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POINTWISE ERROR ESTIMATES FOR SCALAR CONSERVATION LAWS WITH PIECEWISE SMOOTH SOLUTIONS*

EITAN TADMOR† AND TAO TANG‡

Abstract. We introduce a new approach to obtain sharp *pointwise* error estimates for viscosity approximation (and, in fact, more general approximations) to scalar conservation laws with piecewise smooth solutions. To this end, we derive a transport inequality for an appropriately *weighted* error function. The key ingredient in our approach is a one-sided interpolation inequality between classical L^1 error estimates and Lip^+ stability bounds. The one-sided interpolation, interesting for its own sake, enables us to convert a global L^1 result into a (nonoptimal) local estimate. This, in turn, provides the necessary bounds on the coefficients of the above-mentioned transport inequality. Estimates on the weighted error then follow from the maximum principle, and a bootstrap argument yields optimal pointwise error bound for the viscosity approximation.

Unlike previous works in this direction, our method can deal with finitely many waves with possible collisions. Moreover, in our approach one does not follow the characteristics but instead makes use of the energy method, and hence this approach could be extended to other types of approximate solutions.

Key words. conservation laws, error estimates, viscosity approximation, optimal convergence rate, transport inequality

AMS subject classifications. 35L65, 65M10, 65M15

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1. Introduction.

We study the convergence of vanishing viscosity solutions governed by the single conservation law

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \qquad x \in \mathbf{R},\ t > 0, \epsilon > 0, \tag{1.1}$$

and subject to the initial condition prescribed at $t = 0$,

$$u^\epsilon(x,0) = u_0(x). \tag{1.2}$$

We are interested in the pointwise convergence rate of u^ϵ toward the inviscid solution, u, of the corresponding inviscid conservation law

$$u_t + f(u)_x = 0, \qquad x \in \mathbf{R},\ t > 0, \tag{1.3}$$

which is subject to the same initial conditions

$$u(x,0) = u_0(x). \tag{1.4}$$

We will investigate the *pointwise* convergence rate of u^ϵ toward u, assuming that the inviscid solution u has finitely many shocks or rarefaction waves. This is the generic situation [19, 23].

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†Department of Mathematics, UCLA, Los Angeles, CA 90095 (tadmor@math.ucla.edu).

‡Department of Mathematics, Simon Fraser University, Vancouver, BC, Canada V5A 1S6. Current address: Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong (ttang@math.hkbu.edu.hk). Part of the research was carried out while this author was visiting UCLA.

It is well known that $u^\epsilon(\cdot,t)$ converges strongly in L^1 to $u(\cdot,t)$, where $u(\cdot,t)$ is the unique, so-called entropy solution of (1.3)–(1.4) (e.g., [9, 11]). The L^1 convergence rate in this case is upper bounded by

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1} \leq \text{const} \cdot \sqrt{\epsilon}. \tag{1.5}$$

Consult [10] and [18] for the discrete analogue of monotone difference schemes. Although the L^1 convergence rate of order $\mathcal{O}(\sqrt{\epsilon})$ is optimal [17, 24], *in practice* one obtains an L^1 rate of order $\mathcal{O}(\epsilon)$, when applied to convex conservation laws,

$$f'' \geq \beta > 0, \tag{1.6}$$

with finitely many shock or rarefaction discontinuities (and these are the only solutions that can be computed!). For Riemann problems with a convex flux, this first-order estimate has been obtained by Bakhvalov [1] (and Harabetian [7] proved an $\mathcal{O}(\epsilon|\ln\epsilon|)$ rate for Riemann problems with a rarefaction wave). Teng and Zhang [26] provided an $\mathcal{O}(\epsilon)$ rate for Riemann problems with finitely many shocks; Fan [2] established the L^1 convergence rate for the Godunov scheme. The general first-order estimates in L^1 were obtained recently by Tang and Teng [25]. It is proved that for convex conservation laws whose entropy solution consists of finitely many discontinuities, the L^1 error between the viscosity solution, u^ϵ, and its inviscid limit, u, is bounded by $\mathcal{O}(\epsilon|\ln\epsilon|)$. If neither central rarefaction waves nor spontaneous shocks occurs, the error bound is improved to $\mathcal{O}(\epsilon)$.

It is understood that the L^1 error estimate is a global one, while in many practical cases we are interested in the *local* behavior of $u(x,t)$. Consequently, when the error is measured by the L^1 norm, there is a loss of information due to the poor resolution of shock waves in $u(x,t)$. Several authors have investigated pointwise error estimates: For a system of conservation laws, Goodman and Xin [6] proved that the viscosity methods approximating piecewise smooth solutions with finitely many *noninteracting* shocks have a local $\mathcal{O}(\epsilon)$ error bound away from the shocks. A general convergence theory for 1D scalar convex conservation laws was developed by Tadmor and coauthors; see, e.g., [13, 14, 15, 21]. They proved that when measured in the weak $W^{-1,1}$ topology, the convergence rate of the viscous solution is of order $\mathcal{O}(\epsilon)$ in the case of rarefaction-free initial data [13, 21] and is of order $\mathcal{O}(\epsilon|\ln\epsilon|)$ in the general case [15]. These weak $W^{-1,1}$ estimates are then converted into the usual L^1 error bounds of order one-half, and, moreover, pointwise error estimates of order one-third, $\mathcal{O}(\epsilon^{1/3})$, are derived. Pointwise error analysis for finite difference methods to scalar and system of conservation laws was given recently by Engquist and Yu [4] and Engquist and Sjögreen [3].

In this work, we will provide the *optimal* pointwise convergence rate for the viscosity approximation. The previous results for the optimal order-one convergence rates, in both L^1 and L^∞ spaces, are all based on a matching method and traveling wave solutions; see, e.g., [4, 6, 25]. Another approach, based on matching the Green function of the linearized problem, was initiated by Liu [12] in his study of pointwise error estimates of viscous shock waves; consult [20] for the corresponding statement on viscous rarefactions. In this work, however, we avoid the use of matching inner expansions; instead, our arguments are based on energylike estimates, refining the approach initiated in [21]. The proof of our results is based on two ingredients:

(1) a Lip^+ boundedness along [21] which enables us to "convert" a global result into a local estimate for all but finitely many $\mathcal{O}(\epsilon)$ neighborhoods of discontinuities;

(2) a weighted quantity of the error satisfying a transport inequality such that the maximum principle applies. Unlike previous work on pointwise estimates, this framework can deal with finitely many shocks with *possible collisions*.

We recall that as long as the solutions of (1.3)–(1.4) are smooth, the first-order pointwise error estimates can be easily established. We briefly demonstrate the proof. It follows from (1.1) and (1.3) that, for $e(x,t) := u^\epsilon(x,t) - u(x,t)$,

$$e_t + f'(u^\epsilon)e_x = \epsilon e_{xx} - f''(\bullet)u_x e + \epsilon u_{xx}, \tag{1.7}$$

where $\bullet$ is an intermediate value between $u(x,t)$ and $u^\epsilon(x,t)$. Now, if $u \in C^2(\mathbf{R} \times [0,T])$, then the classical maximum principle yields the standard first-order error bound:

$$|e(x,t)| \le C\epsilon, \qquad (x,t) \in \mathbf{R} \times [0,T]. \tag{1.8}$$

In case the solution of (1.3)–(1.4) consists of shock and rarefaction discontinuities, the above argument fails. In this work, instead of considering directly the error function $u^\epsilon(x,t) - u(x,t)$, we construct a weighted error function, $E(x,t) := (u^\epsilon(x,t) - u(x,t))\rho(x,t)$, where ρ is a *distance* function to the singular supports of $u(x,t)$. By properly choosing the distance function, we will show that the classical maximum principle can be applied to a differential equation for E. This leads to an $\mathcal{O}(\epsilon)$ bound for $E(x,t)$ in $\mathbf{R} \times [0,T]$. The key idea which enables us to apply the maximum principle is to employ a (nonoptimal) local estimate, obtained by using the Lip^+ boundedness and the L^1 error bound. This local estimate plays an important role in upper bounding one of the key coefficients in the transport inequality for the weighted error function.

