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Compactification of a Set of Matrices with Convergent Infinite Products

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COMPACTIFICATION OF A SET OF MATRICES WITH CONVERGENT INFINITE PRODUCTS

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To Gil, a great friend and advisor

ABSTRACT. We generalize and unify some aspects of the work of Daubechies and Lagarias [2] on a set Σ of matrices with right-convergent products (RCP). We show that most properties of an RCP set Σ pass to its compactification $\overline{\Sigma}$ (i.e. its closure in the matrix space). Results on finite RCP sets generally hold for compact RCP sets, among which is the existence of a König chain. We reproduce some important classical results in the context of König chains and compactification invariance.

1. INTRODUCTION

Let Σ be a set of square matrices (real or complex). Σ is said to be an RCP (right-convergent-product) set if Σ is *bounded*, and for any sequence $(A_1, A_2, \cdots) \in \Sigma^\infty$,

$$\lim_{n\to\infty} A_1 A_2 \cdots A_n$$

exists.

Examples of RCP sets of matrices are distributed in several areas. Following the pioneering work of Daubechies and Lagarias in [2], we mention here these four fields: the dynamics of non-stationary Markov chains, the iterated function systems, the exact computation of compactly supported wavelets (instead of the subdivision algorithm), and the iterations of random matrices. Daubechies and Lagarias [2] made a matrix-theoretic unification of these applied fields and many general results were established without actually referring to the underlying background. The beautiful work of Berger

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and Wang [1] completed the former one by proving affirmatively the two conjectures left there. Based on the common efforts of these authors, we have now gained deeper insights into the relation between the convergence of matrix products and the concept of joint spectral radius, introduced by Rota and Strang [3] almost four decades ago.

The literature on this part of "matrix calculus" is still green but steadily growing because of its many fascinating properties and its significance in applications. The most recent work can be found in Shih [4] on simultaneous Schur stability.

In this paper, we generalize and unify further some aspects of the work of Daubechies and Lagarias in [2], with a special emphasis on compactness. We show that if Σ is an RCP set, then its compactification, i.e. the closure $\overline{\Sigma}$ of Σ in the matrix space, shares almost all the properties. The most remarkable profit from compactness is the generalization of the following result of Daubechies and Lagarias.

Result 1 (In the proof of Theorem 3.1 of [2]). *Any finite set Σ has a König chain.*

A König chain of Σ is a sequence $(A_1, A_2, \cdots) \in \Sigma^\infty$ such that

$$\|A_1 A_2 \cdots A_m\|^{1/m} \geq \hat{\rho}, \qquad m = 1, 2, \cdots.$$

Here, $\hat{\rho} = \hat{\rho}(\Sigma)$ denotes the Rota-Strang joint spectral radius [3]

$$\hat{\rho}(\Sigma) = \limsup_{m \to \infty} \|\Sigma_m\|^{1/m},$$

$$\Sigma_m = \{A_1 A_2 \cdots A_m \mid A_i \in \Sigma\},$$

$$\|\Lambda\| = \sup_{B \in \Lambda} \{\|B\| \mid B \in \Lambda\}.$$

Result 1 has played a central role in the evolution of their paper. We show later that it can extend to compact sets. But it is generally not true for bounded sets. (For those who have read or will read both [1] and [2] can find that perhaps this is the only non-trivial occasion where compactness makes difference. Also see Section 3 below.)

With this extension and the boundedness result of Berger and Wang [1], we will give a more systematic (if it is fair to say so) proof of the uniformly vanishing theorem (Lemma 5.2 in Daubechies and Lagarias [2] and Theorem 1(b) in Berger and Wang [1]).

Result 2. *A bounded set Σ is a vanishing RCP set if and only if $\hat{\rho}(\Sigma) < 1$.*

An RCP set Σ is said to be *vanishing* if for any matrix sequence $(A_1, A_2, \cdots) \in \Sigma^\infty$,

$$\lim_{m \to 0} A_1 A_2 \cdots A_m = 0.$$

The interest on vanishing RCP sets is enhanced by the fact that vanishing RCP sets can completely characterize another important class of RCP sets —*uniform RCP sets* (see Daubechies and Lagarias [2] or below).

