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A MODULATION METHOD FOR SELF-FOCUSING IN THE PERTURBED CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract

In this Letter we introduce a systematic perturbation method for analyzing the effect of small perturbations on critical self focusing by reducing the perturbed critical nonlinear Schrödinger equation (PNLS) to a simpler system of modulation equations that do not depend on the transverse variables. The modulation equations can be further simplified, depending on whether PNLS is power conserving or not. An important and somewhat surprising result is that various small defocusing perturbations lead to a canonical form for the modulation equations, whose solutions have slowly decaying focusing-defocusing oscillations.

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1 Introduction

The perturbed critical nonlinear Schrödinger equation (PNLS)

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon F(\psi, \psi_z, \nabla_\perp\psi, \ldots) = 0\ , \quad \psi(0,x,y) = \psi_0(x,y) \tag{1}$$

$$\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ , \qquad 0 < \epsilon \ll 1\ ,$$

arises in various physical models in nonlinear optics[1], plasma physics and fluid dynamics (e.g. Table 1). When $\epsilon = 0$ eq. (1) reduces to the critical nonlinear Schrödinger equation (CNLS)

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi = 0\ . \tag{2}$$

We recall that for the nonlinear Schrödinger equation with a general nonlinearity σ and transverse dimension D

$$i\psi_z + \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_D^2}\right)\psi + |\psi|^{2\sigma}\psi = 0\ ,$$

we distinguish between three different cases: 1) When $\sigma D < 2$, the subcritical case, diffraction always dominates and focusing singularities do not form. 2) In the supercritical case $\sigma D > 2$, there is a large

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[1] The amplitude ψ may depend on additional variables, such as t in the case of time-dispersion.

class of smooth initial amplitudes for which a focusing singularity forms in finite distance z. Since in supercritical self-focusing the nonlinearity dominates over diffraction, addition of small perturbations to the equation has a small effect. 3) In the critical case $\sigma D = 2$ [as in the case of eq. (2)], solutions can also become singular in a finite z. However, in this borderline case between subcritical and supercritical self-focusing, singularity formation is characterized by a near-balance between the focusing nonlinearity and diffraction. As a result, critical self-focusing is extremely sensitive to small perturbations, which can have a large effect and can even lead to the arrest of collapse.

Self-focusing is a genuinely nonlinear phenomenon and standard linearization methods cannot be used to analyze singularity formation in eqs. (1) and (2). In addition, methods such as the inverse scattering transform (IST), which is so successful in the $1D$ cubic subcritical case, cannot be applied to eq. (2), because (2) is not integrable. Self-focusing in (1) or (2) is, moreover, a local phenomenon which cannot be accurately captured by global estimates. For these reasons, despite considerable progress the present theory of critical self-focusing in the presence of small perturbations is still far from complete.

In this Letter we present a general method for analyzing the effect of a any deterministic or random perturbation on critical self-focusing. In this method PNLS (1) is reduced to a simpler system of modulation equations which do not depend on the transverse variables. The reduced system is much easier to analyze and to simulate, and it provides insights that are hard to get directly from PNLS.

2 Review of critical self-focusing

CNLS (2) has two important conserved quantities: The power

$$N := \frac{1}{2\pi}\int |\psi|^2\, dxdy$$

and the Hamiltonian

$$H(\psi) := \frac{1}{2\pi}\left(\int |\nabla_\perp \psi|^2\, dxdy - \frac{1}{2}\int |\psi|^4\, dxdy\right)\ , \quad \nabla_\perp = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}\right)\ .$$

A sufficient condition for singularity formation in (2) is

$$H(\psi_0) < 0$$

while a necessary condition is

$$N(\psi_0) \geq N_c \cong 1.86\ .$$

CNLS has waveguide solutions of the form

$$\psi = \exp(it)R(r)\ , \quad r = (x^2+y^2)^{1/2}\ ,$$

where $R(r)$ satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)R - R + R^3 = 0\ , \quad R'(0) = 0\ , \quad \lim_{r\to\infty} R(r) = 0\ . \tag{3}$$

