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INTERFACE PRESERVING COARSENING MULTIGRID FOR ELLIPTIC PROBLEMS WITH HIGHLY DISCONTINUOUS COEFFICIENTS

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Abstract. We propose an interface preserving coarsening, a new technique for solving discontinuous coefficient partial differential equations by multigrid. It selects coarse grid points so that all the coarse grids are aligned with the interfaces for regular interface problems on structured grids, and that the interfaces are resolved as much as possible for irregular interface problems. As a result, multigrid with linear interpolation is sufficient to obtain fact convergence. We show by one-dimensional and two-dimensional interface problems that the convergence rate of the resulting multigrid method is independent of the mesh size and the size of the jump at the interfaces.

1. Introduction. Multigrid methods have been widely used for solving second order elliptic partial differential equations (PDEs). Their convergence rate is often independent of the mesh size, and such optimal convergence theory can be found, for instance, in [5, 6, 7, 17, 22, 25, 30, 31]. However, in practice, the convergence rate may depend on the PDE coefficients. Typically the convergence deteriorates as the coefficients become rougher. For example, if the coefficients have large jumps [1, 4, 10, 11, 17, 19], are highly oscillatory [14, 21, 28] or are anisotropic [12, 17, 26], standard multigrid methods will converge very slowly. Special techniques are needed to handled some of the problems. In this paper, we consider PDEs whose coefficients have jumps of several orders of magnitude. Such problems arise in various applications such as flow in heterogeneous porous media [2], neutron transport [1], biophysics [18], etc. The finite element/difference discretization has to be handled carefully [3, 20, 27]. Despite this, suppose the discretization is properly done. We want to derive a multigrid method which is as insensitive to the jumps as possible.

A typical approach of improving the convergence is to define a sophisticated interpolation such as in black box multigrid [10, 11]. The idea is to capture the discontinuous behavior of the derivative of the solution along the interfaces, or equivalently to preserve the continuity of the flux across the interfaces [1, 10, 11, 17]. However, these methods require a setup phase for the computation of the interpolation operator and extra storage for it. Besides, they usually only apply to structured grid problems. The robust energy-minimizing interpolation proposed by Wan, Chan and Smith [29] may apply to both structured and unstructured grids but the additional cost for the construction of the interpolation operator is still required. In this paper, we devise an efficient multigrid method which still utilizes the simple linear interpolation.

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A successful multigrid method depends not just on interpolation but on all components as a whole. Our key observation is that coarsening can play a crucial role for interface problems. Specifically, the coarse grid points should resolve the shape of the interface in a certain sense to be described later. Intuitively speaking, experiences from the literature [1, 15, 17] indicated that the parts of the solution on the regions of different constant coefficients behave independently and are glued together through a Neumann boundary condition on the interfaces. Theoretically speaking, convergence analysis for interface problems [8, 13, 30] often require that the discontinuities are preserved on all coarser grids. In view of these, we propose an interface preserving coarsening algorithm so that all the coarse grids will align with the interfaces for regular interface problems on structured grids, and that the interfaces are resolved as much as possible for irregular interface problems. Consequently, linear interpolation is sufficient to obtain fast multigrid convergence.

We remark that special coarsening techniques are quite common for anisotropic problems. However, to the best of our knowledge, no coarsening strategy has been studied specifically for discontinuous coefficient problems. In algebraic multigrid [24], coarsening is also done specially according to the notion of *strong coupling*. However, their motivation is purely algebraic, and no geometric interpretation is given. We also remark that much work has been done on domain decomposition for discontinuous coefficient problems; see, for example [9, 25]. For many of these methods, the subdomains are naturally divided by the interfaces. Thus the true interfaces coincide with the computational subdomain interfaces. In multigrid, however, we do not usually have large subdomains and hence the interfaces do not generally align with the coarse grids.

Another approach for discontinuous coefficient problems is to use conjugate gradient preconditioned by diagonal scaling [16, 23]. It is very simple and yet quite effective. Graham and Hagger [15] analyzed such approach theoretically, and extended the analysis to additive Schwarz preconditioners. They showed that the largest eigenvalues of the preconditioned system are bounded and that only a finite number of small eigenvalues approach to zero as the size of the jump increases. Thus the preconditioned conjugate gradient method converges with a number of iterations which grows only logarithmically in the size of the jump. In their analysis, the interfaces are not assumed to align with the coarse grids. Another related approach by Ashby et. al. [2] is to use multigrid preconditioned conjugate gradient.