I intend to give a self-contained exploration. Some parallels to [1] and [2] shall be expected. Together, I hope our efforts will clarify old problems and point more focusively to new ones.

The layout of paper is as follows. Section 2 contains a brief review on RCP sets. The limit set of a U-RCP set is characterized. Notations are conveniently developed. Section 3 centers about the basic properties of the compactification of an RCP set. The existence of a König chain in a compact set is proved in Section 4. Our results are applied to reproduce some important results in [1] and [2].

2. RCP: A Brief Review on Basics

2.1. **RCP, V-RCP, U-RCP.** Throughout this paper, Σ, Λ denote *bounded* sets of square matrices (real or complex) of the same size. Since all what follow are invariant under similarity transforms, it is sometimes advantageous to think them geometrically as a subset of $gl(V_d)$, the general algebra of all linear transforms on a d-dimensional space V_d. If so, *all the matrices are supposed to act from the right side on row vectors*, unless otherwise the rule is denied explicitly. Consequently, a vector is always assumed to be a *row vector.*

For each positive integer m, symbol Σ^m and Σ_m denote

$$\Sigma^m = \Sigma \times \Sigma \times \cdots \times \Sigma = \{\mathbf{t} = (A_1, \cdots, A_m) \mid A_i \in \Sigma\},$$

$$\Sigma_m = \{(\mathbf{t}) = A_1 \cdots A_m \mid \mathbf{t} = (A_1, \cdots, A_m) \in \Sigma^m\}.$$

Here $(\bullet)$ is the "lowering" operator that maps the set Σ^m of finite sequences to the set Σ_m of finite products.

The definition of Σ^m brings no problem to Σ^∞. Besides, we can define the operator $(\bullet)_m$ from Σ^∞ to Σ_m. For any $\mathbf{t} = (A_1, A_2, \cdots) \in \Sigma^\infty$, define

$$(\mathbf{t})_m = A_1 A_2 \cdots A_m.$$

To introduce Σ_∞, one needs the condition RCP.

Definition 1 (RCP set). A bounded set Σ is said to be an RCP (right-convergent-product) set if for any sequence $\mathbf{t} = (A_1, A_2, \cdots) \in \Sigma^\infty$,

$$\lim_{m\to\infty} (\mathbf{t})_m = \lim_{m\to\infty} A_1 A_2 \cdots A_m$$

exists.

If Σ is an RCP set, then Σ_∞ naturally denotes the collection of all the above limits. An element in Σ_∞ is usually denoted by $(\mathbf{t})$, with $\mathbf{t} \in \Sigma^\infty$.

The black-faced $\mathbf{t}$ and the lowering operators $(\bullet)$ and $(\bullet)_m$ will bring many notational advantages, as one shall see throughout the paper.

Definition 2 (V-RCP set). An RCP set Σ is said to be *vanishing* if $\Sigma_\infty = \{0\}$. A vanishing RCP set is simply referred to as a V-RCP set.

Definition 3 (U-RCP set). An RCP set Σ is said to be *uniform* if for any $\epsilon > 0$, one can find $N > 0$, such that, for all $n \geq N$, $m \geq 0$, and $\mathbf{t} \in \Sigma^n$, $\mathbf{s} \in \Sigma^m$,

$$\|(\mathbf{t})(Id - (\mathbf{s}))\| = \|(\mathbf{t}) - (\mathbf{t})(\mathbf{s})\| \leq \epsilon.$$

A uniform RCP set is also called a U-RCP set.

2.2. **The limit set of a U-RCP set.** Here we characterize the limit set Σ_∞ of a U-RCP Σ.

Definition 4 (Right-absorbing). A matrix set Λ is said to be right-absorbing if

$$PQ = P, \qquad \text{for any} \quad P, Q \in \Lambda.$$

The similar definition goes to a *left-absorbing* set. A right-absorbing set can be completely characterized geometrically.

