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In turbulence, fluctuations are developed over a wide range of scales and amplitudes as a result of strong nonlinear interactions. As fluctuations cascade, both mechanical law and statistical law are important to determine the overall structures. The mechanical law manifests in the development of the so-called most intermittent structures, while the statistical law expresses itself in terms of a certain symmetry leading to the organization of the fluctuations into a whole hierarchy. This view of the Hierarchical Structure model (She & Leveque, *Phys. Rev. Lett.*, **72**, 336, 1994) of turbulence is discussed in detail here with some recent analytical development. It is hoped that the model provides inspirations for a renewed phenomenology of turbulence guiding the development of a more complete theory of turbulence.

§1. Introduction

The recent development of physics has undoubtedly been focused on the study of complex dynamics at mesoscopic and macroscopic levels. Systems at such scales involve a large number of degrees of freedom nonlinearly coupled which exhibit complicated or chaotic patterns. The underlying first principles governing the dynamics of these systems are either classical conservation laws of mechanics (e.g. for fluid motions) or classical equations of electrodynamics. Yet, the degree of complexity of the motions at the macro-scales in terms of macro variables have rendered the description extremely difficult, if not impossible. It is believed that some noval, synthetic descriptions of such complex motions are urgently needed.

Turbulence is an outstanding example of such complex motions which have resisted analytical and quantitative description for over a centruy [1]. Motions of fluids at small viscosity are very unstable to perturbations, and fluctuations generated by instabilities may further excite motions at even smaller scales through the nonlinear interactions. These interactions are constrained by the mechanical laws such as the conservation of energy; nonetheless, there are a wide range of admissible solutions to the problem so that the usefulness of the classical fluid mechanical approach is doubteful. The remedy was proposed and followed by researchers from another end of spectrum, the statistical physics [2].

Statistical mechanics was pioneered nearly a centruy ago by Boltzmann and Gibbs, with strong support of Planck, Einstein, etc. It was realized at that time that the fundamental mechanical conception of the nature was doubteful when such phenomena as the heat radiation are considered. The physics appeared to need to discover new laws. Law of probability [3] (e.g. the law of maximum probability, or the law of maximum entropy) was then born in the study of the equilibrium thermodynamics. The law of probability is independent of the accepted laws of mechanics which had dominated the scientific view previously over two centruies. Since, the new conception of the statistical physics has been one of the foundational blocks of the twentieth centruy modern physics.

The equilibrium statistical mechanics has begun with the kinetic description of gas molecules. The kinetic analogy of turbulent fluctuations has also a long history, dating back to Reynolds [4], if not earlier, where the concept of eddies plays similar role as gas molecules. Mathematically, it is founded on the method of spectral decomposition of a turbulent field. The difficulty is that turbulent motions of fluids are at a non-equilibrium state; that is, "eddies" of different sizes are not equivalent, and their distribution resulted from a non-equilirium interaction has no similarity to the usual law of maximum probability in the equilibrium situation. Consequently, the statistical description of turbulent states has been lucking a sound phenomenology. The modern statistical theory of turbulence has not had support from both the statistical physics community and from the fluid mechanical community, the later has always been skeptical to a pure statistical approach.

More recently, it is observed experimentally that turbulence is generally intermittent [5]; moreover, the development of coherent intermittent structures accompany the development of turbulence. These structures stand as outstanding fluctuations of much larger amplitude than the root-mean-square fluctuations, and are presumably maintained by special mechanisms. It is believed that the intermittent structures and large amplitude fluctuations are key features of turbulence at a far-from-equilibrium state which should needs both mechanical and statistical descriptions. The co-existence of coherent intermittent structures with widely spread disordered motions may imply new mechanisms of organizations.

In this paper, we present a few new ideas aiming at establishing a basis for a dual mechanical-statistical approach. The ideas presented throughout this paper form what will be referred to as the Hierarchical Structure model [6]. The basic assumption was first proposed in She & Leveque [6] in 1994, and further developed by Dubruelle [7] and by She & Waymire [8]. Here, the key ideas will be discussed under a newly developed analytical framework, with a group transformation method. Note that there exists a large body of evidence from experimental and computational studies that supports the predictions of the Hierarchical Structure model on the scaling properties of turbulence [9]. Detailed comparison between the theory and experiments will not be the subject of discussion here; instead we will focus on the study of the logical consistency and the rationality of the model, to establish a convincing and fruitful basis for further development of a good non-equilibrium statistical mechanics theory of turbulence. It is hoped that the present understanding of the statistical and mechanical features of fluid turbulence may be suggestive to the study of a variety of other strongly fluctuating medium as well.

§2. The Hierarchical Structure Model of Turbulence

The main physical picture is that turbulence is composed of a hierarchy of structures of variable sizes, variable flucutating amplitudes, and variable degree of coherence (and/or degree of space-filling). It becomes a hierarchy because of the existence of a new equivalence principle which links structures of various amplitudes together. A consequence is the manifestation of a specific type of multi-scaling behavior of turbulent fields: log-Poisson multifractal.

