = \begin{pmatrix} -\Lambda_R^I & 0 \\ 0 & \Lambda_R^{II} \end{pmatrix}, \quad \Lambda_R^I > 0,\ \Lambda_R^{II} > 0 \text{ diagonal}, \tag{7}$$

$$S_L^{-1} A_L S_L = \Lambda_L = \begin{pmatrix} -\Lambda_L^I & 0 \\ 0 & \Lambda_L^{II} \end{pmatrix}, \quad \Lambda_L^I > 0,\ \Lambda_L^{II} > 0 \text{ diagonal}. \tag{8}$$

We treat the cases

$$m =: \dim \Lambda_R^I + \dim \Lambda_L^{II} = \begin{cases} n \\ or \\ n+1 \end{cases}. \tag{9}$$

Here $\dim \Lambda = q$ if Λ is a $q \times q$ matrix. The number m denotes the number of characteristics which enters the transition region around $x = 0$. Therefore, the second case corresponds to a shock.

Connected with (1) is the eigenvalue problem

$$\varphi_{xx} - \Big(A(x)\varphi\Big)_x = \mu\varphi, \quad \|\varphi\|^2 =: \int_{-\infty}^{\infty} |\varphi|^2 dx < \infty. \tag{10}$$

A necessary condition for linear (and therefore also for nonlinear) stability is

Assumption 1 *There are no eigenvalues for* $Re\,\mu \geq 0$, $\mu \neq 0$.

For $\mathrm{Re}\,\mu \geq 0$, $\mu \neq 0$, the eigenvalue problem is well defined. To first approximation we can, for large x, replace the differential equation by

$$\varphi_{xx} - A_R\varphi_x = \mu\varphi. \tag{11}$$

The general solution of **(11)** is of the form

$$\varphi = \sum \sigma_j e^{\kappa_j x}\varphi_{0j},$$

where κ_j, φ_{0j} are normalized solutions of the characteristic equation

$$(\kappa^2 I - A_R\kappa - \mu I)\varphi_0 = 0.$$

σ_j are free parameters. We shall prove that

$$|Re\,\kappa| \neq 0 \text{ if } Re\,\mu \geq 0,\ \mu \neq 0.$$

Thus, $\|\varphi\| < \infty$ implies that $\sigma_j = 0$ if $\mathrm{Re}\,\kappa_j > 0$, i.e.,

$$\varphi = \sum_{Re\,\kappa_j < 0} \sigma_j e^{\kappa_j x}\varphi_{0j}. \tag{12}$$

The corresponding representation is also valid for $x << -1$. In the usual way (see, for example, [8]), we can replace the infinite interval by a bounded interval $-l \leq x \leq l$, l sufficiently large. The boundary conditions at $x = \pm l$ are linear relations between φ and φ_x which guarantee that $\sigma_j = 0$ for growing modes. Thus, (10) is equivalent with

$$\varphi_{xx} - \Big(A(x)\varphi\Big)_x = \mu\varphi, \quad -l \leq x \leq l \tag{13}$$

with $2n$ relation

$$L_0\varphi + L_1\varphi_x = 0 \quad \text{at} \quad x = \pm l. \tag{14}$$

We shall now discuss the possibility that $\mu = 0$ is an eigenvalue. For $\mu = 0$, we have to solve

$$\varphi_{xx} - \Big(A(x)\varphi\Big)_x = 0, \quad \|\varphi\| < \infty, \quad -\infty < x < \infty. \tag{15}$$

Integrating the differential equation gives us

$$\varphi_x - A\varphi = D, \quad \|\varphi\| < \infty, \quad D = const. \tag{16}$$

$\varphi \in L_2$ implies

$$D = 0. \tag{17}$$

We make

Assumption 2 *If $m = n$, then (16),(17) has only the trivial solution. If $m = n + 1$, then (16),(17) has a nontrivial solution φ_0 and the dimension of the eigenspace is exactly* 1.

We have also to exclude generalized eigenvalues and eigenspaces. For $\mu \neq 0$, $\operatorname{Re}\mu \geq 0$, the eigenvalue problem on the whole line is equivalent with (13),(14) which, for $\mu \to 0, l \to \infty$ converges to

$$\begin{aligned} \varphi_{xx} - \Big(A(x)\varphi\Big)_x &= 0, \quad -\infty < x < \infty, && (18) \\ \bar{L}_0\varphi + \bar{L}_1\varphi_x &= 0, \quad x = \pm\infty. && (19) \end{aligned}$$

Therefore,

$$\varphi_x - A\varphi = D. \tag{20}$$

In this case we do not assume a priori that $\varphi \in L_2$. Instead, we allow bounded solutions. Therefore, we cannot directly conclude that $D = 0$. If (19),(20) have a bounded solution φ with $D \neq 0$, we call $\mu = 0$ a generalized eigenvalue, because $\varphi \notin L_2$. We make

Assumption 3 *$\mu = 0$ is not a generalized eigenvalue .*

If $m = n + 1$, then (19),(20) have a solution $\varphi_0 \in L_2$ with $D = 0$. By Assumption 2, the eigenspace contains no other functions belonging to L_2. However, the eigenspace in the generalized sense has dimension $r > 1$ if

$$\varphi_{xx} - \Big(A(x)\varphi\Big)_x = \alpha\varphi_0, \quad -\infty < x < \infty,$$

with boundary conditions (19), has a bounded solution with $\alpha \neq 0$. We make

Assumption 4 *If $m = n + 1$, the generalized eigenspace has dimension $r = 1$, i.e., $\alpha = 0$.*

In Section 4 we shall express assumptions 3 and 4 in algebraic form. We can now formulate our main result

Theorem 2 *If Assumptions 1–4 are valid and the estimates (3)–(6) hold, then the problem (1)–(2) is nonlinearly stable.*

In another paper we shall generalize this result to systems in many space dimensions.

The stability of steady state solutions of viscous conservation laws has been studied many times before. See [1]–[5],[7]–[13]. However, these papers are mostly concerned with scalar equations or weak shocks, while we want to remove that restriction.