Proposition 1. *A set of matrices Λ is right-absorbing if and only if the following two hold:*

(i) Each $P \in \Lambda$ is a (skew) projection: $P^2 = P$.

(ii) Any two projections $P, Q \in \Lambda$ project onto a same space.

We leave this easy proof to our readers. Remind you that we have assumed any matrix acts on row vectors.

For a right-absorbing set Λ, denote by $\mathbf{E}(\Lambda)$ the common projection space. Then $\mathbf{E}(\Lambda)$ can be seen as the inertial manifold of Λ. Or, in terms of the 1-eigenspaces $\mathbf{E}_1(P)$, $P \in \Lambda$,

$$\mathbf{E}(\Lambda) = \mathbf{E}_1(P) = \bigcap_{Q \in \Lambda} \mathbf{E}_1(Q).$$

Another interesting property is the duality principle.

Proposition 2 (Duality). *A matrix set Λ is right-absorbing if and only if $I - \Lambda = \{I - P \mid P \in \Lambda\}$ is left-absorbing.*

The limit set of a U-RCP set can be understood completely via this concept.

Proposition 3. *The limit set Σ_∞ of a U-RCP Σ is right-absorbing.*

Proof. For any $\mathbf{t}, \mathbf{s} \in \Sigma^\infty$,

$$(\mathbf{t})(\mathbf{s}) = \lim_{n,m \to \infty} (\mathbf{t})_n (\mathbf{s})_m = \lim_{m \to \infty} (\mathbf{t})_m (\mathbf{s})_m.$$

By the definition of U-RCP, for any $\epsilon > 0$, as m becomes large enough,

$$\|(\mathbf{t})_m - (\mathbf{t})_m(\mathbf{s})_m\| \leq \epsilon.$$

Since ϵ is arbitrary,

$$\lim_{m\to\infty} (\mathbf{t})_m(\mathbf{s})_m = \lim_{m\to\infty} (\mathbf{t})_m = (\mathbf{t}).$$

This completes the proof. □

Hence there is a common projection space associated to Σ_∞. This, in return, requires

Corollary 1. *Suppose Σ is a U-RCP set. Then for any $A, B \in \Sigma$,*

$$\mathbf{E}_1(A) = \mathbf{E}_1(B).$$

Proof. From the convergence of $\lim_m A^m = A^\infty$ and $\lim_m B^m = B^\infty$, it is easy to see that (1) $\lambda = 1$ is geometrically simple for both A and B (i.e. without Jordan blocks of size ≥ 2); (2) all the rest eigenvalues of A and B are strictly inside the unit circle. Hence,

$$\mathbf{E}_1(A) = \mathbf{E}_1(A^\infty), \qquad \mathbf{E}_1(B) = \mathbf{E}_1(B^\infty).$$

We have already shown above that

$$\mathbf{E}_1(A^\infty) = \mathbf{E}_1(B^\infty) = \mathbf{E}(\Sigma_\infty).$$

This complete the proof. □

It says that elements of a U-RCP set must share a common inertial subspace. This result was first shown in Daubechies and Lagarias [2]. Characterizing this common inertial subspace by the right-absorbing property of the limit set, however, is an innovation of this paper.

We hope this warms us up enough for both the background and notations of the RCP subject.

3. Properties of Compactification: $\Sigma \to \overline{\Sigma}$

In this section, we show that various RCP-related properties are invariant as one goes from Σ to its compactification $\overline{\Sigma}$.

Theorem 1 (Compactification invariance). *Let Σ be a bounded set of matrices, and $\overline{\Sigma}$ denote its closure or compactification in the matrix space. Then*

(-1) $\overline{\Sigma}_m = \overline{\Sigma_m}$. Hence, $\|\overline{\Sigma}_m\| = \|\Sigma_m\|$.
(-2) $\hat{\rho}(\overline{\Sigma}) = \hat{\rho}(\Sigma)$.
(-3) Σ is an RCP if and only if $\overline{\Sigma}$ is.
(-4) Σ is a V-RCP if and only if $\overline{\Sigma}$ is.
(-5) Σ is a U-RCP if and only if $\overline{\Sigma}$ is.