2.1. Cascade of Fluctuations

Frist, it is important to clarify the concept of cascade. When incompressible turbulence in three dimensions is excited at large scales, fluctuations at small scales are developed during a transient time because of nonlinear mode-mode coupling, and excitations in velocity fourier components propogate from large to small scales. This conventional picture of the energy cascade is of a very dynamical nature and is uni-directional from large to small scales. However, as the turbulence reaches a matured stage of a statistically stationary state with the presence of an inertial range (at very small viscosity), the above dynamical picture of the cascade must include a constant feedback effect of small scales onto the large scales. The cascade dynamics discussed here include both the forward and backward interactions which maintain the inertial range as the whole entity.

This holistic view of the inertial range is expressed by a specific way in which small-scale fluctuations (at ℓ') are related to the large-scale ones (at ℓ), that is, statistically by a random linear mapping or a random multiplicative process $\mathcal{W}_{\ell\ell'}$:

$$\delta v_{\ell'} \overset{\text{law}}{=} \mathcal{W}_{\ell\ell'} \delta v_{\ell}. \tag{2·1}$$

We generally refer to the events of $\mathcal{W}_{\ell\ell'}$ as structures. A small set of these events $\mathcal{W}_{\ell\ell'}^{(MI)}$ will be qualified as the most intermittent (MI) structures, which have the largest amplification rate and occupy the smallest volume:

$$\mathcal{W}_{\ell\ell'}^{(MI)} \propto \left(\frac{\ell}{\ell'}\right)^{h_{max}}, \quad P\left(\mathcal{W}_{\ell\ell'}^{(MI)}\right) \propto \left(\frac{\ell}{\ell'}\right)^{D_{min}}. \tag{2·2}$$

It is assumed that $\mathcal{W}_{\ell\ell'}^{(MI)}$ are the only special structures relating small-scale fluctuations to the large ones. Other structures of smaller amplitude and smaller amplification rate are statistically "controled" by the MI structures, and the controle is expressed by some invariance principle described later.

We should stress that the statistical description of the cascade in terms of $\mathcal{W}_{\ell\ell'}$ implies an emancipation of the cascade concept from the conventional pictures of either a migration of fourier excitations or a deformation of spatial patterns. This is a decisive step. It stems from the realization that in either of the above pictures, the statistical law governing the development of the fluctuations of the energy flux across length scales take, both qualitatively and quantitatively, more complicated forms. What is the most important is, however, the (statistical) symmetries respected by the process of cascade, and thus the relevant cascade process needs to take a convenient mathematical form (such as the linear map) and needs to develop a physical interpretation accordingly. This change of the point of view marks a new way of the physical thinking in turbulence studies which starts to be detached from a purely mechanical view or a purely stochastic view.

2.2. Weighted Distribution Functions

In the conventional study of a turbulent field $v(x)$, one constructs the so-called velocity structure functions. This involves in fact three steps. The first is the coarse-graining of the field, which defines the notion of the length scale. One familiar example is to take a difference of the field value at two points separated by a length scale ℓ: $\delta v_\ell(x) = v(x+\ell) - v(x)$. This operation generates a (continuous) family of random fields (or variables) called increments. The second operation is to define a powered field: $(\delta v_\ell)^p(x)$. The third operation is to take the expectation value with respect to a distribution function $dQ(\delta v_\ell)$. Combining the three operations together, we obtain the pth-order moment of the increment field across the length scale ℓ (in the homogeneous situation, we omit the x-dependence)

$$E_Q(\delta v_\ell^p) = \int \delta v_\ell^p dQ(\delta v_\ell) \qquad (2{\cdot}3)$$

The reason to emphasize the three steps is that at each step a parameter can be introduced. The parameters are obvious for the first two steps, that is, ℓ and p. For the third step, the parameter will be introduced below.

The field $v(x)$ is said to be a scaling field when the moments vary with the length scale in power law:

$$E_Q(\delta v_\ell^p) = A_p \ell^{\zeta_p^{(Q)}} \qquad (2{\cdot}4)$$

The novalty of the approach described here involves specifically a Q-dependent scaling exponent. We will see below how this consideration leads to naturally an expansion for the determination of the scaling exponents.

If $\zeta_p^{(Q)} = hp$ is a linear function in p, then we say that the field $v(x)$ is a mono-scaling field, because it is globally self-similar under the transformation

$$v(\lambda x) = \lambda^h v(x). \qquad (2{\cdot}5)$$

When $\zeta_p^{(Q)}$ is a nonlinear function of p, the field $v(x)$ is generally a multi-scaling field, sometimes called a multifractal field. It is believed that the 3-D incompressible turbulent velocity field satisfying the Navier-Stokes equation is a multi-scaling vector field. One of the major goals of the theory of turbulence is to predict the nonlinear scaling law $\zeta_p^{(Q)}$.