2 Constant coefficients

In this section we assume that $A = A_R = A_L$ is a constant matrix. We start with linear stability, i.e., $\varepsilon_1 = \varepsilon_2 = 0$ and consider

$$\begin{aligned} u_t + Au_x &= u_{xx} + F_x, \\ u(x,0) &= 0. \end{aligned} \tag{21}$$

We use the notation

$$\|u(\cdot,t)\|_{2,p}^2 = \sum_{j=0}^{p} \int_{-\infty}^{\infty} |D_x^j u(x,t)|^2 dx, \quad \|u(\cdot,t)\|_{1,p} = \sum_{j=0}^{p} \int_{-\infty}^{\infty} |D_x^j u(x,t)| dx.$$

(If $p = 0$, we suppress it.) We want to prove

Lemma 1 *For any $p = 1, 2, \ldots$, there are constants R_p independent of T and F such that the solutions of (21) satisfy the estimate*

$$\begin{aligned} &\int_0^T \|u(\cdot,t)\|_{2,p+1}^2 + \|u_t(\cdot,t)\|_{2,p-1}^2 dt \\ &\le R_p \left((\int_0^T \|F(\cdot,t)\|_{1,p} dt)^2 + \int_0^T \|F(\cdot,t)\|_{2,p}^2 dt \right). \end{aligned} \tag{22}$$

Proof. Denote by

$$\begin{aligned} \hat{u}(\omega,s) &= \int_0^{\infty} \int_{-\infty}^{\infty} e^{-st} e^{i\omega x} u(x,t) dx\, dt, \\ s &= i\xi + \eta, \ \xi, \eta \text{ real}, \ \eta > 0 \end{aligned}$$

the Fourier-Laplace transform of u. It solves

$$\Big((s+\omega^2)I - i\omega A\Big)\hat{u} = i\omega\hat{F}.$$

By assumption, there is a transformation S such that

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} =: \Lambda, \quad \lambda_j \ real.$$

Therefore,

$$\Big((s+\omega^2)I - i\omega A\Big)^{-1} = S^{-1}\Big((s+\omega^2)I + i\omega\Lambda\Big)^{-1}S$$

exists for $\eta > 0$ and

$$|\hat{u}(\omega,s)|^2 \leq |S^{-1}|\,|S|\,|\hat{F}|^2 \sum_{k=1}^{n} \frac{\omega^2}{(\eta+\omega^2)^2 + (\xi-\lambda_k)^2}. \tag{23}$$

We write $\hat{u} = \hat{u}^I + \hat{u}^{II}$ where

$$\hat{u}^I = \begin{cases} \hat{u} & \text{for } |\omega| \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad \hat{u}^{II} = \begin{cases} 0 & \text{for } |\omega| \leq 1 \\ \hat{u} & \text{for } |\omega| > 1 \end{cases}.$$

The inequality (23) holds with $\hat{u}, \hat{F}$ replaced by $\hat{u}^I, \hat{F}^I$ or by $\hat{u}^{II}, \hat{F}^{II}$. Since

$$|\hat{F}^I(\omega,s)|^2 = |\int_0^\infty \int_{-\infty}^\infty e^{-st}e^{-i\omega x}F(x,t)dx\,dt|^2 \leq \left(\int_0^\infty \|F(\cdot,t)\|_1 dt\right)^2$$

Parseval's relation gives us

$$\begin{aligned} &\int_0^\infty e^{-2\eta t}\|u^I(\cdot,t)\|^2 dt \\ \leq\ & |S^{-1}|\,|S| \left(\int_0^\infty \|F(\cdot,t)\|_1 dt\right)^2 \\ &\times \sum_{k=1}^{n} \int_{-1}^{1}\left(\int_{-\infty}^{\infty} \frac{\omega^2}{(\eta+\omega^2)^2+(\xi-\lambda_k)^2} d\xi\right) d\omega. \end{aligned}$$

Since

$$\begin{aligned} \int_{-1}^{1}\left(\int_{-\infty}^{\infty} \frac{\omega^2}{(\eta+\omega^2)^2+(\xi-\lambda_k)^2} d\xi\right) d\omega &\leq \int_{-1}^{1}\left(\int_{-\infty}^{\infty} \frac{\omega^2}{\omega^4+\xi^2} d\xi\right) d\omega \\ &\leq \int_{-1}^{1}\left(\int_{-\infty}^{\infty} \frac{d\xi}{\xi^2+1}\right) d\omega \leq const. \end{aligned}$$

it follows that there is a constant C_1 such that

$$\int_0^\infty \|u^I(\cdot,t)\|_2^2 dt \le C_1 \left(\int_0^\infty \|F(\cdot,t)\|_1 dt\right)^2.$$

Observing that $|\omega| \le 1$ and that future does not effect the past, we also have

$$\int_0^T \|u^I(\cdot,t)\|_{2,1}^2 dt \le 2C_1 \left(\int_0^T \|F(\cdot,t)\|_1 dt\right)^2. \tag{24}$$

We shall estimate u^{II}. Since $\hat{u}^{II} = 0$ for $|\omega| \le 1$, we obtain from (23) that there is a constant C_2 such that

$$(1+\omega^2)|\hat{u}^{II}(\omega,s)|^2 \le C_2 |\hat{F}^{II}|^2.$$

By Parseval's relation, we now obtain

$$\int_0^T \|u^{II}(\cdot,t)\|_{2,1}^2 dt \le C_2 \int_0^T \|F(\cdot,t)\|_2^2 dt.$$

Thus,

$$\int_0^T \|u(\cdot,t)\|_{2,1}^2 dt \le 2C_1 \left(\int_0^T \|F(\cdot,t)\|_1 dt\right)^2 + C_2 \int_0^T \|F(\cdot,t)\|_2^2 dt. \tag{25}$$

Differentiating (21) and using it to express the time derivative in terms of space derivatives, the desired estimate follows from (25). This proves the lemma.

The estimate (22) tells us that our problem is linearly stable. We shall now prove that it is also nonlinearly stable. We will use the notation

$$E = R_p \left(\left(\int_0^T \|F(\cdot,t)\|_{1,p} dt\right)^2 + \int_0^T \|F(\cdot,t)\|_{2,p}^2 dt \right).$$

Local existence for the solution of (1),(2) is no problem and in a sufficiently small time interval $0 \le t \le T_\varepsilon$, $T_\varepsilon > 0$, the solution of the nonlinear problem satisfies the estimate

$$\int_0^{T_\varepsilon} \|u(\cdot,t)\|_{2,p+1}^2 + \|u_t(\cdot,t)\|_{2,p-1}^2 dt \le 4E. \tag{26}$$

There are two possibilities.

1. $T_\varepsilon = \infty$. In this case $|u(\cdot,t)|_\infty \to 0$.

2. T_ε is finite and equality holds in (26).

By choosing $\varepsilon_1^2 + \varepsilon_2^2$ sufficiently small and p sufficiently large, we will show that the second case cannot happen.