In Section 2, we first discuss the issues of coarsening for discontinuous coefficient problems and explain why sometimes it is mandatory to select coarse grid points in a special way. We illustrate the idea in one dimension in Section 3.1. In two dimensions, we discuss the regular and irregular interface case separately in Section 3.2.1 and Section 3.2.2, respectively. The extension of the algorithms to three dimensions is possible but is not discussed in the present paper. In Section 4, the effectiveness of the interface preserving coarsening is demonstrated by one-dimensional and two-dimensional examples. Finally, concluding remarks are made in Section 5.

2. Failure of Multigrid with Standard Coarsening. As mentioned in the previous section, robust interpolations have been the key for improving multigrid conver-

gence for discontinuous coefficient problems. However, in some circumstances, especially in unstructured grid computations, robust interpolation is not enough. Sometimes, it is mandatory to employ special coarsening strategy.

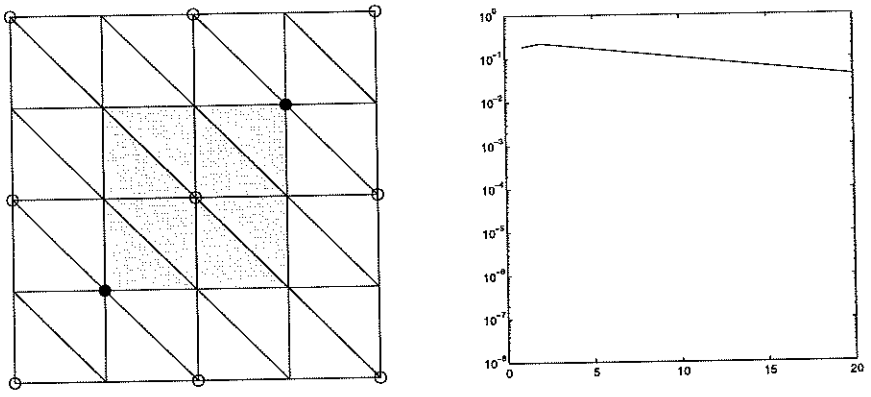

FIG. 1. *(a) A 5 × 5 grid with a jump in coefficient shown by the shaded area. Coarse grid points are denoted by ∘. (b) Standard multigrid convergence.*

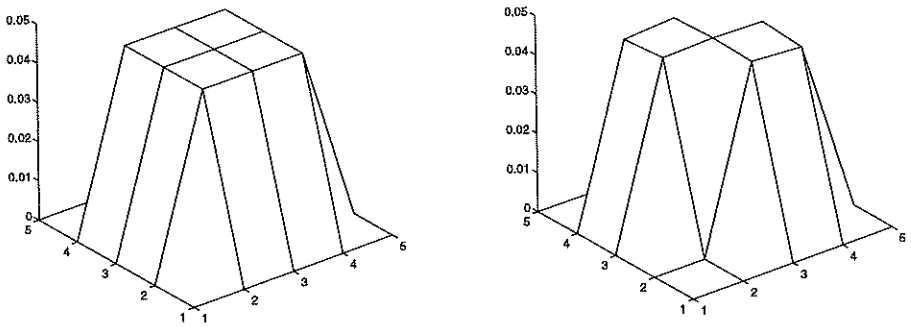

FIG. 2. *(a) Error after smoothing. (b) Error given by interpolation with standard coarsening.*

Consider a square interface on a 5 × 5 triangular mesh as shown in Figure 1(a). The circles denote the coarse grid points chosen by standard coarsening. The jump in coefficient is indicated by the shaded area, and zero Dirichlet boundary condition is used. Since it is a triangular mesh, it is natural to interpolate the noncoarse grid points by the two (and only two) connected coarse grid points. The robust energy-minimizing interpolation described in [29] is used along with the Gauss-Seidel smoothing. Other robust interpolation can be used as well. The slow multigrid convergence is shown in Figure 1(b). The key observation is that the interpolated values at the two noncoarse grid points marked by • are far from the true values of the errors. Figure 2(a) and (b) show the error after smoothing and the error given by interpolation. Due to the large jump in coefficient, the error has larger values at the region with large coefficient and smaller values at the region with small coefficient. On the other hand, the interpolated values at the two corners are zero since their connected coarse grid points are on the

boundary where the values are zero. In fact, even if other more sophisticated interpolations were used, the interpolated values had to be zero as long as they were interpolated only by the two connected coarse grid points on the boundary and the interpolation method preserves constant. Thus the approximation of the error on the coarse grid is very poor which leads to a slow multigrid convergence.