The paper is organized as follows. In section 3 we first consider the case when there is only one shock; i.e., the set of shocks S consists of only one smooth curve. We first show that

$$\text{dist}(x,S)|u(x,t) - u^\epsilon(x,t)| \approx C\epsilon. \tag{1.9}$$

It implies that $|u(x,t) - u^\epsilon(x,t)| \le C(h)\epsilon$ for (x,t) with $\mathcal{O}(h)$ distance away from the set of shocks. The result (1.9) will be generalized to finitely many shocks with *possible collisions*. In section 4 we consider piecewise smooth solutions with rarefaction waves. In this case, due to the initial positive jumps, the uniform boundedness of the Lip^+ estimate does not hold. Instead the entropy solution is characterized by Oleinik's E-condition with a *singular* Lip^+ bound as $t \downarrow 0$. Special attention is paid to construct a proper smoothness region inside which the maximum principle will apply. Our main result in this section asserts

$$|u(x,t) - u^\epsilon(x,t)| \le C(h)\epsilon|\log\epsilon|^2 \tag{1.10}$$

for (x,t) with $\mathcal{O}(h)$ distance away from the sets of rarefactions.

2. A one-sided interpolation inequality. In this section we derive an interpolation inequality which enables us to convert a *global* L^1 error estimate into a *local* error estimate. To begin with, we let $\| \bullet \|_{Lip^+}$ denote the Lip^+ seminorm

$$\|w\|_{Lip^+} := \operatorname*{ess\,sup}_{x \ne y} \left[\frac{w(x) - w(y)}{x - y} \right]^+,$$

where $[w]^+ = H(w)w$, with $H(\bullet)$ the Heaviside function.

The following lemma due to Nessyahu and Tadmor [13, section 2] is at the heart of matter.

LEMMA 2.1. *Assume that* $v \in L^1 \cap Lip^+(\mathcal{I})$, *and* $w \in C^1_{loc}(x-\delta, x+\delta)$ *for an interior* x *such that* $(x-\delta, x+\delta) \subset \mathcal{I}$. *Then the following estimate holds:*

$$|v(x)-w(x)| \leq \text{Const} \cdot \left[\frac{1}{\delta}\|v-w\|_{L^1} + \delta\{\|v\|_{Lip^+(x-\delta,x+\delta)} + |w|_{C^1_{loc}(x-\delta,x+\delta)}\}\right]. \tag{2.1}$$

In particular, if the size of the smoothness neighborhood for w *can be chosen so that*

$$\delta \sim \|v-w\|^{1/2}_{L^1(\mathcal{I})} \cdot (\|v\|_{Lip^+} + |w|_{C^1_{loc}})^{-1/2} \leq \frac{1}{2}|\mathcal{I}|, \tag{2.2}$$

then the following estimate holds:

$$|v(x)-w(x)| \leq \text{Const} \cdot \|v-w\|^{1/2}_{L^1(\mathcal{I})} \cdot \left[\|v\|_{Lip^+} + |w|_{C^1_{loc}(x-\delta,x+\delta)}\right]^{1/2}. \tag{2.3}$$

Thus, (2.3) tells us that if the global L^1 error $\|v-w\|_{L^1}$ is small, then the pointwise error $|v(x)-w(x)|$ is also small whenever w_x is bounded. This does not require the C^1 boundedness of v; the weaker one-sided Lip^+ bound will suffice.

For completeness, we now present the proof along the lines of [21, section 2].

Proof. For any $C^1_0(-1,1)$-unit mass mollifier, $\psi_\delta(x) = \frac{1}{\delta}\psi(x/\delta)$, we obtain

$$|(v*\psi_\delta)(x) - (w*\psi_\delta)(x)| \leq \text{Const} \cdot \frac{1}{\delta}\|v-w\|_{L^1}\ . \tag{2.4}$$

Since $w \in C^1(x-\delta, x+\delta)$, we have

$$|w(x) - (w*\psi_\delta)(x)| \leq \text{Const} \cdot \delta \cdot |w|_{C^1_{loc}(x-\delta,x+\delta)}. \tag{2.5}$$

Combining (2.4) and (2.5) we get the bound for the distance between the modified v and $w(x)$

$$|(v*\psi_\delta)(x) - w(x)| \leq C\delta \cdot |w|_{C^1_{loc}(x-\delta,x+\delta)} + C\frac{1}{\delta}\|v-w\|_{L^1}\ . \tag{2.6}$$

The above estimate holds for *any* $C^1_0(-1,1)$-unit mass mollifier of the form $\psi_\delta(x) = \frac{1}{\delta}\psi(x/\delta)$. Let $\psi^{(+)}_\delta(x) = \frac{1}{\delta}\psi^{(+)}(x/\delta)$, where $\psi^{(+)} \in C^1_0(0,1)$; that is, $\psi^{(+)}$ is supported on $(0,1)$. Consider

$$\begin{aligned} v(x) - (v*\psi^{(+)}_\delta)(x) &= \int_{-\infty}^{\infty} \Big(v(x)-v(x-y)\Big)\psi^{(+)}_\delta(y)dy \\ &= \int_0^\delta \left(\frac{v(x)-v(x-y)}{y}\right)\frac{y}{\delta}\psi^{(+)}\left(\frac{y}{\delta}\right)dy. \end{aligned} \tag{2.7}$$

Since v is assumed to be Lip^+ bounded, (2.7) yields

$$v(x) - (v*\psi^{(+)}_\delta)(x) \leq \delta\,\|v\|_{Lip^+}\,. \tag{2.8}$$

Next we decompose the difference between v and w

$$v(x) - w(x) \equiv \Big(v(x) - (v*\psi^{(+)}_\delta)(x)\Big) + \Big((v*\psi^{(+)}_\delta)(x) - w(x)\Big). \tag{2.9}$$

Setting $\psi_\delta = \psi_\delta^{(+)}$ in (2.6), we get the upper bound for the last term above. This together with the upper bound (2.8) yields

$$v(x) - w(x) \le C\delta\Big[|w|_{C^1_{loc}(x-\delta,x+\delta)} + \|v\|_{Lip^+}\Big] + C\frac{1}{\delta}\|v-w\|_{L^1}\ . \tag{2.10}$$

On the other hand, let $\psi_\delta^{(-)}(x) = \frac{1}{\delta}\psi^{(-)}(x/\delta)$, where $\psi^{(-)} \in C_0^1(-1,0)$. We can verify that

$$(v * \psi_\delta^{(-)})(x) - v(x) = \int_{-\delta}^0 \left[\frac{v(x-y)-v(x)}{-y}\right]\frac{-y}{\delta}\psi^{(-)}\left(\frac{y}{\delta}\right)dy.$$

For $y \in (-\delta, 0)$, the term $(-y/\delta)\psi^{(-)}(y/\delta)$ is nonnegative. Therefore, Lip^+ boundedness implies

$$(v * \psi_\delta^{(-)})(x) - v(x) \le \delta\|v\|_{Lip^+}\ . \tag{2.11}$$

Now replacing ψ_δ by $\psi_\delta^{(-)}$ in (2.6) and combining the resulting inequality and (2.11) lead to

$$w(x) - v(x) \le C\delta\cdot\Big[|w|_{C^1_{loc}(x-\delta,x+\delta)} + \|v\|_{Lip^+}\Big] + C\frac{1}{\delta}\|v-w\|_{L^1}\ . \tag{2.12}$$

The desired estimate (2.1) follows from the upper bound (2.10) and the lower bound (2.12) for $v(x) - w(x)$. ☐

Remark. A special case of the above inequality, with $w \equiv 0$, reads

$$|v(x)| \le \text{Const}\Big[\frac{1}{\delta}\|v\|_{L^1(\mathcal{I})} + \delta\|v\|_{Lip^+(\mathcal{I})}\Big], \qquad (x-\delta, x+\delta) \subset \mathcal{I}. \tag{2.13}$$

In particular, if the interval $\mathcal{I}$ is large enough relative to the ratio $\|v\|_{L^1}/\|v\|_{Lip^+}$, one finds

$$|v(x)| \le \text{Const}\cdot\|v\|^{1/2}_{L^1(\mathcal{I})}\ \|v\|^{1/2}_{Lip^+(\mathcal{I})}\ . \tag{2.14}$$

This is the *one-sided* analogue of a Gagliardo–Nirenberg inequality. A general treatment of these one-sided interpolation estimates is presented in [22]. In classical Gagliardo–Nirenberg inequalities, one interpolates between weak and strong norms, say, between the L^1 and $W^{1,\infty}$ norms (see, e.g., [5, Theorem 9.3]). In (2.14), however, only the *one-sided* bound (of the first derivative) is assumed. Such local error estimates in the presence of one-sided bounds were first used in [21].