We consider $\varepsilon_1(Bu)_x + \varepsilon_2 g_x$ as part of the forcing. By Lemma 1

$$\begin{aligned}
&\int_0^{T_\varepsilon} \|u(\cdot,t)\|_{2,p+1}^2 + \|u_t(\cdot,t)\|_{2,p-1}^2 dt \\
\leq\ & R_p \left(\left(\int_0^{T_2} \|F + \varepsilon_1 Bu + \varepsilon_2 g\|_{1,p} dt \right)^2 + \int_0^{T_\varepsilon} \|F + \varepsilon_1 Bu + \varepsilon_2 g\|_{2,p}^2 dt \right) \\
\leq\ & E + K_1(\varepsilon_1^2 + \varepsilon_2^2) \Bigg(\left(\int_0^{T_\varepsilon} \|Bu\|_{1,p} dt \right)^2 + \int_0^{T_\varepsilon} \|Bu\|_{2,p}^2 dt \\
+\ & \left(\int_0^{T_\varepsilon} \|g\|_{1,p} dt \right)^2 + \int_0^{T_\varepsilon} \|g\|_{2,p}^2 dt \Bigg).
\end{aligned}$$

By (4), there is a constant K_0 such that

$$\begin{aligned}
\left(\int_0^{T_\varepsilon} \|Bu\|_{1,p} dt \right)^2 &+ \int_0^{T_\varepsilon} \|Bu\|_{2,p}^2 dt \\
&\leq K_0 \left(\left(\int_0^{T_\varepsilon} \|B\|_{2,p} dt + 1 \right) \cdot \int_0^{T_\varepsilon} \|u\|_{2,p}^2 dt \right).
\end{aligned}$$

By Sobolev inequalities, we can use (26) to estimate $|D^j u|_\infty$, $j \leq p-2$, in terms of E. Since, by (6), the nonlinear term g is quadratic in u, we can, in the usual way, also estimate

$$\left(\int_0^{T_\varepsilon} \|g\|_{1,p} dt \right)^2 + \int_0^{T} \|g\|_{2,p}^2 dt$$

in terms of E, provided p is sufficiently large. (For more details, see [6].) Choosing $\varepsilon_1^2 + \varepsilon_2^2$ sufficiently small we will never reach equality in (26). Thus, (26) holds for all times and the problem is nonlinearly stable. This proves Theorem 1.

3 Auxilary estimates

In this section we shall collect a number of elementary estimates. We use the notation

$$|y|_\infty = \sup_{0\le x<\infty} |y(x)|, \quad \|y\|_1 = \int_0^1 |y(x)|dx, \quad \|y\|^2 = \int_0^1 |y(x)|^2 dx.$$

Lemma 2 *Consider the scalar differential equation*

$$\frac{dy}{dx} = \lambda y + F(x), \quad x \ge 0, \tag{27}$$

where λ with $Re\,\lambda > 0$ is a constant and $F(x)$ is a smooth function with compact support. It has a unique bounded solution satisfying the estimates

$$|y|_\infty \le \|F\|_1, \quad \|y\| \le \frac{1}{\sqrt{|Re\,\lambda|}}\|F\|_1, \quad \|y\|_1 \le \frac{1}{|Re\,\lambda|}\|F\|_1. \tag{28}$$

Proof. We can write down the solution of (27) explicitly. It is given by

$$y(x) = -\int_x^\infty e^{\lambda(x-\xi)}F(\xi)d\xi.$$

Therefore,

$$|y(x_0)| \le \int_{x_0}^\infty e^{Re\,\lambda(x_0-\xi)}|F(\xi)|d\xi \le \|F\|_1.$$

Since

$$\int_x^\infty e^{Re\,\lambda(x-\xi)}|F(\xi)|d\xi \le \|F\|_1,$$

we obtain the second inequality:

$$\begin{aligned}
\|y\|^2 &= \int_0^\infty |y(x)|^2 dx \le \int_0^\infty \Big(\int_x^\infty e^{Re\,\lambda(x-\xi)}|F(\xi)|d\xi\Big)^2 dx \\
&\le \|F\|_1 \cdot \int_0^\infty |F(\xi)| \cdot \int_x^\infty e^{Re\,\lambda(x-\xi)} dx d\xi \\
&\le \frac{1}{Re\,\lambda}\|F_1\|^2.
\end{aligned}$$

The third inequality follows from

$$\begin{aligned}
\|y\|_1 &\le \int_0^\infty \int_x^\infty e^{Re\,\lambda(x-\xi)}|F(\xi)|d\xi\, dx \\
&\le \int_0^\infty \int_0^\infty e^{-Re\,\lambda\xi'}|F(x+\xi')|d\xi'\, dx \le \frac{1}{Re\,\lambda}\|F\|_1.
\end{aligned}$$

This proves the lemma.

We consider (27) for $Re\,\lambda < 0$ and give boundary conditions

$$y(0) = y_0. \tag{29}$$

In this case we have

Lemma 3 *Let $Re\,\lambda < 0$ and consider the scalar equation with boundary conditions (29) Its solutions satisfy the modified estimates*

$$\begin{aligned} |y|_\infty &\le |y(0)| + \|F\|_1, \\ \|y\| &\le \frac{1}{\sqrt{|Re\,\lambda|}}(|y(0)| + \|F\|_1) \\ \|y\|_1 &\le \frac{1}{|Re\,\lambda|}(|y(0)| + \|F\|_1). \end{aligned} \tag{30}$$

Proof. The solution of (27),(29) can be written as

$$y = y_1 + y_2$$

where

$$\frac{dy_1}{dx} = \lambda y_1 + F(x), \quad y_1(0) = 0, \tag{31}$$

$$\frac{dy_2}{dx} = \lambda y_2, \quad y_2(0) = y_0. \tag{32}$$

By the same argument as above we obtain the desired estimate for (31). The solution of (32) is given by

$$y_2(x) = e^{\lambda x} y_0.$$

Therefore,

$$\begin{aligned} |y_2(x_0)| &\le |y_0|, \\ \|y\|^2 &= \int_0^\infty e^{2Re\,\lambda\,x} dx \cdot |y_0|^2 = \frac{1}{2|Re\,\lambda|} \cdot |y_0|^2, \\ \|y\|_1 &= \int_0^\infty e^{Re\,\lambda x} dx \cdot |y_0| = \frac{1}{|Re\,\lambda|}|y_0|, \end{aligned} \tag{33}$$

and the lemma follows.

Now we consider a system

$$y' = \Lambda y + e^{-\alpha x} B(x) y + F, \quad x \geq 0. \tag{34}$$

Here

$$\Lambda = \begin{pmatrix} -\Lambda^I & 0 \\ 0 & \Lambda^{II} \end{pmatrix}, \quad \Lambda^I + (\Lambda^I)^* > 0, \quad \Lambda^{II} + (\Lambda^{II})^* > 0,$$

is a complex valued constant diagonal matrix, $\alpha > 0$ is a constant and $B(x), F(x)$ with $\|F\|_1 < \infty$ are bounded smooth functions of x. We want to prove

Lemma 4 *Consider (34) with boundary conditions*

$$y^I(0) = y_0^I \tag{35}$$

and assume that

$$\int_0^\infty e^{-\alpha x} |B| dx < \frac{1}{2}. \tag{36}$$

Then (35),(36) has a unique bounded solution with

$$\begin{aligned} |y|_\infty &\leq 2(\|F\|_1 + |y_0^I|), \\ \|y\| &\leq |(\Lambda + \Lambda^*)^{-1}|^{1/2}(\|F\|_1 + |y_0^I|), \\ \|y\|_1 &\leq |(\Lambda + \Lambda^*)^{-1}|(\|F\|_1 + |y_0^I|). \end{aligned} \tag{37}$$

Proof. Existence and uniqueness cause no difficulties. To obtain the estimates we consider $e^{-\alpha x} By$ as part of the forcing and apply Lemma 2 and Lemma 3

$$|y|_\infty \leq |y_0^I| + \|F\|_1 + \|e^{-\alpha x} B(x) y\|_1 \leq |y_0^I| + \|F\|_1 + \frac{1}{2}|y|_\infty.$$

Therefore, the first estimate follows. We have

$$\begin{aligned} \|y\| &\leq \frac{1}{2}\left|(\Lambda + \Lambda^*)^{-1}\right|^{1/2} \left(|y_0^I| + \|F\|_1 + \|e^{-\alpha x} B(x) y\|_1\right) \\ &\leq \frac{1}{2}\left|(\Lambda + \Lambda^*)^{-1}\right|^{1/2} \left(|y_0^I| + \|F\|_1 + \frac{1}{2}|y|_\infty\right), \end{aligned}$$

and the second estimate follows from the first. The third estimate follows in the same manner.

