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SPECTRAL VANISHING VISCOSITY METHOD FOR NONLINEAR CONSERVATION LAWS *

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Abstract We propose a new Spectral Viscosity (SV) scheme for the accurate solution of nonlinear conservation laws. It is proved that the SV solution converges to the unique entropy solution under appropriate reasonable conditions. The proposed SV is implemented directly on high modes of the computed solution. This should be compared with the original non-periodic SV scheme introduced by Maday, Ould Kaber and Tadmor in [20], where spectral viscosity is activated on the *derivative* of the SV solution. The new proposed SV method could be viewed as a correction of the former, and it offers an improvement which is confirmed by our numerical experiments. A post-processing is implemented to recover the spectral accuracy from the computed SV solution. The numerical results show the efficiency of the new method.

Keywords. Spectral method, vanishing viscosity, conservation law.

Subject classification. AMS(MOS): 65M70, 35L65, 35L50.

1 Introduction

Spectral methods employ various orthogonal systems of infinitely differentiable functions to represent an approximate projection of the exact solution sought for. The resulting high accuracy of spectral algorithms was a main motivation behind their rapid development in the past three decades, e.g., Gottlieb and Orszag [10], Canuto et. al.[4], Bernardi and Maday [3], and Guo [14]. The high accuracy of the spectral algorithm hinges on the *global* smoothness of the underlying solution.
Here we discuss spectral approximations to nonlinear conservation laws whose solutions may develop spontaneous jump discontinuities, i.e., shock waves. In this context, "physically relevant" entropy solutions must be admitted. Due to the presence of shock discontinuities, spectral approximations of entropy solutions experience spurious Gibbs' oscillations, which in turn lead to two related difficulties — to loss of accuracy overall the computational domain and, in the nonlinear case, to instabilities. To solve both difficulties, the Spectral Viscosity (SV) method was introduced in the context of Fourier

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approximation to nonlinear conservation laws by Tadmor [26]. The main ingredient of the SV method is the use of high frequencies diffusion which stabilize the spectral computation *without* sacrifice of spectral accuracy. Further results on the periodic SV method can be found in [19, 27, 28, 6]. A more robust periodic (hyper-)SV based on hyper-diffusion of high-frequencies was introduced in [29]. Maday, Ould Kaber and Tadmor [20] were the first to consider the nonperiodic Legendre pseudospectral viscosity method for an initial-boundary value problem, and Ma [17, 18] recently developed the non-periodic Chebyshev-Legendre approximation, based on the ideas of hyper-Spectral Viscosity. For recent applications consult [2, 16, 22, 8].

In this paper we propose a new form for the nonperiodic Spectral Viscosity method. The proposed SV, presented in §2, is implemented directly on high modes of the computed solution. In their original non-periodic SV scheme, Maday et. al [20], advocated a spectral viscosity which is activated on the *derivative* of the SV solution. Here we point out a correction to [20]. Indeed, compared with the spectral viscosity operator in [20], the correction proposed here offers an improvement which is confirmed by the numerical result in §4.

The question of convergence addressed in §3 below, deals with the second difficulty of spectral methods mentioned above — the issue of stability. We conclude our introduction referring to the first difficulty regarding their loss of accuracy in the presence of shock discontinuities. As we argued before [20, §2.1], the SV solution should be viewed as a more faithful approximation to the *projection* of the exact solution, rather than the solution itself. The computations of Shu and Wong in [24] confirm the high accuracy of the computed SV solution as an approximation to the appropriate projection of the exact solution. The spectral content of SV solutions in the context of propagating singularities in *linear* transport equations, was proved in Abarbanel et. al. [1]. Thus, the convergence rate of the SV solution is limited to the first-order convergence rate of the oscillatory exact projections. To accelerate their convergence, thus recovering the full content of the exact solution with spectral accuracy, one needs to *post-process* the SV solution at its final stage. Such post-processing filters were devised in [13, 21] away from the edges of the shocks, consult [20, §2.1] for the non-periodic framework, and [12, 11] up to the shocks. For a recent study which combines an effective edge detectors together with spectral post-processing we refer to [7, 8]. In §4 we use the Gegenbauer polynomial partial sum advocated in [11] to post-process the SV solution, so that spectral accuracy can be recovered.

2 The Spectral Viscosity scheme

2.1 Entropy solutions of nonlinear conservation laws

We consider the nonlinear scalar conservation law over the finite interval $\Lambda := (-1, 1)$,

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \qquad (x,t) \in \Lambda \times [0,T], \tag{2.1}$$

with $H^1_{\rm loc}[0,T]$ boundary values prescribed at the inflow boundary points along $\{\pm 1\} \times [0,T]$,

$$u(\pm 1, t) = g_\pm(t), \quad \pm f'(u(\pm 1, t)) < 0, t > 0, \tag{2.2}$$

and subject to $H^1(\Lambda)$-initial conditions given at $t = 0$

$$u(x,0) = u_0(x), \qquad x \in (-1,1), \quad x \in \{\pm 1\}. \tag{2.3}$$

An entropy weak solution of (2.1) is sought, i.e., a bounded measurable $u(x,t)$, which assumes the prescribed initial and boundary data in the proper sense, and admits the following entropy condition. For all convex entropy pairs, (U,F), $U''(\cdot) \geq 0$ satisfying the compatibility relation $F'(\cdot) = U'(\cdot)f'(\cdot)$, there holds

$$\partial_t U(u(x,t)) + \partial_x F(u(x,t)) \leq 0, \quad (x,t) \in \Lambda \times [0,T]. \tag{2.4}$$

The entropy inequality (2.4) is sufficient, in the scalar case, to single out a unique, physically relevant solution. This so called entropy solution, could be realized by the vanishing viscosity limit, $u = \lim_\varepsilon u^\varepsilon(x,t)$, where u^ε satisfies the regularized vanishing viscosity equation

$$\partial_t u^\varepsilon(x,t) + \partial_x f(u^\varepsilon(x,t)) = \varepsilon \partial_x (D \partial_x u^\varepsilon(x,t)), \qquad D > 0.$$

We note in passing that the regularized viscosity equation admits an equivalent weak formulation, namely, for all $\phi \in C_0^\infty(\Lambda \times \mathbb{R}_t^+)$

$$\int_{\Lambda \times \mathbb{R}_t^+} u^\varepsilon(x,t)\partial_t\phi(x,t) + f(u^\varepsilon(x,t))\partial_x\phi(x,t) + \varepsilon D \partial_x u^\varepsilon(x,t)\partial_x\phi(x,t)\; dxdt = 0. \tag{2.5}$$

For the classical theory of such entropy solutions we refer to Lax [15] and Smoller [25]. Tartar [31] introduced compensated compactness arguments to study the existence and stability of such solutions. In this context, one seeks a sequence of approximate weak solutions with entropy production compact in $H^{-1}_{\mathrm{loc}}(\Lambda \times [0,T])$; an L^∞ weak-star convergence of the corresponding fluxes then follows. Following Tadmor in [26],[20], we shall use compensated compactness arguments to answer the stability question of the SV method discussed in this paper.

