UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

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Jianhong Shen

February 1999

CAM Report 99-7

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90095-1555

ON SOME QUANTUM MECHANICAL AND MATHEMATICAL ASPECTS OF FOURIER TRANSFORMS OF FRACTIONAL ORDERS (FTFO)

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ABSTRACT. Following the efforts of several authors, we continue to develop the theory and applications of Fourier transforms of fractional orders (FTFO). Quantum mechanics considerations reveal the common root, advantages and disadvantages of both the classical Fourier transform and FTFO's. Most operational analysis of FTFO in existence is summarized into a beautiful formula which associates an FTFO to a simple rotation in the phase space. The application of FTFO in reducing the orders of differential equations is studied from a wider context. We introduce for the first time the *singular perturbation* method to "zoom into" the process of reduction of orders, and discover the phenomenon of *resonance*. Namias' integral representation of FTFO is decomposed into the product of scaling operators, pure phase factors, and the ordinary Fourier transform. The nonlinear phase factor allows the application of the *stationary phase* method. Classical results such as the Paley-Wiener theorem and Heisenberg uncertainty principle are generalized to FTFO's. We also propose a fast numerical implementation scheme for FTFO based on the powerful Fast Fourier Transform.

To Hung Cheng, a master of asymptotics.

1. Introduction

The concept of Fourier transforms of fractional orders (FTFO) was first introduced by physicist Namias in 1980. His starting point was the eigenvalues of the Fourier transform, $e^{in\frac{\pi}{2}}$, $n = 0, 1, \cdots$. The associated eigenvectors are the well-known Hermite functions (i.e. the product of $e^{\frac{-x^2}{2}}$ with the normalized Hermite polynomials.) Fourier transform is diagonalized by this complete set of eigen-functions. The intuition of the physicist led him to consider new transforms which commute with Fourier transform, but are not too wild. He chose a new transform to have the eigenvalues $e^{in\alpha}$, $n = 0, 1, \cdots$. It was denoted by $\mathcal{F}_\alpha$, and called a Fourier transform with the fractional order α. FTFO embeds the classical Fourier transform in a one-parameter (i.e. α) family of unitary operators.

By utilizing the basic properties of the Hermite polynomials (mostly those of orthogonal polynomials), such as the three-term recursion relation, and the reproducing kernel formula,

Namias was able to generalize many results in the classical Fourier transform to FTFO. The successful applications of FTFO in certain types of evolutionary or stationary Schrödinger equations in quantum mechanics clearly showed the potential power of this tool. Though some of his derivations were in a formal level, Namias set up the right framework for further studies of the topic.

His work was supplemented by McBride and Kerr [4] in 1987, who constructed the related necessary mathematical foundation. It was, however, a pity that some of the quantum mechanical intuitions were lost in the latter work. The physics trace leading to some important mathematical results was erased and hence it was more difficult to understand only the results themselves.

It was quite recently that Dattoli, Torre and Mazzacurati [3] re-emphasized the right angle to view FTFO. Namias noticed this approach earlier as one could see from the examples he demonstrated. Dattoli, et al. treated the one-parameter family of unitary operators $\mathcal{F}_\alpha$ as the time evolution of a free harmonic oscillator, and α was understood as the "time" parameter.

The quantum mechanics point of view also allowed them to borrow the two different but equivalent pictures of Schrödinger and Heisenberg: the evolution of states and the evolution of observables (or operators). The authors yet did not seem to explore them deep enough endowed with this new degree of freedom.

The work of Dattoli et al. successfully smoothed away the historical difference between Fourier transform and FTFO. It convincingly showed that the family of FTFO was as natural as the classical Fourier transform, being viewed from the underlying physics.

FTFO has also stimulated interesting speculations in quantum optics (see the references in Dattoli et al. [3]).

In spite of all the developments and achievements mentioned above, there are still many theoretical or practical questions about FTFO that should be properly asked and answered. In order to develop FTFO into a mature theory and a tool potentially as powerful as the classical Fourier transform, we have to study FTFO in a much deeper level and wider context, both in mathematics and physics. It is this goal that our paper is aimed at. Together, with the efforts of all the authors mentioned above, we shall push this interesting topic to a proper position in mathematics and physics that it deserves.

The paper is organized as follows.

Section 2 introduces the FTFO family as a time evolution operator in quantum mechanics, following the approach of Dattoli, et al. The periodictiy of $\mathcal{F}_\alpha$ on α is related to the fact that the Hamiltonian of a harmonic oscillator is an *integral observable*.

In Section 3, the correspondence of the classical mechanics and quantum mechanics enables us to establish explicitly the rotational interpretation of $\mathcal{F}_\alpha$ in the phase plane. The periodic parameter α is shown to be exactly the angle of rotation. FTFO is also connected to the two most important operators in the quantum theory of harmonic oscillators – the raising and lowering operators g_+ and g_- .

Section 4 and 5 jointly discuss the mathematical theory behind the well-practised technique (Namias [6] and McBride and Kerr [4]) of reduction of orders for certain classes of differential equations by FTFO.

In Section 4, the philosophical difference between the classical Fourier transform and general FTFO is explained through the concept of *coupling* of position and momentum. The coupling property of FTFO interprets its role in the reduction of orders for differential equations. A general result on reduction of orders is established for some types of high order differential equations.