Proof. Invariant properties (-1) and (-2) are easy to show by definitions. For (-3), it suffices to show that $\overline{\Sigma}$ is an RCP if Σ is. We shall apply the proved (by Berger and Wang [1]) Daubechies-Lagarias' boundedness conjecture:

Result 3. *If Σ is a (bounded) RCP, then there exists a constant C such that for any $m = 1, 2, \cdots$, $\|\Sigma_m\| \le C$. Especially, $\|\overline{\Sigma}_m\| \le C$.*

Take an arbitrary $\mathbf{s} = (B_1, B_m, \cdots) \in \overline{\Sigma}_\infty$. We shall show that $(\mathbf{s}) = \lim_m (\mathbf{s})_m$ exists. For any $\epsilon > 0$, there exists a $\mathbf{t}^\epsilon = (A_1^\epsilon, A_2^\epsilon, \cdots) \in \Sigma^\infty$, such that

$$\|B_m - A_m^\epsilon\| \le \frac{\epsilon}{2^m}, \qquad m = 1, 2, \cdots.$$

Hence for any m,

$$\begin{aligned}\|(\mathbf{s})_m - (\mathbf{t}^\epsilon)_m\| &= \|B_1 \cdots B_m - A_1^\epsilon \cdots A_m^\epsilon\| \\ &\le \sum_{j=0}^{m-1} \|B_1 \cdots B_j (B_{j+1} - A_{j+1}^\epsilon) A_{j+2}^\epsilon \cdots A_m^\epsilon\| \\ &\le \sum_{j=0}^{m-1} \|B_1 \cdots B_j\| \, \|B_{j+1} - A_{j+1}^\epsilon\| \, \|A_{j+2}^\epsilon \cdots A_m^\epsilon\| \\ &\le C^2 \sum_{j=0}^{m-1} \frac{\epsilon}{2^{j+1}} \\ &\le C^2 \epsilon. \qquad (1)\end{aligned}$$

This leads to

$$\limsup_{m\to\infty} \|(\mathbf{s})_m - (\mathbf{t}^\epsilon)\| \le C^2 \epsilon, \qquad (2)$$

from which it is easy to see that $\{(\mathbf{s})_m\}$ is a Cauchy sequence. Therefore, $(\mathbf{s}) = \lim_m (\mathbf{s})_m$ does converge and (-3) is proved.

Now suppose Σ is a V-RCP. In Eq. (2), $(\mathbf{t}^\epsilon) = 0$. Then $(\mathbf{s})$ must be zero since ϵ is arbitrary, which means that $\overline{\Sigma}$ is also a V-RCP. This proves (-4).

For (-5), from the telescoping inequality (1),

$$\|(\mathbf{s})_{n+m} - (\mathbf{s})_n\| \le \|(\mathbf{t}^\epsilon)_{n+m} - (\mathbf{t}^\epsilon)_n\| + 2C^2\epsilon.$$

Suppose that Σ is a U-RCP set. Then there exists N, independent of $\mathbf{s}$, such that for all $n \ge N$,

$$\|(\mathbf{t}^\epsilon)_{n+m} - (\mathbf{t}^\epsilon)_n\| \le \epsilon.$$

Hence, for all $n \ge N$, and $m \ge 0$

$$\|(\mathbf{s})_{n+m} - (\mathbf{s})_n\| \le (2C^2 + 1)\epsilon.$$

Since N does not depend on the choice of $\mathbf{s}$, $\overline{\Sigma}$ is a U-RCP. □

Remark 1. The telescoping technique (1) first appeared in Daubechies and Lagarias [2] in a different context. As we see now, it is the throat passage through which all properties of RCP sets pass to their compactifications.

