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# Stability of solutions to the Cauchy problem of symmetric hyperbolic systems

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**Abstract**

In this report we investigate the linear and nonlinear stability of stationary, constant solutions to the Cauchy problem of quasilinear, symmetric hyperbolic systems of equations. Writing the problem as

$$u_t = \sum_{j=0}^{d} \big(A_{0j} + \varepsilon A_{1j}(x,t,u,\varepsilon)\big) D_j u + \big(B_0 + \varepsilon B_1(x,t,u,\varepsilon)\big) u + F(x,t),$$

with

$$u(t=0) = f(x),$$

we say that the problem is non-linearly stable if, for $\varepsilon$ small enough, the solution $u$ stays smooth for all $t \geq 0$ and its maximum norm tends to zero for $t \to \infty$. In this report we give sufficient conditions for non-linear stability.

This work generalizes a recent work by Kreiss, Kreiss and Lorenz where analogous stability results are shown for systems of conservation laws. This generalization is necessary for some applications such us the theories of relativistic dissipative fluids.

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## 1. Introduction

We want to study the linear and non-linear stability of the Cauchy problem in the whole space of the following quasi-linear symmetric hyperbolic system of equations.

$$u_t = \sum_{j=0}^{d} \big(A_{0j} + \varepsilon A_{1j}(x,t,u,\varepsilon)\big) D_j u + \big(B_0 + \varepsilon B_1(x,t,u,\varepsilon)\big) u + F(x,t), \qquad (1.1)$$

with initial condition

$$u(t=0) = f(x).$$

Here $x \in I\!R^d$, $t \geq 0$, $u$ and $f$ take values in $I\!R^n$, sub-index $t$ denotes partial time derivative and $D_j$ denotes partial derivative with respect to $x_j$.

As usual we denote as

$$|u|^2 = \sum_{j=1}^{n} |u_j|^2 \quad \text{and} \quad |A| = \max_u \{|Au| : |u| = 1\},$$

the sqared Euclidean norm of $u$ and the corresponding matrix norm of $A$. $(u,v)$ denotes the $L_2$-inner product for $u, v \in L_2(I\!R^d, I\!R^n)$, $\|u\| = (u,u)^{1/2}$ denotes the $L_2$-norm, and $\|u\|_{H^p}$ denotes the $H^p$-Sobolev norm for $u \in H^p(I\!R^d, I\!R^n)$.

In the different theorems we will use various assumptions that we state now.

**Assumption 1** a) *The matrices $A_{0j}$ and $B_0$ are constant and symmetric, while $A_{1j}$ are symmetric matrices.*

*b) The matrices $A_{1j}$ and $B_1$ vanish linearly with $u$ pointwise.*

When we study linear stability, instead of Assumption 1b, we will require

*b') Both*

$$\sum_j \int_0^\infty \|A_{1j}(\cdot,t,\varepsilon)\|^2 \, dt, \ \text{and} \ \int_0^\infty \|B_1(\cdot,t,\varepsilon)\|^2 \, dt$$

*are finite.*

We need to specify regularity conditions for the coefficients and source of (1.1). We will use the following notation for the squared $L_1$-norm in space and time

$$M(F,T) := \Big( \int_0^T \int_{I\!R^d} |F(x,t)| \, dx dt \Big)^2.$$

**Assumption 2** a) *Given $C > 0$, for every $p = 0,1,2,\ldots$ there is a constant $K(C,p)$ such that both $|D_x^\alpha D_u^\beta A_{1j}(x,t,u,\varepsilon)| + |D_x^\alpha D_u^\beta A_{1jt}(x,t,u,\varepsilon)|$ and $|D_x^\alpha D_u^\beta B_1(x,t,u,\varepsilon)| + |D_x^\alpha D_u^\beta B_1 t(x,t,u,\varepsilon)|$ are bounded by $K(C,p)$, for all multi-indices $\alpha$ and $\beta$ with $|\alpha| + |\beta| = p$.*

*b) The source $F$ is such that*

$$M(F,\infty) < \infty \quad \text{and} \quad \int_0^\infty \|F(\cdot,t)\|_{H^p}^2 \, dt \leq \infty, \ \ p = 0,1,2,...$$

*and the initial data function $f$ satisfies*

$$K_f := \sup_{|\omega| \le \omega_0} \{|\hat{f}(\omega)|^2\} < \infty, \text{ and } \|f\|^2_{H^p} < \infty, \; p = 0, 1, 2, ...$$

*here $\hat{f}$ denotes the Fourier transform of $f$.*

Clearly, $K_f$ is finite if $f \in L_1(\mathbb{R}^d)$. We introduce notation for the first-order partial differential operators:

$$P_0 := \sum_{j=1}^{d} A_{0j} D_j, \text{ and } P_1 := \sum_{j=1}^{d} A_{1j}(x, t, u, \varepsilon) D_j.$$

The symbol of the constant coefficients system is given by

$$\hat{P}_0(i\omega) + B_0 = \sum_{j=1}^{d} i\omega_j A_{0j} + B_0. \tag{1.2}$$

The following assumption is the main assumption and will be proved to characterize both linear and non-linear stability.

**Assumption 3** *The symbol of the constant coefficients system satisfies the "Stability Eigenvalue Condition", i.e. there exists a positive constant $\delta$ such that all eigenvalues $\lambda$ of (1.2) satisfy,*

*a)* $\operatorname{Re} \lambda(\omega) \le -(\delta/\omega_0^2)|\omega|^2$, *for* $|\omega| \le \omega_0$;

*b)* $\operatorname{Re} \lambda(\omega) \le -\delta$, *for* $|\omega| > \omega_0$.