In the following sections, we will derive a transport inequality for an appropriately weighted error function. The local error estimate in Lemma 2.1 will be used to upper bound the coefficients of that transport inequality (outlined in (3.6) below), which in turn enables us to obtain an optimal local error estimate using a bootstrap argument.

3. Piecewise smooth solutions with shocks. In this section, we assume that the entropy solution of (1.3)–(1.4) is *piecewise smooth*, with finitely many shock discontinuities. Thus, if we let $S(t)$ denote the singular support of $u(\bullet, t)$, then it consists of finitely many shocks $S(t) := \{(x,t)\,|\,x = X_k(t)\}$, each of which satisfies the Rankine–Hugoniot and the Lax conditions

$$X_k' = \frac{[f(u(X_k,t))]}{[u(X_k,t)]}, \tag{3.1}$$

$$f'(u(X_k(t)-,t)) > X_k'(t) > f'(u(X_k(t)+,t))\,. \tag{3.2}$$

We note in passing that many practical initial data lead to a finite number of shocks (see, e.g., [19, 23]).

Owing to the convexity of the flux f, the viscosity solutions of (1.1) satisfy a Lip^+ stability condition, similar to the familiar Oleinik E-condition [16], which asserts an a priori upper bound for the Lip^+ seminorm of the viscosity solution

$$\frac{u^\epsilon(x,t)-u^\epsilon(y,t)}{x-y} \le \|u^\epsilon(\cdot,t)\|_{Lip^+} \le \frac{1}{\|u_0\|^{-1}_{Lip^+}+\beta t}\,, \tag{3.3}$$

where u^ϵ is the solution of (1.1)–(1.2), and β is the convexity constant of the flux f given by (1.6). Consult, e.g., [21]. The above result suggests that if the initial data do not contain non-Lipschitz increasing discontinuities, then the viscosity solution of (1.1) will keep the same property. The same is true for the entropy solution of (1.3)–(1.4).

Equipped with (2.3), together with the global error bound (1.5) and the Lip^+ boundedness (3.3), we obtain the following pointwise error bound (see also [13]):

$$|u^\epsilon(x,t)-u(x,t)| \le C\,\sqrt[4]{\epsilon} \quad \text{for} \quad dist(x,S(t)) \ge \sqrt[4]{\epsilon}. \tag{3.4}$$

Although the above pointwise local estimate is not optimal, it will suffice to derive the optimal error bound by a bootstrap argument which employs the transport equation outlined in (3.6) below.

The basic framework of our approach for dealing with the pointwise error estimates proceeds in the following three steps.

- *Step* 1. Set

  $$E(x,t) := (u^\epsilon(x,t)-u(x,t))\rho(x,t), \tag{3.5}$$

  where ρ is a suitably defined *distance* function to the shock sets $S(t)$. We will also choose a suitable domain of smoothness D such that the following differential equation holds:

  $$E_t + h(x,t)E_x - \epsilon E_{xx} = p(x,t)E + q(x,t)\epsilon, \qquad (x,t)\in D. \tag{3.6}$$

  Here h, p, and q are smooth functions in D.
- *Step* 2. The functions p and q in (3.6) can be (uniformly) upper bounded and bounded, respectively:

  $$p(x,t) \le \text{Const}, \qquad |q(x,t)| \le \text{Const} \quad \text{for all} \quad (x,t)\in D. \tag{3.7}$$

- *Step* 3. Let ∂D denote the usual boundary for this domain of smoothness; it will be shown that

  $$\max_{(x,t)\in\partial D} |E(x,t)| \le C\epsilon. \tag{3.8}$$

The inequality (3.8) together with the maximum principle for (3.6)–(3.7) yield $|E(x,t)| \le C\epsilon$, for all $(x,t)\in D$, which in turn implies the pointwise estimate $|u^\epsilon(x,t)-u(x,t)| \le C\epsilon$ for (x,t) away from the shock set.

In Step 1 mentioned above, the function E is a weighted error function which is continuous for $(x,t)\in \mathbf{R}\times(0,T]$. The key point in this step is to introduce the *distance* function ρ, which satisfies $\rho\to 0$ as $dist(x,S(t))\to 0$ and $\rho\sim\mathcal{O}(1)$ when $dist(x,S(t))\sim\mathcal{O}(1)$. The proof for Step 2 is based on the pointwise error bound (3.4) and the Lax entropy condition (3.2). The third step depends on the choice of the weighted distance function ρ.

3.1. Pointwise error for the case of one shock. We assume that there is a smooth curve $S(t) := \{(x,t)\,|\, x = X(t)\}$, so that $u(x,t)$ is C^2 smooth at any point $x \neq X(t)$. There are two smooth regions $x > X(t)$ and $x < X(t)$. We first consider the pointwise error estimate in the region $x > X(t)$. Let $e(x,t) := u^\epsilon - u$ and set the weighted error

$$E(x,t) = e(x,t)\,\phi(x - X(t))\,.$$

Here, $\phi(x - X(t))$ is a weighted distance to the shock set. The function $\phi(x) \in C^2([0,\infty))$ satisfies

$$\phi(x) \sim \begin{cases} x^\alpha & \text{if} \quad 0 \le x \ll 1, \\ 1 & \text{if} \quad x \gg 1 \end{cases} \tag{3.9}$$

with $\alpha \geq 1$ to be determined later. More precisely, the function ϕ satisfies

$$\phi(0) = 0, \qquad \phi'(x) > 0, \qquad \phi(x) \le x^\alpha \quad \text{for} \quad x > 0, \tag{3.10}$$

$$x\phi'(x) \le \alpha\phi(x) \quad \text{for} \quad x \ge 0, \tag{3.11}$$

$$|\phi^{(k)}(x)| \le \text{Const}, \qquad x \ge 0; \tag{3.12}$$

e.g., $\phi(x) = (1 - e^{-x})^\alpha$. Roughly speaking, the weighted function behaves like $\phi(x) \sim \min(|x|^\alpha, 1)$. Direct calculations using the definition of E give us[1]

$$E_t + f'(u^\epsilon)E_x - \epsilon E_{xx} = \underbrace{\Big(e_t + f'(u^\epsilon)e_x - \epsilon e_{xx}\Big)\phi}_{I_1} + \underbrace{\Big(-X'(t) + f'(u^\epsilon)\Big)\phi' e}_{I_2} - 2\epsilon\phi' e_x - \epsilon\phi'' e. \tag{3.13}$$

It follows from the viscosity (1.1) and the limit (1.3) that

$$\begin{aligned} I_1 &= \Big(-f'(u^\epsilon)u_x + f'(u)u_x + \epsilon u_{xx}\Big)\phi \\ &= -\phi f''(\bullet)(u^\epsilon - u)u_x + \epsilon\phi u_{xx} \\ &= -f''(\bullet)u_x\, E + \epsilon\phi u_{xx}\,, \end{aligned} \tag{3.14}$$

where here (and below) $\bullet$ denotes an intermediate value between $-\|u_0\|_\infty$ and $\|u_0\|_\infty$. Let $u_\pm(t) = u(X(t) \pm 0, t)$ and let

$$I_3(t) = -X'(t) + f'(u_+).$$

Observing that $u - u_+ = u_x(\zeta_1)(x - X(t))$, where ζ_1 is an intermediate value between x and $X(t)$, we obtain

$$\begin{aligned} I_2 &= \Big(I_3 - f'(u_+) + f'(u) - f'(u) + f'(u^\epsilon)\Big)\phi' e \\ &= \phi' e I_3 + \phi' e f''(\bullet)(u - u_+) + \phi' f''(\bullet)e^2 \\ &= \Big(I_3 + f''(\bullet)e\Big)\frac{\phi'}{\phi}\,E + f''(\bullet)u_x(\zeta_1)\frac{(x - X(t))\phi'}{\phi}E, \end{aligned} \tag{3.15}$$

[1]For ease of notation, ϕ denotes $\phi(x - X(t))$ in the remainder of this subsection.

where in the last step we have used the fact $E = e\phi$. It is noted that $e_x = (E_x - \phi' e)/\phi$. This together with (3.13)–(3.15) yield the first desired result (3.6):

$$E_t + h(x,t)E_x - \epsilon E_{xx} = p(x,t)E + q(x,t)\,\epsilon\,,$$

where the coefficient of the convection term is given by

$$h(x,t) = f'(u^\epsilon) + 2\epsilon\frac{\phi'}{\phi}\,, \tag{3.16}$$

and the functions $p := p_1 + p_2$ and q are given by

$$p_1(x,t) := I_3\frac{\phi'}{\phi} + f''(\bullet)e\frac{\phi'}{\phi} + 2\epsilon\left(\frac{\phi'}{\phi}\right)^2, \tag{3.17}$$

$$p_2(x,t) := -f''(\bullet)\,u_x + f''(\bullet)u_x(\zeta_1)\frac{(x-X(t))\phi'}{\phi}, \tag{3.18}$$

$$q(x,t) := \phi u_{xx} - \phi''\,e\,. \tag{3.19}$$

We have then completed Step 1.