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Theorem 2. *Suppose Σ is an RCP set, then $\overline{\Sigma}_\infty \subset \overline{\Sigma_\infty}$. If Σ is a U-RCP set, then $\overline{\Sigma}_\infty = \overline{\Sigma_\infty}$.*

Proof. The first statement follows from the telescoping inequality (1), from which, we easily deduce that for any $s \in \overline{\Sigma}^\infty$ and $\epsilon > 0$, there exists $\mathbf{t}^\epsilon \in \Sigma^\infty$, such that

$$\|(\mathbf{s}) - (\mathbf{t}^\epsilon)\| \le \epsilon C^2.$$

This implies that $(\mathbf{s}) \in \overline{\Sigma_\infty}$.

For the second statement, it suffices to show that $\overline{\Sigma_\infty} \subset \overline{\Sigma}_\infty$.

Suppose that $D \in \overline{\Sigma_\infty}$, which means there exist $\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \cdots \in \Sigma^\infty$, such that

$$\lim_n (\mathbf{t}^{(n)}) = D.$$

Suppose

$$\mathbf{t}^{(n)} = (A_1^{(n)}, A_2^{(n)}, \cdots), \qquad n = 1, 2, \cdots.$$

By the compactness of $\overline{\Sigma}$, it is easy to show (similar to the consecutive selection technique of Helly in functional analysis) that there exist $\mathbf{s} = (B_1, B_2, \cdots) \in \overline{\Sigma}^\infty$ and an indices sequence $n_1 < n_2 < \cdots$, such that

$$\lim_{k \to \infty} A_j^{(n_k)} = B_j, \qquad j = 1, 2, \cdots.$$

Hence,

$$(\mathbf{s})_j = \lim_{k \to \infty} (\mathbf{t}^{(n_k)})_j, \qquad j = 1, 2, \cdots.$$

On the other hand, by the U-RCP assumption and the preceding theorem on compactification invariance, for any given $\epsilon > 0$, there exists $J > 0$, such that for any $j \ge J$ and n_k,

$$\|(\mathbf{t}^{(n_k)})_j - (\mathbf{t}^{(n_k)})\| \le \frac{\epsilon}{3}, \qquad \|(\mathbf{s})_j - (\mathbf{s})\| \le \frac{\epsilon}{3},$$

and for n_k large enough,

$$\|(\mathbf{t}^{(n_k)}) - D\| \le \frac{\epsilon}{3}.$$

Then for each $j \geq J$,

$$\|(\mathbf{s})_j - D\| \leq \limsup_k \left[\|(\mathbf{s})_j - (\mathbf{t}^{(n_k)})_j\| + \|(\mathbf{t}^{(n_k)})_j - (\mathbf{t}^{(n_k)})\| + \|(\mathbf{t}^{(n_k)}) - D\|\right]$$
$$\leq 0 + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Therefore,

$$\|(\mathbf{s}) - D\| \leq \|(\mathbf{s}) - (\mathbf{s})_j\| + \|(\mathbf{s})_j - D\| \leq \epsilon,$$

and $D \in \overline{\Sigma}_\infty$, which completes the proof. □

Corollary 2. *Let Σ be an RCP (or, U-RCP, V-RCP) set. Define*

$$\Lambda = \Sigma_1 \bigcup \Sigma_2 \cdots \bigcup \Sigma_\infty.$$

Then Λ is also an RCP (or, U-RCP, V-RCP, respectively) set.

Proof. Let $S = \Sigma_1 \bigcup \Sigma_2 \bigcup \cdots$ (not including Σ_∞). Then S is an RCP (or, U-RCP, V-RCP) set provided that Σ is and $S_\infty = \Sigma_\infty$. So is the closure $\overline{S}$ according to the compactification invariance. Noticing that S is a semigroup, we have

$$\overline{S} \supset S \bigcup S_\infty = \Lambda.$$

Therefore, Λ is an RCP (or, U-RCP, V-RCP) set. □

Remark 2. A direct proof by definition is more involved than this approach.