In Section 5, through the demonstration of a typical example, which can be easily generalized, we "recover" the lost order reduced by FTFO. Our tool is the theory of singular perturbation for differential equations. The reduction of orders by FTFO is connected to the physical phenomenon of resonance in a formal level.

Section 6 discusses the integral representation of FTFO and some important issues that have been left out in the literature. The quadratic phase transform residing inside an FTFO allows the application of the stationary phase method for certain range of parameters. The Paley-Wiener theorem and the uncertainty principle are generalized to FTFO's. We also discuss briefly the issue of fast numerical implementation of FTFO's in the end.

The conclusion and author's opinions on some future research topics are presented in Section 7.

2. Quantization, Periodicity and FTFO

It has been a familiar fact in classical Fourier analysis that discretizing uniformly a function of position is equivalent to periodizing its Fourier transform in the momentum space. In other words, the Fourier transform of a uniform discrete signal is a periodic function. While it is clear in the context of position and momentum, the fact has received less attention in the duality of time and energy.

In quantum mechanics, there are two equivalent points of view of the evolutionary world, namely, the Schrödinger picture in which a state evolves according to the celebrated Schrödinger equation, and the Heisenberg picture in which not the state, but the concerned observable (a Hermitian operator) evolves. The ring linking these two pictures is the *time evolution operator*

$$\mathbf{U}_t = e^{it\mathcal{H}},$$

where $\mathcal{H}$ is the Hamiltonian of the system, and the Planck's constant $\hbar$ has been taken to be 1 for our convenience. The time evolution operator is unitary since the Hamiltonian is an Hermitian operator.

The time evolution operator $\mathbf{U}_t$ can be understood as the Fourier transform of energy (in the sense of duality). It is therefore expected that the discretization–periodization correspondence mentioned in the beginning should find its parallelism here. If fact, in the time-energy context, it is better to be described as the quantization–periodization connection. Quantization is the physics' way of discretization.

Definition 2.1 (Integral Observables). Let $\mathcal{A}$ be an observable in a quantum system. $\mathcal{A}$ is said to be *integral* if there exist constant a_0 and a_1 such that for any observed value (eigenvalues) a of $\mathcal{A}$,

$$\frac{a - a_0}{a_1}$$

is an integer.

The Hamiltonian of a harmonic oscillator and the z-component angular momentum J_z are the two well-known integral observables.

Let α be the real scalar variable dual to an integral observable $\mathcal{A}$. Then the "Fourier transform"

$$\mathcal{A}_\alpha = e^{i\alpha(\mathcal{A}-a_0)}$$

is a one-parameter of unitary operators. $\mathcal{A}_\alpha$ is in fact periodic with respect to the parameter α. Therefore, with a proper scaling, $\mathcal{A}_\alpha$ is a homeomorphism from $\mathbf{S}^1$ to the group of unitary operators.

FTFO is directly related to the Hamiltonian of the harmonic oscillator

$$\mathcal{H}_0 = -\frac{1}{2}\frac{d}{dx^2} + \frac{1}{2}x^2,$$

which has been de-dimensionalized. The energy spectra of an harmonic oscillator system are all positive half integers

$$\{\frac{1}{2}, \frac{3}{2}; \cdots\}.$$

Hence $\mathcal{H}_0$ is an integral observable. Define a new Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 - \frac{1}{2},$$

so that the spectra are shifted downward to become all non-negative integers. The FTFO associated to parameter α was defined as

$$\mathcal{F}_\alpha = e^{i\alpha\mathcal{H}}.$$

Proposition 2.1. *$\mathcal{F}_\alpha$ is 2π-periodic with respect to α, and*

$$\mathcal{F}_\alpha = \sum_{n=0}^{\infty} e^{in\alpha} \left|n\right\rangle\left\langle n\right|,$$

where the "Ket" $|n\rangle$ denotes the n-th normalized (in $L^2(\mathbb{R})$) Hermite function.

The last formula was the initial definition of FTFO by Namias [6]. And the evolutionary point of view was first discussed by Dattoli et al. [3].

It is easy to see that the ordinary Fourier transform (noticing that here we take e^{ixy} instead of e^{-ixy})

$$\mathcal{F}f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{ixy}\, dy$$

corresponds to $\alpha = \frac{\pi}{2}$, since

$$\mathcal{F}\,|n\rangle = i^n\,|n\rangle, \qquad n = 0, 1, \cdots.$$

The wide applicability of the ordinary Fourier transform in science and many engineering fields raises two natural questions:

Question 1. Why is $\alpha = \pi/2$ so special and lucky in history? Or, what is the real advantage for α being $\pi/2$? And if FTFO's are truly necessary for one reason or another, what profits can they make?

Question 2. Can we generalize most results in the classical Fourier analysis to FTFO?

The second question has attracted more attention than the first one. Most of the work in existence has been focused on it. Our paper is devoted to answering both of them, and aimed at broadening and deepening the current literature on FTFO's.

Since $\mathcal{F}_\alpha = \mathcal{F} \circ \mathcal{F}_{\alpha - \pi/2}$, modulating $\mathcal{F}, \mathcal{F}^2, \mathcal{F}^3$ (all the Fourier transforms of integral orders), we shall assume that $\alpha \in [0, \pi/2]$ in the coming discussions.