Let $\hat{u}(\omega, t)$ denote the Fourier tranform of $u(x, t)$. We introduce the splitting in high and low frequencies of $u$ as follows,

$$u(x,t) = u^I(x,t) + u^{II}(x,t), \quad \text{were } \hat{u}^I(\omega,t) = \begin{cases} \hat{u}(\omega,t) & \text{if } |\omega| \le \omega_0, \\ 0 & \text{if } |\omega| > \omega_0. \end{cases}$$

Along this work $C_{0l}$ and $C_{0l}(p)$, $l = 1, 2, ...$ will denote positive constants that may depend on $P_0$, $B_0$, $\omega_0$ (and $p$ when indicated) but are independent of $P_1$, $B_1$, $\varepsilon$, $F$ and $f$.

During this work we will closely follow the work of Kreiss, Kreiss and Lorenz [1]. The main difference between this work and [1] is that we do not require the system of equations to be in conservation form. However, we need the dimension of space to be bigger than two. Also, we include the initial data function into the estimates instead of making it vanish by means of a transformation.

A main motivation for this work was to generalize the results in [1] to include non conservative hyperbolic systems. This is necessary for some applications such us the theories of relativistic dissipative fluids [2]. Stability for the Cauchy problem of these fluid theories, under periodic boundary conditions, was shown in a previous work [3].

In section 2 in this work we will derive estimates for the solution of the linear equation (Eq. (1.1) with coefficients independent of $u$) and will prove linear stability. In section 3 we will use the linear estimates together with Sobolev's inequalities to prove global existence and stability in the nonlinear case; we will give a proof of the next theorem

**Theorem 1** *Consider the Cauchy problem for the nonlinear equation (1.1), let Assumptions 1, 2 and 3 hold, and let the dimension of space be $d \geq 3$. Then there exists $\varepsilon_0 > 0$ such that the solution $u$ is $C^\infty$ and exists globally in time, provided that $|\varepsilon| \leq \varepsilon_0$. Furthermore,*

$$\lim_{t\to\infty} |u(\cdot,t)|_\infty = 0.$$

## 2. Estimates and stability in the linear case

Consider the Cauchy problem for the linear, symmetric hyperbolic system of equations

$$\begin{aligned} u_t &= \sum_{j=0}^{d} \big(A_{0j} + \varepsilon A_{1j}(x,t,\varepsilon)\big) D_j u + \big(B_0 + \varepsilon B_1(x,t,\varepsilon)\big) u + F(x,t), \\ u(t=0) &= f(x) \end{aligned} \tag{2.1}$$

Application of the Fourier-Laplace transformation to the constant coefficient system, (2.1) with $\varepsilon = 0$, gives

$$(sI - \hat{P}_0(i\omega) - B_0)\tilde{u}(\omega, s) = \tilde{F}(\omega, s) + \hat{f}(\omega) \tag{2.2}$$

with $s = \eta + i\xi$, $\eta > 0$.

*Estimates for $u^I$ (low frequencies)*

Notice that Assumption 3a implies $B_0 \leq 0$, then

$$(\hat{P}_0(i\omega) + B_0) + (\hat{P}_0(i\omega) + B_0)^* = 2B_0 \leq 0,$$

where the symmetry was used. Consequently $|e^{(\hat{P}_0 + B_0)t}| \leq 1$ (see Lemma 2.1.4 in [4]), and we can apply the Kreiss' Matrix Theorem (see [4], Theorem 2.3.2). There are two constants $K_1$ and $K_2$ such that for each $\omega$ there is a transformation $S(\omega)$ such that,

$$S^{-1}(\hat{P}_0 + B_0)S = \begin{pmatrix} \lambda_1 & b_{12} & \dots & b_{1n} \\ & \lambda_2 & \dots & b_{2n} \\ & & \ddots & \vdots \\ 0 & & & \lambda_n \end{pmatrix}$$

with $|S^{-1}| + |S| \leq K_1$ and $|b_{ij}| \leq K_2 |\operatorname{Re}\lambda_i|$, $1 \leq i < j \leq n$. Thus, one obtains

$$\left|\Big((\eta + i\xi)I - (\hat{P}_0 + B_0)\Big)^{-1}\right|^2 \leq K_3 \sum_j \frac{1}{(\eta - \operatorname{Re}\lambda_j)^2 + (\xi - \operatorname{Im}\lambda_j)^2}.$$

By Assumption 3a, $\eta - \operatorname{Re}\lambda_j \geq (\delta/\omega_0^2)|\omega|^2$. Then, integrating the previous inequality, one finds

$$\int_{-\infty}^{\infty} \left|\Big((\eta + i\xi)I - (\hat{P}_0 + B_0)\Big)^{-1}\right|^2 d\xi \leq \frac{C_{01}}{|\omega|^2}. \tag{2.3}$$

We are now ready to prove the estimate for the low frequencies part of the solution in the linear case. We begin with the constant coefficient system.

**Lemma 1** *Let $u$ solve the constant coefficients Cauchy problem (2.1) with $\varepsilon = 0$. If Assumptions 1a, 2b and 3a hold, and the dimension of space is $d \geq 3$, then for all $p = 0, 1, 2, \ldots$ there exist a constant $C_0(p)$ independent of $T$, $F$ and $f$ such that*

$$\int_0^T \|u^I(\cdot, t)\|_{H^p}^2 dt \leq C_0(p)\Big(K_f + M(F,T)\Big).$$

*Proof.* From the Fourier-Laplace transformed system (2.2) we get

$$\begin{aligned} |\tilde{u}|^2 \leq & 2|(sI - \hat{P}_0 - B_0)^{-1}|^2|\tilde{F}|^2 \\ & + 2|(sI - \hat{P}_0 - B_0)^{-1}|^2|\hat{f}|^2 \end{aligned} \tag{2.4}$$