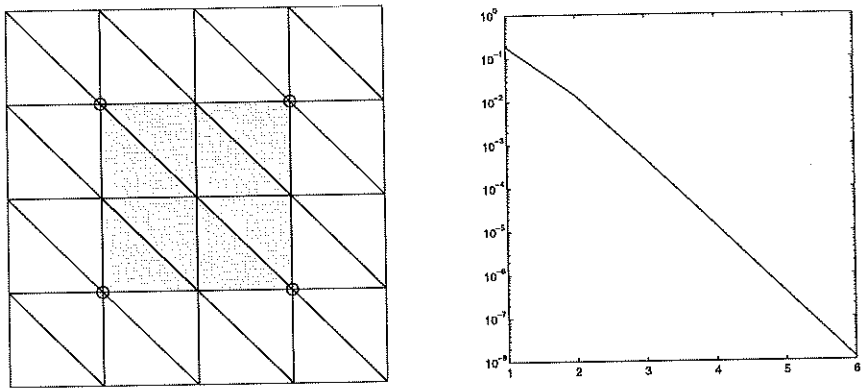

FIG. 3. *(a) Nonstandard coarsening, denoted by ∘, at the four corners of the square interface. (b) Standard multigrid convergence.*

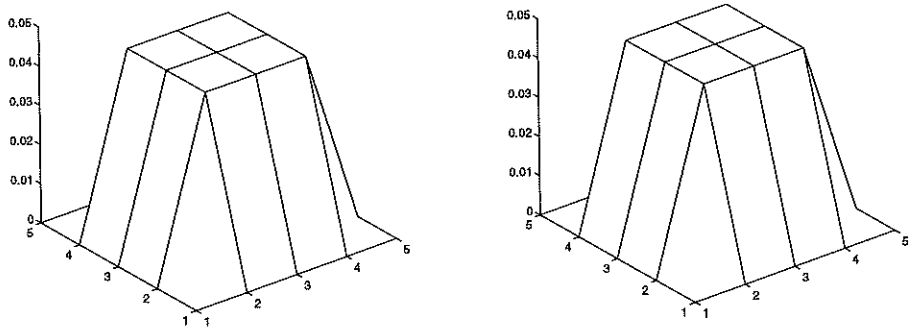

FIG. 4. *(a) Error after smoothing. (b) Error given by interpolation with nonstandard coarsening.*

Based on this observation, a natural remedy is to select coarse grid points so that the previous situation will not happen. For instance, if we choose the coarse grid points shown in Figure 3(a), along with linear interpolation and Gauss-Seidel smoothing, we obtain the usual rapid multigrid convergence (Figure 3(b)). As shown in Figure 4(a) and (b), the true error and the interpolated error are essentially the same, and hence the fine grid errors can be corrected accurately by the coarse grid. In conclusion, a special coarsening is required to maintain the efficient multigrid convergence in this case.

3. Interface Preserving Coarsening. Let Ω be a convex polygon in $\Re^d$, $d = 1, 2$. We consider the following elliptic discontinuous coefficient problem:

$$
\begin{aligned}
-\nabla \cdot a \nabla u &= f, && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where the coefficient a is discontinuous. For the ease of exposition, we assume that a is piecewise constant, i.e.,

$$a(x) \equiv a_i \qquad x \in \Omega_i,$$

where $\Omega_i \subset \Omega$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, and $\bar{\Omega} = \cup_i \overline{\Omega_i}$.

Standard multigrid converges slowly when the a_i's differ by orders of magnitudes. In the following sections, we describe an interface preserving coarsening technique whose resulting multigrid convergence rate can be shown to be independent of the differences of the a_i's.

3.1. One Dimension. The idea of the interface preserving coarsening is to select coarse grid points so that the interfaces align with all the coarse grids. In one dimension, we simply assign the points at the interfaces to be the coarse grid points. After that, we perform standard coarsening for the remaining points; we select every other points as coarse grid points (Figure 5). The idea can be easily extended recursively to coarser grids (Figure 6). The algorithm is described mathematically as follows.