The solution of eq. (3) with the lowest power ('ground state'), sometimes called the Townes soliton, has an important role in self-focusing theory. This positive, monotonically decreasing solution has exactly the critical power for self-focusing

$$\int_0^\infty R^2\, r dr = N_c \tag{4}$$

and its Hamiltonian is equal to zero

$$H(R) = 0\ . \tag{5}$$

Analysis of blowup in CNLS is based on the assumption (which is supported by numerical and analytical evidence) that near the singularity the solution is roughly a modulated Townes soliton:

$$\psi \sim \psi_R \ , \tag{6}$$

where

$$\psi_R := \frac{1}{L}R(\rho)\exp(iS) \ , \quad \rho = \frac{r}{L} \ , \quad S = \zeta + \frac{L_z}{L}\frac{r^2}{4} \ , \tag{7}$$

and

$$\frac{d\zeta}{dz} = \frac{1}{L^2} \ . \tag{8}$$

More precisely, near the singularity, ψ_s, the inner part of the solution[2], whose power is slightly above critical, collapses towards the singularity in a quasi-self-similar fashion,

$$\psi_s \sim \frac{1}{L}V(\zeta, \rho)\exp(iS) \ ,$$

where

$$V \to R \qquad \text{as} \quad z \to Z_c$$

and Z_c is the location of the singularity. Based on this modulation ansatz, it was shown that near the singularity self-focusing can be described by the reduced system [1, 2, 3]

$$\beta_z = -\frac{\nu(\beta)}{L^2} \ , \tag{9}$$

$$L_{zz} = -\frac{\beta}{L^3} \ , \tag{10}$$

where

$$\nu(\beta) \sim c\exp\left(\frac{\pi}{\beta^{1/2}}\right) \ , \quad c \cong 45.1 \ .$$

In order to motivate the system (9, 10), we note that the modulation variable L is the radial width as well as 1/amplitude of the focusing part ψ_s, and that β is proportional to the excess power above critical of ψ_s [4]. Therefore, at the point of blowup $L(Z_c) = \beta(Z_c) = 0$. The $\nu(\beta)$ term arises from radiation effects (power losses of ψ_s) during self-focusing. Since near the singularity

$$0 \leq \beta(z) \ll 1 \ ,$$

$\nu(\beta)$ is exponentially small and self-focusing is essentially adiabatic.

2.1 Adiabatic approach

Originally, the reduced system (9, 10) was analyzed by solving (9) to leading order near Z_c and then using (10). This leads to the *loglog law* for the rate of critical blowup [1, 2, 3]. However, it turns out that the loglog law does not become valid even after amplification of the peak amplitude by a factor of a billion or more, which is long after the nonlinear Schrödinger equation ceases to be physically relevant. Fortunately, this can be 'fixed' by solving (9,10) using an adiabatic approach. Since changes in β (i.e. the power of ψ_s) are exponentially small compared with the focusing rate, we first solve (10) with β constant, and only then add non-adiabatic effects (9) as the next-order correction. Application of this approach leads to an *adiabatic law* for critical self-focusing which is valid almost from the onset of self-focusing [5].

[2] A possible definition is $\psi_s = \psi$ for $0 \leq \rho \leq \rho_c$, with $1 \ll \rho_c$ constant.

3 Modulation theory for self-focusing in the perturbed CNLS

The adiabatic law, which provides an accurate description of critical self-focusing in domain of physical interest, is obtained in two stages: 1) Derivation of the reduced modulation equations (9,10) which do not depend on the transverse variables and 2) Solving these equations with the radiation term $\nu(\beta)$ neglected to leading order (adiabatic approach). In this section we extend this approach to self-focusing in PNLS: 1) The modulation ansatz (6) is used in Proposition 3.1 to reduce eq. (1) to the system (11) and 2) The reduced system is analyzed with the adiabatic approach (Propositions 3.2–3.3). More details, as well as proof of results are published elsewhere[3] [6].

For modulation theory to be valid, the following *three conditions* must hold:

1. The focusing part of the solution is close to the asymptotic profile [eq. (6)].
2. The power of the focusing part is close to critical
$$\left|\frac{1}{2\pi}\int |\psi_s(z,x,y)|^2\,dxdy - N_c\right| \ll 1\ ,$$
or equivalently,
$$|\beta(z)| \ll 1\ .$$
3. The perturbation is small:
$$|\epsilon F| \ll |\Delta_\perp \psi| \quad \text{and} \quad |\epsilon F| \ll |\psi|^3\ .$$

In general, at the onset of self-focusing only condition 3 holds. As the solution approaches Z_c (the blowup point in the absence of the perturbation), conditions 1–2 are also satisfied and modulation theory becomes valid.