Now,

$$\begin{aligned} |\tilde{F}|^2 &\leq \left| \int_0^\infty \int_{I\!R^d} e^{-i\omega\cdot x} e^{-\eta t} F(x,t) dx dt \right|^2 \\ &\leq \left( \int_0^\infty \int_{I\!R^d} e^{-\eta t} |F(x,t)| dx\, dt \right)^2 \leq M(F, \infty). \end{aligned}$$

So that, using Parseval's relation,

$$\int_0^\infty e^{-2\eta t}\|u^I\|^2 dt = \frac{1}{(2\pi)^{d+1}} \int_{|\omega|\leq\omega_0} \int_{-\infty}^\infty |\tilde{u}(\omega, \eta + i\xi)|^2 d\xi\, d\omega,$$

and estimates (2.3) and (2.4) we get

$$\begin{aligned} \int_0^\infty \|u^I\|^2 dt &\leq C_{01} M(F,\infty) \int_{|\omega|\leq\omega_0} \frac{d\omega}{|\omega|^2} + C_{01} \int_{|\omega|\leq\omega_0} \frac{|\hat{f}(\omega)|^2}{|\omega|^2} d\omega \\ &\leq C_{02}\Big(K_f + M(F,\infty)\Big), \end{aligned} \tag{2.5}$$

where the last inequality holds because, as the dimension of space is $d \geq 3$, the integral of $|\omega|^{-2}$ is finite. To estimate the space derivatives notice that $D^\alpha u$ satisfies the same constant coefficients PDE than $u$, also notice that

$$|\widetilde{D^\alpha u}(\omega, s)| \leq |\omega|^{|\alpha|} |\tilde{u}(\omega, s)| \leq \omega_0^{|\alpha|} |\tilde{u}(\omega, s)|.$$

Then, using estimate (2.5), we get

$$\begin{aligned} \int_0^\infty \|u^I\|_{H^p}^2 dt &\leq \frac{1}{(2\pi)^{d+1}} \sum_{|\alpha|\leq p} \int_0^\infty \int_{|\omega|\leq\omega_0} |\widetilde{D^\alpha u}(\omega, \eta + i\xi)|^2 d\xi\, d\omega \\ &\leq C_0(p)\Big(K_f + M(F,\infty)\Big). \end{aligned} \tag{2.6}$$

Finally, as the values of $u^I(x,t)$, $t \leq T$ do not depend on the values of $F(x,t)$, $t > T$, we can replace $\infty$ by a finite time $T$ in the previous estimate. This proves the lemma.

Consider now the Cauchy problem for the variable coefficient system (2.1). We can apply the estimate (2.6) to obtain an estimate in this case, for $p \geq 1$. To this end we define, for a solution $u(x,t)$ of (2.1),

$$G(x,t) = F(x,t) + \varepsilon \sum_j A_{1j}(x,t,\varepsilon) D_j u(x,t) + \varepsilon B_1(x,t,\varepsilon) u(x,t). \tag{2.7}$$

First notice that

$$\begin{aligned} M(G,\infty) &\leq 2M(F,\infty) + 2\varepsilon^2 \left( \int_0^\infty \int_{\mathbb{R}^d} \sum_j |A_{1j}||D_j u| dx\, dt \right)^2 \\ &\quad + 2\varepsilon^2 \left( \int_0^\infty \int_{\mathbb{R}^d} |B_1||u| dx\, dt \right)^2 \\ &\leq 2M(F,\infty) + 2\varepsilon^2 C_1 \int_0^\infty \|u\|_{H^1}^2 dt. \end{aligned}$$

where

$$C_1 = \int_0^\infty \Big( \sum_j \|A_{1j}\|^2 + \|B_1\|^2 \Big) dt. \tag{2.8}$$

Thus, application of (2.6) with $F$ replaced by $G$ gives

$$\begin{aligned} \int_0^\infty \|u^I\|_{H^p}^2 dt &\leq C_0(p) K_f + 2C_0(p) M(F,\infty) + 4\varepsilon^2 C_0(p) C_1 \int_0^\infty \|u^I\|_{H^1}^2 dt \\ &\quad + 4\varepsilon^2 C_0(p) C_1 \int_0^\infty \|u^{II}\|_{H^1}^2 dt, \end{aligned}$$

where $u = u^I + u^{II}$ was used. Then, choosing $\varepsilon$ small enough and repeating the argument for replacing $\infty$ for a finite time $T$, we have proven

**Lemma 2** *Let $u$ solve (2.1). If Assumptions 1a, 1b', 2 and 3a hold, and the dimension of space is $d \geq 3$, then for $p = 1, 2, \ldots$*

$$\int_0^T \|u^I\|_{H^p}^2 dt \leq 2C_0(p) K_f + 4C_0(p) M(F,T) + 4\varepsilon^2 C_0(p) C_1 \int_0^T \|u^{II}\|_{H^1}^2 dt, \tag{2.9}$$

*where $C_0(p)$ is the same constant of Lemma 1, $C_1$ is given by (2.8), and it is assumed that $4\varepsilon^2 C_0(p) C_1 \leq 1/2$.*

*Estimates for $u^{II}$ (high frequencies)*

Given $u \in L_2(\mathbb{R}^d, \mathbb{R}^n)$ (resp. $u \in H^p(\mathbb{R}^d, \mathbb{R}^n)$) and using the decomposition in low and high frequencies for $u$ given in the Introduction, we introduce the decomposition $L_2 = L_2^I \oplus L_2^{II}$ (resp. $H^p = H^{pI} \oplus H^{pII}$) such that $u^I \in L_2^I$ and $u^{II} \in L_2^{II}$.