We consider now the homogeneous system

$$y' = \Lambda y + e^{-\alpha x} B(x) y \tag{38}$$

with boundary conditions (35) and assume that (36) holds. For every $y^I(0)$, we can obtain a unique bounded solution. In particular, $y^{II}(0)$ is determined. Thus, we can consider (35),(38) as a linear mapping

$$y^{II}(0) = Q y^I(0). \tag{39}$$

We can prove

Lemma 5 *Assume that (36) holds. Then*

$$|Q| \leq 2 \int_0^\infty e^{-\alpha x} |B(x)| dx = 2 \| e^{-\alpha x} |B(x)| \, \|_1. \tag{40}$$

Proof. We first solve the auxilary problem

$$\begin{aligned} v' &= \Lambda v, \\ v^I(0) &= y^I(0), \quad |v|_\infty < \infty. \end{aligned}$$

Its solution is given by

$$\begin{aligned} v^I &= e^{-\Lambda^I x} y_0^I, \quad v^{II} \equiv 0, \quad \textit{i.e.}, \\ |v^I|_\infty &\leq |y_0^I|, \quad |v^{II}|_\infty = 0. \end{aligned}$$

The difference $w = y - v$ solves

$$\begin{aligned} w' &= \Lambda w + e^{-\alpha x} B(x) w + e^{-\alpha x} B(x) v, \\ & w^I(0) = 0. \end{aligned}$$

By Lemma 4,

$$|w|_\infty \leq 2 \| e^{-\alpha x} B(x) \|_1 |y_0^I|.$$

Therefore, the desired estimate follows.

We consider again (38) with boundary conditions (35) but we now assume that

$$B = B(x, \sigma) = \sum_{\nu=0}^{\infty} \sigma^\nu B_\nu(x)$$

is an analytic function of a complex valued parameter σ. We want to prove

Lemma 6 *Assume that (36) holds for $|\sigma| \leq \sigma_0$, $\sigma_0 = const. > 0$. Then, for $|\sigma| < \sigma_0$, the mapping (39) is analytic, i.e., $Q(\sigma)$ is an analytic function.*

Proof. Differentiating (38) with respect to σ gives us

$$\begin{aligned}
(\frac{\partial y}{\partial \sigma})' &= \Lambda \frac{\partial y}{\partial \sigma} + e^{-\alpha x} B(x,\sigma)\frac{\partial y}{\partial \sigma} + e^{-\alpha x}\frac{\partial B}{\partial \sigma} y, \\
&\frac{\partial y^I(0)}{\partial \sigma} = 0.
\end{aligned}$$

By Lemma 4, we have uniform bounds for $\partial y/\partial \sigma$ and the lemma follows.

We consider the case that both Λ and B are analytic functions of σ, i.e., we consider

$$\begin{aligned}
y' &= (\Lambda_0 + \sigma\tilde{\Lambda}(\sigma))y + e^{-\alpha x}B(x,\sigma)y, \\
&y^I(0) = y_0^I.
\end{aligned} \tag{41}$$

We can prove

Lemma 7 *Assume that, for $|\sigma| \leq \sigma_0$,*

$$\int_0^\infty e^{-\frac{\alpha}{2}x}|B|dx \leq \frac{1}{2}, \quad -\frac{\alpha}{4}I + \sigma\tilde{\Lambda} \leq 0, \quad -\frac{\alpha}{4}I - \sigma\tilde{\Lambda} \leq 0.$$

Then the corresponding mapping (39) is analytic in σ for $|\sigma| < \sigma_0$.

Proof. Introducing a new variable by $y = e^{\sigma\tilde{\Lambda}x}\tilde{y}$ into (41) gives us

$$\begin{aligned}
\tilde{y}' &= \Lambda_0\tilde{y} + e^{-\frac{\alpha}{2}x}\tilde{B}(x,\sigma)\tilde{y}, \\
&\tilde{y}^I(0) = y_0^I.
\end{aligned}$$

Here

$$|\tilde{B}(x,\sigma)| = |e^{-(\frac{\alpha}{4}I+\sigma\tilde{\Lambda})x}B(x,\sigma)e^{-(\frac{\alpha}{4}I-\sigma\tilde{\Lambda})x}| \leq |B(x,\sigma)|.$$

Therefore, the lemma follows from the previous lemma.

4 Variable coefficients

In this section we assume that $A(x)$ has the form (3). We start again with linear stability when $\varepsilon_1 = \varepsilon_2 = 0$. Laplace transform in time and Fourier transform in space give us

$$\hat{u}_{xx} - \Big(A(x)\hat{u}\Big)_x - s\hat{u} = \hat{F}_x, \quad \|\hat{u}\| =: \int_{-\infty}^{\infty} |\hat{u}|^2 dx < \infty. \tag{42}$$

We shall derive some properties of the corresponding eigenvalue problem

$$\varphi_{xx} - \Big(A(x)\varphi\Big)_x = \mu\varphi, \quad \|\varphi\| < \infty. \tag{43}$$

For large x, we can write

$$A(x) = A_R + e^{-x}\tilde{A}_R(x)$$

and, to first approximation, we can replace (43) by

$$\varphi_{xx} - A_R\varphi_x = \mu\varphi.$$

Introducing $\tilde{\varphi} = S_R^{-1}\varphi$ as a new variable we obtain the scalar equations

$$\tilde{\varphi}_{jxx} - \lambda_j\varphi_{jx} = \mu\varphi_j, \quad j = 1, 2, \ldots, n.$$

Here λ_j are the eigenvalues of A_R. The general solution is of the form

$$\tilde{\varphi}_j = \sigma_1 e^{\kappa_{1j}x} + \sigma_2 e^{\kappa_{2j}x},$$

where κ_{1j}, κ_{2j} solve the characteristic equation

$$\kappa^2 - \lambda_j\kappa - \mu = 0. \tag{44}$$

We need

Lemma 8 *The solutions of (44)*

$$\kappa_{1j} = \frac{\lambda_j}{2}\left(1 + \sqrt{1 + \frac{4}{\lambda_j^2}\mu}\right), \quad \kappa_{2j} = \frac{\lambda_j}{2}\left(1 - \sqrt{1 + \frac{4}{\lambda_j^2}\mu}\right),$$

have the following properties for $Re\,\mu \geq 0$, $\mu \neq 0$ *:*

1. *There is exactly one* κ *with* $Re\,\kappa > 0$ *and one* κ *with* $Re\,\kappa < 0$.