The main result of modulation theory is the following:

Proposition 3.1 *If conditions 1–3 hold and if F is an even function in x and y, self-focusing in PNLS (1) is given to leading order by the reduced system*

$$\beta_z(z) + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_z - \frac{2\epsilon}{M} f_2\ , \quad L_{zz}(z) = -\frac{\beta}{L^3}\ . \tag{11}$$

The auxiliary functions f_1 and f_2 are given by

$$f_1(z) = 2L(z)\,Re\left[\frac{1}{2\pi}\int F(\psi_R)\exp(-iS)[R(\rho)+\rho\nabla_\perp R(\rho)]\,dxdy\right] \tag{12}$$

$$f_2(z) = Im\left[\frac{1}{2\pi}\int \psi_R^* F(\psi_R)\,dxdy\right] \tag{13}$$

where

$$M = \frac{1}{4}\int_0^\infty r^3 R(r)\,dr \cong 0.55\ .$$

We note that assuming that we can carry out the transverse integration, f_1 and f_2 are known functions of the modulation variables L, β, ζ and their derivatives.

[3] The paper is available at the web sites www.math.tau.ac.il/~fibich and http://georgep.stanford.edu.

3.1 Conservative and non-conservative perturbations

Considerable simplification is achieved if we distinguish between conservative perturbations i.e. those for which the power remains conserved

$$\frac{d}{dz}\int |\psi(z,x,y,\cdot)|^2\, dxdy \equiv 0$$

in (1), and non-conservative perturbations:

Proposition 3.2 *Let conditions 1–3 hold.*

1. *If F is a conservative perturbation, i.e.*

$$Im\int \psi^* F(\psi)\, dxdy \equiv 0\ ,$$

then $f_2 \equiv 0$, and to leading order (11) reduces to

$$-L^3 L_{zz} = \beta_0 + \frac{\epsilon}{2M} f_1\ , \quad \beta_0 = \beta(0) - \frac{\epsilon}{2M} f_1(0)\ , \tag{14}$$

where β_0 is independent of z.

2. *If F is a non-conservative perturbation, i.e.*

$$Im\int \psi^* F(\psi)\, dxdy \not\equiv 0\ ,$$

then to leading order (11) reduces to

$$\beta_z = -\frac{2\epsilon}{M} f_2\ , \quad L_{zz} = -\frac{\beta}{L^3}\ . \tag{15}$$

Note that in both cases, non-adiabatic effects disappear from the leading order behavior of (11).

3.2 Canonical effect of conservative perturbations

It has been observed that various seemingly different small perturbations have the same effect: Arrest of collapse, followed by focusing-defocusing cycles (see Figure 1A). In the next Proposition we use modulation theory to explain this observation, by showing that all conservative perturbations for which f_1 is of the form[4]

$$f_1 \sim -\frac{C_1}{L^2}\ \ ,\ C_1 = \text{constant} \tag{16}$$

have the same qualitative effect on self-focusing.

Proposition 3.3 *When self-focusing is given by (14) and f_1 is given by (16) then $y := L^2$ satisfies the canonical oscillator equation*

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m)\ , \tag{17}$$

[4] e.g. those marked in Table 1 with †

where

$$y_M = \frac{\sqrt{\beta_0^2+\epsilon C_1 H_0/M^2}+\beta_0}{-2H_0/M} = \frac{M\beta_0}{-H_0}\left[1+O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right] \tag{18}$$

$$y_m = \frac{\epsilon C_1}{2M}\frac{1}{\sqrt{\beta_0^2+\epsilon C_1 H_0/M^2}+\beta_0} = \frac{\epsilon C_1}{4M\beta_0}\left[1+O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right] , \tag{19}$$

$$\beta_0 = \beta(0)+\frac{\epsilon C_1}{2ML^2(0)} , \quad H_0 \sim H(0)+\frac{\epsilon C_1}{4}\frac{1}{L^4(0)} .$$

Let us define

$$L_m := y_m^{1/2} , \quad L_M := y_M^{1/2} .$$

1. *If the perturbation is* <u>*defocusing*</u>*, i.e.*

$$\epsilon C_1 > 0 , \tag{20}$$

   then it will arrest blow-up in (14), i.e. L (and y) will remain positive for all z.

   (a) *If in addition to (20), $\beta_0 > 0$ and $H_0 < 0$, then*

$$0 < L_m < L_M$$

   and L will go through periodic oscillations between L_m and L_M (Figure 1A). The period of the oscillations is

$$\Delta Z = 2\sqrt{\frac{My_M}{-H_0}}\,E\left(1-\frac{y_m}{y_M}\right) , \tag{21}$$

   where $E(m)=\int_0^{\pi/2}(1-m\sin^2\theta)^{1/2}\,d\theta$ is the complete elliptic integral of the second kind.