3. Coupling, Lowering and Raising

Define

$$D = \frac{d}{dx} \quad \text{and} \quad p = -iD.$$

Then p is the momentum operator.

Define

$$\mathbf{q} = \begin{pmatrix} x \\ p \end{pmatrix}$$

to be the vector operator corresponding to a phase point in classical mechanics.

In addition, denote by R_α the rotation matrix of the phase plane

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Then the results in Dattoli et al. [3], McBride et al. [4] and Namias [6] can be organized in a neat way.

Proposition 3.1. *For any parameter α,*

$$\mathcal{F}_\alpha \mathbf{q} \mathcal{F}_\alpha^\dagger = R_\alpha \mathbf{q}. \tag{1}$$

This gives, in the Heisenberg picture, the "time" evolution of the phase operator $\mathbf{q}$ in a quantum system described by $\mathcal{H}$. Because of the quantization of energy, the "time" parameter α becomes periodic and is better understood as an angle of rotation, as is crystally clear from the preceding proposition. That probably explains why no one in the above three mentioned papers ever used the symbol t for α.

Before us, people studied the individual effect of x and D. This is misleading. It is the physics that leads to considering the right objects: the momentum p (instead of D) and the phase point operator $\mathbf{q}$.

For ordinary Fourier transform, the angle $\alpha = \frac{\pi}{2}$. Hence x and p are decoupled: x is transformed to p, while p to $-x$. For a general FTFO, it does not hold any more. x and p become *coupled.* We are therefore led to the concept of *eigenphases.*

Definition 3.1 (Eigenphase). A nonzero phase point operator $g = ax + bp$ for some *complex* scalars a and b is called the eigenphase of $\mathcal{F}_\alpha$, if

$$\mathcal{F}_\alpha g \mathcal{F}_\alpha^\dagger = \lambda g, \tag{2}$$

for some complex scalar λ.

Theorem 3.1. *Define*

$$g_- = x + ip = x + D \qquad \textit{and} \qquad g_+ = x - ip = x - D.$$

Then up to a multiplicative scalar, these are the only two eigenphases for any $\mathcal{F}_\alpha$.

Proof. From the preceding proposition, $g = ax + bp$ is an eigenphase if and only if (a, b) is a left eigenvector of the rotation matrix R_α. And the λ in (2) is exactly the corresponding eigenvalue of R_α. The two eigenpairs for R_α are: $(1, -i)$ for $\lambda = e^{i\alpha}$, and $(1, i)$ for $\lambda = e^{-i\alpha}$. Hence the theorem follows. □

In other words, we have

$$\mathcal{F}_\alpha \, g_- \, \mathcal{F}_\alpha^\dagger = e^{-i\alpha} \, g_- \, , \tag{3}$$

$$\mathcal{F}_\alpha \, g_+ \, \mathcal{F}_\alpha^\dagger = e^{i\alpha} \, g_+ \, . \tag{4}$$

But only one is independent since $g_-{}^\dagger = g_+$.

Taking derivative with respect to α and evaluating at $\alpha = 0$, we obtain

$$[\mathcal{H}, \, g_+ \,] = \, g_+ \, , \tag{5}$$

$$[\mathcal{H}, \, g_- \,] = - \, g_- \, . \tag{6}$$

These are the two familiar bracket relations in quantum mechanics. In fact, from them, the harmonic oscillator eigen-problem can be solved completely and symbolically. g_+ and g_- are like the eigenvectors of $\mathcal{H}$ in the bracket algebra. Physicists call g_+ and g_- the *raising* and *lowering* operators, since they shift the eigenstates by

$$g_- \ |n\rangle = c_n^- \, |n-1\rangle, \tag{7}$$

$$g_+ \ |n\rangle = c_n^+ \, |n+1\rangle, \tag{8}$$

for some real constants c_n^- and c_n^+, $n = 0, 1, \cdots$. Here we assume $|-1\rangle = 0$.

Relation (1) (or equivalently, Eq. (5) and (6)), is the physics that determines the transforms of x and d/dx under FTFO mathematically. Most of the operational calculus carried out in [3, 4, 6] can be derived from it.

4. Reduction of Orders by FTFO

Why has the ordinary Fourier transform been so powerful a tool in analysis? It is mostly because that when $\alpha = \pi/2$, x and p (or D) are completely decoupled. Therefore, a differential operator

$$L = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots$$

with constant coefficients becomes under Fourier transform

$$\mathcal{F} \, L \mathcal{F}^\dagger = (-ix)^n + a_1(-ix)^{n-1} + a_2(-ix)^{n-2} + \cdots .$$

This reduces the order of the differential equation to 0. A linear high order differential operator like L simply becomes a multiplier in the Fourier domain.

However, this advantage becomes a disadvantage if we apply the ordinary FT to an operator like

$$L_1 = D^2 + x^2,$$

or

$$L_2 = D^2 - ixD + x^2,$$

in which the coefficients are (polynomial) functions of x. They are transformed to

$$-\mathcal{F}\, L_1\, \mathcal{F}^\dagger = x^2 + D^2 = L_1 \qquad \text{and} \qquad -\mathcal{F}\, L_2\, \mathcal{F}^\dagger = x^2 + D^2 - ixD - i = L_2 - i.$$

The order of the differential operators remains unchanged. Even worse, the form of the operators does not change either essentially. Therefore the ordinary FT fails to yield simple representations for differential operators with non-constant coefficients such as L_1 and L_2.