Next we move to Step 2, verifying the boundedness of the coefficients p and q inside a suitable domain. To choose a proper domain of smoothness D inside the region $x > X(t)$, we let

$$D := \Big\{(x,t)\ \Big|\ x \geq X(t) + \epsilon^{1/4}\,,\ 0 \leq t \leq T\Big\}. \tag{3.20}$$

Using Lax geometrical entropy condition (3.2), $u_+(t) \leq u_-(t)$, and the convexity of f, it follows that I_3 is nonpositive:

$$\begin{aligned} I_3(t) = -X'(t) + f'(u_+) &= \int_0^1 \Big[f'(u_+) - f'(\theta u_+ + (1-\theta)u_-)\Big]d\theta \\ &= \int_0^1 f''(\bullet)(1-\theta)d\theta\,(u_+ - u_-) \leq 0. \end{aligned}$$

For $(x,t) \in D, x > X(t) + \sqrt[4]{\epsilon}$, and hence by the property (3.11) of the weighted distance function ϕ we have

$$0 \leq \frac{\phi'}{\phi} \leq \frac{C}{x - X(t)} \leq C\epsilon^{-1/4} \quad \text{for} \quad (x,t) \in D.$$

The last two upper bounds, together with (3.4), lead to the following estimate for p_1:

$$p_1 \leq 0 + C\epsilon^{1/4}\epsilon^{-1/4} + C\epsilon\epsilon^{-1/2} \leq C \quad \text{for} \quad (x,t) \in D\,. \tag{3.21}$$

By the property of ϕ, $(x - X(t))\phi'(x - X(t))/\phi(x - X(t)) \leq \text{Const}$ and the regularity of u, $|u_x| \leq \text{Const}$ for all $(x,t) \in D$, we obtain that p_2 is also upper bounded. Again, due to the C^2 smoothness assumption on u, q is bounded in the domain of smoothness D. This completes Step 2.

Finally, we need to verify Step 3, upper bounding E on ∂D. We first check that the maximum value for E on the left boundary is bounded by $\mathcal{O}(\epsilon)$. On the left boundary, we have $x - X(t) = \epsilon^{1/4}$; hence, since $|\phi| \leq x^\alpha$ and since by (3.4) $|e(x,t)| = \mathcal{O}(\epsilon^{1/4})$, we have

$$|E(x,t)| \leq \epsilon^{\alpha/4}|e(x,t)| \leq C\epsilon^{\alpha/4}\,\epsilon^{1/4}\,.$$

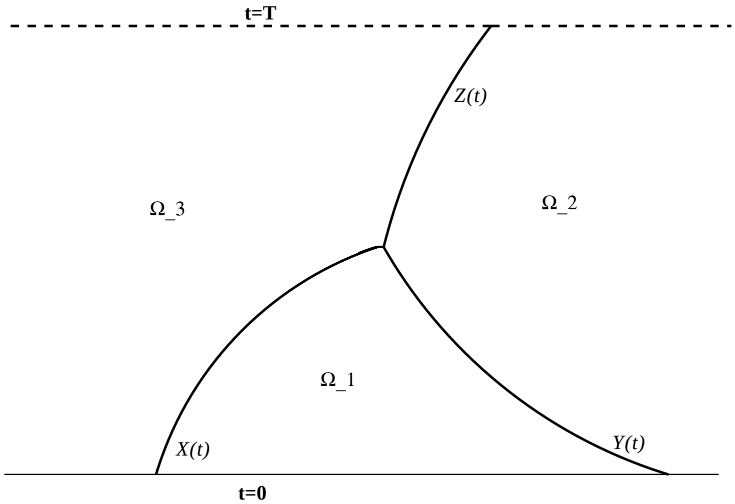

FIG. 1. *Solution structure for two shocks.*

Choosing $\alpha = 3$, we have $E(x,t) = \mathcal{O}(\epsilon)$ on the left boundary of the domain D. On the right and bottom boundaries of D, $E(x,t)$ vanishes. This completes Step 3. Hence, the maximum principle gives

$$|E(x,t)| \leq C\epsilon \quad \text{for} \quad (x,t) \in D.$$

This implies that the *weighted* error $u^\epsilon(x,t) - u(x,t)\,\phi(x - X(t))$ is bounded by $\mathcal{O}(\epsilon)$. In particular, for (x,t) bounded away from the shock set $S(t)$ we have an $\mathcal{O}(\epsilon)$ *pointwise error* bound. Similarly, we can show that the same holds when (x,t) is on the left side of the shock.

We summarize what we have shown by stating the following.

ASSERTION 3.1. *Let* $u^\epsilon(x,t)$ *be the viscosity solutions of* (1.1)–(1.2), and let $u(x,t)$ be the entropy solution of (1.3)–(1.4). If the entropy solution has *only one* shock discontinuity $S(t) = \{(x,t)|x = X(t)\}$, then the following error estimate holds:

- For a weighted distance function ϕ, $\phi(x) \sim \min(|x|^3, 1)$,

$$|(u^\epsilon - u)(x,t)|\phi(|x - X(t)|) = \mathcal{O}(\epsilon)\,. \tag{3.22}$$

- In particular, if (x,t) is away from the singular support, then

$$|(u^\epsilon - u)(x,t)| \leq C(h)\epsilon\,, \qquad dist(x, S(t)) \geq h. \tag{3.23}$$

Remark. In the proof of Assertion 3.1, with the choice of the domain of smoothness (3.20) the coefficients p_1, p_2, and q in (3.17)–(3.19) can be easily bounded, except the second term in p_1. This term, $f''(\bullet)\,e\,\phi'/\phi$, is unlikely to be uniformly upper bounded in D, unless a local error estimate such as (3.4) is available.

3.2. Pointwise error for two shocks with possible interactions. Consider two initial shocks which after collision merge to a single shock; its structure is demonstrated by Figure 1. Let $X_1(t) = X(t) \bigcup Z(t)$ and $X_2(t) = Y(t) \bigcup Z(t)$. The shock set is then $S(t) = \{(x,t)\,|\,x = X_1(t), \text{ or } x = X_2(t)\}$. Using the same techniques that were used in the previous subsection yield

$$\left|\phi\Big(X_1(t)-x\Big)\left(u^\epsilon(x,t)-u(x,t)\right)\right| \le C_T\epsilon \quad \text{for} \quad (x,t)\in\Omega_3\,, \tag{3.24}$$

$$\left|\phi\Big(x-X_2(t)\Big)\left(u^\epsilon(x,t)-u(x,t)\right)\right| \le C_T\epsilon \quad \text{for} \quad (x,t)\in\Omega_2\,. \tag{3.25}$$

In other words, pointwise error estimates for points in the domains Ω_2 and Ω_3 are obtained by following the same techniques used in section 3.1. We now turn to the estimate of the pointwise error $e(x,t)=u^\epsilon(x,t)-u(x,t)$ for $(x,t)\in\Omega_1$. Set

$$E(x,t)=e(x,t)\,\phi(x-X(t))\phi(Y(t)-x), \qquad (x,t)\in\Omega_1, \tag{3.26}$$

to be the *weighted error* function, where the new weighted function is the product of the distance functions $\phi(x-X(t))$ and $\phi(Y(t)-x)$. We abbreviate $\phi_1:=\phi(x-X(t))$ and $\phi_2:=\phi(Y(t)-x)$. Direct calculation using the definition of E in (3.26) gives

$$\begin{aligned} E_t+f'(u^\epsilon)E_x-\epsilon E_{xx} &= \underbrace{\Big(e_t+f'(u^\epsilon)e_x-\epsilon e_{xx}\Big)\phi_1\phi_2}_{J_1} \\ &\quad+\underbrace{\Big(-X'(t)+f'(u^\epsilon)\Big)\phi_1'\phi_2 e}_{J_2} \\ &\quad+\underbrace{\Big(Y'(t)-f'(u^\epsilon)\Big)\phi_1\phi_2' e}_{J_3}-2\epsilon\left(\phi_1'\phi_2-\phi_1\phi_2'\right)e_x \\ &\quad-\epsilon\Big(\phi_1''\phi_2-2\phi_1'\,\phi_2'+\phi_1\phi_2''\Big)e, \qquad (x,t)\in\Omega_1. \end{aligned} \tag{3.27}$$