   (b) *If in addition to (20), $\beta_0 > 0$ and $H_0 > 0$, then*

      i. *If $L_z(0) < 0$, self-focusing is arrested when $L = L_m > 0$, after which L will increase monotonically to infinity (Figure 1B).*

      ii. *If $L_z(0) > 0$, L will increase monotonically to infinity.*

2. *If the perturbation is* <u>*focusing*</u>*, i.e.*

$$\epsilon C_1 < 0$$

   and if in addition $\beta_0 > 0$ and one of the following two conditions holds (1) $H_0 > 0$ and $L_z(0) < 0$ or (2) $H_0 < 0$, then the solution of (14) will blow up in a finite distance (Figure 1C), i.e.

$$\exists Z_* \quad \text{such that} \quad 0 < Z_* < \infty \quad \text{and} \quad L(Z_*) = 0 .$$

3.3 Non-adiabatic effects

The results of the previous section show that the exponentially small radiation term $\nu(\beta)$ disappears from the leading order behavior of perturbed CNLS. In the non-conservative case the effect of $\nu(\beta)$ is even smaller than the $(f_1)_z$ term which is also ignored. However, in the conservative case when $\beta_0 > 0$ and $H_0 < 0$, a defocusing perturbation can lead to periodic oscillations (as in Proposition 3.3-1a). In this case, the non-adiabatic radiation effect $\nu(\beta)$ provides the only mechanism for decay of the oscillations. It can be shown that if ϵ is moderately small the total power loss during one oscillation is small and the focusing-defocusing oscillations are slowly decreasing, but that for sufficiently small ϵ the quasi-periodic picture breaks down and focusing is completely arrested after a few oscillations [7]. Further analysis of non-adiabatic effects in (17) can be found in [4].

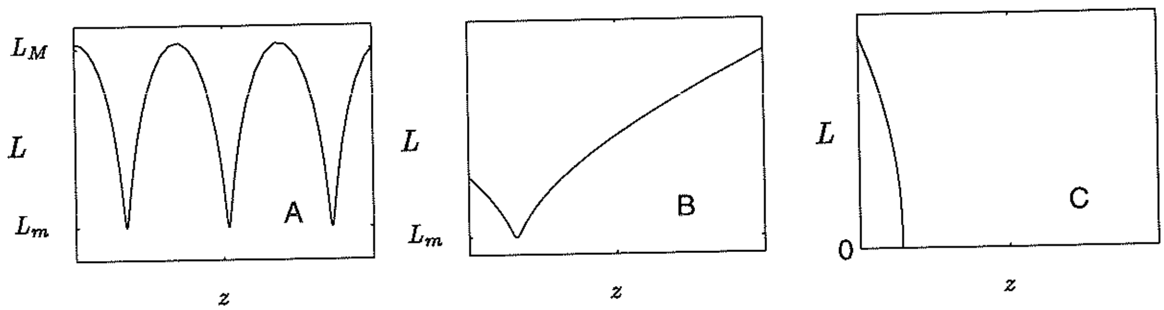

Figure 1: The leading order effect of the canonical conservative perturbation (16) A: Defocusing perturbation and $H_0 < 0$ (Proposition 3.3 -1.a) B: Defocusing perturbation, $H_0 > 0$ and $L_z(0) < 0$ (Proposition 3.3-1bi) C: Focusing perturbation and $L_z(0) < 0$ (Proposition 3.3 -2). In all cases $\beta_0 > 0$ (i.e. power above critical).

3.4 Modulation theory for multiple perturbations

In some cases, one is interested in the combined effect of several small perturbations e.g. randomness and quintic nonlinearity or time-dispersion and nonparaxiality (see Table 1). Modulation theory can easily handle these cases, since the modulation equations are linear in F. Therefore, one simply adds the contribution of each perturbation to the modulation equations.

4 Applications

The modulation approach was used by Malkin to study the effect of a small defocusing fifth power nonlinearity [4]. In [8], Fibich, Malkin and Papanicolaou analyzed the effect of small normal time-dispersion, using for the first time a systematic approach that is generalized in this Letter. A similar approach was also used by Fibich to analyze the effect of beam nonparaxiality [7] and the unperturbed CNLS [5] and by Fibich and Papanicolaou to analyze the combined effect of time-dispersion and nonparaxiality [9]. Additional applications, listed in Table 1, are derived in [6].

Direct numerical confirmation of the validity of the modulation equations and the adiabatic approach was carried out in the case of the unperturbed NLS [5] and in the case of small normal time dispersion [8]. In many other cases, there is qualitative agreement between the predictions of the modulation equations and the results of numerical simulations of the corresponding PNLS. For example, the behavior of decaying focusing-defocusing oscillations was observed numerically for fiber arrays [10], saturating nonlinearities [11, 12] and nonparaxiality [13].