2.2 The discrete framework

We let $\mathbb{P}_N$ denote the space of algebraic polynomials of degree $\leq N$, and we let $(L_k)_{k\geq 0}$ denote the orthogonal family of Legendre polynomials in this space

$$(L_j, L_k) = \frac{2}{2k+1}\delta_{jk}.$$

Here, $(\cdot,\cdot)$ and $\|\cdot\|$ represent the usual $L^2(\Lambda)$-inner product and norm. Next we let $\{\xi_j\}_{j=0}^N$ denote the zeroes of $(1-x^2)L_N'(x)$ with $\xi_0 = -1 < \xi_1 < \ldots < \xi_N = 1$. In the sequel we shall use the Legendre Gauss-Lobatto quadrature rule, stating that there exists weights, ω_j, such that $\forall \phi \in \mathbb{P}_{2N-1}(\Lambda)$ we have, see e.g. [4]

$$\int_{-1}^{1} \phi(x)dx = \sum_{j=0}^{N} \omega_j \phi(\xi_j). \tag{2.6}$$

This suggests to define a discrete inner product, $(.,.)_N$,

$$(\phi,\psi)_N = \sum_{j=0}^{N} \omega_j \phi(\xi_j)\psi(\xi_j),$$

and we let $\|\cdot\|_N$ denote the corresponding discrete norm. Indeed, this discrete norm is equivalent with the usual L^2-norm over $\mathbb{P}_N(\Lambda)$:

$$\|\phi\| \leq \|\phi\|_N \leq \sqrt{2 + \frac{1}{N}}\|\phi\|, \quad \forall \phi \in \mathbb{P}_N \tag{2.7}$$

and of course, due to (2.6) we obtain

$$(\phi, \psi) = (\phi, \psi)_N, \quad \text{if } \deg\phi + \deg\psi \leq 2N - 1. \tag{2.8}$$

Associated with the $N+1$ points of the Legendre Gauss-Lobatto quadrature rule, $\{\xi_j\}_{j=0}^N$, is a unique $\mathbb{P}_N$-interpolant which we denote by $\mathcal{I}_N$:

$$\mathcal{I}_N(\phi)(x) \equiv \sum_{k=0}^{N} \frac{(\phi, L_k)_N}{\|L_k\|_N^2} L_k(x), \qquad \mathcal{I}_N(\phi)(\xi_j) = \phi(\xi_j) \quad j = 0, 1, \ldots, N.$$

The projection $\mathcal{I}_N$ can be viewed as an 'approximate identity' in the $\mathbb{P}_N$-space; in this context we recall the result of [3] which provides us with the estimate

$$\|\frac{\partial}{\partial x}\mathcal{I}_N\phi\| + N \cdot \|\phi - \mathcal{I}_N\phi\| \leq C\|\frac{\partial}{\partial x}\phi\|. \tag{2.9}$$

We note in passing that similar estimates hold for some other 'approximate identities' in the $\mathbb{P}_N$-space. Clearly, (2.9) applies to P_N – the usual $L^2(\Lambda)$ projection into $\mathbb{P}_N$. For instance, (2.9) remains valid if we replace $\mathcal{I}_N\phi$ with $\mathcal{J}_N\phi$,

$$\mathcal{J}_N\phi := \int_{-1}^{x} P_{N-1}\frac{\partial}{\partial x}\phi dx$$

Indeed, using standard estimates of the latter (consult [4]), we obtain

$$\|\frac{\partial}{\partial x}\mathcal{J}_N\phi\| + N \cdot \|\phi - \mathcal{J}_N\phi\| \leq C\|\frac{\partial}{\partial x}\phi\| \tag{2.10}$$

Finally, using (2.8) with $\psi \equiv \mathcal{I}_{N-1}\psi + (\psi - \mathcal{I}_{N-1}\psi)$ followed by (2.9), imply that the error of Gauss quadrature for $\mathbb{P}_{2N}$-polynomials does not exceed

$$|(\phi, \psi) - (\phi, \psi)_N| \leq C\|\psi - \mathcal{I}_{N-1}\psi\|\|\phi\| \leq \frac{C}{N}\|\partial_x\psi\| \cdot \|\phi\|, \quad \forall \phi, \psi \in \mathbb{P}_N(\Lambda) \tag{2.11}$$

2.3 The spectral viscosity scheme

We seek an N-degree approximate solution, $u_N(x,t)$, which approximates the interpolant of the exact entropy solution, $I_N u(x,t)$. Initially, we set $u_N(x,0) = I_N U_0(x)$. To evolve in time, we introduce the following Spectral Viscosity operator, Q. Expressed in terms of the Legendre expansion $v = \sum_{l=0}^{\infty} \hat{v}_l L_l$, the spectral viscosity operator, Q, takes the form

$$Qv(x) := \sum_{l=0}^{N} \hat{q}_l \hat{v}_l L_l(x), \qquad v = \sum_{l=0}^{\infty} \hat{v}_l L_l(x). \tag{2.12}$$

Here, $\hat{q}_l$ are the so called viscosity coefficients,

$$\begin{cases} \hat{q}_l = 0, & \text{for } l \leq m, \\ \hat{q}_l \geq 1 - \dfrac{m^2}{l^2}, & \text{for } m < l \leq N, \end{cases} \tag{2.13}$$

which are at our disposal. Observe that the spectral viscosity operator is activated only the high mode numbers, $\geq m$. In particular, if we let $m \uparrow \infty$, then the SV operator is spectrally small (in the sense that $\|Qv\|_{H^{-s}} \leq cm^{-s}\|v\|$). We shall occasionally highlight the dependence of the SV operator on this cut off of high wave numbers, writing $Q = Q_m$.