Again, using the viscosity (1.1) and its inviscid limit (1.3) gives

$$J_1=-f''(\bullet)u_xE+\epsilon\phi_1\phi_2u_{xx}. \tag{3.28}$$

In order to estimate J_2 and J_3, we let

$$J_{41}=-X'(t)+f'(u(X(t)+0,t)), \qquad J_{42}=Y'(t)-f'(u(Y(t)-0,t)).$$

As in the last subsection, we can show that

$$\begin{cases} J_2=\Big(J_{41}+f''(\bullet)\,e\Big)\frac{\phi_1'}{\phi_1}E+f''(\bullet)u_x(\zeta_1)\frac{(x-X(t))\phi_1'}{\phi_1}E, \\ J_3=\Big(J_{42}-f''(\bullet)\,e\Big)\frac{\phi_2'}{\phi_2}E+f''(\bullet)u_x(\zeta_2)\frac{(Y(t)-x)\phi_2'}{\phi_2}E. \end{cases} \tag{3.29}$$

Using the above results, together with the definition of E, we end up with the transport equation similar to (3.6),

$$E_t+h(x,t)E_x-\epsilon E_{xx}=p(x,t)E+q(x,t)\,\epsilon.$$

Here, the coefficient of the convection term is given by

$$h(x,t)=f'(u^\epsilon)+2\epsilon\left(\frac{\phi_1'}{\phi_1}-\frac{\phi_2'}{\phi_2}\right), \tag{3.30}$$

and the functions $p := p_1 + p_2$ and q are defined by

$$p_1 = \Big(J_{41} + f''(\bullet)\, e\Big)\frac{\phi_1'}{\phi_1} + \Big(J_{42} - f''(\bullet)\, e\Big)\frac{\phi_2'}{\phi_2} + 2\epsilon\left(\frac{\phi_1'}{\phi_1} - \frac{\phi_2'}{\phi_2}\right)^2, \tag{3.31}$$

$$p_2 = -f''(\bullet)u_x + f''(\bullet)\, u_x(\zeta_1)\frac{(x - X(t))\phi_1'}{\phi_1} + f''(\bullet)u_x(\zeta_2)\frac{(Y(t) - x)\phi_2'}{\phi_2}, \tag{3.32}$$

$$q(x,t) = \phi_1\phi_2 u_{xx} - \Big(\phi_1''\phi_2 + 2\phi_1'\,\phi_2' + \phi_1\phi_2''\Big). \tag{3.33}$$

Next we consider a subdomain of Ω_1, which has $\epsilon^{1/4}$ distance away from the shock curves $x = X(t)$ and $x = Y(t)$:

$$D := \Big\{(x,t) \;\Big|\; X(t) + \epsilon^{1/4} \le x \le Y(t) - \epsilon^{1/4}\,,\, 0 \le t \le T\Big\}\,. \tag{3.34}$$

We have then finished Step 1.

Following the same arguments as used in section 3.1 we can show that p_1, p_2 are uniformly upper bounded, and q is uniformly bounded. Moreover, we have that $\max_{(x,t)\in\partial D} |E(x,t)| = \mathcal{O}(\epsilon)$. In other words, we have also verified Steps 2 and 3. Therefore, $\mathcal{O}(\epsilon)$ pointwise error bound is obtained for points inside Ω_1 which are away from the two shock curves.

We summarize what we have shown by stating the following.

ASSERTION 3.2. *Let $u^\epsilon(x,t)$ be the viscosity solutions of* (1.1)–(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)–(1.4). *If the entropy solution has* only two *shock discontinuities, $S(t) = \{(x,t) \mid x = X_1(t),\, x = X_2(t)\}$, then the following error estimate holds:*

- *For a weighted distance function ϕ, $\phi(x) \sim \min(|x|^3, 1)$,*

$$|(u^\epsilon - u)(x,t)|\phi(|x - X_1(t)|)\phi(|x - X_2(t)|) = \mathcal{O}(\epsilon)\,. \tag{3.35}$$

- *In particular, if (x,t) is away from the singular support of u, then*

$$|(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad dist(x, S(t)) \ge h. \tag{3.36}$$

3.3. Finitely many shocks with possible interactions. In this general case, we define the weighted distance function as

$$\rho(x,t) = \prod_{k=1}^{K} \phi\Big(|x - X_k(t)|\Big). \tag{3.37}$$

Consider the weighted error function $E(x,t) = (u^\epsilon(x,t) - u(x,t))\,\rho(x,t)$. We can apply the same techniques as used in section 3.2 to obtain the transport (3.6), to upper bound the coefficient function p and to bound the coefficient function q, and to bound the weighted error on boundaries.

Finally, combining Assertions 3.1 and 3.2 and the discussions above, we conclude the following.

THEOREM 3.1. *Let $u^\epsilon(x,t)$ be the viscosity solutions of* (1.1)–(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)–(1.4). *If the entropy solution has* finitely *many shock discontinuities, $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$, then the following error estimates hold:*

- *For a weighted distance function ϕ, $\phi(x) \sim \min(|x|^3, 1)$,*

$$|(u^\epsilon - u)(x,t)| \prod_{k=1}^{K} \phi\Big(|x - X_k(t)|\Big) = \mathcal{O}(\epsilon)\,. \tag{3.38}$$

- *In particular, if* (x,t) *is away from the singular support of* u, *then*

$$|(u^\epsilon - u)(x,t)| \leq C(h)\epsilon\,, \qquad dist(x, S(t)) \geq h. \tag{3.39}$$

We close this section with a revisit of our convergence rate argument. Our above derivation of pointwise error estimates is based on two ingredients: a global error bound and a Lip^+-stability result. In fact, for convex conservation laws with piecewise smooth solutions the global L^1 error bound, order 1/2 in (1.5), can be improved to order $\mathcal{O}(\epsilon)$ [25], which in turn leads to a refinement of our "shock layer" estimate. More generally, let's consider the case of one shock with a general L^1 error estimate of order $\|u^\epsilon - u\|_{L^1} = \mathcal{O}(\epsilon^\gamma)$. This, together with the interpolation inequality (2.1) yields $|(u^\epsilon - u)(x,t)| \leq C\{\epsilon^\gamma/\delta + \delta\}$ for all δ's, $\delta = \delta(x) \geq dist(x, S(t))$, and hence

$$|u^\epsilon(x,t) - u(x,t)| \leq C\,\epsilon^{\gamma/2} \quad \text{for} \quad dist(x, S(t)) \geq \epsilon^{\gamma/2}.$$

This local error bound suggests that the domain of smoothness will be chosen as those x's such that $dist(x, S(t)) \geq \epsilon^{\gamma/2}$. It will guarantee the upper boundedness of the function p in the transport (3.6). We still choose $\phi \sim \min(x^\alpha, 1)$, which gives

$$\max_{\partial D} |(u^\epsilon - u)(x,t)|\phi(|x - X(t)|) \leq C\epsilon^{\gamma/2}\epsilon^{\alpha\gamma/2}. \tag{3.40}$$

The weighted error function $E = (u^\epsilon - u)\phi$ is now upper bounded by its boundary values, which do not exceed the right-hand side of (3.40) and the last forcing term in (3.6). Thus to optimize the error bound for E, we let $\gamma/2 + \alpha\gamma/2 = 1$, which yields

$$\alpha = 2/\gamma - 1.$$

This analysis can be easily extended to the case of finitely many shocks.

The general L^1 convergence rate estimate of order $\gamma = \frac{1}{2}$ led to our above choice of $(\alpha, \gamma) = (3, 1)$. It is shown in [25] that when the entropy solution has only finitely many shock discontinuities, then the L^1 rate of convergence is precisely of order $\gamma = 1$, which in turn leads to $\alpha = 1$. Hence, we arrive at the following improved version of Theorem 3.1.