The Hierarchical Structure model predicts a special multi-scaling turbulent field. It starts with introducing a (continuous) family of weighted distribution functions $dQ_q(y)$ parameterized by $q \in R^+$:

$$dQ_q(\delta v_\ell) = \frac{1}{E_{Q_0}(\delta v_\ell^q)} |\delta v_\ell|^q dQ_0(\delta v_\ell), \qquad (2{\cdot}6)$$

We will hereafter denote the expectation with respect to dQ_q by E_q. Below, we will consider a change in dQ_q expressed by a translation $q \to q' = q + \Delta q$ ($\Delta q > 0$); this translation shifts the weight towards high-amplitude fluctuations. Therefore, the physical meaning of the family of the distribution functions can be better described as a "microscope" for exploring scaling properties of low versus high amplitude fluctuations in δv_ℓ.

2.3. Group Transformation Law

The group transformation method defines a semi-group in the above triplet parameter space (ℓ, p, q). Specifically, let $E_q(\delta v_\ell^p) = \int \delta v_\ell^p dQ_q(\delta v_\ell)$ and define the function (or physical observable)

$$X(T,p,q) = \log E_q(\delta v_\ell^p). \tag{2·7}$$

With this observable, we may consider the translation in both p and q. Similarly, we redefine a variable $T = -\log \ell$ such that a rescaling $\ell \to \ell' = \lambda\ell$ ($\lambda < 1$) amounts to a translation $T' \to T + \Delta T$ where $\Delta T = -\log \lambda$. We now examin the possible scenario in this triplet group structure $(T,p,q) \to (T',p',q')$.

By the definition (2·6), we have an important algebric identity:

$$X(T_0, p_0 + p, q_0) = X(T_0, p_0, q_0) + X(T_0, p, q_0 + p_0). \tag{2·8}$$

That is, the function X may be additive in the middle argument if the measure is shifted from $q_0 \to q_0 + p_0$. In different words, X forms a translation group in the middle argument if we add to it a corresponding shift in the third argument. The identity (2·8) is valid for aribitrary values of the arguments.

It is immediately clear in (2·8) that $X(T_0, p_0, q_0)$ is invariant with respect to a translation in T_0. This is a feature of the power-law range in which the change in the amplitude of the moments depends only on the ratio of the two scales, i.e. $\Delta T = T - T_0$. We are then led to consider $X(\Delta T, p_0, q_0) = X(T_0 + \Delta T, p_0, q_0) - X(T_0, p_0, q_0)$. From the definition (2·8), we obtain, in general,

$$X(\Delta T, p_0 + p, q_0) = X(\Delta T, p_0, q_0) + X(\Delta T, p, q_0 + p_0). \tag{2·9}$$

$X(\Delta T, p_0, q_0)$ describes the change of the moments of order p_0 with respect to the distribution function dQ_{q_0} (hereafter denoted as the measure-q_0), under a rescaling by a factor $\lambda = e^{-\Delta T}$. Note that the relation (2·9) is an identity.

When the field $v(x)$ is a scaling field satisfying (2·4), then

$$X(\Delta T, p_0, q_0) = V(p_0, q_0)\Delta T, \tag{2·10}$$

where $V(p_0, q_0)$, or $\zeta_{p_0}^{(q_0)}$ (in (2·4)) is the scaling exponent of the moment of order p_0 with respect to the measure-q_0, and can also be regarded as the rate of change similar to the speed in the kinematic context (where X and T are position and time respectively). We then obtain the general result:

$$V(p_0 + p, q_0) = V(p_0, q_0) + V(p, q_0 + p_0). \tag{2·11}$$

The relation (2·11) indicates that the rate of change at a higher order $p_0 + p$ can be obtained by the one at the order p_0 plus that at the order p order while the later is evaluated with respect to a shifted measure from q_0 to q_0+p_0. The relation (2·11) is an exact property under a translation group in p_0 and q_0. It is the most important relation in the present context, and will hereafter referred to as the fundamental identity.

Generalizing (2·11) to N displacements in p_0 yield a more general identity:

$$V\left(\sum_{i=0}^{N} p_i, q_0\right) = V(p_0, q_0) + \sum_{i=1}^{N} V\left(p_i, q_0 + \sum_{j=0}^{i-1} p_j\right). \qquad (2\cdot12)$$

Concerning the nature of these identities, let us note again that they are from the definition of the scaling formula and from the definition of the measure dQ_q. Therefore, certain properties resulted from those definitions (such as the moment log-concavity property, etc.) may manifest more explicitely here. The exploration of the consequences of these identities below is meaningful in this regard.

2.4. Monofractal Field and the Most Intermittent Structures

Before proceding further, let us consider a special solution to (2·11), the monofractal field. Assume that the rate $V(p_0, q_0)$ does not depend on the weighted distribution function, i.e. q_0. Then, (2·11) leads to

$$V(p_0+p, q_0) = H(p_0+p) = H(p_0) + H(p). \qquad (2\cdot13)$$

The solution to (2·13) is simply

$$H(p_0) = p_0 h. \qquad (2\cdot14)$$

One example of the stochastic field $v(x)$ giving rise to (2·14) is the fractional Brownian motion with scaling exponent h which is a mono-scaling field satisfying

$$\delta v_{\lambda\ell} \stackrel{\text{law}}{=} \lambda^h \delta v_\ell. \qquad (2\cdot15)$$

This is essentially the model of turbulence of Kolmogorov in 1941 [10], yielding a non-intermittent inertial range. Note, however, that the fractional Brownian motion is not the only field satisfying (2·14). Any field for which the probability distribution function of δv_ℓ is invariant with respect to the rescaling transformation satisfies the property (2·14).