Why? Because x and D are decoupled under the Fourier transform, and they do not communicate and coordinate to each other to have one order reduced.

In contrast, FTFO can be expected to overcome this difficulty because of its coupling of x and D.

We shall proceed with a typical example, whose formal form first appeared in Namias' original work, and was made rigorous in the paper of McBride and Kerr [4]. Then a general result will be established in the end.

Example 1. Find one special solution to

$$u_{xx}(x) + x^2 u(x) = 0.$$

The exact solutions to this equation can be explicitly expressed by Bessel functions J_ν (see Bender and Orszag [1], for example).

Notice that if we know one special solution, then the classical method of reduction of orders will find another linearly independent one.

The associated differential operator is L_1 defined above, or in terms of the momentum operator p,

$$L_1 = x^2 - p^2.$$

Therefore, L_1 is transformed by $\mathcal{F}_\alpha$ to

$$\begin{aligned}\widehat{L}_1 &= (x\cos\alpha + p\sin\alpha)^2 - (-x\sin\alpha + p\cos\alpha)^2 \\ &= -\cos(2\alpha)p^2 + \cos(2\alpha)x^2 + 2\sin(2\alpha)xp - i\sin(2\alpha).\end{aligned}$$

If we choose $\alpha = \pi/4$, then in the FTFO domain, the original second order equation becomes a first order one:

$$2x\hat{u}_x(x) + \hat{u}(x) = 0.$$

And the solution is

$$\hat{u}(x) = \frac{C}{\sqrt{|x|}}.$$

The inverse transform eventually gives one special solution to the equation:

$$u = C\sqrt{|x|}\ J_{-1/4}(\frac{x^2}{2}),$$

where J_ν is the Bessel function. This agrees to the classical results. □

By applying FTFO again, can we reduce one-order further? The answer is no. This is because $\mathcal{F}_\alpha \circ \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta}$ and generally it is impossible to eliminate two terms by one parameter.

From now on, we assume that L is a differential operator in the non-commutative ring $\mathbb{C}[x, D]$; that is, a differential operator with polynomial coefficients.

Definition 4.1 (D-dominant and the Leading Term). Suppose

$$L = L_0 + L_1 + \cdots$$

is a differential operator of n-th order with

$$\begin{aligned}L_0 &= a_0 D^n + a_1 x D^{n-1} + a_2 x^2 D^{n-2} + \cdots, \qquad a_0 \neq 0, \\ L_1 &= b_0 D^{n-1} + b_1 x D^{n-2} + b_2 x^2 D^{n-3} + \cdots, \\ &\cdots.\end{aligned}$$

An operator L allowing such a decomposition is said to be *D-dominant*, and L_0 is called the *leading term* of L.

Do not confuse the structure of a leading term with Cauchy type of operators in which each monomial is like $x^i D^i$.

The *leading symbol* of a D-dominant operator L of order n is the following homogeneous polynomial in x and p, with complex coefficients

$$\zeta(x,p) = a_0 i^n p^n + a_1 i^{n-1} x p^{n-1} + a_2 i^{n-2} x^2 p^{n-2} + \cdots,$$

provided that the leading term of L is given above. Notice that here x and p are considered as scalar variables, rather than observables (i.e. operators). This switching should cause no confusion in the appropriate context.

Theorem 4.1 (Reduction of Orders). *Let $\zeta(x,p)$ be the leading symbol of a D-dominant differential operator L. If $\zeta(x,p)$ contains one zero in the real projective line $\mathbb{RP}^1$, then, there exists an angle α whose associated FTFO $\mathcal{F}_\alpha$ reduces one order of L.*

Proof. It is easy to see that L_0 determines the order of $\widehat{L} = \mathcal{F}_\alpha L \mathcal{F}_\alpha^\dagger$ completely, since L is D-dominant. On the other hand, just as shown in the example, the coefficient attached to D^n in $\widehat{L}_0$ is exactly $(-i)^n \zeta(sin\alpha, \cos\alpha)$. Since $\zeta(x,p)$ contains one zero in $\mathbb{RP}^1$, one can find some α, such that $\zeta(sin\alpha, \cos\alpha) = 0$. The FTFO associated to this α therefore reduces one order of L. □

We apply this general theorem to another example.

Example 2. Reduce one order of the following equation

$$u_{xxxx} + 4x^2 u_{xx} + 3x^4 u = 0.$$

First we compute the leading symbol

$$\zeta(x,p) = (ip)^4 + 4x^2(ip)^2 + 3x^4 = p^4 + 3x^4 - 4x^2 p^2.$$

By setting $\zeta(x,p) = 0$, we obtain in fact four angles whose associated FTFO's can all reduce one order of the equation:

$$\alpha_{1,2} = \pm\frac{\pi}{4} \qquad \text{and} \qquad \alpha_{3,4} = \pm\frac{\pi}{6}.$$

We shall see in the coming section, that these angles do reveal some important information about the solution space of the differential equation.

5. Reduction of Orders – Singular Perturbation Theory

One important question is not answered (and even asked) in the previous section. That is "how is exactly one order of the equation lost under some FTFO's?"