It was shown in [5] that if the system (2.1) satisfies the Assumption 3b, then there exists a constant $C_0'$ and a time independent pseudo-differential operator $S$ acting on $L_2^{II}(\mathbb{R}^d, \mathbb{R}^n)$, both depending only on $P_0 + B_0$, with the following properties

(1') $S$ is selfadjoint and bounded, i.e. $\|S\|_{L_2^{II}} \leq C_0'$.
(2') $I+S$ is positive definite with $\|I+S\|_{L_2^{II}} + \|(I+S)^{-1}\|_{L_2^{II}} \leq C_0'$.
(3') $2\,\mathrm{Re}\,(I+S)(P_0+B_0) \leq -\delta\,(I+S)$.
(4') $S$ is a smoothing operator, i.e. its symbol satisfies $|\hat{S}(\omega)| = C_0'/|\omega|$, for $|\omega| \geq \omega_0$.
We now extend the action of $S$ to $L_2$ by defining $S = I$ on $L_2^I$. Thus, if we denote $H := I + S$, it is obvious that there exists a constant $C_0$ that depends only on $P_0 + B_0$ such that
(1) $H$ is selfadjoint and bounded, i.e. $\|H\| \leq C_0$.
(2) $H$ is positive definite with $\|H\| + \|H^{-1}\| \leq C_0$.
(3) $2\,\mathrm{Re}\,H(P_0+B_0) \leq -\delta\,H$ when restricted to $L_2^{II}$.
Also, it is easy to show using Parseval's relation and property (4')
(4) $|(u, D_j v)_{\mathcal{H}} - (u, D_j v)| \leq C_0\|u\|\,\|v\|$, for all $u \in L_2$, $v \in H^1$ and $j = 1, 2, ..., d$.

Using the operator $H$ we introduce a new inner product and norm, first in $L_2$ and then in $H^p$.

$$(u,v)_{\mathcal{H}} := (u, Hv), \text{ and } \|u\|_{\mathcal{H}} := (u,u)_{\mathcal{H}}^{1/2}.$$

And

$$(u,v)_{p,\mathcal{H}} := \sum_{|\alpha|\leq p} (D^\alpha u, H D^\alpha v), \text{ and } \|u\|_{p,\mathcal{H}} := (u,u)_{p,\mathcal{H}}^{1/2}.$$

Properties (1) and (2) of $H$ imply the equivalence

$$\frac{1}{C_0}\|u\|_{H^p} \leq \|u\|_{p,\mathcal{H}} \leq C_0\|u\|_{H^p}, \text{ for } p = 0, 1, 2, ... \tag{2.10}$$

We now use the $\mathcal{H}$-norm to get an energy estimate for $u^{II}$.

$$\begin{aligned} \frac{d}{dt}\|u^{II}\|_{\mathcal{H}}^2 &= 2\,\mathrm{Re}\,(u^{II}, u_t^{II})_{\mathcal{H}} \\ &= 2\,\mathrm{Re}\,(u^{II}, u_t)_{\mathcal{H}} \\ &= 2\,\mathrm{Re}\,(u^{II}, (P_0+B_0)u)_{\mathcal{H}} + 2\varepsilon\,\mathrm{Re}\,(u^{II}, P_1 u)_{\mathcal{H}} \\ &\quad + 2\varepsilon\,\mathrm{Re}\,(u^{II}, B_1 u)_{\mathcal{H}} + 2\,\mathrm{Re}\,(u^{II}, F)_{\mathcal{H}}. \end{aligned} \tag{2.11}$$

We estimate the four terms above separately.

i) Using the property (3) we get

$$\begin{aligned} 2\,\mathrm{Re}\,(u^{II}, (P_0+B_0)u)_{\mathcal{H}} &= 2\,\mathrm{Re}\,(u^{II}, (P_0+B_0)u^{II})_{\mathcal{H}} \\ &\leq (u^{II}, 2\,\mathrm{Re}\,H(P_0+B_0)u^{II}) \\ &\leq -\delta\|u^{II}\|_{\mathcal{H}}^2 \end{aligned}$$

ii) The second term can be separated as

$$2\varepsilon\,\mathrm{Re}\,(u^{II}, P_1 u)_{\mathcal{H}} = 2\varepsilon\,\mathrm{Re}\,(u^{II}, P_1 u^I) + 2\varepsilon\,\mathrm{Re}\,(u^{II}, P_1 u^{II}).$$

But

$$
\begin{aligned}
|2\,\mathrm{Re}\,(u^{I\!I}, P_1 u^{I})_{\mathcal{H}}| &\le 2\|u^{I\!I}\|\|HP_1u^{I}\| \\
&\le 2C_0\|u^{I\!I}\|\|\sum_j A_{1j}D_ju^{I}\| \\
&\le 2C_0|A_1|_\infty \omega_0\|u^{I}\|\,\|u^{I\!I}\|.
\end{aligned}
$$

where $|A_1|_\infty = \sum_j |A_{1j}|_\infty$ and $\|D_ju^I\| \le \omega_0\|u^I\|$ were used. Also

$$
P_1u^{I\!I} = \sum_j D_j(A_{1j}u^{I\!I}) - (\sum_j D_jA_{1j})u^{I\!I}.
$$

So that

$$
\begin{aligned}
|2\,\mathrm{Re}\,(u^{I\!I}, P_1u^{I\!I})| &\le \sum_j |2\,\mathrm{Re}\,(u^{I\!I}, D_j(A_{1j}u^{I\!I}))_{\mathcal{H}}| + |2(u^{I\!I}, (\sum_j D_jA_{1j})u^{I\!I})_{\mathcal{H}}| \\
&\le \sum_j |2\,\mathrm{Re}\,(u^{I\!I}, D_j(A_{1j}u^{I\!I}))| + 2\sum_j C_0\|u^{I\!I}\|\|A_{1j}u^{I\!I}\| \\
&\quad + 2\|u^{I\!I}\|\|H(\sum_j D_jA_{1j})u^{I\!I}\| \\
&\le \sum_j |(u^{I\!I}, (D_jA_{1j})u^{I\!I})| + 2C_0|A_1|_\infty\|u^{I\!I}\|^2 + 2C_0|DA_1|_\infty\|u^{I\!I}\|^2 \\
&\le 2(C_0|A_1|_\infty + (1+C_0)|DA_1|_\infty)\|u^{I\!I}\|^2.
\end{aligned}
$$