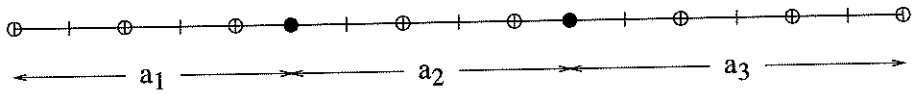

FIG. 5. *One-dimensional interface preserving coarsening without Step 2 (cf. Algorithm 3.1). The regions of different coefficient are denoted by a_1, a_2 and a_3. Coarse grid points are denoted by $\circ$ and the coarse grid points at the interfaces are denoted by $\bullet$.*

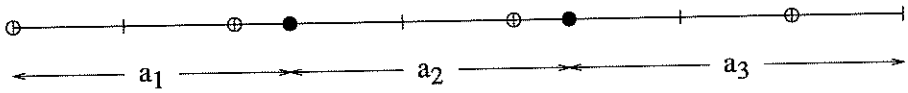

FIG. 6. *One-dimensional interface preserving coarsening on the coarse grid points given by Figure 5.*

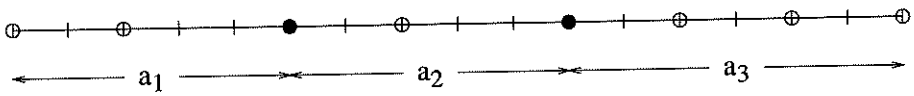

FIG. 7. *One-dimensional interface preserving coarsening with optional Step 2 (cf. Algorithm 3.1) in effect. No two consecutive points are coarse grid points.*

Let $N^k = \{n_1^k, \ldots, n_{m_k}^k\}$ be the set of fine grid points at level k. Let N_C^k be the set of coarse grid points and $N_F^k = N^k \backslash N_C^k$ be the set of noncoarse grid points. The algorithm consists of the following three steps; see Algorithm 3.1.

Step 1 selects the interface points as coarse grid points. Step 2 ensures that no two coarse grid points are adjacent to each other unless they are both interface points. Thus the number of coarse grid points is not more than half of the fine grid points. The difference of applying and not applying Step 2 is shown in Figure 7 and Figure 5, respectively. The former has fewer coarse grid points but more overlaps between basis functions and the opposite for the latter. Our default is not to apply Step 2. Step 3 performs the standard coarsening between the selected coarse grid points from Step 1.

Algorithm 3.1: 1D Interface Preserving Coarsening

1. Set $N_C^k = \{$ interface points $\} \equiv \{n_{i_1}^k, n_{i_2}^k, \ldots, n_{i_p}^k\}$.
   Assume $n_{i_0}^k = n_1^k$ and $n_{i_{p+1}}^k = n_{m_k}^k$.
2. (optional) Set $N_F^k = \{$ neighbors of $j : j \in N_C^k\}$.
3. for $l = 0$ to p
   for $j = n_{i_l}^k$ to $n_{i_{l+1}}^k$
   if $j \notin N_F^k \cup N_C^k$ then
   $N_C^k = N_C^k \cup \{j\}, \quad N_F^k = N_F^k \cup \{$ neighbors of j $\}$
   end if
   end for
   end for

The other multigrid components are standard. We use Gauss-Seidel as smoother and linear interpolation. We remark that the coordinate information of the computational points is needed to perform the linear interpolation due to the nonuniformity in spacing of the coarse grid points.

3.2. Two Dimensions. We describe separately the algorithm for the regular and irregular interface. For the former, we may apply the one-dimensional technique. For the latter, we need another algorithm to maintain low complexity while still being able to resolve the interfaces.

3.2.1. Regular Interfaces. Suppose the grids are regular tensor product grids, and the regions of different coefficients, Ω_i, are also formed by tensor products. In other words, let

$$\begin{aligned} X &= \{x : x \text{ is the x-coordinate of an interface point}\}, \\ Y &= \{y : y \text{ is the y-coordinate of an interface point}\}. \end{aligned}$$

Then $\Omega_i = [x_1, x_2] \times [y_1, y_2]$ for some $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Figure 8 shows an example of a regular interface problem.

Algorithm 3.2: 2D Regular Interface Preserving Coarsening

Let Ω^k be the set of grid points on level k.
1. Set $X^k = \{x : (x, y) \in \Omega^k\}$, $Y^k = \{y : (x, y) \in \Omega^k\}$.
2. Set $N_{Cx}^k =$ coarse grid points obtained from 1D coarsening on X^k.
3. Set $N_{Cy}^k =$ coarse grid points obtained from 1D coarsening on Y^k.
4. Set $N_C^k = \{(x, y) : x \in N_{Cx}^k,\ y \in N_{Cy}^k\}$.