Equipped with the SV operator (2.12)-(2.13), we now turn to construct our Legendre viscosity approximation of the initial-boundary value problem (2.1). To this end we let $u_N(x,t) \in \mathbb{P}_N$ for $t \geq 0$ be determined by the moment condition – a discrete analogue of the weak formulation (2.5) requiring that for all $\phi \in \mathbb{P}_N$, $t > 0$, we have

$$(\partial_t u_N(t) + \partial_x I_N f(u_N(t)), \phi)_N + \varepsilon_N (\partial_x Q u_N(t), \partial_x Q\phi)_N = (B(u_N(t)), \phi)_N, \quad \forall\, \phi \in \mathbb{P}_N. \quad (2.14)$$

Here, $B(\cdot)$ is a penalty boundary operator,

$$B(u_N(t)) = (\lambda(t)(1-x) + \mu(t)(1+x))\, \partial_x L_N(x),$$

where the free pair of "Lagrange-multipliers', (λ, μ), are chosen to match the inflow boundary data, $u_N(x,t) = g_\pm(t)$, prescribed at $x = 1$ whenever $f'(u_N(1,t)) < 0$ and at $x = -1$ whenever $f'(u_N(-1,t)) > 0$.

The spectral viscosity method depends on two free parameters: the vanishing amplitude of the viscosity $\varepsilon = \varepsilon_N$, and the size of viscosity-free spectrum, $m = m_N$. As in [20] we choose

$$\varepsilon = \varepsilon_N \sim cN^{-\alpha}, \quad m = m_N \sim cN^{\beta}, \quad 0 < 4\beta < \alpha \leq 1. \quad (2.15)$$

In particular, an increasing portion of the spectrum of size $m_N \sim N^\beta$ remains viscous free, thus retaining the (formal) spectral accuracy of the SV scheme (2.14) with the underlying conservation law (2.1).

Remark. We do not claim the parameterization in (2.15) to be optimal. In particular, arguing along the lines of [30, 17, 18], one can use hyper-viscosity regularization to increase the size of the viscosity-free modes, m_N, thus obtaining better resolution of the resulting SV scheme.

We close this section by explaining how the SV method (2.14) can be implemented as a collocation method. We first *realize* the spectral viscosity in terms of an N-degree polynomial, $\vee_N$, such that

$$(\partial_x(Qu_N), \partial_x(Q\phi))_N = (\vee_N, \phi)_N, \qquad \forall \phi \in \mathbb{P}_N. \quad (2.16)$$

Recall that the discrete inner product $(\cdot,\cdot)_N$ involves the Gauss-Lobatto weights, $W := diag(\omega_0, \ldots, \omega_N)$. If we let D_Q denote the $(N+1)\times(N+1)$ differentiation matrix associated with the derivative of the SV so that $(D_Q\phi)(\xi_j) = \partial_x(Q\phi)(\xi_j)$, $0 \leq j \leq N$ for all ϕ's $\in \mathbb{P}_N$. Then (2.16), expressed in terms of the corresponding $N+1$ vectors, reads $< D_Q u_N, W D_Q \phi > = < \vee_N, W\phi >$, and hence, $\vee_N = W^{-1} D_Q^T W D_Q u_N$. Actually, we have $D_Q = L_D \tilde{Q} L^T W$, where $\tilde{Q} := diag(\hat{q}_0 \|L_0\|_N^{-2}, \ldots, \hat{q}_N \|L_N\|_N^{-2})$ and L, L_D are the $(N+1)\times(N+1)$ matrixes with the elements:

$$(L)_{jk} = L_k(\xi_j), \qquad (L_D)_{jk} = (\partial_x L_k)(\xi_j), \qquad j,k = 0,1,\cdots,N.$$

Thus, by denoting $\tilde{D} := \tilde{Q} L_D^T W L_D \tilde{Q}$, $\vee_N = L\tilde{D}L^T W u_N$. Since, for $0 \leq k \leq l \leq N$, we have from (2.6) that

$$(L_D^T W L_D)_{kl} = (\partial_x L_l, \partial_x L_k) = (L_l \partial_x L_k)|_{x=-1}^{x=1} = \frac{1}{2}[1 + (-1)^{k+l}]k(k+1), \quad (2.17)$$

it follows that

$$(\tilde{D})_{lk} = (\tilde{D})_{kl} = \tilde{Q}_{kk}\tilde{Q}_{ll}(\partial_x L_l, \partial_x L_k) = \begin{cases} k(k+1)\gamma_k\gamma_l\hat{q}_k\hat{q}_l, & m < k \le l \le N,\ k+l \text{ even}, \\ 0, & \text{otherwise}\ , \end{cases}$$

where $\gamma_k := \|L_k\|_N^{-2} = (k+1/2)$ for $0 \le k < N$ and $\gamma_N = N/2$. Another way to reach this expression is to put $\phi = L_k$ in (2.16) so that, for $u_N(x) = \sum_{l=0}^N \hat{u}_l L_l(x)$,

$$\begin{aligned} \vee_N(x) &= \sum_{k=0}^N \frac{(\partial_x(Qu_N), \partial_x(QL_k))}{\|L_k\|_N^2} L_k(x) = \sum_{k=0}^N \sum_{l=0}^N \frac{(\hat{q}_l\partial_x L_l, \hat{q}_k\partial_x L_k)}{\|L_k\|_N^2} \hat{u}_l L_k(x) \\ &= \sum_{k=0}^N \sum_{l=0}^N \tilde{Q}_{kk}\tilde{Q}_{ll}(\partial_x L_l, \partial_x L_k)\|L_l\|_N^2 \hat{u}_l L_k(x) = \sum_{k=0}^N \sum_{l=0}^N (\tilde{D})_{kl}\|L_l\|_N^2 \hat{u}_l L_k(x) \\ &= (L_0(x), L_1(x), \cdots, L_N(x))\tilde{D}L^T W(u_N(\xi_0), u_N(\xi_1), \cdots, u_N(\xi_N))^T. \end{aligned}$$

Remark. To gain a better insight into the SV operator we observe that the SV operator Q is self-adjoint with respect to the discrete inner-product $(\cdot,\cdot)$, and thanks to (2.8), one can integrate by parts. Consequently, the SV expression on the left of (2.14) takes the form

$$\varepsilon_N(\partial_x(Qu_N), \partial_x(Q\phi))_N = \varepsilon_N\partial_x(Qu_N) \cdot Q\phi|_{x=-1}^{x=1} - \varepsilon_N(Q\partial_{xx}^2(Qu_N), \phi)_N. \tag{2.18}$$

The realization of the SV operator here shows that $\vee_N$ is an approximation to $Q\partial_{xx}^2(Qu_N)$ which takes into account the boundary terms, thus preventing spurious boundary layers. Specifically, comparing (2.16) with (2.18) with $\phi = \phi_i,\ \phi_i(\xi_j) = \delta_{ij}$ yields

$$\vee(\xi_i) = -Q\partial_{xx}^2(Qu_N)(\xi_i) + \partial_x Qu_N \cdot Q\phi_i|_{-1}^{1}.$$

The SV operator here is different than the original SV method introduced in [20].