We would now explore the possibility of having a mono-scaling "component" other than a fractional Brownian motion in a turbulent field. The fact that $V(p_0, q_0)$ (scaling) is independent of q_0 indicates that any subset of fluctuation events has uniformly the same scaling property (e.g. with exponent h), irrespective of their amplitude (parameterized by q_0). It is believe that in a 3-D incompressible turbulent field, there are a small set of events which keep a certain degree of coherence across scales in such a way that their cascade is characterized by a unique scaling. The maintainance of the degree of coherence means that the probability distribution function on the set keeps an invariant form as scale changes.

The most efficient mechanism to maintain the coherence is by the geometrical confinement. The most intermittent structures occupy therefore a spatial volume which should have the smallest fractal dimension. As we see below, this is indeed a consistent assumption: as $\ell \to 0$, these structures are events of the largest amplitude occupying the thinest set. We call these events the most intermittent structures.

2.5. The Hierarchical Structure Model

In the framework of the Hierarchical Structure model, it is believed that fully developed turbulence contains not only isolated, geometrically recognizable intermittent structures, but also many other irregular patterns having apparantly no structure. The later part is even dominant in probability. On the other hand, it is believed that, in the nonlinear problem like turbulence, the starting point should not be a featureless field like a Quasi-Gaussian (or Quasi-Normal) field; it should be around something like the most intermittent structure which is a characteristic feature of the nonlinear problem. Under these consideration, a generalization of (2·14) to a multifractal scaling field is sought to be

$$V(p_0, q_0) = H(p_0) + F(p_0)G(q_0), \tag{2·16}$$

where $H(p_0)$ denotes the q_0-independent part, and the measure-dependent factor is assumed to separable in factors $F(p_0)$ and $G(q_0)$. It may be shown that if $H(p_0)$ satisfies (2·13), then $F(p_0)$ and $G(q_0)$ will satisfy

$$G(p_0 + q_0) = G(q_0)\mathcal{L}(p_0), \tag{2·17}$$

$$F(p_0 + p) = F(p_0) + F(p)\mathcal{L}(p_0), \tag{2·18}$$

where $\mathcal{L}(p_0) = F'(p_0)/F(0)$ $(F'(x) = dF/dx)$. The proof of (2·17) makes use of the fundamental identity in the limit $p \to 0$, whereas the relation (2·18) comes from the application of the fundamental identity at finite p. Let $p \to \infty$ in (2·18), we obtain

$$F(p_0) = C(1 - \mathcal{L}(p_0)), \tag{2·19}$$

where $C = F(\infty)$ is assumed to be finite. The equation (2·19) is a first order differential equation for $F(p)$, and is easily solved to yield a solution: $\mathcal{L}(p) = \beta^p$ $(\beta < 1)$. In summary, under the assumption (2·16), we obtain the solution:

$$H(p) = ph, \tag{2·20}$$

$$G(q) = \beta^q, \tag{2·21}$$

$$F(p) = C(1 - \beta^p), \tag{2·22}$$

with three parameters h, C and β. Combining them yields a scaling formula for $V(p, q)$:

$$V(p, q) = ph + C(1 - \beta^p)\beta^q. \tag{2·23}$$

The corresponding scaling formula for usual structure functions is

$$\zeta_p = V(p, 0) = ph + C(1 - \beta^p). \tag{2·24}$$

This is the original formula of She & Leveque[6].

The properties (2·17) and (2·18) as a result of the assumption (2·16) are particularly striking in the sense that $G(q_0)$ and $F(p_0)$ at any q_0 and p_0 can be determined. This implies a strong symmetry (see the next subsection) which has grouped fluctuations at different levels (parameterized by q) together. This is the concept of the hierarchy, the origin of the name of "the Hierarchical Structure model".

Another remarkable feature of the model (2·16) described above is that it can be superposed of any number of additional structures similar to $F \times G$. Indeed, it can be shown that if one assumes

$$V(p_0, q_0) = H(p_0) + \sum_i^N F_i(p_0)G_i(q_0), \tag{2·25}$$

with N factorizable terms $F \times G$, then as long as the relation (2·13) is valid and

$$G_i(p_0 + q_0) = G_i(q_0)\mathcal{L}_i(p_0), \tag{2·26}$$

$$F_i(p_0 + p) = F_i(p_0) + F_i(p)\mathcal{L}_i(p_0), \tag{2·27}$$

where $\mathcal{L}_i(p_0) = F_i'(p_0)/F_i(0)$, the fundamental identity is satisfied. In other words, F_i, G_i and $\mathcal{L}_i$ like the ones in (2·20)-(2·22) are solutions to the scaling problem. In this case, a more general formula for the scaling exponents ζ_p is:

$$\zeta_p = ph + \sum_i^N C_i(1 - \beta_i^p). \tag{2·28}$$

We have therefore a situation where multiple mechanisms ($N \geq 1$) can act concurrently in linking small-scale fluctuations to the large-scale ones. When $N > 1$, there is a kind of superposition of the multiple $F \times G$ terms; because of this superposition property, it is likely that the solution (2·28) is fairly general and representative to many problems. It is important to clarify the meaning of the superposition here. The superposition addressed here is concerned with those structures entangled and folded together in the same space and time location to yield one unique multiscaling process. The log-Poisson random multiplicative process discussed below may help to develop more intuition about such multi-mechanisms situation. We shall, however, be mostly interested in the special case (2·16) with a single mechanism as She & Leveque[6] originally suggested, and would suggest to call it a minimal scaling model of a field with a mixture of order (intermittent structures) and disorder (random environment).

2.6. Minimal Model: Equivalence Principle

In the special case of the minimal model (a single $F \times G$ term), there is an equivalence principle. Let $\delta v_\ell^{(\infty)} \propto \ell^h$ represent the most intermittent structure. Define the normalized field $\Pi_\ell = \delta v_\ell / \delta v_\ell^{(\infty)}$. Then, the scaling for Π_ℓ under the minimal model will satisfy $V(p, q) = F(p)G(q)$ (without the term $H(p)$). It can be shown that

$$E_q(\Pi_\ell^p) \propto \left(E_q(\Pi_\ell^{p'})\right)^{\frac{F(p)}{F(p')}}. \tag{2·29}$$

This relation indicates that the relative scaling of pth moment versus p'th moment under the measure dQ_q is independent of the measure. Recall that the measure parameterized by q probes the fluctuations of increasing amplitude as q increases. Then, the above independence property suggests that all fluctuation amplitudes are equivalent in some sense. On the other hand, the relative scaling describes how the change of one moment follows another as the probability density function, $P(\Pi_\ell)$, undergoes deformations due to the change of parameters (e.g. ℓ). The fact that the relative scaling does not depend on q implies that the deformation of $P(\Pi_\ell)$ is global and uniform everywhere in the domain of the normalized random variable $\Pi_\ell \in (0,1)$. Put it in another way, $\delta v_\ell^{(\infty)}$ is the only characteristic amplitude; when the normalization is performed, no other amplitude is preferred and hence all measures dQ_q are equivalent in describing the deformation law (or relative scaling). We call this an equivalence principle for the minimal model, a model of only one characteristic intermittent structure.

This equivalence principle puts the minimal model on a particular spot. It is a model of minimal complexity. It suggests that near some threshold, nonlinear dynamics may organize a competition among various chaotic components of the cascade (i.e. various β_i) and select the most dominant one β_{max}. Note that the largest β seem to represent the most efficient mechanism destabilizing the MI structures (small but more frequent Poisson jumps). In principle, several families (hierarchies) of structures may coexist, each of which is described by one $F \times G$ term. The peculiarity of the isotropic turbulence is that such competition seems to lead to one outstanding mechanism, the minimal model[6], with a remarkable accuracy. As the system reach such a minimal complexity state, the scaling law will be described by the original model of She & Leveque[6]. This new simplicity emerging from an overwhelming complexity may be the reason why the model[6] seems to be *over-successful* when contrasting against experimental results[9].

2.7. Random Multiplicative Process

The expansion schema described above in (2·25) corresponds in fact to a particular set of random multiplicative processes known as the log-Poisson process. It was shown in She & Waymire[8] that (2·23) can be exactly realized if the multiplicative factor $\mathcal{W}_{\ell\ell'}$ in (2·1) is set to be

$$\mathcal{W}_{\ell\ell'} = \left(\frac{\ell'}{\ell}\right)^h \beta^n, \tag{2·30}$$

where n is a Poisson random variable with $\langle n\rangle \propto \log \ell'/\ell$. In general, corresponding to (2·25), the multiplicative process is

$$\mathcal{W}_{\ell\ell'} = \left(\frac{\ell'}{\ell}\right)^h \prod_i \beta_i^{n_i}, \tag{2·31}$$

where all n_i's are Poisson random variable with mean $\tau_i = \langle n_i\rangle \propto \log(\ell'/\ell)$. It can be shown by direct calculation of $\langle \mathcal{W}_{\ell\ell'}^p\rangle$ and comparison with (2·28) that $\tau_i = C_i \log(\ell'/\ell)$ where C_i is the parameter introduced in (2·28).

It was indicated in She & Waymire[8] that (2·31) can be motivated by the famous Levy-Khinchin decomposition of the infinitely divisible process when the cascade is represented random-multiplicatively. It is interesting that by a manipulation of the algebraic structure of the scaling formula, we arrive at a similar expansion. This indicates the expansion (2·25) represents a kind of systematic approximation. Further studies should be pursued to fully reveal the amazing power of the algebraic structure (2·11) with (2·25) as its solution.