2.

$$sign(Re\,\kappa_{1j}) = sign\,\lambda_j, \quad sign(Re\,\kappa_{2j}) = -sign\,\lambda_j.$$

3. For $|\mu| << 1$,

$$\kappa_{1j} = \lambda_j + \mathcal{O}(\mu), \quad \kappa_{2j} = -\frac{\mu}{\lambda_j} + \frac{\mu^2}{\lambda_j^3} + \mathcal{O}(\mu^3).$$

4. For every constant $\delta > 0$, there is a constant $\varrho > 0$ such that

$$|Re\,\kappa_{ij}| \geq \varrho, \; i = 1,2 \quad \textit{for all } \mu \textit{ with } |\mu| \geq \delta, \; Re\,\mu \geq 0. \tag{45}$$

Proof. Assume that $\kappa = i\tau \neq 0$ is a purely imaginary root of (44). Then

$$-\tau^2 - i\tau\lambda = \mu, \quad i.e., \; Re\,\mu < 0,$$

and property 1 follows. For $\mu = 1$, property 2 holds. Since the real part cannot change sign, property 2 holds for all $\mu \neq 0$ with $\mathrm{Re}\,\mu \geq 0$. The usual expansions give us property 3. Let $\mu = i\xi + \eta$, ξ, η real. For small $|\mu|$,

$$Re\,\kappa_2 = -\frac{\eta}{\lambda_j} - \frac{\xi^2}{\lambda_j^3} + \mathcal{O}(\eta^2 + \xi^2\eta). \tag{46}$$

Thus, (45) holds for small $|\mu|$. It also holds for large $|\mu|$. Therefore, using a compactness argument, property 4 follows. This proves the lemma.

Lemma 8 gives us

Theorem 3 *Consider (42) for $|s| \geq \delta > 0$, $Re\,s \geq 0$. If assumption 1 holds, there is a constant $K_1 = K_1(\delta)$ such that the solutions of (42) satisfy the estimate*

$$\|\hat{u}\|^2 + \|\hat{u}_x\|^2 \leq K_1(\delta)\|\hat{F}\|^2.$$

Proof. For large $|s|$, the estimate can be obtained by integration by parts. For $|s| \geq \delta$, $|s|$ bounded, one can use a compactness argument. (For details, see [8].)

Thus, we need only consider the case that $|s| << 1$. Corresponding to the discussion in Section 2, we want to derive an estimate

$$\|\hat{u}\|^2 + \|\hat{u}_x\|^2 \leq K_2(\delta)\left(\|\hat{F}\|^2 + \|\hat{F}\|_1^2\right), \quad Re\,s > 0, \; |s| \leq \delta. \tag{47}$$

We first solve an auxiliary problem

$$w_{xx} - \Big(A(x)w\Big)_x = \hat{F}_x, \quad \|w\| < \infty,$$

or

$$w_x - A(x)w = \hat{F}, \quad \|w\| < \infty. \tag{48}$$

We split the interval $-\infty < x < \infty$ into subintervals

$$x \leq -l, \quad -l \leq x \leq l, \quad x \geq l, \quad l >> 1.$$

By (3) and (7), $\tilde{w} = S_R^{-1} w$ satisfies

$$\tilde{w}_x - (\Lambda_R + e^{-x}\tilde{B}_R)\tilde{w} = \tilde{F}, \quad x \geq l-3. \tag{49}$$

Using Lemma 4 and (48) to estimate $\tilde{w}_{Rx}$, we determine a solution of (49) with homogeneous boundary conditions, for $x = l - 3$, which satisfies the estimates

$$\int_{l-3}^{\infty} |\tilde{w}_R|^2 dx \leq 4|\Lambda_R^{-1}| \left(\int_{l-3}^{\infty} |\tilde{F}| dx \right)^2, \tag{50}$$

$$\int_{l-3}^{\infty} |\tilde{w}_{Rx}|^2 dx \leq 2|\Lambda_R + \mathcal{O}(e^{-l})|^2 \int_{l-3}^{\infty} |\tilde{w}|^2 dx + 2\int_{l-3}^{\infty} |\tilde{F}|^2 dx, \tag{51}$$

$$\int_{l-3}^{\infty} |\tilde{w}_R| dx \leq 2|\Lambda_R^{-1}| \int_{l-3}^{\infty} |\tilde{F}| dx, \quad |\tilde{w}_R|_\infty \leq 2\int_{l-3}^{\infty} |\tilde{F}| dx. \tag{52}$$

Let $\varphi_R \in C^\infty$ be a monotone cut-off function with

$$\varphi_R = \begin{cases} 0 & \text{for } x \leq l-2 \\ 1 & \text{for } x \geq l-1 \end{cases}.$$

Then the function

$$w_R = \varphi_R S_R \tilde{w}_R$$

satisfies the same type of estimates.

The corresponding construction and estimates hold for $x < 0$. Thus,

$$w_1 = w - (w_R + w_L)$$

satisfies

$$w_{1x} - A(x)w_1 = F_1, \quad \|w_1\| < \infty. \tag{53}$$

Here

$$F_1 = F - (w_R + w_L) - G(x), \quad G = (\varphi_R)_x S_R \tilde{w}_R + (\varphi_L)_x S_L \tilde{w}_L,$$

has compact support in $-l \leq x \leq l$. Since $G(x)$ has only local support, we can use the maximum norm estimate of (52) to estimate its L_1 and L_2 norms. Therefore, we can estimate the L_1 and L_2 norms of F_1 in terms of the L_1 and L_2 norms of $\hat{F}$.

To solve (53) we first reduce the interval of integration to $-l \leq x \leq l$. The boundary conditions at $x = l$ are provided by (49) with $\tilde{F} \equiv 0$. By Lemma 5, they can be written as

$$\tilde{w}_1^{II}(l) = e^{-l} Q_1(l) \tilde{w}_1^I(l), \quad \tilde{w}_1(l) = S_R w_1(l). \tag{54}$$

Correspondingly,

$$\tilde{w}_1^I(-l) = e^{-l} Q_1(-l) \tilde{w}_1^{II}(-l), \quad \tilde{w}_1(-l) = S_L w_1(-l). \tag{55}$$

Note that the same reduction procedure can be applied to (16) and (17). Therefore, it follows by assumption 2 that we can solve (53) with boundary conditions (54),(55). If $m = n$ (see (11)), then (54),(55) represent n boundary conditions and the solution is unique. If $m = n+1$, then there are $n-1$ boundary conditions and the solution is not unique. We choose the solution such that its L_2 norm is minimal. We have estimates

$$\begin{aligned} \int_{-l}^{l} (|w_1|^2 + |w_{1x}|^2) dx &\leq const. \int_{-l}^{l} |F_1|^2 dx, & (56) \\ \max_{-l \leq x \leq l} |w_1|^2 + \left(\int_{-l}^{l} |w_1| dx\right)^2 &\leq const. \int_{-l}^{l} |F_1|^2 dx. & (57) \end{aligned}$$