Let us go back to Example 1 and deeper into it. We shall study the behavior of the equation under $\mathcal{F}_\alpha$ when α is near the critical angle $\pi/4$. Our major tool is the *singular perturbation* theory for differential equations. The analysis presented here for this typical example applies also to more general equations whose orders can be reduced by certain FTFO's. We explore this new direction in details in our another coming paper.

Set $\alpha = \pi/4 - \epsilon/2$. Then

$$\cos(2\alpha) = \sin\epsilon, \qquad \sin(2\alpha) = \cos\epsilon.$$

Define $\hat{u}^\epsilon = \mathcal{F}_\alpha u$. Assume that $\epsilon \ll 1$. Then to the first order of ϵ, $u_{xx} + x^2 u = 0$ is transformed to

$$\epsilon(\hat{u}^\epsilon_{xx} + x^2\hat{u}^\epsilon) - i(x\hat{u}^\epsilon_x + \hat{u}^\epsilon) = 0. \tag{9}$$

This is a singular perturbation problem with respect to ϵ. The so called "slowly-varying" solution (the leading term) (see Bender and Orszag [1]) is obtained by setting

$$-i(x\hat{u}^\epsilon_x + \hat{u}^\epsilon) = 0,$$

which is the equation we were very happy with in the preceding section. It yields an approximate solution whose ϵ-neighborhood contains an exact solution to (9), asymptotically.

However, this time, we are more interested in the other "lost" linearly independent solution. It is possible to obtain it now since ϵ is non-zero, at least in the asymptotic sense. Hence we must apply the singular perturbation technique to distill this solution.

The homogeneity of the equation allows us to assume that the other solution $\hat{u}$ is of order $O(1)$. To asymptotically simplify Eq. (9), we make a change of variables $x = \delta y$, and $\hat{u}(y) = \hat{u^\epsilon}(x)$. Then

$$\frac{\epsilon}{\delta^2}\hat{u}_{yy} + \epsilon\delta^2 y^2\hat{u} - i(2y\hat{u}_y + \hat{u}) = 0.$$

The balance of dominant terms implies $\delta = \sqrt{\epsilon}$ (assuming $\epsilon > 0$) and the equation for the scaled "rapidly-varying" solution is

$$\hat{u}_{yy} - i(2y\hat{u}_y + \hat{u}) = 0.$$

This equation falls well within the applicable scope of the ordinary Fourier transform. One special solution which allows ordinary Fourier transform is

$$\hat{u}(y) = C|y|^{\frac{1}{2}} \exp(\frac{iy^2}{2}) J_{-\frac{1}{4}}(\frac{y^2}{2}).$$

Since

$$J_\nu(y) = (\frac{y}{2})^\nu \sum_{n=0}^{\infty} \frac{(-y^2/4)^n}{n!\,\Gamma(n+\nu+1)},$$

$\hat{u}(y)$ is continuous at $y = 0$ and $\hat{u}(0)$ is nonzero. The asymptotic formula (see Bender and Orszag [1], for instance) of $J_\nu(y)$ for large y implies that near $y = \infty$, the leading magnitude of $\hat{u}(y)$ is $O(|y|^{-\frac{1}{2}})$. Especially, our assumption that $\hat{u}$ is of order $O(1)$ is valid.

Therefore the singular solution (leading term) to (9) is

$$\hat{u}^\epsilon(x) = \hat{u}(\frac{x}{\sqrt{\epsilon}}) = C|\frac{x}{\sqrt{\epsilon}}|^{\frac{1}{2}} \exp(\frac{ix^2}{2\epsilon}) J_{-\frac{1}{4}}(\frac{x^2}{2\epsilon}).$$

Strictly speaking, this formula holds only for x's such that $x \leq O(1/\sqrt{\epsilon})$ since we have dropped the term $\epsilon x^2 \hat{u}$ in the rapidly-varying approximation. This term is not negligible compared with the $\hat{u}$ term when $|x| \gg 1/\sqrt{\epsilon}$.

We thus find another (approximate) solution to the equation in Example 1, independent to the one already derived there

$$u^\epsilon(x) = \mathcal{F}^{-1}_{\pi/4-\epsilon/2} \hat{u}^\epsilon = \mathcal{F}_{-\pi/4+\epsilon/2} \hat{u}^\epsilon \simeq \mathcal{F}_{-\pi/4} \hat{u}^\epsilon.$$

The last step is due to that fact that $\mathcal{F}_\alpha$ depends on α analytically for $0 < \alpha < \pi$ (see Namias' integral representation in the next section).

This solution is lost at $\alpha = \pi/4$ (or $\epsilon = 0$). We call it the *resonance* phenomenon. It can be explained more vividly as follows. First modify the above $\hat{u}^\epsilon(x)$ by multiplying it with an appropriate normalization constant

$$\hat{u}^\epsilon(x) = \frac{c}{\sqrt{\epsilon}} \hat{u}\left(\frac{x}{\sqrt{\epsilon}}\right),$$

so that $\hat{u}^\epsilon(x) \to \delta(x)$ as $\epsilon \to 0^+$ (since the equation is homogeneous).