The two-dimensional regular interface preserving coarsening is obtained by a "tensor product" of the one-dimensional algorithm; see Algorithm 3.2. A point $x \in X^k$ is an interface point if there exist a region Ω_i such that $\Omega_i = [x, \tilde{x}] \times [y, \tilde{y}]$ for some $\tilde{x} \in X^k$ and $y, \tilde{y} \in Y^k$. Thus $\tilde{x}$ is also an interface point in X^k. The interface points

in Y^k are defined similarly. A result of the regular interface preserving coarsening is illustrated in Figure 8. We note that an extension of the algorithm to three dimensions is straightforward.

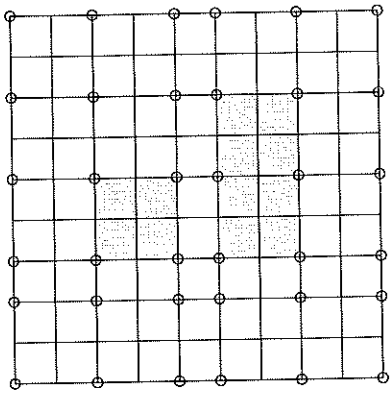

FIG. 8. *Tensor product interface preserving coarsening for a regular two-dimensional interface. Coarse grid points are denoted by* $\circ$.

3.2.2. Irregular Interfaces. We can still apply the regular techniques if the interfaces are not too irregular. For example, for the interface shown in Figure 9(a), we may apply Algorithm 3.2 to obtain a coarsening of the grid points; we collect the x and y coordinates and determine their interface points as before, and then coarsen along the x and y direction independently. Figure 9(a) shows the resulting coarse grid points. For less regular interfaces, however, this tensor product procedure may create excessive number of coarse grid points to preserve the interfaces; see Figure 9(b). Consequently, the overall computational cost may increase.

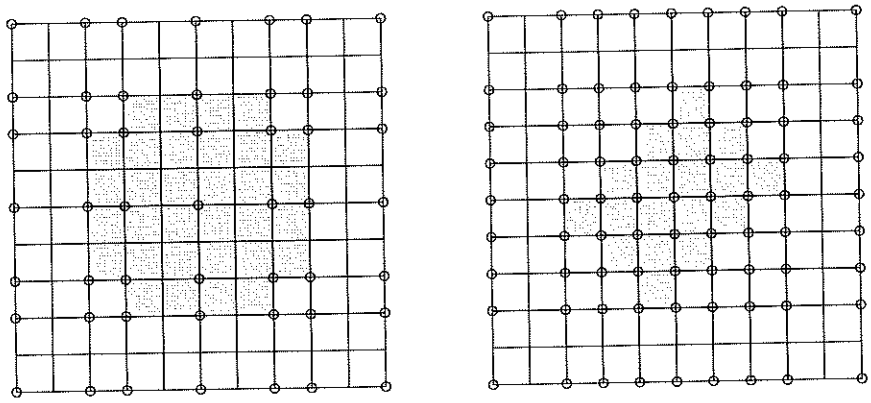

FIG. 9. *Tensor product interface preserving coarsening for (a) a less irregular interface, (b) very irregular interface. Coarse grid points are denoted by* $\circ$.

In general, we want to resolve the shape of the interface while maintaining the coarse to fine grid point ratio to be 1:4 as closely as possible. The idea is as follows. We first identify the points where the interfaces are located. Then we apply standard coarsening to these points and then to the remaining points. We convert some of the noncoarse grid points near the interfaces to coarse grid points so that the shapes of the

interfaces are better resolved. We have not yet derived a rule of conversion which can be justified rigorously. Nevertheless, we suggest two effective heuristics to be discussed later.

The above procedures can be described mathematically as follows. For simplicity, we describe the two regions case only. Let Ω^+, Ω^- be disjoint open subsets of Ω such that $\Omega = \overline{\Omega^+} \cup \Omega^-$. Let $\Gamma = \partial\Omega^+$ be the interface and $\Gamma \cap \partial\Omega = \emptyset$. Let $a(x,y) \equiv a^+$ in Ω^+ and $a(x,y) \equiv a^-$ in Ω^- and $a^- \leq a^+$. The case of multiple interfaces is treated similarly and hence is omitted. The two-dimensional irregular interface preserving coarsening is given in Algorithm 3.3.

Algorithm 3.3: 2D Irregular Interface Preserving Coarsening

1. Determine the set of fine grid points N^+ in $\overline{\Omega^+}$.
2. Full coarsening on N^+.
3. Full coarsening on $N^- = N \backslash N^+$, N=set of fine grid points.
4. Either (a) or (b) is used. Change a noncoarse grid point x_i^h to a coarse grid point, if any of the condition is satisfied.