4. A Compact Set Has a König Chain

Daubechies and Lagarias proved

Result 4 (Theorem 3.1 in [2]). *If Σ is a* finite *RCP set, then $\hat{\rho}(\Sigma) \leq 1$.*

Result 5 (Theorem 4.1 in [2]). *A finite set Σ is a V-RCP if and only if $\hat{\rho}(\Sigma) < 1$.*

Both of them were conveniently deduced from the following result (the essential parts of both proofs in [2]).

Result 6. *A finite set* Σ *has a König chain.*

The name "König chain" is used for the first time in this paper. We find it convenient and appropriate.

Definition 5 (König chain). Let Σ be a matrix set. $\mathbf{t} = (A_1, A_2, \cdots) \in \Sigma^\infty$ is said to be a König chain of Σ if for $m = 1, 2, \cdots$

$$\|(\mathbf{t})_m\|^{1/m} = \|A_1 \cdots A_m\|^{1/m} \geq \hat{\rho}(\Sigma).$$

Daubechies and Lagarias proved Result 6 by using König's infinity lemma on finitely branching trees [2].

In what follows, we show that the finiteness condition in the above three results can be safely replaced by the compactness condition. Through one example, we also demonstrate that the "boundedness" condition is generally not sufficient for the existence of a König chain. Applications of König chains are mainly shown through generalizing the above results.

4.1. **A compact set has a König chain.** First, as an intermediate step, we introduce a weaker concept.

Definition 6 (σ-König chain). Given a set of matrices Σ and a sequence of non-negative numbers $\sigma = (\sigma_1, \sigma_2, \cdots)$. A sequence $\mathbf{t} = (A_1, A_2, \cdots) \in \Sigma^\infty$ is called a *σ-König chain* if for any m,

$$\|(\mathbf{t})_m\|^{1/m} = \|A_1 A_2 \cdots A_m\|^{1/m} \geq \sigma_m.$$

Lemma 1. *If σ is a compact set of matrices, then for any $\sigma = (\sigma_1, \sigma_2, \cdots)$, so that $\sigma_m < \hat{\rho}(\Sigma)$, $m = 1, 2, \cdots$, there exists a σ-König chain.*

For any integer m, we say $\mathbf{t} \in \Sigma^m$ is *above* σ, if

$$\|(t)_k\|^{1/k} \geq \sigma_k, \qquad k = 1, 2, \cdots, m.$$

For a given σ and Σ, let Σ_σ^m denote all $\mathbf{t} \in \Sigma^m$ which are above σ.

Proof of the Lemma. We complete the proof in two steps.

Step 1. We show that for any m, Σ_σ^m is non-empty.

We proceed with the prefix technique in [2]. Suppose otherwise for some integer N, Σ_σ^N is empty. Then, for any $\mathbf{s} \in \Sigma^m$ with $m \gg N$, there shall be a partition of $\mathbf{s}$

$$\mathbf{s} = (\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_k, \mathbf{r}),$$

such that

(i) $\mathbf{t}_j \in \Sigma^{l_j}$ for some $l_j \leq N$, and $\mathbf{r} \in \Sigma^l$ for some $l \leq N-1$. $\sum_j l_j + l = m$.

(ii) $\|(\mathbf{t})_j\|^{1/l_j} \leq \sigma_{l_j}$, for $j = 1, 2, \cdots, k$.