THEOREM 3.2. *Let* $u^\epsilon(x,t)$ *be the viscosity solutions of* (1.1)–(1.2) *and* $u(x,t)$ *be the entropy solution of* (1.3)–(1.4). *If the entropy solution has* finitely *many shock discontinuities,* $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$, *then the following error estimates hold:*

- *For a weighted distance function* ϕ, $\phi(x) \sim \min(|x|, 1)$,

$$|(u^\epsilon - u)(x,t)| \prod_{k=1}^{K} \phi\Big(|x - X_k(t)|\Big) = \mathcal{O}(\epsilon). \tag{3.41}$$

- *In particular, if* (x,t) *is away from the singular support of* u, *then*

$$|(u^\epsilon - u)(x,t)| \leq C(h)\epsilon \quad \textit{for} \quad dist(x, S(t)) \geq h. \tag{3.42}$$

- *Since the weighted function* $\phi(x) \sim |x|$, *it follows from* (2.1) *and* (3.41) *that*

$$|(u^\epsilon - u)(x,t)| \sim \epsilon\, dist(x, S(t))^{-1}. \tag{3.43}$$

This implies that the thickness of the shock layer is of order $\mathcal{O}(\epsilon)$.

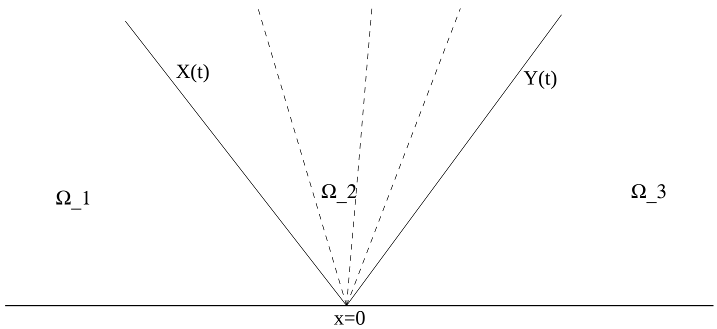

FIG. 2. *Solution structure for one rarefaction wave.*

Remark. With, say, $\gamma = 1$, our above arguments show that (3.43) holds in the restricted range where $\delta(x) := dist(x, S(t)) \geq \sqrt{\epsilon}$. Yet, by Lemma 2.1, the one-sided interpolation estimate in (2.1) tells us that $|(u^\epsilon - u)(x,t)| < \text{Const}\{\epsilon/\delta(x) + \delta(x)\}$ $\forall x$'s, and one recovers (3.43) in the remaining portion of the shock layer where $\delta^2(x) \leq \epsilon$, as asserted. Integration of (3.43) then yields

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1} = \int_{\{x|dist(x,S(t))>\epsilon\}} \frac{\mathcal{O}(\epsilon)}{dist(x,S(t))} dx + \int_{\{x|dist(x,S(t))\leq\epsilon\}} \mathcal{O}(1) dx$$
$$\leq \mathcal{O}(\epsilon|\log\epsilon|).$$

Thus, our pointwise error estimate is sharp enough to recover the underlying L^1 convergence estimate of order (essentially) one—an enjoyable sharpness. (A similar argument applies to the case $\gamma < 1$.)

4. Piecewise smooth solutions with rarefaction waves. In this section we assume that the entropy solution of (1.3)–(1.4) is piecewise C^2 smooth, with finitely many rarefaction waves (due to initial positive jumps). In this case it is proved in [25] that

$$\|u(\bullet,t) - u^\epsilon(\bullet,t)\|_{L^1(\mathbf{R})} \leq C\epsilon\,|\log\epsilon|\,. \tag{4.1}$$

Since the viscosity solution has initial positive jump, $\|u_0\|_{Lip^+} = +\infty$, the Lip^+ estimate (3.3) gives

$$\frac{u^\epsilon(x,t) - u^\epsilon(y,t)}{x-y} \leq \frac{1}{\beta t}\,, \qquad x, y \in \mathbf{R},\ t > 0. \tag{4.2}$$

For simplicity, we consider the case with only one initial positive jump at the origin. The case with finitely many rarefaction waves can be handled using ideas similar to those given in sections 3.2 and 3.3. It is known that there exist two straight lines $x = X(t)$ and $y = Y(t)$, starting from the origin, such that $u(\cdot,t)$ is given as a rarefaction fan

$$u(x,t) = (f')^{-1}\left(\frac{x}{t}\right) \quad \text{for} \quad X(t) < x < Y(t). \tag{4.3}$$

The rarefaction set is denoted by

$$R(t) = \{(x,t)\,|\,x = X(t) \text{ or } x = Y(t)\}.$$

The structure of the solution is given in Figure 2. It follows from (4.2) and (4.3) that $\|u^\epsilon\|_{Lip^+} \sim t^{-1}$ and $u_x \sim t^{-1}$ inside the rarefaction fan. Outside the rarefaction fan, we have $u_x \sim \mathcal{O}(1)$. These observations, together with the interpolation inequality (2.3) and the L^1 bound (4.1), yield

$$|u^\epsilon(x,t) - u(x,t)| \leq C\sqrt{\epsilon|\log\epsilon|}\; t^{-1/2} \quad \text{for} \quad dist(x, R(t)) \geq \sqrt{\epsilon t|\log\epsilon|}\,. \tag{4.4}$$

We now turn to upgrade the error estimate of $e(x,t) = u^\epsilon(x,t) - u(x,t)$ for (x,t) away from the rarefaction set $R(t)$, using a similar bootstrap argument used before.

4.1. Pointwise estimates outside the rarefaction fans. We only consider the right domain of the rarefaction fan Ω_3. The pointwise error estimate for points on the left side of the rarefaction fan, Ω_1, can be obtained similarly. Along the lines for the case of one shock, we define the weighted error function

$$E(x,t) = \phi(x - Y(t))\, e(x,t), \qquad (x,t) \in \Omega_3\,, \tag{4.5}$$

where ϕ satisfies (3.10)–(3.12). By direct calculation we obtain the form (3.6):

$$E_t + h(x,t)\, E_x - \epsilon E_{xx} = p(x,t)E + q(x,t)\epsilon,$$

where the coefficient of the convection term is given by

$$h(x,t) = f'(u^\epsilon) + 2\epsilon\frac{\phi'}{\phi}\,, \tag{4.6}$$

and the functions $p := p_1 + p_2$ and q are given by

$$\begin{cases} p_1(x,t) := f''(\bullet)e\frac{\phi'}{\phi} + 2\epsilon\left(\frac{\phi'}{\phi}\right)^2, \\ p_2(x,t) := -f''(\bullet)\, u_x + f''(\bullet)u_x(\zeta_1)\frac{(x-Y(t))\phi'}{\phi}, \\ q(x,t) := \phi u_{xx} - \phi''\, e\,. \end{cases} \tag{4.7}$$

The *main difference* between the current discussion and the treatment of the shock discontinuity in section 3.1 lies in the different choices for the boundaries of domain of smoothness D. In section 3.1, the boundary of the domain of smoothness consists of x's such that $dist(x, S) \sim C(\epsilon)$. In this subsection, however, the boundary of the domain of smoothness is located at $dist(x, R) \sim C(\epsilon, t)$. The dependence on t in the rarefaction case highlights the singularity of the local error bound (4.4). More precisely, inside the region Ω_3 we let

$$D := \left\{(x,t) \;\middle|\; x \geq Y(t) + \gamma\sqrt{\epsilon\, t\,|\log\epsilon|}\,,\, 0 \leq t \leq T\right\}, \tag{4.8}$$

where γ is a constant to be determined later. For $(x,t) \in D$, it follows from the property of ϕ, $x\phi'(x) \leq \alpha\phi$, that

$$0 \leq \frac{\phi'}{\phi} \leq \frac{\alpha}{x - Y(t)} \leq \frac{\alpha}{\gamma}(\epsilon\, t\,|\log\epsilon|)^{-1/2}\,.$$

This and the pointwise error bound (4.4) lead to

$$p_1(x,t) \leq \left(\frac{C\alpha}{\gamma} + \frac{C\alpha^2}{\gamma^2}\right)t^{-1} \quad \text{for} \quad (x,t) \in \Omega_3.$$

Using the piecewise smoothness assumptions on u and properties for ϕ, we obtain the uniform boundedness of p_2 and q. Hence, for γ sufficiently large,

$$p(x,t) \leq \frac{1}{2}t^{-1} + C, \qquad |q(x,t)| \leq C \quad \text{for} \quad (x,t) \in \Omega_3\,. \tag{4.9}$$

On the left boundary of D, we have $x - Y(t) \sim (\epsilon\, t\, |\log \epsilon|)^{-1/2}$, which together with $|\phi(x)| \leq x^\alpha$ and the local error bound (4.4) lead to

$$|E(x,t)| \leq C(\epsilon\, t\, |\log \epsilon|)^{\alpha/2} \sqrt{\epsilon\, |\log \epsilon|}\; t^{-1/2}.$$

Since E vanishes on the right and bottom boundaries of D, this indicates that *for* $\alpha = 1$,

$$\max_{(x,t)\in\partial D} |E(x,t)| \leq C\epsilon\, |\log \epsilon|. \tag{4.10}$$

It follows from the transport (3.6), the upper bounds (4.9), and the boundary error estimate (4.10) that

$$|(u^\epsilon - u)(x,t)\, \phi(x - Y(t))| \leq C\epsilon\, |\log \epsilon| \quad \text{for} \quad x - Y(t) \geq \mathcal{O}(\sqrt{\epsilon\, t\, |\log \epsilon|})\,. \tag{4.11}$$

Here, $\phi(x) \sim \min(|x|\,, 1)$. Error bounds similar to (4.11) hold for $x \leq X(t) - \mathcal{O}(\sqrt{\epsilon\, t\, |\log \epsilon|})$.