with $|DA_1|_\infty = \sum_j |D_jA_{1j}|_\infty$ and the symmetry of $A_{1j}$ and properties (1) and (4) were used. Thus

$$
|2\varepsilon\,\mathrm{Re}\,(u^{I\!I}, P_1u)_{\mathcal{H}}| \le (\mu + 2|\varepsilon|(C_0|A_1|_\infty + (1+C_0)|DA_1|_\infty))\|u^{I\!I}\|^2 + \varepsilon^2\,\frac{C_0^2|A_1|_\infty^2\omega_0^2}{\mu}\|u^I\|^2
$$

holds for any $\mu > 0$.

iii) By property (1)

$$
\begin{aligned}
|2\varepsilon\,\mathrm{Re}\,(u^{I\!I}, B_1u)_{\mathcal{H}}| &\le 2|\varepsilon|C_0|B_1|_\infty\|u^{I\!I}\|\|u^I\| \\
&\le \mu\|u^{I\!I}\|^2 + \frac{\varepsilon^2C_0^2|B_1|_\infty^2}{\mu}\|u^I\|^2
\end{aligned}
$$

iv) Again, by (1)

$$
|2\,\mathrm{Re}\,(u^{I\!I}, F)_{\mathcal{H}}| \le \mu\|u^{I\!I}\|^2 + \frac{C_0^2}{\mu}\|F\|^2.
$$

Finally, using i, ii, iii, and iv in (2.11), and the equivalence (2.10) for $p = 0$, we have shown

$$
\begin{aligned}
\frac{d}{dt}\|u^{I\!I}\|_{\mathcal{H}}^2 &= \big(-\delta + 3\mu C_0^2 + 2|\varepsilon|C_0^2(C_0|A_1|_\infty + (1+C_0)|DA_1|_\infty)\big)\|u^{I\!I}\|_{\mathcal{H}}^2 \\
&\quad + \varepsilon^2\,\frac{C_0}{\mu}(\omega_0^2|A_1|_\infty^2 + |B_1|_\infty^2)\|u^I\|^2 + \frac{C_0^2}{\mu}\|F\|^2.
\end{aligned}
$$

Then, choosing $\mu = \delta/18C_0^2$ and calling $\varepsilon_1 = \delta/12C_0^2(C_0|A_1|_\infty + (1+C_0)|DA_1|_\infty)$ we have shown

**Lemma 3** *Let $u$ be a solution of (2.1), and let Assumptions 1a, 2 and 3b hold. Then, if $|\varepsilon| \leq \varepsilon_1(A,B)$, we get*

$$\frac{d}{dt}\|u^{II}\|_{\mathcal{H}}^2 = -\frac{2}{3}\delta\,\|u^{II}\|_{\mathcal{H}}^2 + \varepsilon^2 C_{A,B}\|u^I\|^2 + C_0'\|F\|^2. \tag{2.12}$$

*Here, $\varepsilon_1(A,B)$ depends on $P_0+B_0$, $|A_1|_\infty$ and $|DA_1|_\infty$; $C_{A,B}$ depends on $P_0+B_0$, $|A_1|_\infty$ and $|B_1|_\infty$, and $C_0'$ only depends on $P_0+B_0$.*

We will now generalize the estimate in the previous lemma to include derivatives. Applying $D^\alpha$ to (2.1) we get

$$\begin{aligned}\frac{\partial(D^\alpha u)}{\partial t} &= \sum_j (A_{0j}+\varepsilon A_{1j})D_jD^\alpha u + (B_0+\varepsilon B_1)D^\alpha u + \varepsilon R^\alpha + D^\alpha F,\\ D^\alpha u(t=0) &= D^\alpha f,\end{aligned} \tag{2.13}$$

where $R^\alpha = \sum_j\big(D^\alpha(A_{1j}D_ju) - A_{1j}D^\alpha u\big) + D^\alpha(B_1u) - B_1D^\alpha u$ contains only lower order terms, i.e. derivatives of $A_{1j}$, $B_1$ and $u$ up to order $|\alpha|$. We get the following result.

**Lemma 4** *Let $u$ be a solution of (2.1), and let Assumptions 1a, 2 and 3b hold. Then, if $|\varepsilon| \leq \varepsilon_1(A,B,p)$, we get for $p = 1,2,\ldots$*

$$\frac{d}{dt}\|u^{II}\|_{p,\mathcal{H}}^2 = -\frac{2}{3}\delta\,\|u^{II}\|_{p,\mathcal{H}}^2 + \varepsilon^2 C_{A,B,p}\|u^I\|_{H^p}^2 + C_0'\|F\|_{H^p}^2. \tag{2.14}$$

*Here, $\varepsilon_1(A,B,p)$ and $C_{A,B,p}$ are constants that depend $P_0+B_0$ and the $L_\infty$-norms of $A_{1j}$ and $B_1$ and their derivatives up to order $p$, but are independent of $F$. $C_0'$ only depends on $P_0+B_0$.*

*Proof.* For $|\alpha| \leq p$ we apply Lemma 3 to the system (2.13) with $F$ replaced by $D^\alpha F + \varepsilon R^\alpha$ to get

$$\frac{d}{dt}\|D^\alpha u^{II}\|_{\mathcal{H}}^2 = -\frac{2}{3}\delta\,\|D^\alpha u^{II}\|_{\mathcal{H}}^2 + \varepsilon^2 C_{A,B}\|D^\alpha u^I\|^2 + 2\varepsilon^2 C_0'\|R^\alpha\|^2 + 2C_0'\|D^\alpha F\|^2.$$