Define

$$c = \|\Sigma\| = \sup_{A \in \Sigma} \|A\|,$$

$$d = \max_{1 \leq k \leq N} \sigma_k.$$

Then $d < \hat{\rho}(\Sigma) \leq c$.

$$\begin{aligned} \|(\mathbf{s})\| &= \|(\mathbf{t}_1)(\mathbf{t}_2) \cdots (\mathbf{t}_k)(\mathbf{r})\| \\ &\leq \|(\mathbf{t}_1)\| \, \|(\mathbf{t}_2)\| \cdots \|(\mathbf{t}_k)\| \, \|(\mathbf{r})\| \\ &\leq d^{l_1} d^{l_2} \cdots d^{l_k} c^l \\ &= d^m (c/d)^l \leq d^m (c/d)^{N-1}. \end{aligned}$$

This immediately implies that

$$\|\Sigma_m\|^{1/m} \leq d(c/d)^{(N-1)/m}.$$

Let $m \to \infty$. We end up with $\hat{\rho}(\Sigma) \leq d$. A contradiction to $d < \hat{\rho}(\Sigma)$!

Step 2. Σ has a σ-König chain.

Here, compactness shall place the parallel role of the "finitely-branching" property in Daubechies and Lagarias' proof.

Following Step 1, for each m, suppose

$$\mathbf{t}_m = (A_{m,1}, A_{m,2}, \cdots, A_{m,m}) \in \Sigma^m$$

is above σ. From the compactness condition on Σ, there exists a sequence of indices $n_1 < n_2 < \cdots$ such that

$$\lim_{k\to\infty} A_{n_k,j} = B_j \in \Sigma$$

exists for any $j = 1, 2, \cdots$. We claim that $\mathbf{s} = (B_1, B_2, \cdots) \in \Sigma^\infty$ is a σ-König chain. In fact, for any m,

$$\|(\mathbf{s})_m\| = \lim_{k\to\infty} \|(\mathbf{t}_{n_k})_m\|.$$

Since $\mathbf{t}_n$ is above σ, for $n_k \geq m$, we have

$$\|(\mathbf{t}_{n_k})_m\| \geq \sigma_m^m.$$

Hence, $\|(\mathbf{s})_m\| \geq \sigma_m^m$. Step 2 is complete since m is arbitrary, and this proves the lemma. □

Theorem 3. *A compact matrix set Σ has a König chain.*

Proof. Take a team of $\sigma^{(k)} = (\sigma_1^{(k)}, \sigma_2^{(k)}, \cdots)$, so that

(i) $\sigma_j^{(k)} < \hat{\rho}$, for any j, k.

(ii) For any fixed j, $\lim_k \sigma_j^{(k)} = \hat{\rho}$.

For each k, the preceding lemma ensures there is a $\sigma^{(k)}$-König chain, say

$$\mathbf{t}_k = (A_{k,1}, A_{k,2}, \cdots),$$

such that, for any m,

$$\|(\mathbf{t}_k)_m\|^{1/m} \geq \sigma_m^{(k)}.$$

Likewise in the proof of the lemma, the compactness of Σ implies that one can find an indices sequence $n_1 < n_2 < \cdots$, so that

$$\lim_{k\to\infty} A_{n_k,m} = B_m \in \Sigma$$

exists. Now we claim that $\mathbf{s} = (B_1, B_2, \cdots) \in \Sigma^\infty$ is a König chain: for any m,

$$\|(\mathbf{s})_m\|^{1/m} = \lim_k \|(\mathbf{t}_{n_k})_m\|^{1/m} \geq \lim_k \sigma_m^{(n_k)} = \hat{\rho}.$$

□

The following example shows that compactness is essential for the existence of a König chain under a given matrix norm.

EXAMPLE.

Let I_2 denote the 2 by 2 identity matrix. Define

$$\Sigma = \left\{ (1 - \frac{1}{n}) I_2 \mid n = 0, 1, \cdots \right\}.$$

It is easy to see that Σ is an RCP (by monotone convergence). It is not compact since the unique cumulating "point" I_2 does not belong to Σ. First we have

$$\hat{\rho}(\Sigma) = \hat{\rho}(\overline{\Sigma}) = 1,$$

which is not difficult to see due to the presence of I_2 in $\overline{\Sigma}$. For any $\mathbf{t} \in \Sigma^m$, we have $(\mathbf{t}) = \sigma I_2$ for some $\sigma < 1$. Hence,

$$\|(\mathbf{t})\|^{1/m} = \sigma^{1/m} < 1 = \hat{\rho},$$

which implies that Σ has no König chain.