Remark. Again, the main application of the (nonoptimal) local error estimate, (4.4), in the rarefaction case, is to upper bound the expression $f''(\bullet)\, e\, \phi'/\phi$ for p_1.

4.2. Pointwise estimates inside the rarefaction fan. The present case requires a special treatment since unlike our previous discussion in section 4.1, derivatives are no longer *uniformly* bounded in Ω_2. It follows from (4.3), the representative formula for u, that

$$0 < u_x(x,t) = \frac{1}{f''(u)\, t}\,, \qquad |u_{xx}(x,t)| \leq Ct^{-2}, \qquad (x,t) \in \Omega_2\,. \tag{4.12}$$

Another feature for points inside the rarefaction fan is that

$$|x - y| \leq (f'(u_0(0+)) - f'(u_0(0-)))t \quad \text{for any} \quad (x,t), (y,t) \in \Omega_2\,. \tag{4.13}$$

The weighted error function is defined by

$$E(x,t) = \phi(x - X(t))\phi(Y(t) - x)e(x,t), \qquad (x,t) \in \Omega_2,$$

where $e(x,t) = u^\epsilon(x,t) - u(x,t)$. The basic idea of the pointwise error estimate in this subsection is as follows:

- *Step* I. We show that the weighted error function satisfies

$$E_t + h(x,t)E_x - \epsilon E_{xx} = pE + q_1 + q_2\, \epsilon \quad \text{for} \quad (x,t) \in \Omega_2\,. \tag{4.14}$$

- *Step* II. A subdomain $\mathbf{D} \subset \Omega_2$, which satisfies $dist(\partial\mathbf{D}, R(t)) \geq \mathcal{O}(\epsilon)$ will be chosen. Inside this subdomain, it will be shown that

$$p(x,t) \leq t^{-1}, \qquad |q_1(x,t)| \leq C\epsilon\, |\log \epsilon|, \qquad |q_2| \leq C. \tag{4.15}$$

- *Step* III. We show that on the boundary of $\mathbf{D}$,

$$\max_{(x,t)\in\partial\mathbf{D}} |E(x,t)| \le Ct\epsilon\,|\log\epsilon|\,. \tag{4.16}$$

We point out that the upper bound for p in (4.15) cannot be replaced by Ct^{-1} with $C < 1$. Once (4.14)–(4.16) are obtained, the standard Gronwall inequality argument will lead to the desired estimate

$$\max_{(x,t)\in\mathbf{D}} |E(x,t)| \le C\epsilon|\log\epsilon|^2\,. \tag{4.17}$$

To begin with, let $\phi_1 := \phi(x - X(t))$ and $\phi_2 := \phi(Y(t) - x)$. It follows from $\phi(x) \le x^\alpha$ and (4.13) and the fact that

$$\phi(x - X(t)) \le Ct^\alpha, \qquad \phi(Y(t) - x)) \le Ct^\alpha, \qquad (x,t) \in \Omega_2\,. \tag{4.18}$$

Direct calculation using the definition of the function E gives us

$$\begin{aligned} E_t + f'(u^\epsilon)E_x - \epsilon E_{xx} = {} & \underbrace{(e_t + f'(u^\epsilon)e_x - \epsilon e_{xx})\phi_1\phi_2}_{M_1} + \underbrace{(-X'(t) + f'(u^\epsilon))\phi_1'\phi_2 e}_{M_2} \\ & + \underbrace{\big(Y'(t) - f'(u^\epsilon)\big)\phi_1\phi_2' e}_{M_3} - 2\epsilon\left(\phi_1'\phi_2 - \phi_1\phi_2'\right)e_x \\ & - \epsilon\big(\phi_1''\phi_2 + 2\phi_1'\phi_2' + \phi_1\phi_2''\big)e. \end{aligned} \tag{4.19}$$

We now estimate $M_j, 1 \le j \le 3$. Using (1.1) and (1.3) we find

$$\begin{aligned} M_1 &= (-f'(u^\epsilon)u_x + f'(u)u_x + \epsilon\, u_{xx})\phi_1\phi_2 \\ &= -(f'(u^\epsilon) - f'(u))u_x\phi_1\phi_2 + \epsilon\,\phi_1\phi_2 u_{xx} \\ &= -(f''(u)e + \frac{1}{2}f'''(\bullet)e^2)u_x\phi_1\phi_2 + \epsilon\,\phi_1\phi_2 u_{xx} \\ &= -t^{-1}\,e\,\phi_1\phi_2 - \frac{1}{2}f'''(\bullet)e^2u_x\phi_1\phi_2 + \epsilon\,\phi_1\phi_2 u_{xx}\,, \end{aligned}$$

where in the second-to-last step we have used the Taylor expansion for $f'(u^\epsilon)$, and in the last step we have used the first equation in (4.12). Since the curve $x = X(t)$ is a straight line, we have $X'(t) = X(t)t^{-1}$. It follows from the explicit formula, $u(x,t) = (f')^{(-1)}(x/t)$, that

$$\begin{aligned} M_2 &= (-X(t)t^{-1} + xt^{-1} - f'(u) + f'(u^\epsilon))\phi_1'\phi_2\,e \\ &= t^{-1}(x - X(t))\phi_1'\phi_2\,e + f''(\bullet)e^2\phi_1'\phi_2. \end{aligned}$$

Similarly, we can obtain

$$M_3 = t^{-1}(Y(t) - x)\phi_1\phi_2'\,e - f''(\bullet)e^2\phi_1\phi_2'\,.$$

Using the above results, together with the definition of E, we end up with (4.14), where

$$h = f'(u^\epsilon) + 2\epsilon\left(\frac{\phi_1'}{\phi_1} - \frac{\phi_2'}{\phi_2}\right), \tag{4.20}$$

$$p = -t^{-1} + t^{-1}(x - X(t))\frac{\phi_1'}{\phi_1} + t^{-1}(Y(t) - x)\frac{\phi_2'}{\phi_2}\,, \tag{4.21}$$

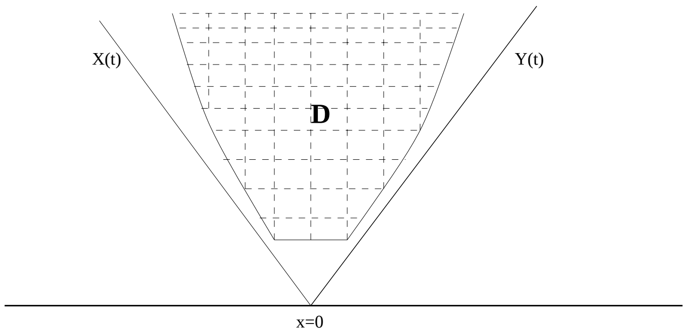

FIG. 3. *Solution structure inside the rarefaction region.*

$$q_1 = f''(\bullet)\, e^2\, \phi_1'\, \phi_2 - f''(\bullet)\, e^2\, \phi_2'\, \phi_1 + 2\epsilon \left(\frac{\phi_1'}{\phi_1} - \frac{\phi_2'}{\phi_2}\right)^2,$$

$$\phi_1\, \phi_2\, e - \frac{1}{2} f'''(\bullet) u_x\, e^2\, \phi_1 \phi_2, \tag{4.22}$$

$$q_2 = \phi_1 \phi_2 u_{xx} - \Big(\phi_1'' \phi_2 - 2\phi_1' \phi_2' + \phi_1 \phi_2''\Big) e\,. \tag{4.23}$$