5 Conclusion

In this Letter we have introduced a modulation theory for analyzing the leading order effects of small perturbations on critical self-focusing. This theory is able to capture the delicate balance between nonlinearity and diffraction in critical self-focusing, because it is based on perturbations of the solution around modulated Townes solitons (ψ_R). We note that the validity of other studies of PNLS in which the derivation of reduced equations is based on modulated Gaussians is questionable, because modulated Gaussians cannot capture the delicate balance in critical self-focusing [e.g. Gaussians cannot simultaneously satisfy (4) and (5)]. Moreover, the derivation of reduced equations with our ψ_R-based modulation theory is just as easy as with modulated Gaussians. In fact, all that is needed is to carry out the transverse integration in evaluating f_1 and f_2.

We have already remarked that modulation theory becomes valid near the blowup point Z_c. For some perturbations (e.g. nonparaxiality, saturating nonlinearities) one can show that the modulation equations remain valid for all z [6, 7]. However, in other cases (e.g. small normal time-dispersion) it is not clear for how long modulation theory remains valid, and further analysis may be needed for the advanced stages of self-focusing.

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Perturbed CNLS	Application	Reduced equation	conser-vative	Derived in
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon\psi_{xxxx} + \frac{2}{5}\epsilon^2\psi_{xxxxxx} = 0$	fiber arrays	$(y_z)^2 = \frac{-4H_0}{My^2}(y_M - y)(y - y_m)(y - y_3)$	yes	[6]
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi - \epsilon\lvert\psi\rvert^4\psi = 0$	quintic nonlinearity	(17) with $C_1 = (4/3)I_6$	yes†	[4]
$i\psi_z + \Delta_\perp\psi + \frac{1}{2\epsilon}(1 - \exp(-2\epsilon\lvert\psi\rvert^2))\psi = 0$	saturating nonlinearity	(17) with $C_1 = (4/3)I_6$	yes†	[6]
$i\psi_z + \Delta_\perp\psi + \frac{\lvert\psi\rvert^2}{1+\epsilon\lvert\psi\rvert^2}\psi = 0$	saturating nonlinearity	(17) with $C_1 = (4/3)I_6$	yes†	[6]
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi - \epsilon\phi_x\psi = 0$, $\alpha\phi_{xx} + \phi_{yy} = -(\lvert\psi\rvert^2)_x$	Davey-Stewartson	(9) , (10)	yes	[6]
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon\psi_{zz} = 0$	nonparaxiality	(17) with $C_1 = 4N_c$	no	[7]
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon(x^2+y^2)h(z)\psi = 0$	h random	$-L^3L_{zz} = \beta_0 + 4\epsilon L^4 h(z)$	yes	[6]
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon_1(x^2+y^2)h(z)\psi - \epsilon_2\lvert\psi\rvert^4\psi = 0$	quintic nonlinearity $+h$ random	$-L^3L_{zz} = \beta_0 + 4\epsilon_1 L^4 h(z) - \frac{2\epsilon_2 I_6}{3M}\frac{1}{L^2}$	yes	[6]
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi - \epsilon\psi_{tt} = 0$	time-dispersion	$\beta_z = \frac{2\epsilon N_c}{M}\zeta_{tt}$, (8) , (10)	no	[8]
$i\psi_z + \Delta_\perp\psi + N\psi = 0$, $\epsilon N_t + N = \lvert\psi\rvert^2$	Debye relaxation	$-L^3L_{zz} = \beta_0 + \frac{\epsilon C_D}{2M}\frac{L_t}{L}$ $C_D = \int(\nabla_\perp R^2)^2 r^3\,dr \cong 6.43$	yes	[6]
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon_1\psi_{zz} + \epsilon_2\left[2i\frac{n_0 c_g}{c}(\lvert\psi\rvert^2\psi)_t - \psi_{zt}\right] - \epsilon_3\psi_{tt} = 0$	time-dispersion+ nonparaxiality	$\beta_z = -\frac{2\epsilon_1 N_c}{M}\left(\frac{1}{L^2}\right)_z - (6c_g\frac{n_0}{c} - 2)\frac{\epsilon_2 N_c}{M}\left(\frac{1}{L^2}\right)_t + \frac{2\epsilon_3 N_c}{M}\zeta_{tt}$, (8), (10)	no	[9]

Table 1: Perturbations of critical NLS and their corresponding modulation equations. Here $I_6 = \int_0^\infty R^6\,rdr$. † - f_1 given by (16)