As all terms in $R^\alpha$ are linear in some derivative of $u$ up to order $|\alpha|$, we can add all the inequalities with $|\alpha| \leq p$ to get

$$\frac{d}{dt}\|u^{II}\|_{p,\mathcal{H}}^2 = -\frac{2}{3}\delta\,\|u^{II}\|_{p,\mathcal{H}}^2 + \varepsilon^2 C_{A,B}\|u^I\|^2 + 2\varepsilon^2 C_0'\tilde{C}_{A,B,p}\|u^I + u^{II}\|_{H^p}^2 + 2C_0'\|F\|_{H^p}^2.$$

Here $\tilde{C}_{A,B,p}$ depends on the $L_\infty$-norms of $A_{1j}$ and $B_1$ and their derivatives up to order $p$. Using the equivalence (2.10) and defining an appropriate $\varepsilon_1 = \varepsilon_1(\tilde{C}_{A,B,p})$ the lemma follows.

Using the estimates for the low frequencies part $u^I$, and the high frequencies part $u^{II}$ of the solution to (2.1) we get the general estimate and stability result for the linear case, stated in the following theorem.

**Theorem 2** *Assume that $u$ solves the linear system (2.1), Assumptions 1a, 1b', 2 and 3 hold and the dimension of space is $d \geq 3$. Then, for any $p = 1, 2, 3, \ldots$ there are positive constants $C_0(p)$ and $\varepsilon_0(A, B, p)$ such that*

$$\int_0^\infty (\|u\|^2_{H^{p+1}} + \|u_t\|^2_{H^p})\, dt \leq C_0(p)\Big[K_f + \|f\|^2_{H^{p+1}} + M(F, \infty) + \int_0^\infty \|F\|^2_{H^{p+1}}\, dt\Big], \quad (2.15)$$

*provided that $|\varepsilon| \leq \varepsilon_0(A, B, p)$. $C_0(p)$ depends only on $P_0 + B_0$ and $p$, while $\varepsilon_0(A, B, p)$ depends also on the constants $C_1$ and $K(0,0), K(0,1), \ldots K(0, p+1)$ of Assumption 2a. Consequently,*

$$\lim_{t\to\infty} |u(\cdot, t)|_\infty = 0.$$

*Proof.* Denote

$$y_1(t) = \|u^I(\cdot, t)\|^2_{H^p}$$
$$y_2(t) = \|u^{II}(\cdot, t)\|^2_{p,\mathcal{H}}$$

The results of Lemmas 2 and 4 can be written as

$$\int_0^\infty y_1\, dt \leq C_{01}(p)K_f + 2C_{01}(p)M(F, \infty) + 2\varepsilon^2 C_{01}(p)C_1 \int_0^\infty \|u^{II}\|^2_{H^1}\, dt \qquad (2.16)$$

and

$$\frac{dy_2}{dt} \leq -\frac{2}{3}\delta y_2 + \varepsilon^2 \tilde{C}_{A,B,p}\, y_1 + C_{02}\|F\|^2_{H^p}.$$

Integrating this last estimate we get

$$\begin{aligned} y_2(t) \leq & y_2(0)e^{-\frac{2}{3}\delta t} + C_{02}\int_0^t e^{-\frac{2}{3}\delta(t-\tau)}\|F(\cdot,\tau)\|^2_{H^p}\, d\tau \\ & + \varepsilon^2 \tilde{C}(A, B, p)\int_0^t e^{-\frac{2}{3}\delta(t-\tau)} y_1(\tau)\, d\tau. \end{aligned} \qquad (2.17)$$

Notice that

$$\int_0^\infty \int_0^t e^{-a(t-\tau)} f(\tau)\, d\tau dt = \int_0^\infty \int_\tau^\infty e^{-a(t-\tau)} f(\tau)\, dt d\tau = \frac{1}{a}\int_0^\infty f(\tau)\, d\tau.$$

Integrating (2.17) and using the previous identity we get

$$\int_0^\infty y_2(t)\, dt \leq \frac{3}{2\delta}y_2(0) + \frac{3C_{02}}{2\delta}\int_0^\infty \|F(\cdot,\tau)\|^2_{H^p}\, dt + \varepsilon^2 \frac{3\tilde{C}_{A,B,p}}{2\delta}\int_0^\infty y_1(\tau)\, d\tau. \qquad (2.18)$$

The equivalence of norms (2.10) implies

$$\int_0^\infty \|u\|^2_{H^p}\, dt \leq 2\int_0^\infty y_1\, dt + 2C_0 \int_0^\infty y_2\, dt.$$

Thus, (2.16) and (2.18) give, for $|\varepsilon| \le \tilde{\varepsilon}_0$,

$$\int_0^\infty \|u\|_{H^p}^2\, dt \le C_{03}(p)\Big[K_f + \|f\|_{H^p}^2 + M(F,\infty) + \int_0^\infty \|F\|_{H^p}^2 dt\Big]. \tag{2.19}$$

Here $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\tilde{C}_{A,B,p}, C_1)$. To estimate $u_t$ and its space derivatives we apply $D^\alpha$ to the equation (2.1) and take norm. Adding all the estimates for $|\alpha| \le p$ we get

$$\|u_t\|_{H^p}^2 \le C_{04}(p)\|u\|_{H^{p+1}}^2 + \varepsilon^2 \tilde{\tilde{C}}_{A,B,p}\|u\|_{H^{p+1}}^2 + C(p)\|F\|_{H^p}^2.$$

Finally, integrating this last estimate and using (2.19) for $p+1$ we get the desired estimate (2.15), with $\varepsilon_0$ conveniently defined. The stability follows from

$$|u(\cdot,T)|_\infty^2 \le C\int_T^\infty (\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^p}^2)\, dt \to 0, \text{ when } T \to \infty.$$

for $p > d/2$. This proves the theorem.