Corresponding to this new $\hat{u}^\epsilon(x)$, the inverse FTFO

$$u^\epsilon(x) \simeq \mathcal{F}^{-1}_{\pi/4}\hat{u}^\epsilon.$$

shall converge to $u^*(x) = \mathcal{F}_{-\pi/4}\delta$ in the asymptotic sense, or

$$\mathcal{F}_{\pi/4}u^*(x) = \delta(x).$$

From Namias' integral representation in the coming section, we have

$$u^*(x) = c\exp\left(i\frac{x^2}{2}\right),$$

for some constant c. This is indeed an approximate solution to the original equation for $x \gg 1$ since

$$u^*_{xx} + x^2u^* = iu^*.$$

For large x, the term iu^* is negligibly small compared to the other two terms on the left hand side. (Notice that in the theory of asymptotic analysis, the leading term of the solution for large x has a factor of $1/\sqrt{|x|}$ (see Bender and Orszag [1]). Our approximate solution u^* has been obtained from the singular perturbation in the FTFO domain and this factor has been lost. However, the phase factors are identical. For resonance, as in physics, the phase information is the most important.)

We say that this approximate solution $u^*(x)$ to

$$u_{xx}(x) + x^2u(x) = 0$$

is *resonant* with $\mathcal{F}_{\pi/4}$. What is happening here can be explained in a more general picture. There exists one direction in the 2-dimensional (or d-dimensional, if the equation is of order d) solution space of the original equation, whose (asymptotic) phase factor is exactly the *conjugate* of the quadratic phase factor (see the integral representation of FTFO in the next section) in $\mathcal{F}_\alpha$. This direction (or one order) is lost when one applies $\mathcal{F}_\alpha$, because its image under $\mathcal{F}_\alpha$ blows up in the ordinary sense (like a δ function).

6. The Integral Representation of FTFO

In this section, we shall discuss the integral representation of FTFO and its properties. The nonlinear (quadratic) phase factor in an FTFO allows the application of the *stationary phase method.* We also generalize the classical Paley-Wiener theorem and Heisenberg uncertainty principle to FTFO's. The numerical issue is discussed briefly in the end.

Define two unitary operators in $L^2(\mathbb{R})$:

$$g_a f(x) := \exp(-i\frac{ax^2}{2})f(x) \qquad \text{and} \qquad s_\lambda f(x) := \sqrt[4]{\lambda} f(\sqrt{\lambda}x).$$

g_a is a pure phase transform and s_λ is an ordinary scaling operator. We assume that both a and λ are positive.

Define

$$G_{a,\lambda} = g_a \circ s_\lambda,$$

and

$$T_{a,\lambda} = G^\dagger_{-a,\lambda} \circ \mathcal{F} \circ G_{a,\lambda} = s_\lambda^{-1} \circ g_a \circ \mathcal{F} \circ g_a \circ s_\lambda,$$

where $\mathcal{F}$ is the ordinary Fourier transform. Then the two-parameter family of transforms $T_{a,\lambda}$ are all unitary transforms in $L^2(\mathbb{R})$ and their integral representation is given by

Proposition 6.1 (Integral Representation of $T_{a,\lambda}$)**.**

$$T_{a,\lambda} f(x) = \frac{1}{\sqrt{2\pi\lambda}} \exp\left(-i\frac{ax^2}{2\lambda}\right) \int_{-\infty}^{\infty} dy\, f(y) \exp\left(-i\frac{ay^2}{2\lambda} + i\frac{xy}{\lambda}\right).$$

The connection of $T_{a,\lambda}$ and $\mathcal{F}_\alpha$ is described by Namias' integral representation formula.

Theorem 6.1 (Namias' Theorem)**.** *For* $\alpha \in (0, \frac{\pi}{2}]$,

$$\mathcal{F}_\alpha = e^{i\frac{1}{2}(\frac{\pi}{2}-\alpha)} \cdot T_{\cos\alpha, \sin\alpha}.$$

Proof. Namias' proof followed immediately from Mehler's formula (Morse and Feschbach [5]) on Hermite polynomials:

$$\exp\left(\frac{x^2+y^2}{2}\right) \sum_{n=0}^{\infty} e^{in\alpha} \,|n\rangle(x)\, \langle n|\,(y) = \frac{1}{\sqrt{1-e^{2i\alpha}}} \exp\left(\frac{2xye^{i\alpha} - e^{2i\alpha}(x^2+y^2)}{1-e^{2i\alpha}}\right),$$

where $|n\rangle(x)$ is the n-th normalized Hermite function. □

From the expression of $T_{a,\lambda}$, $\mathcal{F}_\alpha$ is unitarily similar to $cg_a \circ \mathcal{F} \circ g_a$ for some constant c. The quadratic phase factor g_a allows $\mathcal{F}_\alpha$ to have a larger definition domain than the ordinary Fourier transform. For FTFO, it seems that the phase cancellation mechanism plays an important role. That is, it allows the function being transformed to blow up at ∞ at some mild rate, under the care of the quadratic oscillation cancellation.

A general statement concerning $\mathrm{Dom}(\mathcal{F}_\alpha)$ shall involve the smoothness and growing order at ∞. Here we establish one relatively simpler result, which is usually enough for applications.