We extend this solution to $l \leq x < \infty$ by again using the homogeneous system (49) with the now known boundary values $\tilde{w}_1^I(l)$. By Lemma 4 and using the differential equation, we obtain

$$\begin{aligned} \int_l^\infty |\tilde{w}_1|^2 dx &\leq 2|\Lambda_R^{-1}|\, |\tilde{w}_1(l)|^2, \\ \int_l^\infty |\tilde{w}_{1x}|^2 dx &\leq |\Lambda_R^{-1} + \mathcal{O}(e^{-l})| \int_l^\infty |\tilde{w}_1|^2 dx, \\ \int_l^\infty |\tilde{w}_1| dx &\leq 2|\Lambda_R^{-1}|\, |\tilde{w}_1^I(l)|. \end{aligned}$$

For $x \leq -l$ we proceed correspondingly.

Using (50)-(52), (56),(57), we have proved

Lemma 9 *There is a constant C such that the equation (48) has a solution w which satisfies the estimate*

$$\|w\|^2 + \|w_x\|^2 + \|w\|_1^2 \leq C(\|\hat{F}\|^2 + \|\hat{F}\|_1^2).$$

The function $u_1 = \hat{u} - w$ solves

$$u_{1xx} - \Big(A(x)u_1\Big)_x - su_1 = sw, \quad \|u_1\| < \infty. \tag{58}$$

We write (58) as a first order system

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_x = \begin{pmatrix} A & I \\ sI & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ sw \end{pmatrix} \tag{59}$$

and consider (59) for $x \geq l - 3$. Here we can write

$$\begin{pmatrix} A & I \\ sI & 0 \end{pmatrix} = \begin{pmatrix} A_R & I \\ sI & 0 \end{pmatrix} + e^{-x} \begin{pmatrix} B_R & 0 \\ 0 & 0 \end{pmatrix}.$$

By (7),

$$S_0^{-1} \begin{pmatrix} A & I \\ sI & 0 \end{pmatrix} S_0 = \begin{pmatrix} \Lambda_R & I \\ sI & 0 \end{pmatrix} + e^{-x} \begin{pmatrix} \tilde{B}_R & 0 \\ 0 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} S_R & 0 \\ 0 & S_R \end{pmatrix}.$$

By Lemma 8, we can find a lower triangular transformation

$$S_1 = I - s \begin{pmatrix} 0 & 0 \\ \Lambda_R^{-1} & 0 \end{pmatrix} + \mathcal{O}(s^2), \tag{60}$$

analytic in s, such that

$$\begin{aligned} &S_1^{-1} \left(\begin{pmatrix} \Lambda_R & I \\ sI & 0 \end{pmatrix} + e^{-x} \begin{pmatrix} \tilde{B}_R & 0 \\ 0 & 0 \end{pmatrix} \right) S_1 \\ &= \begin{pmatrix} \mathcal{K}_1 & I \\ 0 & \mathcal{K}_2 \end{pmatrix} + e^{-x} \begin{pmatrix} \tilde{B}_R & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(se^{-x}). \end{aligned}$$

Here

$$\begin{aligned} \mathcal{K}_1 &= \begin{pmatrix} -\mathcal{K}_1^I & 0 \\ 0 & \mathcal{K}_1^{II} \end{pmatrix} = \Lambda_R + \mathcal{O}(s), \\ \mathcal{K}_2 &= \begin{pmatrix} \mathcal{K}_2^I & 0 \\ 0 & -\mathcal{K}_2^{II} \end{pmatrix} = -s\Lambda_R^{-1} + s^2\Lambda_R^{-3} + \mathcal{O}(s^3) \end{aligned}$$

where

$$\mathcal{K}_i^I + (\mathcal{K}_i^I)^* > 0, \quad \mathcal{K}_i^{II} + (\mathcal{K}_i^{II})^* > 0, \quad i = 1, 2\,,$$

are diagonal matrices with eigenvalues κ_{1j} and κ_{2j}, respectively. In particular, there is a constant $d > 0$ such that

$$|(\mathcal{K}_2 + \mathcal{K}_2^*)^{-1}| \leq d|s|^{-2}, \quad Re\, s \geq 0, \; s \neq 0. \tag{61}$$

Finally, there is an upper triangular matrix

$$S_2 = \begin{pmatrix} I & -\Lambda_R^{-1} \\ 0 & I \end{pmatrix} + \mathcal{O}(s),$$

analytic in s, such that

$$\begin{aligned} & S_2^{-1} \left(\begin{pmatrix} \mathcal{K}_1 & I \\ 0 & \mathcal{K}_2 \end{pmatrix} + e^{-x} \begin{pmatrix} \tilde{B}_R & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(se^{-x}) \right) S_2 \\ &= \begin{pmatrix} \mathcal{K}_1 & 0 \\ 0 & \mathcal{K}_2 \end{pmatrix} + e^{-x} B, \quad B = \begin{pmatrix} \tilde{B}_R & -\tilde{B}_R \Lambda_R^{-1} \\ 0 & 0 \end{pmatrix} + sB_1. \end{aligned}$$

We now proceed as before. We make a change of variables

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = S_0 S_1 S_2 \begin{pmatrix} \overline{u}_1 \\ \overline{v}_1 \end{pmatrix}, \tag{62}$$

such that (59) becomes

$$\begin{pmatrix} \overline{u}_1 \\ \overline{v}_1 \end{pmatrix}_x = \left(\begin{pmatrix} \mathcal{K}_1 & 0 \\ 0 & \mathcal{K}_2 \end{pmatrix} + e^{-x} B \right) \begin{pmatrix} \overline{u}_1 \\ \overline{v}_1 \end{pmatrix} + s\tilde{w}, \quad x \geq l-3. \tag{63}$$

We solve (63) with boundary conditions

$$\overline{u}_1^I(l-3) = \overline{v}_1^{II} = 0.$$

By Lemma 4, (61) and (63), for all sufficiently large l,

$$\begin{aligned} & \int_{l-3}^{\infty} (|\overline{u}_1|^2 + |\overline{v}_1|^2) dx \\ \leq \; & \frac{2|s|^2 d}{|s|^2} \left(\int_{l-3}^{\infty} |\tilde{w}| dx \right)^2 = 2d \left(\int_{l-3}^{\infty} |\tilde{w}| dx \right)^2, \end{aligned} \tag{64}$$

$$\begin{aligned} & \int_{l-3}^{\infty} (|\overline{u}_{1x}|^2 + |\overline{v}_{1x}|^2) dx \\ \leq \; & const. \left(\left(\int_{l-3}^{\infty} |\tilde{w}| dx \right)^2 k + |s|^2 \int_{l-3}^{\infty} |\tilde{w}|^2 dx \right), \end{aligned} \tag{65}$$

$$\sup_{x \geq l-3} (|\overline{u}_1|^2 + |\overline{v}_1|^2) \leq 4|s|^2 \left(\int_{l-3}^{\infty} |\tilde{w}| dx \right)^2. \tag{66}$$

We solve the corresponding problem for $x < 0$, piece the two solutions together and subtract these functions from u_1 and v_1 to obtain

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} A & I \\ sI & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + sw_1, \quad w_1 = \begin{pmatrix} w_{11} \\ w_{12} \end{pmatrix}. \tag{67}$$

Here w_1 has support in $-l \le x \le l$. (Observe that terms proportional to $(\varphi_R)_x$, $(\varphi_L)_x$ have only local support and can be estimated with help of (66).)