2.8. **Rationality of the Model**

One may immediately question if (2·25) is an arbitrary choice for satisfying the fundamental identiy (2·11)? The answer may be "Yes" to some extent. However, the property of the model suggests a certain degree of rationality, which we will expand here. The basic foundation of the framework is the usefulness in introducing the class of measure-q, $dQ_q(\delta v_\ell)$, in determining the multi-scaling behavior of the field. The basic idea is that the nature of the nonlinear scaling in a mixed medium like turbulence implies a close link between high and low-amplitude fluctuation events, and consequently certain algebraic structure in q_0 may be explored to reveal this property. It turns out that $V(p,q)$ can have a simple (elegant) structure in its dependence on q and that this structure further allow an unlimited number of possibilities to be superposed. These generous features make the model sound in many ways, and constitute a rational basis in the author's view. Below, we expand this view further with several specific remarks.

First, $H(p)$ should be viewed as the limiting behavior of $V(p,q)$:

$$H(p) = \lim_{q\to\infty} V(p,q). \tag{2·32}$$

This is consistent only if the model solution guarantees that the terms $F_i \times G_i$ remain bounded at large p, which is the case here. It is not necessarily true for all scaling problems. For example, it is not satisfied for the Kraichnan model[11] of a passive scalar advected by a δ-correlated velocity field in time. It is also not satisfied for the log-Normal model of energy cascade[12]. Log-Normal is to be excluded because of unphysically singular behavior $V(p,q) \to -\infty$ as $q \to \infty$. With respect to the Kraichnan model[11], the discrepancy stems from the fact that in the limit $q \to \infty$, there does not exist any q-independent factor in $V(p,q)$ in the Kraichnan model of the passive scalar. In other words, our assumption of $V(p,\infty) < \infty$ is not contradicted by Kraichnan's assumption on the conditional expectation of the scalar dissipation. So the two models are basically different. Which assumption is good can not be decided *a priori*. However, what we have shown here is that if the limit $V(p,\infty)$ is finite, then the linear behavior of $H(p)$ is unavoidable. Physically, it amounts to assess the existence of the most intermittent structures.

Secondly, the $F\times G$ term should be considered as the correction to the $H(p)$ term, the contribution of the most singular structures. So, the present model definitely does not address the deviation from Kolmogorov 1941 model[10], or deviation from a linear Langevin-type random cascade process. Instead, it assesses the dominant role of the most intermittent structures. In physical terms, turbulent cascade is led by the most intermittent structures, rather than by a near-gaussian random field with perhaps a few ordered structures. The later is the basis for usual renormalized perturbation approachs. The model developed here is an intrinsically non-perturbative one.

Thirdly, the deviation from the $H(p)$ contribution is finite and the $F \times G$ terms have a distinctively self-similar group structure which suggests distinctively new mechanisms for the energy cascade from the singular structure contribution $H(p)$. Each term may be related to one specific mechanism. Because of the self-similar group structure (2·17) and (2·18), the form is a sound description over the entire domain of p and q. Further support comes from the work of She & Waymire[8)] (see the last subsection) who showed that they correspond exactly to a Poissonian generation of the "defect" structures during the cascade as the logarithmic length scale is taken as time. Each defect reduces the fluctuation amplitude by a factor β_i, and therefore introduce a deviation further away from the singular structure cascade scenario. Hence, they may be understood as the "chaotic" component of the cascade dynamics as opposed to the "coherent" component due to the most intermittent structure. This picture will be further expanded below to form a non-equilibrium dynamical picture of the energy cascade for turbulence.

Finally, what kind of positive role does the model developed here have on the future analytic theory of turbulence? In the future, there will be undoubtedly other elegent, non-perturbative formalism of a more analytical nature starting from the dynamic equation to be developed to express the scaling solution of the kind of nonlinear problem as turbulence. But we believe that those theories need certain basic assumptions of the same nature as we have developed here. In other words, the concept of the most intermittent structures and the (quantized) deviation, etc. may find their door into new theories. We just cannot guess in which exact form this will happen. Only the time will tell.

2.9. Burgers' Turbulence

In the Burgers' turbulence (Burgers' equation with random force), the most intermittent structures are shocks, and there is an invariant probability distribution function (PDF) for δv_ℓ built around the PDF of the amplitude for the shocks (or the large negative events for $\partial v(x,t)/\partial x$). In this case, the characteristic scaling exponent $h = 0$, that is, the scaling exponent ζ_p (or $V(p)$) is independent of p at large p where the shock structures dominate. This is an example of the field satisfying (2·14) and (2·15) but is not a fractional Brownian motion. Note that $\zeta_p = 0$ is not the solution for the Burgers' turbulence, because it violates the energy conservation property. Physically speaking, shocks can not be everywhere; they can only occupy a small portion of the space. The overall description of the whole field or the whole scaling formula requires the consideration of the other portion of the space. This invites the necessity of finding other solutions than (2·14).