   Criterion set (a):

   i. $x_i^h \in N^+$ and no coarse grid point $x_j^H \in N^+$ is connected to x_i^h.
   ii. $x_i^h \in N^-$ and no coarse grid point $x_j^H \in N^-$ is connected to x_i^h.
   iii. x_i^h is connected to one coarse grid point only.
   iv. Coarse grid points in N^+ do not interpolate those in N^-, and vice versa.

   Criterion set (b):

   i. $x_i^h \in N^+$ and less than two coarse grid points $x_j^H \in N^+$ connected to x_i^h.
   ii. $x_i^h \in N^-$ and less than two coarse grid points $x_j^H \in N^-$ connected to x_i^h.
   iii. $x_i^h \in \partial\Omega$ and no $x_j^H \in \partial\Omega$ is connected to x_i^h.
   iv. Coarse grid points in N^+ do not interpolate those in N^-, and vice versa.

Remarks:

Step 1: Notice that the diagonal entries of the stiffness matrix $\mathcal{A}$ is

$$\mathcal{A}_{ii}^h = a^+ \int_\Omega |\nabla \phi_i^h|^2 dx,$$

for all $x_i^h \in N^+$. If $a^+ \gg a^-$, the set of points in N^+ can be easily identified by the large diagonal entries of $\mathcal{A}_{ii}^h$.

Step 2, 3: They are just standard coarsenings on the points in N^+ first followed by those in N^-.

Step 4: Two sets of heuristic criteria are suggested to ensure that the noncoarse grid points on Γ are properly interpolated so that the discontinuous derivative behavior of the solution is captured. Criterion a(i) & (ii) require noncoarse grid points to be interpolated by at least one coarse grid point in the same region. These conditions eliminate the situation given by the example in Section 2. The purpose of criterion a(iii) is to take care of a special case in unstructured grid computations rather than for resolving

the interfaces. If a noncoarse grid point is connected to one coarse grid only, the interpolated weight at the noncoarse grid point must be 1 due to the constant preserving constraint. In other words, we have a local piecewise constant instead of piecewise linear interpolation there which may affect convergence. Criterion a(iii) eliminates this case by making such noncoarse grid points to coarse grid points. In Criterion set (b), the first two criteria serve a similar purpose as a(i) & (ii), except that they require more coarse grid point connections in the same region. Criterion b(iii) requires that noncoarse grid points on the boundary must be connected to at least one coarse grid point on the boundary. For both criterion sets, we eliminate the influence of coarse grid points from the other region by criterion a(iv) & b(iv); we allow interpolation from coarse grid points to noncoarse grid points in the same region only.

Figure 10 shows an example of the irregular interface preserving heuristics. Since Algorithm 3.3 is based on the graph connection of a matrix, we use here a triangular mesh for a better illustration. Compared to Figure 9(b), we need much fewer coarse grid points. Moreover, within each constant coefficient region, the coarse grid points are selected as if in standard coarsening. More coarse grid points, especially in the case when the criterion set (b) is used, are selected near the interface to resolve its shape.

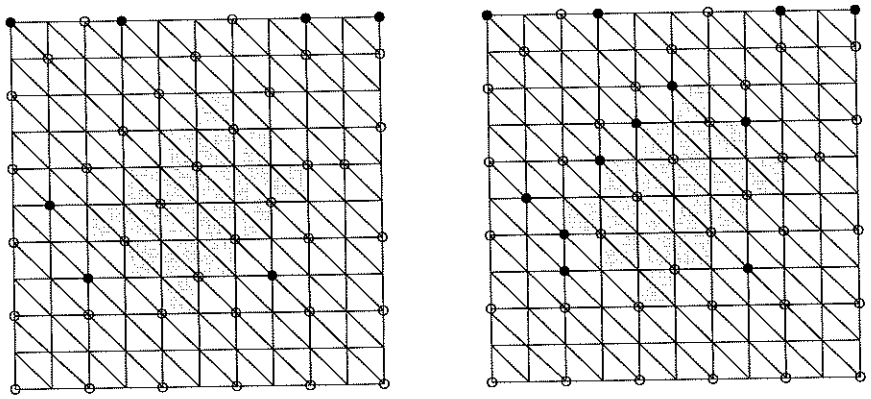

FIG. 10. *Irregular interface preserving coarsening for an irregular interface using (a) Criterion set (a), (b) Criterion set (b), in Algorithm 3.3. Coarse grid points are denoted by* $\circ$. *The coarse grid points selected by either criterion set are denoted by* $\bullet$.