This reduction process can be performed for any sufficiently large l. We fix $l = l_0$. Then there is a constant C such that

$$\int_{-l}^{l} (|w_1|^2 + |w_{1x}|^2)dx \le C(\|w\|^2 + \|w\|_1^2). \tag{68}$$

We now replace the infinite interval $-\infty < x < \infty$ by a finite interval $-l \le x \le l$, $l \ge l_0$ sufficiently large. Introduce the same change of variables as before, see (62). The special form of $e^{-x}B$ and Lemma 5 show that the boundary conditions for $\overline{v}_2$ are of the form

$$\overline{v}_2^I(l) = \mathcal{O}(|s|e^{-l}), \quad \overline{v}_2^{II}(-l) = \mathcal{O}(|s|e^{-l}). \tag{69}$$

Considering $\overline{v}_2$ a given function, the boundary conditions for $\overline{u}_2$ are given by

$$\overline{u}_2^{II}(l) = e^{-l}\left(Q_1(l)\overline{u}_2^I(l) + Q_2(l)\overline{v}_2(l)\right) + \mathcal{O}(|s|e^{-l}), \tag{70}$$

$$\overline{u}_2^I(-l) = e^{-l}\left(Q_1(-l)\overline{u}_2^{II}(-l) + Q_2(-l)\overline{v}_2(-l)\right) + \mathcal{O}(|s|e^{-l}). \tag{71}$$

Here $Q_1(l), Q_1(-l)$ are given in (54) and (55), respectively. $\mathcal{O}(|s|e^{-l})$ stands for linear expressions in $\overline{u}_2$, $\overline{v}_2$ whose coefficients are of order $\mathcal{O}(|s|e^{-l})$.

Before we can estimate the solution of (67), (69)–(71), we need to discuss the corresponding eigenvalue problem. We can write it as the first order system

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_x = \begin{pmatrix} A & I \\ \mu I & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

and reduce the infinite interval to $-l \le x \le l$. By the same procedure, the boundary conditions are given by (69)–(71), with $\bar{u}_2, \bar{v}_2, s$ replaced by $\bar{\varphi}, \bar{\psi}$ and μ, respectively. For $\mu = 0$, we obtain

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_x = \begin{pmatrix} A & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad -\infty < x < \infty, \tag{72}$$

with boundary conditions

$$\bar{\varphi}^{II}(l) = e^{-l}\left(Q_1(l)\bar{\varphi}^I(l) + Q_2(l)\bar{\psi}(l)\right), \quad \bar{\psi}^I(l) = 0, \tag{73}$$

$$\bar{\varphi}^I(-l) = e^{-l}\left(Q_1(-l)\bar{\varphi}^{II}(-l) + Q_2(-l)\bar{\psi}(-l)\right), \quad \bar{\psi}^{II}(-l) = 0. \tag{74}$$

(72)–(74) give us

$$\psi(l) - \psi(-l) = 0, \qquad \bar{\psi}^{I}(l) = 0, \quad \bar{\psi}^{II}(-l) = 0. \tag{75}$$

Inverting the transformation (62), we can express $\bar{\varphi}, \bar{\psi}$ interms of φ, ψ. In particular, since $S_1 = I$ for $\mu = 0$ and S_2 is upper triangular,

$$\bar{\psi}(l) = S_R^{-1}\psi(l), \quad \bar{\psi}(-l) = S_L^{-1}\psi(-l)$$

are independent of φ. Thus, for $m = n$ or $m = n+1$, (75) represents $2n$ or $2n+1$ linear relations for the $2n$ unknowns of $\psi(l)$, $\psi(-l)$. We make

Algebraic assumption 1 *The linear system (75) has only the trivial solution.*

We can now prove

Lemma 10 *Assumption 3 and the Algebraic assumption 1 are equivalent.*

Proof. If the system (75) has only the trivial solution, then $\psi \equiv 0$ and (72)–(74) reduces to (20) with $D = 0$. If (75) has a nontrivial solution, then, for $l \to \infty$, we obtain a solution of (20) with $D \neq 0$. This proves the lemma.

In the case $m = n + 1$ we also need to consider

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_x = \begin{pmatrix} A & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}, \quad -l \le x \le l, \tag{76}$$

with boundary conditions (73) and (74). Here φ_0 is the eigenfunction corresponding to the eigenvalue $\mu = 0$. Now ψ and α must satisfy

$$\psi(l) - \psi(-l) = \alpha \int_{-l}^{l} \varphi_0 dx, \quad \bar{\psi}^{I}(l) = 0, \quad \bar{\psi}^{II}(-l) = 0. \tag{77}$$

This is a system of $2n+1$ linear relations for the $2n+1$ unknowns of $\psi(l)$, $\psi(-l), \alpha$. We make

Algebraic assumption 2 *The system (77) is nonsingular if we replace $\int_{-l}^{l} \varphi_0 dx$ by $\int_{-\infty}^{\infty} \varphi_0 dx$. Thus, if M denotes the coefficient matrix of (77), then M^{-1} is uniformly bounded for all sufficiently large l.*

We want to prove

Lemma 11 *Assumption 4 is equivalent with the Algebraic assumption 2.*

Proof. If (77) has only the trivial solution for $l = \infty$, then $\psi \equiv 0$ and (76),(73)– (74) reduce to (20) with $D = 0$. If (77) has a nontrivial solution for $l = \infty$, then we can construct a nontrivial bounded solution which does not belong to L_2. This proves the lemma.

Now consider (67) with boundary conditions (69)-(71). We start with the case $m = n$. For $s = 0$ the problem is reduced to the eigenvalue problem. By the assumptions the only solution is the trivial solution. Therefore, for sufficiently small $|s|$, the problem (67), (69)–(71) has a unique solution satisfying the estimate

$$\int_{-l}^{l} (|u_2|^2 + |u_{2x}|^2 + |v_2|^2 + |v_{2x}|^2)dx \leq const.\, |s|^2 \int_{-l}^{l} |w_1|^2 dx. \tag{78}$$

As before, we extend the solution to the infinite interval $-\infty < x < \infty$, using the now known values of u_2, v_2 as boundary conditions. By (61) and Lemma 4, we obtain the desired estimate (47).