Let us 'test' ((2.14) against $\phi = \phi_i$, where ϕ_i is the standard characteristic polynomial of $\mathbb{P}_N(\Lambda)$ satisfying $\phi_i(\xi_j) = \delta_{ij},\ 0 \le i,j \le N$. At the interior points we obtain

$$\frac{d}{dt}u_N(\xi_i,t) + \frac{\partial}{\partial x}\mathcal{I}_N f(u_N)(\xi_i,t) = \varepsilon_N \vee_N (\xi_i,t), \quad 1 \le i \le N-1. \tag{2.19}$$

At the outflow boundaries, say at $x = +1$, (2.14) yields

$$\frac{d}{dt}u_N(+1,t) + \frac{\partial}{\partial x}\mathcal{I}_N f(u_N)(+1,t) = \varepsilon_N \vee_N (+1,t). \tag{2.20}$$

We note that the last term on the right of (2.20) defined via (2.16) prevents the creation of a boundary layer. Equations (2.19)-(2.20) together with the prescribed inflow data (say $g_-(t)$ at $x = -1$), furnish a complete equivalent statement of the pseudospectral (collocation) viscosity approximation (2.14).

3 Convergence of the SV method

To establish the necessary a priori estimates for u_N, we first prepare

Lemma 3.1 *Consider the spectral viscosity operator $Q = Q_m$, (2.12) with the parameterization in (2.13). Then for any $\phi \in \mathbb{P}_N$,*

$$\|\partial_x \phi\|^2 \leq 2\|\partial_x(Q\phi)\|^2 + cm^4 \ln N \|\phi\|^2,$$
$$\|\partial_x(Q\phi)\|^2 \leq 2\|\partial_x \phi\|^2 + cm^4 \ln N \|\phi\|^2.$$

Remark.. The lemma shows the equivalence of the H^1 norm before and after application of the spectral viscosity operator, $Q = Q_m$, for moderate size of $m_N << N^{1/4}$. This holds despite the fact that for $m = m_N \sim cN^\beta \uparrow \infty$, the corresponding SV operator, Q_m is spectrally small.

Proof. Let $\hat{\phi}_l$ be the coefficients of the Legendre expansion of $\phi(x)$, and

$$J_{l,N} = \{j \mid l+1 \leq j \leq N, \ l+j \text{ odd}\}.$$

Then by the relation between the coefficients of the Legendre expansions of $\phi(x)$ and those for $\partial_x \phi(x)$ (see [4]),

$$\partial_x \phi(x) = \sum_{l=0}^{N-1} \hat{\phi}_l^{(1)} L_l(x), \quad \hat{\phi}_l^{(1)} = (2l+1) \sum_{j \in J_{l,N}} \hat{\phi}_j.$$

Next set $\hat{r}_l = 1 - \hat{q}_l$, and let R denote the corresponding low modes filter

$$R\phi(x) := \sum_{l=0}^{N} \hat{r}_l \hat{\phi}_l L_l(x).$$

Clearly $\hat{r}_l = 1$ for $l \leq m$, and $\hat{r}_l \leq m^2 l^{-2}$ for $l > m$. Since $\partial_x \phi(x) \equiv \partial_x(Q\phi(x)) + \partial_x(R\phi(x))$, it suffices to prove that

$$\|\partial_x(R\phi)\|^2 \leq cm^4 \ln N \|\phi\|^2.$$

We decompose $\partial_x(R\phi(x)) = A_1(x) + A_2(x)$ where

$$A_1(x) := \partial_x(\sum_{l=0}^{m} \hat{r}_l \hat{\phi}_l L_l(x)), \qquad A_2(x) := \partial_x(\sum_{l=m+1}^{N} \hat{r}_l \hat{\phi}_l L_l(x)).$$

By standard inverse inequality, e.g., [4], $\|\partial_x \phi\| \leq cN^2 \|\phi\|$, $\forall \phi(x) \in \mathbb{P}_N$,, and hence $\|A_1\|^2 \leq cm^4 \|\phi\|^2$. Further let $J_{l,N,m} = \{j \mid j \in J_{l,N}, \ j > m\}$. Then

$$\begin{aligned}
\|A_2\|^2 &= \sum_{l=0}^{N-1} (2l+1)^2 \left(\sum_{j \in J_{l,N,m}} \hat{r}_j \hat{\phi}_j \right)^2 \|L_l\|^2 \\
&\leq 2 \sum_{l=0}^{N-1} (2l+1) \left(\sum_{j \in J_{l,N,m}} |\hat{r}_j|^2 \|L_j\|^{-2} \right) \left(\sum_{j \in J_{l,N,m}} |\hat{\phi}_j|^2 \|L_j\|^2 \right) \\
&\leq cm^4 \|\phi\|^2 \sum_{l=0}^{N-1} (2l+1) \sum_{j \in J_{l,N,m}} j^{-3} \\
&\leq cm^4 \|\phi\|^2 \left(m^{-2} \sum_{l=0}^{m} (2l+1) + \sum_{l=m+1}^{N-1} (2l+1) l^{-2} \right) \\
&\leq cm^4 \ln N \|\phi\|^2,
\end{aligned}$$

and the desired estimates follow. ■

The following lemma is in the heart of matter.