3.2.3. Complexity Issue. Resolving the interface, in general, increases the total number of coarse grid points which in turn increases the overall computational cost. However, if the interface forms a simple piecewise smooth curve, the increase in the number of coarse grid points is at most the total number of grid points on the interface, which is only $O(\frac{1}{h})$ compared to $O(\frac{1}{h^2})$ total number of coarse grid points. Suppose the complexity, η_0, of one V-cycle multigrid with standard coarsening is estimated to be:

$$\eta_0 = O(\frac{1}{h^2} + \frac{1}{4h^2} + \frac{1}{16h^2} + \ldots) = O(\frac{4}{3h^2}).$$

Then, the complexity, η_1, of one V-cycle multigrid with our special coarsening is:

$$\eta_1 = O(\frac{1}{h^2} + (\frac{1}{4h^2} + \frac{1}{h}) + (\frac{1}{4}(\frac{1}{4h^2} + \frac{1}{h}) + \frac{1}{2h}) + (\frac{1}{4}(\frac{1}{4}(\frac{1}{4h^2} + \frac{1}{h}) + \frac{1}{2h}) + \frac{1}{4h}) + \ldots)$$

$$= O(\frac{4}{3h^2} + \frac{8}{3h}).$$

Thus the extra amount of work due to the increase in the number of coarse grid points is asymptotically small compared to the standard one.

4. Numerical Results. We demonstrate the effectiveness of the interface preserving coarsening by three examples. The multigrid settings are standard. We apply two pre- and post- Gauss-Seidel smoothings. Linear interpolation is used for the one-dimensional problem and the two-dimensional regular interface problem. For the irregular interface problem, since the tensor product grid structure is destroyed after one level of coarsening, we have to use unstructured grid multigrid techniques for the subsequent coarser grids. In particular, we use the energy-minimizing interpolation proposed by Wan, Chan and Smith [29]. The multigrid iteration was terminated when the relative residual norm was less than 10^{-6}.

Example 1: We compare the convergence results of the multigrid method with the one-dimensional interface preserving coarsening and linear interpolation and that with the standard coarsening and flux preserving interpolation [17]. The model equation is

$$\begin{aligned} -\frac{d}{dx}a(x)\frac{d}{dx}u(x) &= 1 \qquad \text{in } (0,1) \\ u &= 0 \qquad \text{at } x = 0 \text{ and } x = 1, \end{aligned}$$

where

$$a(x) = \begin{cases} 10^4 & \text{if } x \le 1/4 + h \\ 1 & \text{if } 1/4 + h < x \le 1/2 + h \\ 10^2 & \text{if } x > 1/2 + h. \end{cases}$$

Here, h is the size of the fine grid. It is so designed that the interfaces will not align with any standard coarse grids. The convergence results of the two multigrid methods are shown in Table 1. There is no difference in performance for both methods. Moreover, their convergence results are independent of the mesh size h and the jump in the coefficient.

h	MG Method 1	MG Method 2	Standard MG
1/32	6	6	16
1/64	6	6	17
1/128	6	6	20
1/256	6	6	24

TABLE 1

Convergence of multigrid methods for a 1D discontinuous coefficient problem. Method 1 uses interface preserving coarsening and linear interpolation. Method 2 uses standard coarsening and flux preserving interpolation.

Example 2: We show the effectiveness of the two-dimensional tensor product interface preserving coarsening for regular interfaces. The problem is a modification of Example

I in [1]:

$$-\nabla \cdot a(x,y)\nabla u = 1,$$

where

$$a(x,y) = \begin{cases} a^+ & 0.25 \le x \le 0.75 \quad \& \quad 0.25 \le y \le 0.75 \\ a^- & \text{otherwise.} \end{cases}$$

We fix $a^- = 1$ and vary a^+ from 10 to 10^4. The convergence results of the multigrid method with interface preserving interface and linear interpolation and that with standard coarsening and energy-minimizing interpolation are shown in Table 2. Again, their performances are essentially the same, and their convergences do not depend on the mesh size nor the size of the jump.