Applying the new model (2·16) to the Burgers' turbulence, we can then obtain a consistent overall description for at least $p \geq 1$. Since the shocks are the only dominant structures for $p \geq 1$, we have a strong degeneracy situation: $\beta = 0$, and the corrsponding scaling formula is $\zeta_p = C$ $(p \geq 1)$. In fact, $C = 1$ is obtained by considering the conservation of the energy flux to be scale-independent, $\langle \delta v_\ell^3 \rangle / \ell = \ell^0$. Note that the scaling laws in this degenerate case is identical to the so-called β-model of Frisch, Nelkin and Sulem[13].

It is believed that the infinite-compressibility property of the Burgers' shock is the physical reason for them to be very stable structures and hence leads to strong degeneracy. The 3-D incompressible Navier-Stokes turbulence is considerably more unstable, and we expect a finite β. Experimental and phenomenological estimate is $\beta = (2/3)^{1/3} \approx 0.874$ for the velocity field. It is clear that the Hierarchical Structure model explains the difference between these special cases in a reasonable and physically sound way. We will now further develop some physical picture for the organization of turbulent fluctuations in terms of eddies.

§3. Physical Interpretations: Quantized Eddies

In this section, we develop a "particle" picture of turbulent field, which is intended to develop more intuition about the model described above. It is hoped that such intuition could assist the development of a more dynamical theory later on. As the readers may have noticed, we are pursuing a vigorous phenomenological approach to the problem, instead of an anytical approach based on the known dynamical equations of motion such as the Navier-Stokes equations. The motivation stems from the fact that the large-scale dynamics of turbulence are coupled with small-scale fluctuations so intemately taht if they are isolated (in for example the so-called large-eddy-simulations), their dynamics may have very different nature from what could be derived mathematically with known perturbation techniques. The present phenomenological approach is based on some observations (such as scaling) and then attempt to derive a physical model to be tested against further experiments, both from laboratory and computer simulations. The major effort devoted here is to achieve a theoretical consistent description of the orders and disorders, a unified description of both fluid mechanical patterns like filaments and a vastly distributed arrays of irregular motions. We believe that moderate success has been achieved in this regard. Further research should either connect this picture to an exact system of dynamics, or apply the model to the simulation of real turbulent flows. We hope that the following discussion may be helpful in moving toward either direction.

3.1. A Particle-picture of Turbulence: Eddies

Consider an emsemble of fictious particals which are parameterized by a continuous variable of size (length scale) and a discrete variable of weight (fluctuating amplitude). Denote the size by $s = |\log \ell|$ and the weight by $w = n|\log \beta|$ (n=0, 1, 2,...), and n will be referred to as a quantum number of the particle. Let particles of a fixed size s_0 be introduced into the system randomly at a constant rate and then let their size shrink as if they moves along the s-axis in constant speed. As their size decreases, it also absorbs, from time to time, a quantum so that its quantum number is increased by one and its weight makes a jump. After some time, the system contains particles of variable sizes with variable weights. In summary, there are two stochastic dynamical processes and one deterministic process; the first stochastic process is the introduction of particle and the second is the absorption of quantum; while the deterministic process is the change of size as the time goes. In the $s - w$ plane, particles perform an uni-directional walk composed of horizontal moves and random vertical jumps. The particle path $w(s)$ will be a series of stop and jump. The length of the stops should have an exponential distribution, and thus the number of jumps upto a fixed size s is clearly Poisson distributed. Consequently, when the ensemble of particles of a fixed size s is considered, the distribution of weights $w(s)$ should follow a Poisson statistics.

These particles are not the actual molecules, nor the fluid elements. The later are a restricted area of material regions flowing with the fluid velocities. Instead, these particles are considered as a synthetic representation of turbulent fluctuations appearing in an extensive range of scales, which are measurable experimentally. It is important to note that they are not a local feature in the physical space of the fluid velocity field - a local description may not be significant as far as the cascade dynamics are concerned, due to strong nonlinear coupling and the incompressibility effects. They are rather a form of correlared excitations at some specific scales like a wavelet coefficient. We thus call them "*eddies*" following the fluid mechanical tradition. It is hoped that the distribution of "size" and "weight" of those eddies give a good deal of measurable statistical information about the underlying turbulent field.

3.2. Statistical Laws of Eddies

Let us define the length scale of an eddy to be $\ell = e^{-s}$ and its amplitude $\psi = e^{-w}$. We will further assume that there exists an external field which influences the amplitude of all eddies as their size changes; specifically as the size changes across an inertial range from $s \to s'$ with $\Delta s = s' - s$, the amplitude changes by $e^{-h\Delta s}$. Then, the amplitude of all particles of size s is

$$\psi(s) = \psi_0 e^{-hs-w(s)}, \qquad (3{\cdot}1)$$

where ψ_0 is the initial amplitude of the particle when it was introduced into the system at the initial size $s_0 = 0$. It should become apparent that $\psi(s)$ has the same statistical property as δv_ℓ when the later follows the Hierarchical Structurel model described above:

$$\langle \psi^p(s) \rangle \propto e^{-s\zeta_p} = \ell^{\zeta_p}. \qquad (3{\cdot}2)$$

Each big eddy ψ_0 subsequently becomes small eddy, and as each new big eddy is introduced at some space-time location, it may cover many small eddies at the approximety of the same location. In our picture, there is no splitting of big eddies into small eddies; rather each big eddy follow their own course in cascading from large to small scales. Statistically, this yields the multifractal of the Hierarchical Structure model for the amplitude moments. Each such multifractal cluster (of size ℓ_0) is parameterized by the triplet (h, C, β). The first two parameters describe the properties of the most intermittent structure, and the last one (β) expresses somewhat the extension of the hierarchy: as $\beta \to 0$, Poisson jumps are large; mechanically, the most intermittent structures are relatively more stable since there would be relatively fewer jumps.