Proposition 6.2. *Suppose that $\alpha \in (0, \frac{\pi}{2})$. Then, for all $-1 < \gamma < 1$, $\mathcal{F}_\alpha |x|^\gamma$ is well-defined.*

Proof. The proof is only sketched since it involves only the standard analysis. Since $\alpha \neq \frac{\pi}{2}$, we essentially need show that the integral

$$\int_{-\infty}^{+\infty} |x|^\gamma e^{-i\frac{x^2}{2}+ixa}\, dx$$

is well-defined. The singularity only lies at $\pm\infty$. Hence, by a suitable shifting, it suffices to justify the integral

$$\int_A^{+\infty} x^\gamma e^{-i\frac{x^2}{2}}\, dx$$

for some $A > 0$. By a change of variables from x to $y = x^2/2$, the latter integral becomes

$$c \int_B^{\infty} y^{-b}\, e^{-iy}\, dy,$$

for some positive constants c, b, and B. The last integral is well-defined by Abel's Theorem on summations. □

Besides this expansion of the transform domain, due to the nonlinear phase factor, general FTFO's also allow asymptotic analysis, for example, the technique of stationary phase.

Theorem 6.2 (Stationary Phase). *Suppose that $a, \lambda > 0$ and $A = a/\lambda \gg 1$, and that $f(x)$ satisfies the regularity conditions for the stationary phase approximation. Then we have the asymptotic (with respect to A) leading term equivalence*

$$\mathcal{F} \circ G_{a,\lambda} f(x) \simeq \exp\left(i(\frac{x^2}{2a} - \frac{\pi}{4})\right) \frac{1}{\sqrt{A\sqrt{\lambda}}} f\left(\frac{x}{A\sqrt{\lambda}}\right).$$

Proof.

$$\begin{aligned}
\mathcal{F} \circ G_{a,\lambda} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\ \sqrt[4]{\lambda} f(\sqrt{\lambda} y) \exp\left(-i\frac{ay^2}{2} + iyx\right) \\
&= \frac{1}{\sqrt{2\pi\sqrt{\lambda}}} \int_{-\infty}^{\infty} du\ f(u) \exp\left(-i\frac{Au^2}{2} + i\frac{xu}{\sqrt{\lambda}}\right) \\
&= \exp\left(i\frac{x^2}{2a}\right) \frac{1}{\sqrt{2\pi\sqrt{\lambda}}} \int_{-\infty}^{\infty} du\ f(u) \exp\left(-i\frac{A}{2}(u - \frac{\sqrt{\lambda}}{a}x)^2\right) \\
&\simeq \exp\left(i(\frac{x^2}{2a} - \frac{\pi}{4})\right) \frac{1}{\sqrt{2\pi\sqrt{\lambda}}} \sqrt{\frac{2\pi}{A}} f\left(\frac{\sqrt{\lambda}}{a}x\right) \\
&= \exp\left(i(\frac{x^2}{2a} - \frac{\pi}{4})\right) \frac{1}{\sqrt{A\sqrt{\lambda}}} f\left(\frac{x}{A\sqrt{\lambda}}\right).
\end{aligned}$$

The step at "$\simeq$" is realized by the stationary phase approximation. □

Apply this theorem to $\mathcal{F}_\alpha$ with a small positive angle α. Since $a = \cos\alpha \simeq 1$ and $\lambda = \sin\alpha \simeq \alpha$, we have

Corollary 6.1 (Stationary Phase). *For small α, the leading term of $\mathcal{F}_\alpha f(x)$ is*

$$\mathcal{F}_\alpha f(x) \simeq \exp\left(i\frac{\alpha}{2}(x^2+1)\right)\ f(x).$$

Especially, as $\alpha \to 0$, this asymptotic leading term also confirms that $\mathcal{F}_0$ is the identity transform.

The integral representation and the structure of $T_{a,\lambda}$ make it possible to generalize many results in the classical Fourier analysis to FTFO. Some work has been done in [3, 4, 6]. Here we shall address on certain important aspects of FTFO that have been missed in the limited literature. Our first result generalizes the classical Paley-Wiener theorem in Fourier analysis.

Theorem 6.3 (Paley–Wiener Theorem for FTFO). *Fix an angle $\alpha \in (0, \frac{\pi}{2}]$. For any function $f(x)$, let $\hat{f}(z)$ denote its $\mathcal{F}_\alpha$ transform.*

(i) Suppose that $f(x)$ is a compactly supported C^∞ function. Then $\hat{f}(z)$ is an entire function in the complex plane; and there exist real numbers $a < b$, such that for any positive integer n,

$$|\hat{f}(z)| \le C_n(1+|z|)^n \exp(\mathrm{Re}z\ \mathrm{Im}z\ \cot\alpha) \begin{cases} e^{-a\ \mathrm{Im}\ z/\sin\alpha} & \text{if Im } z \ge 0 \\ e^{-b\ \mathrm{Im}\ z/\sin\alpha} & \text{if Im } z \le 0. \end{cases} \tag{10}$$

Here C_n is a positive constant only depending on n.

(ii) Conversely, any entire function satisfying the bounds of Eq. (10) for some real numbers $a < b$ is the $\mathcal{F}_\alpha$ transform of a C^∞ function with support in $[a, b]$.