Remark 3 (Norm dependence). Since the definition of a König chain involves the norms of finite products, this concept is norm-dependent. It can be seen more clearly as follows. Suppose we have two equivalent matrix norms $\| \bullet \|_1$ and $\| \bullet \|_2$ so that $\|A\|_1 \equiv 2\|A\|_2$ for any matrix A. Assume $\hat{\rho}(\Sigma) = 1$, which is fortunately norm-independent. Define

$$\sigma = (2^{-1}, 2^{-1/2}, 2^{-1/3}, \cdots).$$

Suppose $\mathbf{t} = (A_1, A_2, \cdots) \in \Sigma^\infty$ is a σ-König chain with respect to $\| \bullet \|_2$, i.e.

$$\|A_1 A_2 \cdots A_m\|_2^{1/m} \geq 2^{-1/m},$$

for $m = 1, 2, \cdots$. Then,

$$\|A_1 A_2 \cdots A_m\|_1^{1/m} \geq 1 = \hat{\rho}!$$

This means that $\mathbf{t}$ is a König chain with respect to $\|\bullet\|_1$! In the above example, we have silently accepted any vector-norm-induced matrix-norm so that $\|I_2\| = 1$, which is usual in vector-matrix analysis.

4.2. **Applications of compactification and König chains.** Some applications are in order now.

Proposition 4. *If Σ is a compact RCP set, then $\hat{\rho}(\Sigma) \leq 1$.*

Proof. Let $\mathbf{t} \in \Sigma^\infty$ be a König chain. Since Σ is an RCP, $(\mathbf{t}) = \lim_m (\mathbf{t})_m$ exists. Hence,

$$\hat{\rho}(\Sigma) \leq \liminf_m \|(\mathbf{t})_m\|^{1/m} = \lim_m \|(\mathbf{t})\|^{1/m} \leq 1.$$

□

Corollary 3. *If Σ is a (bounded) RCP set, then $\hat{\rho}(\Sigma) \leq 1$.*

This follows from the compactification invariance (Section 3) and the preceding proposition.

It extends Daubechies and Lagarias's Result 4 in the beginning of this section. It also can be proved by the result of Berger and Wang [1] on equivalent spectral radii. Our proof is more elementary.

Proposition 5. *A compact matrix set Σ is a V-RCP if and only if $\hat{\rho}(\Sigma) <$ 1.*

Proof. The direction from $\hat{\rho} < 1$ to V-RCP is trivial. Assume Σ is a V-RCP. Suppose otherwise $\hat{\rho} \geq 1$. let $\mathbf{t} \in \Sigma^\infty$ be a König chain. Then

$$\|(\mathbf{t})\| = \lim_m \|(\mathbf{t})_m\| \geq 1,$$

which is impossible since $(\mathbf{t}) \in \Sigma_\infty = \{0\}$. □

Corollary 4. *A bounded set Σ is a V-RCP if and only if $\hat{\rho}(\Sigma) < 1$.*

This again follows readily from the compactification invariance and the preceding proposition.

It generalizes Daubechies and Lagarias Result 5 in the beginning of the section. Daubechies and Lagarias also proved it later in [2] in a different way.

As a result, we have

Corollary 5. *A V-RCP set must also be a U-RCP set.*

Finally, before ending this paper, let us mention that the close relation between V-RCP and U-RCP sets is not a coincidence. In fact, Daubechies and Lagarias [2] characterized U-RCP sets completely by V-RCP sets. We state it here in a slightly different way, which is more geometrical and corresponds to the projection operators discussed in Section 2.

Result 7. *A bounded set of matrices (or linear transforms on row vectors) Σ is a U-RCP if and only if there exists a (skew) projection P (i.e. $P^2 = P$) such that*

(i) $P\Sigma = P$.

(ii) $\Sigma_P = (I-P)\Sigma(I-P)^T$ *is a V-RCP.*

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