**Remark:** *Global existence for the linear problem consider in this section is well-known, and the solution can diverge at most exponentially. Thus the Laplace transformation in $t$ is justified for $\eta \ge \eta_0$ with $\eta_0$ large enough. The estimates obtained obtained show that no singularities occur for $\eta \ge 0$. Thus, the contour of integration in the complex plane can be deformed to $\eta = 0$ as we have done here.*

## 3. Nonlinear stability

Using some of the estimates obtained for the solution of the linear equation (3.1) and the Sobolev's inequalities we will estimate the nonlinear solution and prove nonlinear stability.

*Proof of Theorem 1.* Local (in time) existence is well known for quasilinear symmetric hyperbolic systems; let $u$ denote such solution to our problem for a given initial data function $f$ and Source $F$. We will show that, for $\varepsilon$ small enough,

$$\|f\|_{H^{p+1}}^2 + \int_0^{t_1} (\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^p}^2)\, dt, \ \ p = d+3,$$

remains bounded by a constant $\kappa$ sufficiently large, even when $t_1 \to \infty$; thus the solution exists globally. Suppose this is not the case; then there exists a time $T(\kappa,\varepsilon)$ such that

$$\|f\|_{H^{p+1}}^2 + \int_0^T (\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^p}^2)\, dt = \kappa. \tag{3.1}$$

In what follows $C_\kappa^l$, $l = 1,2,\ldots$ will denote constants that are independent of $\varepsilon$ and $T$, but may depend on $\kappa$.

Recalling that, for all $f(t) \in H^1([0,T], I\!R)$,

$$\max_{0\le t\le T} |f(t)|^2 \le \min_{0\le t\le T} |f(t)|^2 + \int_0^T (|f(t)|^2 + |f'(t)|^2)\, dt,$$

Sobolev's inequality and (3.1) imply, for $|\alpha| + \frac{d}{2} < p$,

$$\begin{aligned} \|D^\alpha u\|^2_{\infty,T} := \sup_{0\le t\le T} |D^\alpha u(\cdot,t)|^2_\infty &\le C\Big(\|f\|^2_{H^p} + \int_0^T (\|u\|^2_{H^p} + \|u_t\|^2_{H^p})\,dt\Big) \\ &\le C\,\kappa, \end{aligned} \tag{3.2}$$

where $C$ is a constant independent of $\varepsilon$, $T$ and $\kappa$.

To apply the estimates obtained for the linear case we rewrite (1.1) as

$$\begin{aligned} &u_t = (P_0 + B_0)u + \varepsilon \sum_j \tilde{A}_{1j}(x,t,\varepsilon) D_j u + \tilde{B}_1(x,t,\varepsilon)u + F(x,t), \\ &u(t=0) = f(x), \end{aligned} \tag{3.3}$$

with

$$\tilde{A}_{1j}(x,t,\varepsilon) = A_{1j}(x,t,u(x,t),\varepsilon)$$

and

$$\tilde{B}_1(x,t,\varepsilon) = B_1(x,t,u(x,t),\varepsilon).$$

To estimate $u^I$ we apply the Lemma 1 with $F$ replaced by $F + \varepsilon\sum_j \tilde{A}_{1j}D_j u + \varepsilon B_1 u$. We get

$$\begin{aligned} \int_0^T \|u^I(\cdot,t)\|^2_{H^{p+1}} dt \le & C_{01}(p)K_f + 2C_{01}(p)\Big(M(F,T) \\ & + \varepsilon^2 M(\sum_j \tilde{A}_{1j}D_j u, T) + \varepsilon^2 M(B_1 u, T)\Big). \end{aligned}$$

We want to estimate in terms of $\kappa$ the last two terms. Notice that,

$$\begin{aligned} M(\sum_j \tilde{A}_{1j}D_j u, T) &= \Big(\sum_j \int_0^T \int_{\mathbb{R}^d} |\tilde{A}_{1j}D_j u|\,dxdt\Big)^2 \\ &\le \big(K(C\kappa, 0)\big)^2 \Big(\sum_j \int_0^T \int_{\mathbb{R}^d} |u|\,|D_j u|\,dxdt\Big)^2 \\ &\le C^1_\kappa, \end{aligned}$$

where (3.2) and Assumptions 1a and 2a were used. We get a similar result for $M(B_1 u, T)$. Thus

$$\int_0^T \|u^I(\cdot,t)\|^2_{H^{p+1}} dt \le C_{01}(p)K_f + 2C_{01}(p)M(F,T) + \varepsilon^2 C^2_\kappa. \tag{3.4}$$

To estimate $u^{II}$ we use the $\mathcal{H}$-norm. Lemma 3 applied to (3.3) gives

$$\frac{d}{dt}\|u^{II}\|^2_{\mathcal{H}} \le -\frac{2}{3}\delta\|u^{II}\|^2_{\mathcal{H}} + \varepsilon^2 C^3_\kappa \|u^I\|^2 + C_{02}\|F\|^2, \tag{3.5}$$

To estimate the derivatives of $u^{II}$ we first apply $D^\alpha$, $1 \leq |\alpha| \leq p+1$, to (3.3)

$$(D^\alpha u)_t = (P_0 + B_0)D^\alpha u + \varepsilon \sum_j \tilde{A}_{1j} D_j D^\alpha u + \varepsilon D^\alpha(\tilde{B}_1 u) + D^\alpha F + \varepsilon R^\alpha, \tag{3.6}$$

with

$$R^\alpha = \sum_j (D^\alpha(\tilde{A}_{1j} D_j u) - \tilde{A}_{1j} D_j D^\alpha u) = \sum_j R_j^\alpha.$$