	MG Method 1			MG Method 2			Standard MG		
h	10	10^2	10^4	10	10^2	10^4	10	10^2	10^4
1/16	5	5	6	6	5	5	14	*	*
1/32	5	6	6	6	6	6	14	*	*
1/64	6	6	6	6	6	6	14	*	*
1/128	6	6	6	7	6	6	14	*	*

TABLE 2

Convergence of multigrid methods for a 2D discontinuous coefficient problem. Method 1 uses interface preserving coarsening and linear interpolation. Method 2 uses standard coarsening and energy-minimizing interpolation. The numbers $10, 10^2, 10^4$ *indicate the values of* a^+. $*$ *denotes convergence more than 100 iterations.*

Example 3: Finally, we present the results of an irregular interface problem. The star-shaped interface is shown in Figure 11(a), and the irregular interface coarsening with criterion set (b) on the 16×16 triangular mesh is shown in Figure 11(b). Since the coefficient is assumed to be constant on each triangle, a wrinkleness is resulted on the interface. We can see that more coarse grid points are selected near the interface and the remaining parts are coarsened in a standard way.

The convergence results of the multigrid method using the irregular interface preserving coarsening and energy-minimizing interpolation are given in Table 3. (Note that we cannot use the standard linear interpolation in this case since the regular nested grid structure is destroyed after one level of coarsening.) First, the convergence is independent of the jump. Second, for a fixed number of levels, the convergence is independent of the mesh size. However, the convergence rate does deteriorate with the number of levels. To alleviate this problem, we do not go through many levels, if we can afford to solve exactly, for instance, a coarse grid problem of size 100. In that case, it only needs about 20 iterations to convergence.

The "number of nodes" column shows the number of nodes at each level. As discussed in Section 3.2.2, it is important to maintain a reasonable ratio between the fine and coarse grid points so that the computational cost is not much higher than that of standard multigrid. From the column, we see that the ratio is between 1:3 and 1:4, which is close to the standard ratio 1:4.

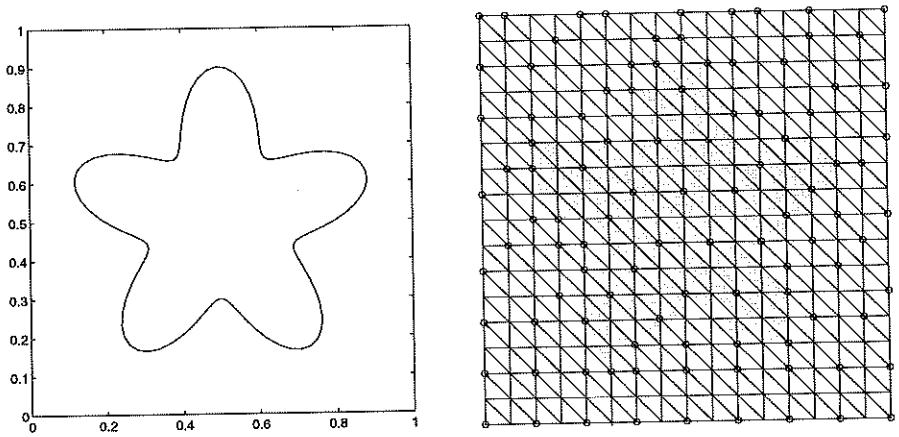

FIG. 11. *(a) A star-shaped interface is formed by:* (x, y), $x = 0.5 + r\cos\theta$, $y = 0.5 + r\sin\theta$ *where* $r = 0.3 + 0.1\sin(5\theta)$, $0 \le \theta \le 2\pi$. *(b) The star-shaped interface in (a) on a* 16×16 *triangular mesh.*

Grid size	Level	# of nodes	$a^+ = 10$	$a^+ = 10^2$	$a^+ = 10^3$	$a^+ = 10^4$
1/32	2	344	12	13	14	14
	3	116	14	14	15	15
	4	40	20	22	22	22
1/64	2	1261	13	13	13	13
	3	367	14	14	14	14
	4	102	18	18	18	18
	5	28	30	37	39	39
1/128	3	1258	16	16	16	16
	4	319	19	19	19	19
	5	86	27	28	28	28
	6	25	43	52	54	54

TABLE 3

Convergence of the interface preserving coarsening multigrid for the star-shaped interface problem.

5. Concluding Remarks. We have demonstrated numerically that coarsening can be an efficient alternative for robust interpolation to solve discontinuous coefficient problems using multigrid. In fact, we have shown by an example that special coarsening sometimes is mandatory no matter what the interpolation is. We have shown that multigrid with the proposed interface preserving coarsening and the simple linear interpolation is an effective solution method for discontinuous coefficient problems on structured grids. In general, we need to combine the interface preserving coarsening and an unstructured grid interpolation for irregular interface problems on general domains, and we have shown by an example how this can be done.

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