This completes our Step I. We introduce two curves

$$\Gamma_1 : x = X(t) + \sqrt{\epsilon\, t\, |\log \epsilon|}\,, \quad \Gamma_2 : x = Y(t) - \sqrt{\epsilon\, t\, |\log \epsilon|}\,, \quad 0 \le t \le T\,, \tag{4.24}$$

and define the following points:

$$P_1 : \Big(X(t) + \sqrt{\epsilon\, t\, |\log \epsilon|},\, t\Big), \quad P_2 : \Big(Y(t) - \sqrt{\epsilon\, t\, |\log \epsilon|},\, t\Big) \quad \text{with} \quad t = 4\epsilon/(\gamma^2\, |\log \epsilon|),$$

$$P_3 : \Big(Y(t) - \sqrt{\epsilon\, t\, |\log \epsilon|},\, t\Big), \quad P_4 : \Big(X(t) + \sqrt{\epsilon\, t\, |\log \epsilon|},\, t\Big) \quad \text{with} \quad t = T.$$

Here γ is defined by

$$\gamma := f'(u_0(0+)) - f'(u_0(0-)) > 0. \tag{4.25}$$

These four points, together with curves $\Gamma_1, \Gamma_2, t = 4\epsilon/(\gamma^2\, |\log \epsilon|)$, and $t = T$, form a domain, denoted by $\mathbf{D}$; see Figure 3. In order that the domain $\mathbf{D}$ is meaningful, the following must hold:

$$Y(t) - \sqrt{\epsilon\, t\, |\log \epsilon|} > X(t) + \sqrt{\epsilon t\, |\log \epsilon|} \quad \text{for} \quad 4\epsilon/(\gamma^2\, |\log \epsilon|) < t < T.$$

In fact, the above inequality is valid since $Y(t) - X(t) = \gamma t$.

Using the fact that $x\phi'(x) \le \alpha\phi$ we find

$$p(x,t) \le (2\alpha - 1) t^{-1} \quad \text{for} \quad (x,t) \in \mathbf{D}. \tag{4.26}$$

It follows from the local error estimate (4.4), the bounds in (4.18) asserting $\phi_1 \le Ct^\alpha, \phi_2 \le Ct^\alpha$, and the properties of the weighted function ϕ, that the following hold:

$$|q_1(x,t)| \le C\epsilon\, |\log \epsilon|\, t^{-1+\alpha} + C\epsilon \left\{ \left(\frac{\phi_1'}{\phi_1}\right)^2 + \left(\frac{\phi_2'}{\phi_2}\right)^2 \right\} \phi_1\, \phi_2 \sqrt{\epsilon\, |\log \epsilon|}\; t^{-1/2}$$

$$
\begin{aligned}
&+C\epsilon\,|\log\epsilon|\,t^{-2+2\alpha}\\
&\le C\epsilon\,|\log\epsilon|\,t^{-1+\alpha}+C\epsilon\left\{\frac{\phi_1'}{\phi_1}\phi_1'\,\phi_2+\frac{\phi_2'}{\phi_2}\phi_1\phi_2'\right\}\sqrt{\epsilon\,|\log\epsilon|}\;t^{-1/2}\\
&\quad+C\epsilon\,|\log\epsilon|\,t^{-2+2\alpha}\\
&\le C\epsilon\,|\log\epsilon|\,t^{\alpha-1}+C\epsilon\left\{\frac{1}{(x-X(t))}+\frac{1}{(Y(t)-x)}\right\}t^{\alpha}\sqrt{\epsilon\,|\log\epsilon|}\;t^{-1/2}\\
&\quad+C\epsilon\,|\log\epsilon|t^{-2+2\alpha}.
\end{aligned}
$$

Inside the domain $\mathbf{D}$ where $dist(x,R(t))\ge\sqrt{\epsilon t\,|\log\epsilon|}$, the above inequality yields

$$|q_1(x,t)|\le C\epsilon\,|\log\epsilon|\,(t^{\alpha-1}+t^{2\alpha-2}). \tag{4.27}$$

Using the properties of ϕ and the fact that $u_{xx}\sim t^{-2}$ we find

$$|q_2(x,t)|\le Ct^{2\alpha-2}+C\quad\text{for}\quad(x,t)\in\mathbf{D}. \tag{4.28}$$

The above bounds suggest that the required estimate (4.15) in Step II is satisfied by choosing $\alpha=1$. Finally, we observe that

$$|E(x,t)|\le C(x-X(t))^{\alpha}\,(Y(t)-x)^{\alpha}\,\sqrt{\epsilon\,|\log\epsilon|}\,t^{-1/2}\,.$$

On either the left or right boundaries of $\mathbf{D}$ we have

$$
\begin{aligned}
|E(x,t)|&\le C(\sqrt{\epsilon t\,|\log\epsilon|})^{\alpha}\,t^{\alpha}\sqrt{\epsilon\,|\log\epsilon|}\,t^{-1/2}\\
&\le Ct^{(3\alpha-1)/2}(\epsilon\,|\log\epsilon|)^{(1+\alpha)/2}\,.
\end{aligned}
$$

The same result holds for the bottom of $\mathbf{D}$. Therefore,

$$\max_{(x,t)\in\partial\mathbf{D}}|E(x,t)|\le Ct^{(3\alpha-1)/2}\,(\epsilon\,|\log\epsilon|)^{(1+\alpha)/2}\,. \tag{4.29}$$

Step III is then verified for $\alpha=1$. We conclude that the pointwise error bound of order $\mathcal{O}(\epsilon|\log\epsilon|^2)$ holds for the weighted function inside the rarefaction fan.

Remark. Inside the rarefaction fan, the main application of the (nonoptimal) local error estimate (4.4) is to bound the q_1 term in (4.22).

Combining the results in sections 4.1 and 4.2, we arrive a pointwise error bound for $u^\epsilon-u$ away from the singular support of u. Using an idea similar to that in section 3.3, this error bound can also be extended to the case of finitely many rarefaction waves. We summarize what we have shown by stating the following.

THEOREM 4.1. *Let $u^\epsilon(x,t)$ be the viscosity solutions of* (1.1)–(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)–(1.4). *If the entropy solution has* finitely *many rarefaction waves, $R(t)=\{(x,t)|x=X_k(t)\}_{k=1}^K$, then the following error estimates hold:*

- *For a weighted distance function ϕ, $\phi(x)\sim\min(|x|,1)$,*

$$|(u^\epsilon-u)(x,t)|\prod_{k=1}^{K}\phi\Big(|x-X_k(t)|\Big)=\mathcal{O}(\epsilon\,\log^2\epsilon). \tag{4.30}$$

- *In particular, if (x,t) is away from the singular support of u, then*

$$|(u^\epsilon-u)(x,t)|\le C(h)\epsilon\,\log^2\epsilon\qquad\textit{for}\quad dist(x,R(t))\ge h. \tag{4.31}$$

- *Since the weighted function $\phi(x)\sim|x|$, it follows from* (4.30) *that*

$$|(u^\epsilon-u)(x,t)|\sim dist(x,R(t))^{-1}\epsilon\,\log^2\epsilon\,. \tag{4.32}$$

This implies that the thickness of the rarefaction layer is of order $\mathcal{O}(\epsilon\,\log^2\epsilon)$.

5. Concluding remarks. In this work we have used an innovative idea which enables us to convert a *global* L^1 error estimate into a *local* error estimate. Using this local error estimate and a bootstrap argument we have proved that the viscosity approximation satisfies a *pointwise* error estimate of order $\mathcal{O}(\epsilon)$ for all but finitely many neighborhoods of shock discontinuities, each of width $\mathcal{O}(\epsilon)$. Similarly, an $\mathcal{O}(\epsilon|\log\epsilon|^2)$ pointwise estimate holds outside finitely many rarefaction neighborhoods of width $\mathcal{O}(\epsilon|\log\epsilon|)$ (optimal error estimates *inside* these neighborhoods can be found in [8]). In particular, integrating these estimates overall in the computational domain we find that our pointwise error estimates are sharp enough to recover (an almost) first-order, $\mathcal{O}(\epsilon\log\epsilon)$ L^1 error estimate.

Finally we note that the framework introduced in this work applies, in principle, to finite difference schemes and relaxation approximations, which will be considered in a future work.

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