The chain rule implies that each $R_j^\alpha$ is a sum of terms of the form

$$r_j(x,t,\alpha,\sigma) D^{\sigma_1} u \ldots D^{\sigma_r} u \tag{3.7}$$

where, for all $|\alpha| \leq p+1$,

$$|\sigma_1| + \ldots + |\sigma_r| \leq p+2, \quad |\sigma_l| \leq p+1, \quad l = 1, \ldots, r$$

and $r_j(x,t,\alpha,\sigma)$ are partial derivatives $D_x^\beta D_u^\gamma A_{1j}$, $|\beta| + |\gamma| \leq p+1$. On the one hand, all factors $D^{\sigma_l} u$ in (3.7), with the exception of at most one of them, can be bounded in the infinite norm, otherwise there would be two factors with

$$|\sigma_l| + \frac{d}{2} \geq p, \quad |\sigma_m| + \frac{d}{2} \geq p;$$

consequently,

$$p + 2 + d \geq |\sigma_l| + |\sigma_m| + d \geq 2p,$$

which would contradict the choice $p = d+3$. On the other hand, the factors $r_j(x,t,\alpha,\sigma)$ can be bounded in terms of the constants $K(C_K, 0), K(C_K, 1), \ldots, K(C_K, p+1)$ of Assumption 2. Therefore,

$$\|R^\alpha\|^2 \leq C_\kappa^4 \|u\|_{H^{|\alpha|}}^2.$$

Similarly

$$\|D^\alpha(B_1 u)\| \leq C_\kappa^5 \|u\|_{H^{|\alpha|}}^2.$$

Application to (3.6) of Lemma 3 with $F$ replaced by $D^\alpha F + \varepsilon R^\alpha + D^\alpha(\tilde{B}_1 u)$ gives

$$\frac{d}{dt}\|D^\alpha u^{II}\|_{\mathcal{H}}^2 \leq -\frac{2}{3}\delta \|D^\alpha u^{II}\|_{\mathcal{H}}^2 + \varepsilon^2 C_\kappa^6 \|D^\alpha u^I\|^2 + C_{02}\|D^\alpha F + \varepsilon R^\alpha + \varepsilon D^\alpha(\tilde{B}_1 u)\|^2.$$

So, adding these estimates for $1 \leq |\alpha| \leq p+1$ to (3.5), and using the previous bounds for $R^\alpha$ and $D^\alpha(B_1 u)$ and the equivalence (2.10) we get

$$\frac{d}{dt}\|u^{II}\|_{p+1,\mathcal{H}}^2 \leq -\frac{1}{3}\delta \|D^\alpha u^{II}\|_{\mathcal{H}}^2 + \varepsilon^2 C_\kappa^7 \|u^I\|_{H^{p+1}}^2 + C_{03}\|F\|_{H^{p+1}}^2.$$

Integrating like in the proof of Theorem 2 and using the equivalence (2.10) we get

$$\int_0^T \|u^{II}\|_{p+1,\mathcal{H}}^2 \, dt \leq C_{04}\left[K_f + \|f\|_{H^{p+1}}^2 + \int_0^T \|F\|_{H^{p+1}}^2 \, dt\right] + \varepsilon^2 C_\kappa^8.$$

Using the equivalence (2.10) again and adding with (3.4) we get

$$\int_0^T \|u\|_{H^{p+1}}^2\, dt \le C_{05}(p)\Big[K_f + \|f\|_{H^{p+1}}^2 + M(F,T) + \int_0^T \|F\|_{H^{p+1}}^2\, dt\Big] + \varepsilon^2 C_\kappa^9 \qquad (3.8)$$

if $|\varepsilon| \le \varepsilon_2(\kappa)$.

We now use the differential equation (1.1) to estimate $u_t$ in the $H^p$-norm. First notice that

$$\|D^\alpha u_t\|^2 \le \|(P_0+B_0)D^\alpha u\|^2 + \varepsilon^2\|D^\alpha((P_1+B_1)u)\|^2 + \|D^\alpha F\|^2.$$

The highest derivative of $u$ in each of the first two terms is of order $|\alpha|+1$. We can treat the second term as we did before with $R^\alpha$. Adding the resulting estimates for $0 \le |\alpha| \le p$ we get

$$\|u_t\|_{H^p}^2 \le C_{06}(p)\|u\|_{H^{p+1}}^2 + \|F\|_{H^p}^2 + \varepsilon^2 C_\kappa^{10}\|u\|_{H^{p+1}}^2.$$

Integrating in time, using (3.8) and adding $\|f\|_{H^{p+1}}^2$ we find

$$\|f\|_{H^{p+1}}^2 + \int_0^T (\|u\|_{H^{p+1}}^2 + \|u_t\|_{H^p}^2)\, dt \le C_{07}(p)\Big[K_f + \|f\|_{H^{p+1}}^2 + M(F,\infty) + \int_0^\infty \|F\|_{H^{p+1}}^2\, dt\Big] + \varepsilon^2 C_\kappa^{11}.$$

Therefore, choosing

$$\kappa = 1 + C_{07}(p)\Big[K_f + \|f\|_{H^{p+1}}^2 + M(F,\infty) + \int_0^\infty \|F\|_{H^{p+1}}^2\, dt\Big],$$

and if $|\varepsilon| \le \varepsilon_0(\kappa)$ is conveniently defined, we arrive at a contradiction and the time $T$ can not exist. Then the solution exists globally in time and standard arguments show that it is $C^\infty$. Stability is shown as in the proof of Theorem 2.

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