In summary, we come to a picture that a fully developed turbulent field can be represented by a number of eddies of variable sizes and weights. Each eddy has a mother eddy ψ_0 (at the integral scales ℓ_0) which has its own distribution $P(\psi_0)$. What we have left unaddressed is the dynamics $s(t)$, from which one may derive the the composition of whole field in terms of a distribution of many eddies $\{s_i(t), w_i(t)\}$. The actual construction of the full field from this set of eddies is an advanced topic which will be left for future study.

As described above, the statistical family of eddies are characterized by (h, C, β). We call the ensemble of eddies which belong to the same family a local hierarchy. Among the three parameters, β seem to reflect the level of disorders near the intermittent structure. In an inhomogeneous turbulence, the parameters h and C may be subject to variations in space. Even in homogeneous turbulence, due to fluctuations, a finite space-time ensemble may also show fluctuation in the parameter h and C. However, there is experimental evidence that β holds a higher degree of the universality.

§4. Conclusion

Let us summarize the main assumptions that the Hierarchical Structure model has put forward to describe turbulent fluctuations. First, it assumes the existence of some dominant structures at small scales (the existence of $H(p)$) which further defines an asymptotic ($\ell \to 0$) fractal set of a definite dimension. In mathematical terms, the probability of finding such structures as $\ell \to 0$ scales in power law with the length scale, and the amplitude of the structure vary also in power law with the length scale. Strictly speaking, this is valid only within the inertial range. The so-called absolute scaling (in ℓ) formula predicted by the model is also applicable only to the inertial range. Secondly, it assumes the existence of an additional mechanism which are called defect Poissonian jumps. This mechanism has an elegant algebraic structure describing the nonlinear scaling laws, and has a remarkble property that multiple jumps may act concurrently without interference. This last property renders the model very plausible as a leading approximation of a general class of nonlinear scaling problems.

With the successful prediction of the scaling laws, one naturally asks: where are we in pursuing an adequate description of turbulence? Let us offer some comments in this regard. First, classical approach from a fluid mechanical standpoint has attempted to relate the properties of turbulence to special local solutions or perturbed solutions of the dynamical equations. Special solutions once found are often randomly superposed to constitute a stochastic field to evaluate statistical correlations (see, e.g.[14]). On the other hand, statistical approach often starts with the equations for the correlation functions derived from the dynamical equations and then introduces purely *statistical* assumptions to simplify the equations. Both approaches need guidance from a certain phenomenological framework. For example, the hidden phenomenology behind the random superposition of local flow structures is the finite extent of spatial coupling of fluid structures in turbulent environment. This is questionable for scales smaller than the integral scale because within an eddy of the integral scales, small-scale structures (e.g. of the vorticity) are both very rich (i.e. neither laminar nor regular) and strongly coupled. In the statistical approach, a phenomenology often followed is the so-called "maximum stochasticity" first mentionned by Kraichnan[15]. That is, one seeks for a stochastic field with the least structural features which can accomplish the work of transfering kinetic energy from large to small scales. More recently, an approach was suggested in terms of a "fusion rule"[16]; further clarification of its physical meaning is needed.

In author's view, the description of turbulence requires a hydrid use of mechnical (deterministic) law and statistical (probabilitic) law. Throughout this text, we have shown that the Hierarchical Structure model seems to contain the both ingredients: the most intermittent structures and the quantized defect Poissonian jumps act simultaneously to give a whole picture of the energy cascade. The Burgers' turbulence is a special case where the most intermittent structures, shocks, are highly degenerate, or stable. The 3-D incompressible turbulence is definitely different; experimental and numerical observations[9] have shown that the minimal model (the original She-Leveque model) is an accurate description of the scaling laws. It is fair to conclude that the present theoretical proposal has received good experimental support.

The elegance in the algebraic structure of the model presented in this paper has enhanced the confidence about model, and has de-mystified the success of the model when it is applied to analyze a variety of situations such as magnetohydrodynamics[17], solar wind turbulence[18], the GOY shell model[19], etc. On the other hand, it has not been possible yet to incorporate the ideas presented here to a theoretically constructive treatment of the dynamics in such a way leading to at least some approximate evaluation of the model parameters. This is to say that presently the model has not constituted a complete phenomenological framework to guide the development of a further analytical theory. It is hoped that the increased understanding provided here will contribute to this eventual goal.

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