Proof. The proof follows readily from the classical Paley–Wiener theorem and Namias' integral representation theorem. □

Another important aspect of FTFO is its associated *uncertainty principle*. Recall first the uncertainty principle in classical Fourier analysis. Let $\phi(x) \in L^2(\mathbb{R})$ be a "wave" function, i.e., $\|\phi\|_{L^2} = 1$, and $\hat{\phi}(y)$ its Fourier transform. Then

$$\mathrm{Var}(\phi)\mathrm{Var}(\hat{\phi}) \geq \frac{1}{4},$$

where

$$\mathrm{Var}(f) := \int_{-\infty}^{\infty} (x - \overline{x})^2 |f(x)|^2 \, dx,$$
$$\overline{x} := \int_{-\infty}^{\infty} x |f(x)|^2 \, dx.$$

This celebrated Heisenberg uncertainty principle is conventionaly explained as "you cannot expect that both ϕ and $\hat{\phi}$ have very small supports." Its influence in harmonic analysis can be clearly seen in wavelet theory nowadays, in which a wavelet, in certain sense, realizes the optimal simultaneous localization.

Let us fix an $\alpha \in [0, \pi/2]$. Still denote by $\hat{\phi}(y)$ the $\mathcal{F}_\alpha$ transform of a wave function $\phi(x)$. Then

Theorem 6.4 (Uncertainty Principle for FTFO). *For any $\alpha \in [0, \pi/2]$,*

$$\mathrm{Var}(\phi)\mathrm{Var}(\hat{\phi}) \geq \frac{\sin^2 \alpha}{4}.$$

Proof. As in the classical Fourier analysis, the simplest proof is to utilize the properties of the transform, instead of applying the integral formula directly.

Define two operators $A = x$ and $B = x \cos \alpha - p \sin \alpha$, where $p = -id/dx$ is the momentum operator. By Proposition 3.1, it is easy to check that

$$\mathcal{F}_\alpha B \mathcal{F}_\alpha^\dagger = y.$$

Therefore

$$\begin{aligned}\mathrm{Var}(\hat{\phi}(y)) &= \int_{-\infty}^{\infty} (y-\overline{y})^2 |\hat{\phi}(y)|^2 \, dy, \\ &= \langle (y-\overline{y})\hat{\phi}(y),\ (y-\overline{y})\hat{\phi}(y) \rangle, \\ &= \langle (B-\overline{B})\phi(x),\ (B-\overline{B})\phi(x) \rangle, \\ &= \mathrm{Var}_{\phi}(B).\end{aligned}$$

The key step above relies on the unitary property of FTFO. Hence

$$\mathrm{Var}(\phi)\mathrm{Var}(\hat{\phi}) = \mathrm{Var}_{\phi}(A)\mathrm{Var}_{\phi}(B).$$

That is, the uncertainty of FTFO is essentially one between the two "observables" A and B. Since

$$[A, B] = AB - BA = -i \sin\alpha,$$

The standard quantum computation gives (see Strichartz [9] or Das and Mellissinos [2])

$$\mathrm{Var}_{\phi}(A)\mathrm{Var}_{\phi}(B) \geq \frac{\sin^2\alpha}{4}.$$

This completes the proof. □

The uncertainty reaches the maximum when $\alpha = \pi/2$, or in the case of classical Fourier transform.

The third issue, from the more practical point of view, is the computational aspect. By Theorem 6.1 and the structure of $T_{a,\lambda}$, fast computation for $\mathcal{F}_{\alpha}$ is possible based on the popular fast numerical transformer–FFT. The only additional pre-processings are (1) a uniform scaling s_{λ}, and (2) a multiplication by a quadratic phase factor. Similar comments hold for the post-processings. Fortunately, both the pre- and post-processings are easily to be fast implemented because they do not require matrix-vector multiplications. Therefore, such an implementation of $\mathcal{F}_{\alpha}$ via FFT is essentially as fast as FFT itself. This is definitely good news for any computational applications of FTFO.

7. Conclusions

Following the efforts of several authors, this paper addresses some important issues on Fourier transforms of fractional orders. Both the physics (mostly classical and quantum mechanics) and mathematics (reduction of order, singular perturbations, stationary phase, and so on) of the nature of FTFO are discussed. Most results presented here are new and await further research efforts.

Before ending this paper, the author would like to mention two topics that he thinks may have potential importance for future research. (1) The possible application of FTFO in signal processing. The author, based on his experience and knowledge in signal processing, believes that the pure quadratic phase transform has intrinsic meaning in signal processing. It may be very useful for processing certain types of signals. See, for example, Shen and Strang [7, 8] for the role of pure phase transforms. (2) The futher applications of FTFO in differential equations. This is an important direction. Our paper has carried out only two aspects, i.e., the general rule of reduction of orders and the singular perturbation theory. There must exist other interesting aspects or more general theories, that have escaped the sightfield of the authors of [3, 4, 6], as well as that of this paper.

Acknowledgment

The author wishes to thank Professor Gilbert Strang and Gian-Carlo Rota for their generous help and constant support on the research. The author is also pleased to acknowledge the help on quantum mechanics from Professor Eric D'Hoker in the physics department of UCLA. The asymptotic methods of stationary phase and singular perturbation have been inspired by Professor Joseph Keller, who was teaching a seminar course on asymptotic methods in wave propagations at UCLA during the time when the author was working on this paper.

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CAM PROFESSOR, COMPUTATIONAL AND APPLIED MATHEMATICS, 7354 MATH. SCIENCE BUILDING, UCLA, LOS ANGELES, CA 90095, USA. EMAIL: *jhshen@math.ucla.edu.*