We now consider the case that $m = n + 1$. We split the forcing

$$s \begin{pmatrix} w_{11} \\ w_{12} \end{pmatrix} = s \begin{pmatrix} w_{11} \\ w_{12} - \alpha\varphi_0 \end{pmatrix} + s \begin{pmatrix} 0 \\ \alpha\varphi_0 \end{pmatrix}.$$

We want to determine α in such a way that we can estimate u_2, v_2 and α in terms of $\|w_1\|$. Consider first the auxiliary problem

$$\begin{pmatrix} y \\ z \end{pmatrix}_x = \begin{pmatrix} A & I \\ sI & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} G \\ H - \alpha\varphi_0 \end{pmatrix}, \quad -l \leq x \leq l, \tag{79}$$

with boundary conditions (69)-(71). Here G, H have support in $-l_0 \leq x \leq l_0$, $l_0 \leq l$. We start with $s = 0$. The boundary conditions for z are independent of y. Using the differential equation we obtain

$$z(l) - z(-l) + \alpha \int_{-l}^{l} \varphi_0 dx = \int_{-l}^{l} H\, dx, \quad \tilde{z}^{I}(l) = \tilde{z}^{II}(-l) = 0. \tag{80}$$

For all $l \geq l_1$, l_1 sufficiently large it follows by the Algebraic assumption 2 that there is a unique solution $z(l)$, $z(-l)$ and α. Further, there is a constant C_1 such that,

$$|z(l)| + |z(-l)| + |\alpha| \leq C_1 \int_{-l}^{l} |H| dx \leq C_1 l_0^{1/2} \|H\|. \tag{81}$$

Also, If (81) holds, then $z(x) = z(-l) + \int_{-l}^{x}(H - \alpha\varphi_0)d\tilde{x}$ is uniquely determined and satisfies the estimates

$$|z|_\infty \leq |z(-l)| + \int_{-l}^{l} |H| dx + \alpha \int_{-l}^{l} |\varphi_0| dx \leq const. l_0^{1/2} \|H\|,$$
$$\int_{-l}^{l} (|z|^2 + |z_x|^2) dx \leq const. l_0 (l+1) \|H\|^2.$$

We can now solve

$$y_x = Ay + z + G, \quad -l \leq x \leq l, \tag{82}$$

with boundary conditions

$$\bar{y}^{II}(l) = e^{-l}\left(Q_1(l)\bar{y}^I(l) + Q_2(l)\bar{z}(l)\right), \tag{83}$$
$$\bar{y}^I(-l) = e^{-l}\left(Q_1(-l)\bar{y}^{II}(-l) + Q_2(-l)\bar{z}(-l)\right), \tag{84}$$

Inverting the transformation (62), we can write the boundary conditions (83),(84) as relations for $\tilde{y}(l) = S_R^{-1}y(l)$ and $\tilde{y}(-l) = S_L^{-1}y(-l)$, respectively.

In the usual way we can make the boundary conditions homogeneous and obtain from Lemma 9 that (82)- (84) has a solution y_P with

$$\int_{-l}^{l} (|y_P|^2 + |(y_P)_x|^2) dx \leq C \left(\int_{-l}^{l} (|z|^2 + |G|^2) dx + |z|_\infty^2 \right).$$

Since (83),(84) are only $n-1$ conditions, the solution is not unique. All solutions are of the form

$$y = y_P - \tilde{\alpha}\varphi_0.$$

We choose $\tilde{\alpha}$ such that

$$z_+ - z_- + \tilde{\alpha} \int_{-l}^{l} \varphi_0 dx = \int_{-l}^{l} y_p \, dx, \quad \tilde{z}^I(l) = \tilde{z}^{II}(-l) = 0.$$

Observe that we use this procedure to determine only $\tilde{\alpha}$. The vectors z_+ and z_- are not used. We have the estimate

$$|\tilde{\alpha}|^2 \leq C_1^2 \left(\int_{-l}^{l} |y_P| dx \right)^2 \leq C_1^2 l \int_{-l}^{l} |y_P|^2 dx.$$

By adding the above side conditions for α and $\tilde{\alpha}$, we can solve our auxiliary problem (79) uniquely for $s = 0$. The solution satisfies the estimates (81) and

$$\int_{-l}^{l} (|y|^2 + |z|^2 + |y_x|^2 + |z_x|^2)dx \leq C_2(l)(\|G\|^2 + \|H\|^2).$$

Here C_2 is a constant which depends on l (algebraically). Therefore, we can also solve the problem for $|s| \leq \delta = \delta(l)$, δ sufficiently small. The estimates become

$$\begin{aligned} |z(l)| + |z(-l)| + |\alpha| &\leq C_1 l_0^{1/2}\|H\| + \mathcal{O}(s)(\|H\| + \|G\|), \qquad (85) \\ \int_{-l}^{l} (|y|^2 + |z|^2 &+ |y_x|^2 + |z_x|^2)dx \\ &\leq (C_2(l) + \mathcal{O}(s))\left(\|H\|^2 + \|G\|^2\right). \qquad (86) \end{aligned}$$

This shows that the first term in the splitting generates a solution satisfying (85),(86) with $\|H\|^2 + \|G\|^2$ replaced by $|s|^2\|w_1\|^2$. Thus, we can, by Lemma 4 and (61), extend the solution to $-\infty < x < \infty$ such that it satisfies an estimate of type (47).

It remains to solve (67) with sw_1 replaced by the second term in the splitting.

Instead of using the boundary conditions (69) –(71), we consider (67) on the infinite interval and write it as a second order equation

$$y_{xx} - (Ay)_x - sy = \alpha sF, \quad -\infty < x < \infty,$$

where

$$F = \begin{cases} \varphi_0 & \text{for } |x| \leq l \\ 0 & \text{for } |x| > l \end{cases}.$$

Thus, $u^{(1)} = y - \alpha\varphi$ satisfies

$$u_{xx}^{(1)} - (Au^{(1)})_x - su^{(1)} = \alpha s w^{(1)}, \quad -\infty < x < \infty, \qquad (87)$$

where

$$w^{(1)} = \begin{cases} 0 & \text{for } |x| \leq l \\ \varphi_0 & \text{for } |x| > l \end{cases}.$$

(87) is of the same type as (58) and we can repeat the process. Since

$$\|w^{(1)}\|^2 + \|w^{(1)}\|_1^2 \leq Ce^{-l}(\|w\|^2 + \|w\|_1^2),$$

the iteration process converges, provided l is sufficiently large and $|s|$ sufficiently small.

This concludes the proof of (47). Since

$$\|\hat{F}\|_1 = \int_0^\infty |\hat{F}(x,s)|dx \leq \int_0^\infty \int_0^\infty |F(x,t)|dx\,dt,$$

the estimate (47) together with the estimate of Theorem 3 gives us, by Parseval's relation,

$$\int_0^T (\|u\|^2 + \|u_x\|^2)dt \leq K\left(\int_0^T \|F\|^2 dt + \left(\int_0^T \|F\|_1 dt\right)^2\right).$$

Now we can proceed as in Section 2 and show Theorem 2 of the introduction.

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