Lemma 3.2 *Consider the SV scheme (2.14) with $H^1[0,T]$ boundary values, (2.2), and H^1 initial conditions, (2.3). Assume that the SV solution remains uniformly bounded,*

$$\max_{0\le t\le T} \|u_N(\cdot,t)\|_{L^\infty} \le A_\infty. \tag{3.1}$$

Then there exists a constant (depending on A_∞) such that the following H^1-bound holds

$$\varepsilon_N\left[\|\partial_t u_N\|^2_{L^2([0,T],L^2(\Lambda))} + \partial_x\|u_N\|^2_{L^2([0,T],L^2(\Lambda))}\right] \le Const. \tag{3.2}$$

Proof. To simplify the presentation, we shall deal with the prototype case where one boundary, say $x=-1$, is an inflow boundary, while $x=1$ is an outflow one. Then

$$B\left(u_N(t)\right) = \lambda(t)(1-x)\partial_x L_N(x).$$

Recall that ξ_j are the zeros of $\partial_x L_N(x), 1\le j\le N-1$, so that the boundary operator $B(u_N)$ vanishes at all but the inflow boundary point $x=-1$, where it involves the corresponding values of $\omega^{(0)} = 2/N(N+1)$ and $\partial_x L_N(-1) = (-1)^{N+1}N(N+1)/2$. Thus

$$\left(B\left(u_N(t)\right), v\right)_N = 2(-1)^{N+1}\lambda(t)v(-1,t).$$

Let $\phi\equiv 1$ in (2.14). Since $I_N f\left(u_N(x,t)\right)\in \mathbb{P}_N$, we deduce in view of (2.8) that

$$\partial_t\left(u_N(t),1\right) + f\left(u_N(1,t)\right) - f\left(u_N(-1,t)\right) = 2(-1)^{N+1}\lambda(t). \tag{3.3}$$

Consequently,

$$|\lambda(t)| \le \frac{1}{\sqrt{2}}\|\partial_t u_N(t)\| + \max\left(|f(u_N(1,t))|, |f(u_N(-1,t))|\right). \tag{3.4}$$

Further, set $\eta(t) := \int_0^t \lambda(s)\,ds$; then, integration of (3.3) yields for $t\le T$,

$$|\eta(t)| \le \frac{1}{\sqrt{2}}\|u_N(t)\| + \frac{1}{\sqrt{2}}\|u_N(0)\| + t\max_{|z|\le A}|f(z)|, \quad \eta(t) := \int_0^t \lambda(s)\,ds. \tag{3.5}$$

Next we recall the SV parameterization in (2.15), $\varepsilon_N\sim cN^{-\alpha}, \quad m\sim cN^{\beta}, \quad 0<4\beta<\alpha\le 1$. To get the desired H^1-energy bound, we integrate the SV scheme against u_N. That is, we set $\phi = u_N$ in (2.14). Let $F(u) = \int^u wf'(w)dw$ denote entropy flux corresponding to the quadratic entropy, $U(u) = u^2/2$. Using (2.11) followed by Lemma 3.1 we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_N(t)\|_N^2 + F(u_N(1,t)) - F(u_N(-1,t)) \\
&\quad + \varepsilon_N(\partial_x(Qu_N(t)), \partial_x(Qu_N(t))) + 2(-1)^N\lambda(t)u_N(-1,t) \\
&= (\partial_x f(u_N(t)), u_N(t)) - (\partial_x I_N f(u_N(t)), u_N(t))_N \\
&= -\left((I-I_N)f(u_N(t)), \partial_x u_N(t)\right) \\
&\le \frac{c}{N}\|\partial_x f(u_N(t))\|\cdot\|\partial_x u_N(t)\| \le \frac{c_A}{N}\left(\|\partial_x(Qu_N(t))\|^2 + m^4\ln N\|u_N(t)\|_N^2\right).
\end{aligned}$$

Here and below, c_A stand for various constants depending of the uniform bound $A_\infty := \max \|u_N\|_{L^\infty}$. Thus for any $t \leq T$,

$$\|u_N(t)\|_N^2 + 2\left(\varepsilon_N - \frac{c_A}{N}\right) \int_0^t \|\partial_x(Qu_N(s))\|^2\, ds$$
$$\leq c_0 + 2\int_0^t \left(\frac{c_A m^4 \ln N}{N}\|u_N(s)\|^2 + 2\max_{|z|\leq A}|F(z)| - 2(-1)^N \lambda(s) g_-(s)\right) ds.$$

By (3.5),

$$\left|\int_0^t \lambda(s) g_-(s)\, ds\right| = \left|g_-(t)\eta(t) - g_-(0)\eta(0) - \int_0^t g_-'(s)\eta(s)\, ds\right|$$
$$\leq |g_-(t)| \left(\|u_N(t)\| + c_A + c_0\right) + \int_0^t |g_-'(s)| \left(\|u_N(s)\| + c_A + c_0\right) ds.$$

Putting the two estimates above together, we have from (3.1) and the SV parameterization in (2.15) that

$$\|u_N(t)\|^2 + \varepsilon_N \int_0^t \|\partial_x(Qu_N(s))\|^2\, ds \leq c_A\left(\|g_-\|^2_{H^1(0,t)} + t + 1\right) + c_0.$$

Using Lemma 3.1 and (2.15) again, we arrive at

$$\varepsilon_N \|\partial_x u_N\|^2_{L^2([0,T];L^2(\Lambda))} \leq c_A\left(\|g\|^2_{H^1(0,T)} + T + 1\right) + c_0. \tag{3.6}$$

Next, we set $\phi = \partial_t u_N$ in the SV weak formulation (2.14). By (2.9), (2.7), and (3.4),

$$\|\partial_t u_N(t)\|^2 + \frac{\varepsilon_N}{2}\frac{d}{dt}\|\partial_x(Qu_N(t))\|^2 \leq c_A\|\partial_x u_N(t)\|^2 + \frac{1}{2}\|\partial_t u_N(t)\|^2 + c_A\left|\frac{d}{dt}g_-(t)\right|^2 + c_A.$$

Temporal integration of the above inequality followed by (3.6), implies

$$\|\partial_t u_N\|^2_{L^2([0,T];L^2(\Lambda))} \leq c_A\left(\|\partial_x u_N\|^2_{L^2((0,T);L^2(\Lambda))} + \|g_-\|^2_{H^1(0,T)} + 1\right)$$
$$\leq \frac{c_A}{\varepsilon_N}\left(\|g_-\|^2_{H^1(0,T)} + T + c_0 + 1\right). \tag{3.7}$$

The inequalities (3.6) and (3.7) conclude the proof. ■

Equipped with the H^1 bound Lemma 3.2 we are now ready to the main stability result of this paper, stating

Theorem 3.1 *Let u_N be the solution of the Spectral Viscosity scheme (2.14)-(2.15). Assume that it remains uniformly bounded so that (3.1) holds. Then u_N tends (strongly in $L^p_{\text{loc}}(Q)$ $1 \leq p < \infty$) to a weak solution, u, of the initial-boundary value problem (2.1).*
If, in addition, the SV amplitude is set $\varepsilon_N \sim N^{-\alpha}$ with $\alpha < 1$, then u is the unique entropy solution.

Remark. One can follow the lines of Tadmor [28], and Chen et. al. [6] to derive the uniform bound assumed in (3.1).

Proof. Let $\Omega = \Lambda \times [0, T]$ and define $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$ as before. Let

$$(v, w)_{\Omega,N} = \int_0^T (v(t), w(t))_N \, dt.$$

We still use c_A to denote, as before a constant depending on A_∞ with possible dependence on T.

We want to show that the entropy production of u_N is compact in $H^{-1}_{\rm loc}(\Omega)$. To this, we consider for an arbitrary convex entropy pair, (U, F),

$$(\partial_t U(u_N) + \partial_x F(u_N), w)_\Omega \equiv \sum_{j=1}^{5} G_j\left(U'(u_N)\phi\right). \tag{3.8}$$

We decompose the entropy production to the five terms on the right given by

$$\begin{aligned}
G_1(\psi) &= (\partial_t u_N + \partial_x f(u_N), \psi - \psi_N)_\Omega,\\
G_2(\psi) &= (\partial_x f(u_N) - \partial_x I_N f(u_N), \psi_N)_\Omega,\\
G_3(\psi) &= (\partial_t u_N + \partial_x I_N f(u_N), \psi_N)_\Omega - (\partial_t u_N + \partial_x I_N f(u_N), \psi_N)_{\Omega,N},\\
G_4(\psi) &= \varepsilon_N(\partial_x(Ru_N), \partial_x\psi_N)_\Omega + \varepsilon_N(\partial_x u_N, \partial_x(R\psi_N))_\Omega,\\
G_5(\psi) &= -\varepsilon_N(\partial_x u_N, \partial_x\psi_N)_\Omega - \varepsilon_N(\partial_x(Ru_N), \partial_x(R\psi_N))_\Omega.
\end{aligned}$$

The last identity holds for *arbitrary* $\psi_N \in \mathbb{P}_N$. Following Maday et. al. [20, §5] we specify

$$\psi_N = \mathcal{J}_N\psi := \int_{-1}^{x} P_{N-1}\partial_y\psi(y,t)dy.$$

This specific choice will play an essential role in derivation of the entropy condition below. Observe that $\psi_N \in \mathbb{P}_N$ with $\psi_N(-1,t) = 0$. We recall that the operator R above denote the complement operator, $Q + R = Id$, associated with symbols $\hat{r}_l = 1 - \hat{q}_l$. We proceed with upper bound on the five terms on the right of (3.8).

By (2.10), (3.6), and (3.7),

$$|G_1(\psi)| \le \frac{c_A}{\sqrt{\varepsilon_N}}\|\psi - \psi_N\|_D \le \frac{c_A}{N\sqrt{\varepsilon_N}}\|\partial_x\psi\|_\Omega.$$

According to (3.6),

$$|G_2(\psi)| = |(f(u_N) - I_N f(u_N), \partial_x\psi_N)_\Omega| \le \frac{c}{N}\|\partial_x f(u_N)\|_\Omega\|\partial_x\psi_N\|_\Omega \le \frac{c_A}{N\sqrt{\varepsilon_N}}\|\partial_x\psi_N\|_\Omega.$$

By virtue of (2.11) and (3.7),

$$|G_3(\psi)| = |(\partial_t u_N, \psi_N)_\Omega - (\partial_t u_N, \psi_N)_{\Omega,N}| \le \frac{c_A}{N}\|\partial_t u_N\|_\Omega\|\partial_x\psi_N\|_\psi \le \frac{c_A}{N\sqrt{\varepsilon_N}}\|\partial_x\psi_N\|_\Omega.$$

To proceed, we utilize Lemma 3.1, $\|\partial_x(R\phi)\|_\Omega \le cm^2\sqrt{\ln N}\|\phi\|_\Omega$. Applied with $\phi = u_N, \psi_N$ we find

$$\begin{aligned}
|G_4(\psi)| &\le \varepsilon_N cm^2\sqrt{\ln N}\|u_N\|_\Omega\|\partial_x\psi_N\|_\Omega + \varepsilon_N\|\partial_x u_N\|_\psi\|\partial_x(R\psi_N)\|_\Omega\\
&\le c_A m^2\sqrt{\ln N}(\varepsilon_N\|\partial_x\psi_N\|_\Omega + \sqrt{\varepsilon_N}\|\psi_N\|_\Omega).
\end{aligned} \tag{3.9}$$

Similarly

$$|G_5(\psi)| \leq \sqrt{\varepsilon_N} c_A \|\partial_x \psi_N\|_\Omega + \varepsilon_N m^4 \ln N \|\psi_N\|_\Omega.$$

The previous statements, with $\psi = U'(u_N)\phi$, tell us that

$$\begin{aligned}
&\left|\sum_{j=1}^{5} G_j\left(U'(u_N)\phi\right)\right| \\
&\leq c_A\left(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m^2\sqrt{\ln N} + \sqrt{\varepsilon_N}\right)\left(\|\partial_x u_N\|_\Omega \|\phi\|_{L^\infty(\Omega)} + c_A\|\partial_x\phi\|_\Omega\right) \\
&\leq c_A\left(\|\phi\|_{L^\infty(\Omega)} + \left(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m^2\sqrt{\ln N} + \sqrt{\varepsilon_N}\right)\|\partial_x\phi\|_\Omega\right).
\end{aligned}$$

Thus, the entropy production $\partial_t U(u_N) + \partial_x F(u_N)$ can be written as a sum of two terms – the first tends to zero in $H^{-1}(\Omega)$ and the second is bounded in $L^1(\Omega)$. In view of Murat lemma, consult [5], bounded sequences in $W^{-1,p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $2 < p < \infty$ form a compact subset of $H^{-1}(\Omega)$. We conclude that the entropy production of the SV solution is H^{-1}-compact, which in turn, by compensated compactness arguments, [31], implies that u_N converges strongly (in $L^p_{\text{loc}}(\Lambda \times \mathbb{R}_t^+), p < \infty$) to a weak solution, u, of the conservation law (2.1).

It remains to show that the u is indeed the unique entropy solution. To verify the entropy condition for $\varepsilon_N \sim cN^{-\alpha}$, $\alpha < 1$, we proceed as follows. It is easy to see that

$$\begin{aligned}
\sum_{j=1}^{3}\left|G_j\left(U'(u_N)\phi\right)\right| &\leq \frac{c_A}{N\sqrt{\varepsilon_N}}\left\|\partial_x\left(U'(u_N)\phi\right)_N\right\|_\Omega \\
&\leq \frac{c_A}{N\sqrt{\varepsilon_N}}\left(\|\partial_x u_N\|_\Omega\|\phi\|_{L^\infty(\Omega)} + \|u_N\|_{L^\infty(\Omega)}\|\partial_x\phi\|_\Omega\right) \\
&\leq c_A\left(N^{\alpha-1}\|\phi\|_{L^\infty(\Omega)} + N^{\frac{\alpha}{2}-1}\|\partial_x\phi\|_\Omega\right) \longrightarrow 0.
\end{aligned}$$

By the choice of the SV parameters in (2.15), together with (3.6) and (3.10),

$$\begin{aligned}
\left|G_4\left(U'(u_N)\phi\right)\right| &\leq c_A\varepsilon_N m^2\sqrt{\ln N}\left\|\partial_x\left(U'(u_N)w\right)\right\|_\Omega + c_A\sqrt{\varepsilon_N}m^2\sqrt{\ln N}\left\|U'(u_N)\phi\right\|_\Omega \\
&\leq c_A\sqrt{\varepsilon}M^2\sqrt{\ln N}\left(\|\phi\|_\Omega + \|\partial_x\phi\|_\Omega\right) \longrightarrow 0.
\end{aligned}$$

Finally, let $(U'(u_N)\phi)_N$ denote our usual projection, $(U'(u_N)\phi)_N = \mathcal{J}_N(U'(u_N)\phi$. It is here that we take advantage of our special choice of projection, $\mathcal{J}_N$. Indeed, for any nonnegative test function, $\phi(x) \geq 0$, we find

$$\begin{aligned}
G_5\left(U'(u_N)\phi\right) &= -\varepsilon\left(\partial_x u_N, P_{N-1}\partial_x\left(U'(u_N)\phi\right)\right)_\Omega - \varepsilon_N\left(\partial_x(Ru_N), \partial_x\left(R(U'(u_N)\phi)_N\right)\right)_\Omega \\
&= -\varepsilon_N\left(\partial_x u_N, U''(u_N)w\partial_x u_N\right)_\Omega - \varepsilon_N\left(\partial_x u_N, U'(u_N)\partial_x w\right)_\Omega \\
&\quad -\varepsilon_N\left(\partial_x(Ru_N), \partial_x\left(R(U'(u_N)\phi)_N\right)\right)_\Omega \\
&\leq -\varepsilon_N\left(\partial_x u_N, U'(u_N)\partial_x\phi\right)_\Omega - \varepsilon_N\left(\partial_x(Ru_N), \partial_x\left(R(U'(u_N)\phi)_N\right)\right)_\Omega \\
&\leq c_A\sqrt{\varepsilon_N}\|\partial_x\phi\|_\Omega + c_A\varepsilon_N m^4\ln N\|\phi\|_\Omega \longrightarrow 0.
\end{aligned}$$

It follows that u satisfies the entropy inequality (2.4) in the sense of distribution, and so it is the unique entropy solution. ■

Remark. Maday, Ould Kaber, and Tadmor [20] introduced the non-periodic SV scheme with spectral viscosity of the form

$$\varepsilon\left(Q\partial_x u_N(t), \partial_x \phi\right)_N,$$

parameterized with viscosity coefficients [20, equation 2.2]

$$\begin{cases} \hat{q}_l = 0, & \text{for } l \le m, \\ \hat{q}_l \ge 1 - \dfrac{m^4}{l^4}, & \text{for } m < l \le N. \end{cases} \tag{3.10}$$

Observe that the spectral viscosity operator is applied here *once* to the first derivative of the SV solution. Thus the 'amount' of high-modes smoothing introduced in (3.10) is comparable to the amount of SV introduced here in (2.13) which is activated *twice* – before and after differentiation. The main difference between these two approaches, however, lies in the activation of high-modes diffusion to the SV solution rather than to its first derivative as in (3.10).

In the proof of the main result of that paper, Corollary 3.2 plays an important role, analogous to Lemma 3.1. According to [20, Corollary 3.2], for any $\phi \in \mathbb{P}_N$, we have

$$\|\partial_x \phi\|^2 \le \|\partial_x \phi\|_Q^2 + cm^4 \ln N \|\phi\|^2$$

where $\|\partial_x \phi\|_Q^2$ stands for the weighted norm $\|\partial_x \phi\|_Q^2 = (Q\partial_x \phi, \partial_x \phi)$. The argument based on dyadic decomposition of ϕ, fails, however, precisely because the additional terms introduced by differentiation of each dyadic bloc. Indeed, let us take $\phi(x) = L_N(x)$, so that

$$\partial_x \phi(x) = \sum_{\substack{l=0 \\ l+N \text{ odd}}}^{N-1} (2l+1) L_l(x),$$

and let $\hat{q}_l = 1$, for $m < l \le N$. Then on the one hand,

$$\|\partial_x \phi\|^2 - \|\partial_x \phi\|_Q^2 = \sum_{\substack{l=0 \\ l+N \text{ odd}}}^{m} (2l+1)^2 \|L_l\|^2 = 2 \sum_{\substack{l=0 \\ l+N \text{ odd}}}^{m} (2l+1) = O(m^2);$$

on the other hand, however, $\|\phi\|^2 = (N + \frac{1}{2})^{-1}$. To establish the a priori estimates (3.6) and (3.7), therefore requires $m_N^4 \le c \ln N$ which is much stronger than condition (2.15) imposed in [20]. Thus the result of the present paper is a correction and an improvement of the result in Maday et. al. [20].

4 Numerical Results

In this section, we give some numerical results of the scheme (2.19)-(2.20). We consider the Hopf equation (or inviscid Burgers' equation)

$$\partial_t u(x,t) + \partial_x u^2(x,t)/2 = 0, \qquad (x,t) \in \Omega,$$

with initial values

$$u(0,x) = 1 + \frac{1}{2}\sin \pi x, \quad x \in [-1,1]$$

and boundary conditions $u(-1,t) = g(t)$, where the inflow data are taken from the outflow boundary, i.e., $g(t) = u(1,t)$.

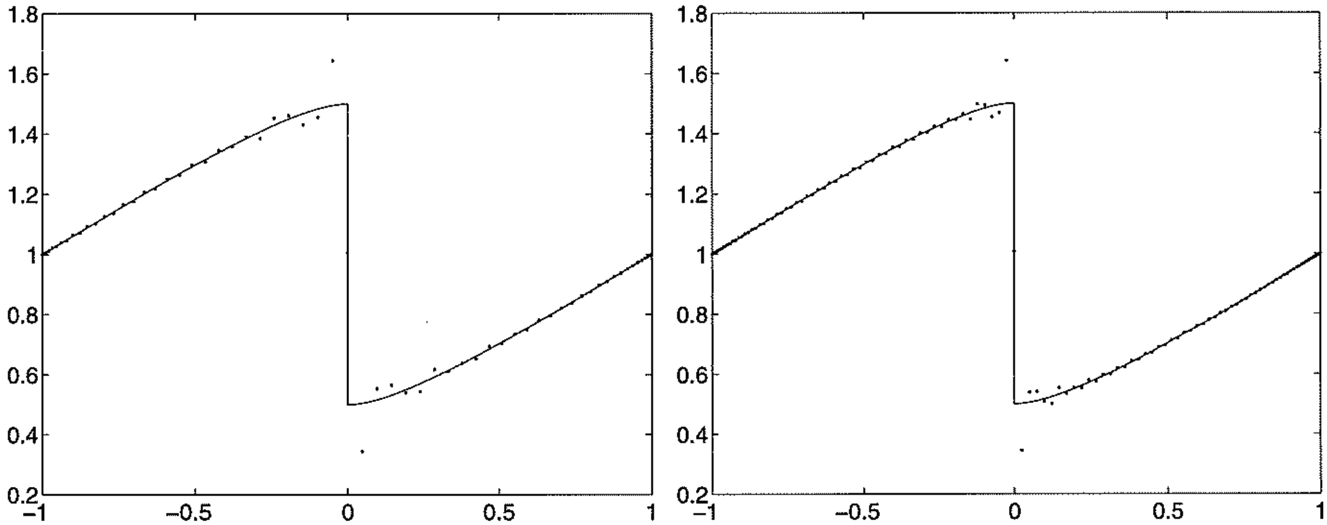

Figure 4.1: Solution of the pseudospectral viscosity method with (a) $N = 64$ modes on the left and (b) $N = 128$ mode on the right.

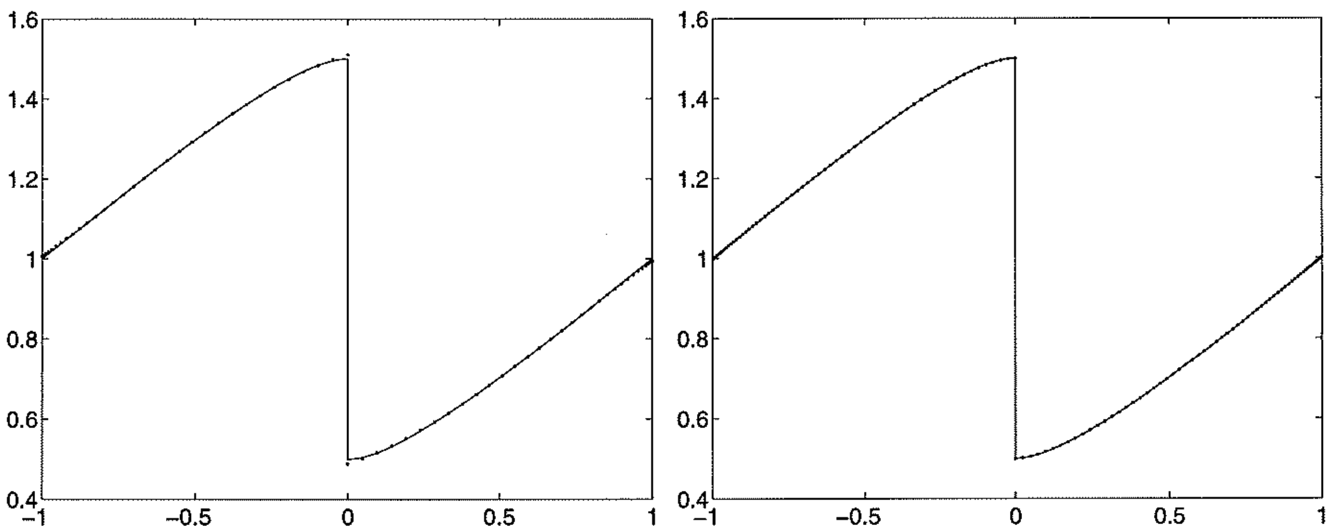

Figure 4.2: The SV solution in Figure 4.1 after the post-processing.

This is the example presented in Maday, Ould Kaber, and Tadmor [20]. We compute the same problem for the purpose of comparison. For time discretization, we use the forth-order Runge-Kutta scheme with time step $\Delta t = 10^{-5}$. After the Legendre pseudospectral viscosity solution is obtained at time $t = 1$, it is post-processed using the Gegenbauer reconstruction method to recover the accuracy . As described in Gottlieb and Shu [11], we expand u_N in the Gegenbauer series based on the Gegenbauer polynomials $C_n^\lambda(x)$, for $0 \le n \le m$ in the smooth regions $[-1, 0]$ and $[0, 1]$.

In Figure 4.1 (a) and (b), we show the numerical results of the Legendre viscosity method with $N = 64$ and $N = 128$, respectively. The parameters in the viscosity term are taken as $\varepsilon \simeq N^{-1}$ and $M \simeq N^{0.25}$. Figure 4.2 (a) and (b) are the corresponding results after the post-processing with $\lambda = m \simeq 0.05N$. It is clearly seen that the results given here enjoy better resolution than those reported in [20].

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